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VIABILITY AND ARBITRAGE UNDER KNIGHTIAN UNCERTAINTY[1]

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We reconsider the microeconomic foundations of financial economics under Knightian Uncertainty. We remove the (implicit) assumption of a common prior and base our analysis on a common order instead. Economic viability of asset prices and the absence of arbitrage are equivalent. We show how the different versions of the Efficient Market Hypothesis are related to the assumptions one is willing to impose on the common order. We also obtain a version of the Fundamental Theorem of Asset Pricing using the notion of *sublinear* pricing measures. Our approach unifies recent versions of the Fundamental Theorem under a common framework.

KEYWORDS: Viability, Knightian Uncertainty, No Arbitrage, Robust Finance.

1. INTRODUCTION

Recently, a large and increasing body of literature has focused on decisions, markets, and economic interactions under uncertainty. Frank Knight's pioneering work (Knight (1921)) distinguishes *risk* – a situation that allows for an objective probabilistic description – from *uncertainty* – a situation that cannot be modeled by a single probability distribution.

In this paper, we discuss the foundations of no–arbitrage pricing and its relation to economic equilibrium under Knightian Uncertainty.

Asset pricing models typically take a basic set of securities as given and determine the range of option prices that is consistent with the absence of arbitrage. From an economic point of view, it is crucial to know if modeling security prices directly is justified; an asset pricing model is called viable if its security prices can be thought of as (endogenous) equilibrium outcomes of a competitive economy.

Under risk, this question has been investigated in Harrison and Kreps' seminal work (Harrison and Kreps (1979)). Their approach is based on a common prior (or reference probability) that determines the null sets, the topology, and the order of the model. The common prior assumption is made in almost all asset pricing models.

In recent years, it has become clear that many of the standard financial models used in practice face Knightian uncertainty, the most salient examples being stochastic volatility, term structure and credit risk models. If Knightian uncertainty is recognized in these models, a reference measure need not exist, see, e.g.,

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Epstein and Ji (2013). Under Knightian uncertainty, we thus have to forego the assumption of a common prior.

We replace the common prior with a *common order* with respect to which agents' preferences are monotone. We thus assume that market participants share a common view of when one contract is better than another. This assumption is far weaker than the assumption of a common prior. In particular, it allows to include Knightian uncertainty.

Our main result shows that the absence of arbitrage and the (properly defined) economic viability of the model are equivalent. In equilibrium, there are no arbitrage opportunities; conversely, for arbitrage–free asset pricing models, it is possible to construct a heterogeneous agent economy such that the asset prices are equilibrium prices of that economy.

The main result is based on a number of other results that are of independent interest. To start with, in contrast to risk, it is no longer possible to characterize viability through the existence of a single linear pricing measure (or equivalent martingale measure). Instead, it is necessary to use a suitable *nonlinear* pricing expectation, that we call a sublinear martingale expectation. A sublinear expectation has the common properties of an expectation including monotonicity, preservation of constants, and positive homogeneity, yet it is no longer additive. Indeed, sublinear expectations can be represented as the supremum of a class of (linear) expectations, an operation that does not preserve linearity[1]. Nonlinear expectations arise naturally for *preferences* in decision–theoretic models of ambiguity–averse preferences (Gilboa and Schmeidler (1989), Maccheroni, Marinacci, and Rustichini (2006)). It is interesting to see that a similar nonlinearity arises here for the *pricing* functional[2].

The common order shapes equilibrium asset prices. We study various common orders and how they are related to versions of the *Efficient Market Hypothesis*. The original (strong) version of the Efficient Market Hypothesis (Fama (1970)) states that properly discounted expected returns of assets are equal under the common prior. We show that the original Efficient Market Hypothesis holds true under the very strong assumption that the common order is induced by expected payoffs under a common prior, i.e. when agents' preferences are monotone with respect to expected payoffs under a common prior.

When the common order is given by the almost sure ordering under a given prior, we obtain the weak form of the Efficient Market Hypothesis; it states that expected returns are equal under some probability measure that is *equivalent* to the common prior. This order allows agents to use risk–adjusted probabilities (or stochastic discount factors based on the marginal rate of substitution)

[1] In economics, such a representation theorem appears first in Gilboa and Schmeidler (1989). Sublinear expectations also arise in Robust Statistics, compare Huber (1981), and they play a fundamental role in theory of risk measures in Finance, see Artzner, Delbaen, Eber, and Heath (1999) and Föllmer and Schied (2011).

[2] Beißner and Riedel (2016)) develop a general theory of equilibria with such nonlinear prices under Knightian uncertainty.

to price financial claims (compare Cochrane (2001) and Rigotti and Shannon (2005)). We thus obtain the classic version of the Fundamental Theorem of Asset Pricing (Harrison and Kreps (1979); Harrison and Pliska (1981); Duffie and Huang (1985); Dalang, Morton, and Willinger (1990); Delbaen and Schachermayer (1998)).

Under conditions of Knightian uncertainty, our main results lead to new versions of the Efficient Market Hypothesis.

If the market orders payoffs by considering a family of expected payoffs for a set of possible priors in the spirit of the incomplete expected utility model of Bewley (2002), we obtain a generalization of the strong Efficient Market Hypothesis under Knightian uncertainty. In viable markets, a sublinear martingale expectation exists that is linear on the subspace of mean–ambiguity–free payoffs (securities that have the same expectation under all priors). For the market restricted to this subspace we still get the classical strong form of the EMH. For mean–ambiguous securities, this conclusion need not be true.

Under Knightian uncertainty, one is naturally led to study sets of probability measures that are not dominated by one common prior (Epstein and Ji (2014), Vorbrink (2014), e.g.). It is then natural to take the quasi–sure ordering as the common order of the market. A claim dominates quasi–surely another claim if it is almost surely greater or equal under all considered probability measures. If the class of probability measures describing Knightian uncertainty is not dominated by a single probability measure, the quasi–sure ordering is more incomplete than any almost sure ordering. If we merely assume that the common ordering is the *quasi–sure* order induced by a set of priors, we obtain a weak version of the efficient market hypothesis under Knightian uncertainty. Bouchard and Nutz (2015) and Burzoni, Frittelli, and Maggis (2016) discuss the absence of arbitrage in such a setting. We complement their analysis by giving a precise economic equilibrium foundation.

We also study the consequences for asset pricing when the common order is induced by smooth ambiguity preferences as introduced by Klibanoff, Marinacci, and Mukerji (2005). Knightian uncertainty is modeled by a second-order prior over the class of multiple priors. We show that in this case, one can identify a stochastic discount factor that is used for asset pricing. The EMH holds true in the sense that *average* discounted asset prices are equal where the average is taken over the expected returns under all priors with the help of the second-order prior.

The above examples show that asset pricing under Knightian uncertainty (or a common order) leads to weaker conclusions for expected returns. In this sense, our paper shows that the EMH has to be interpreted in a careful way. In the early days, the EMH has frequently been identified with the *Random Walk Hypothesis*, the claim that asset returns are independent and identically distributed, maybe even normally distributed. It has been amply demonstrated in empirical research[3]

[3]Compare Lo and MacKinlay (2002) and Akerlof and Shiller (2010) for empirical results on

that the random walk hypothesis is rejected by data. Even the weak form of the EMH based on a common prior does allow for time-varying and correlated expected returns, though. Our paper shows that without a common prior, an even wider variety of "expected returns" is consistent with equilibrium and the absence of arbitrage[4].

We conclude this introduction by discussing the relation to some further literature.

The relation of arbitrage and viability has been discussed in various contexts. For example, Jouini and Kallal (1995) and Jouini and Kallal (1999) discuss models with transaction cost and other frictions. Werner (1987) and Dana, Le Van, and Magnien (1999) discuss the absence of arbitrage in its relation to equilibrium when a finite set of agents is fixed a priori. Our approach is based on the notion of a common weak order. Cassese (2017) considers the absence of arbitrage in an order-theoretic framework derived from coherent risk measures. It is shown that a price system is coherent if and only if pricing by expectation is possible.

Knightian uncertainty is also closely related to robustness concerns that play an important role in macroeconomics. Rational expectations models have recently been extended to take the fear of model misspecification into account see, e.g., Hansen and Sargent (2001, 2008). In this literature, tools from robust control are adapted to analyze how agents should cope with fear of model misspecification, that is modelled by putting a penalty term on the discrepancy between the agent's and the true model. Such fear of model misspecification is a special case of ambiguity aversion. Our analysis thus sheds also new light on the foundations of asset pricing in robust macroeconomic models.

Riedel (2015) works in a setting of complete Knightian uncertainty under suitable topological assumptions. Absence of arbitrage is equivalent to the existence of full support martingale measures in this context. We show that one can obtain this result from our main theorem when all agents use the pointwise order and consider contracts as relevant if they are nonnegative and positive in some state of the world. Several notions given in robust finance are also covered in our setting by properly choosing the set of relevant sets. Indeed, the definition given in the initial paper of Acciaio, Beiglböck, Penkner, and Schachermayer (2016) uses a small class of relevant contracts and Bartl, Cheridito, Kupper, and Tangpi (2017) considers only the contracts that are uniformly positive as relevant. A comparative summary of these studies is given in Subsection 4 below. Hence our approach provides a unification of different notions in this context as well.

The paper is set up as follows. Section 2 describes the model and the two main contributions of this paper in concise form. The assumptions of our model and their relation to previous modelling is discussed in Section 3. Section 4 derives various classic and new forms of the Efficient Market Hypothesis. Section 5 is devoted to the proofs of the main theorems. The appendix contains a detailed

deviations from the random walk hypothesis. Malkiel (2003) reviews the EMH and its critics.

[4]The exact determination of the impact of Knightian uncertainty on asset returns is beyond the scope of this study.

study of general discrete time markets when the space of contingent payoffs consists of bounded measurable functions. It also discusses further extensions as, e.g., the equivalence of absence of arbitrage and absence of free lunches with vanishing risk, or the question if an optimal superhedge for a given claim exists.

2. THE MODEL AND THE TWO MAIN THEOREMS

A non-empty set Ω contains the states of the world; the σ–field $\mathcal{F}$ on Ω collects the possible events.

The commodity space (of contingent claims) $\mathcal{H}$ is a vector space of $\mathcal{F}$-measurable real-valued functions containing all constant functions. We will use the symbol c both for real numbers as well as for constant functions. $\mathcal{H}$ is endowed with a metrizable topology τ and a pre-order $\leq$ that are compatible with the vector space operations.

The pre-order $\leq$ is interpreted as the common order of all agents in the economy; it replaces the assumption of a common prior. We assume throughout that the preorder $\leq$ is consistent with the order on the reals for constant functions and with the pointwise order for measurable functions. A consumption plan $Z \in \mathcal{H}$ is negligible if we have $0 \leq Z$ and $Z \leq 0$. $C \in \mathcal{H}$ is nonnegative if $0 \leq C$ and positive if in addition not $C \leq 0$. We denote by $\mathcal{Z}$, $\mathcal{P}$ and $\mathcal{P}^+$ the class of negligible, nonnegative and positive contingent claims, respectively.

We also introduce a class of *relevant contingent claims* $\mathcal{R}$, a convex subset of $\mathcal{P}^+$. We think of the relevant claims as being the contracts that allow to finance desirable consumption plans. In most examples below, we will take $\mathcal{R} = \mathcal{P}^+$. The introduction of $\mathcal{R}$ allows to subsume various notions of arbitrage that were discussed in the literature.

The financial market is modelled by the set of *net trades* $\mathcal{I} \subset \mathcal{H}$, a convex cone containing 0. $\mathcal{I}$ is the set of payoffs that the agents can achieve from zero initial wealth by trading in the financial market.

An *agent* in this economy is described by a preference relation (i.e. a complete and transitive binary relation) on $\mathcal{H}$ that is

- *weakly monotone with respect to* $\leq$, i.e. $X \leq Y$ implies $X \preceq Y$ for every $X, Y \in \mathcal{H}$;
- *convex*, i.e. the upper contour sets $\{Z \in \mathcal{H} : Z \succeq X\}$ are convex;
- τ*-lower semi–continuous*, i.e. for every sequence $\{X_n\}_{n=1}^{\infty} \subset \mathcal{H}$ converging to X in τ with $X_n \preceq Y$ for $n \in \mathbb{N}$, we have $X \preceq Y$.

The set of all agents is denoted by $\mathcal{A}$.

A financial market $(\mathcal{H}, \tau, \leq, \mathcal{I}, \mathcal{R})$ is *viable* if there is a family of agents $\{\preceq_a\}_{a \in A} \subset \mathcal{A}$ such that

- 0 is optimal for each agent $a \in A$, i.e.

$$\forall \ell \in \mathcal{I} \quad \ell \preceq_a 0, \tag{2.1}$$

- for every relevant contract $R \in \mathcal{R}$ there exists an agent $a \in A$ such that

$$0 \prec_a R\,. \tag{2.2}$$

We say that $\{\preceq_a\}_{a\in A}$ *supports the financial market* $(\mathcal{H}, \tau, \leq, \mathcal{I}, \mathcal{R})$.

A net trade $\ell \in \mathcal{I}$ is an arbitrage if there exists a relevant contract $R^* \in \mathcal{R}$ such that $\ell \geq R^*$. More generally, a sequence of net trades $\{\ell^n\}_{n=1}^{\infty} \subset \mathcal{I}$ is a *free lunch with vanishing risk* if there exists a relevant contract $R^* \in \mathcal{R}$ and a sequence $\{e_n\}_{n=1}^{\infty} \subset \mathcal{H}$ of nonnegative consumption plans with $e_n \xrightarrow{\tau} 0$ satisfying $e_n + \ell_n \geq R^*$ for all $n \in \mathbb{N}$. We say that the financial market is *strongly free of arbitrage* if there is no free lunch with vanishing risk.

Our first main theorem establishes the equivalence of viability and absence of arbitrage.

THEOREM 2.1 *A financial market is strongly free of arbitrage if and only if it is viable.*

If there is a common prior $\mathbb{P}$ on $(\Omega, \mathcal{F})$, a financial market is viable if and only if there exists a linear pricing measure in the form of a risk-neutral probability measure $\mathbb{P}^*$ that is equivalent to $\mathbb{P}$. In the absence of a common prior, we have to work with a more general, sublinear notion of pricing. A functional

$$\mathcal{E} : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$$

is a *sublinear expectation* if it is monotone with respect to $\leq$, translation-invariant, i.e. $\mathcal{E}(X + c) = \mathcal{E}(X) + c$ for all constant contracts $c \in \mathcal{H}$ and $X \in \mathcal{H}$, and sublinear, i.e. for all $X, Y \in \mathcal{H}$ and $\lambda > 0$, we have $\mathcal{E}(X + Y) \leq \mathcal{E}(X) + \mathcal{E}(Y)$ and $\mathcal{E}(\lambda X) = \lambda\mathcal{E}(X)$. $\mathcal{E}$ has *full support* if $\mathcal{E}(R) > 0$ for every $R \in \mathcal{R}$. Last not least, $\mathcal{E}$ has the *martingale property* if $\mathcal{E}(\ell) \leq 0$ for every $\ell \in \mathcal{I}$. We say in short that $\mathcal{E}$ is a sublinear martingale expectation with full support if all the previous properties are satisfied.

It is well known from decision theory that sublinear expectations can be written as upper expectations over a set of probability measures. In our more abstract framework, probability measures are replaced by suitably normalized functionals. We say that $\varphi \in \mathcal{H}'_+$[5] is a *martingale functional*[6] if it satisfies $\varphi(1) = 1$ (normalization) and $\varphi(\ell) \leq 0$ for all $\ell \in \mathcal{I}$. In the spirit of the probabilistic language, we call a linear functional absolutely continuous if it assigns the value zero to all negligible claims. We denote by $\mathcal{Q}_{ac}$ the set of absolutely continuous martingale functionals.

The notions that we introduced now allow us to state the general version of the fundamental theorem of asset pricing in our order-theoretic context.

THEOREM 2.2 (Fundamental Theorem of Asset Pricing) *The financial market is viable if and only if there exists a lower semi–continuous sublinear martingale expectation with full support.*

[5] $\mathcal{H}'$ is the topological dual of $\mathcal{H}$ and $\mathcal{H}'_+$ is the set of positive elements in $\mathcal{H}'$.

[6] In this generality the terminology *functional* is more appropriate. When the dual space $\mathcal{H}'$ can be identified with a space of measures, we will use the terminology *martingale measure*.

In this case, the set of absolutely continuous martingale functionals $\mathcal{Q}_{ac}$ is not empty and

$$\mathcal{E}_{\mathcal{Q}_{ac}}(X) := \sup_{\phi \in \mathcal{Q}_{ac}} \phi(X)$$

is the maximal lower semi–continuous sublinear martingale expectation with full support.

REMARK 2.3

1. In the above theorem, we call $\mathcal{E}_{\mathcal{Q}_{ac}}(X)$ maximal in the sense that any other lower semi–continuous sublinear martingale expectation with full support $\mathcal{E}$ satisfies $\mathcal{E}(X) \leq \mathcal{E}_{\mathcal{Q}_{ac}}(X)$ for all $X \in \mathcal{H}$.
2. Under nonlinear expectations, one has to distinguish martingales from symmetric martingales; a symmetric martingale has the property that the process itself and its negative are martingales. When the set of net trades $\mathcal{I}$ is a linear space as in the case of frictionless markets, a net trade ℓ and its negative $-\ell$ belong to $\mathcal{I}$. In this case, sublinearity and the condition $\mathcal{E}_{\mathcal{Q}_{ac}}(\ell) \leq 0$ for all $\ell \in \mathcal{I}$ imply $\mathcal{E}_{\mathcal{Q}_{ac}}(\ell) = 0$ for all net trades $\ell \in \mathcal{I}$. Thus, the net trades ℓ are symmetric $\mathcal{E}_{\mathcal{Q}_{ac}}$-martingales.

3. DISCUSSION OF THE MODEL

Common order instead of common prior

Under risk it is natural to assume that all market participants consider a payoff X better than another payoff Y if X is greater or equal Y almost surely under a certain reference measure $\mathbb{P}$. As we aim to discuss financial markets under Knightian uncertainty, we forego any explicit or implicit assumption of a common prior $\mathbb{P}$. Instead, we base our analysis on a *common order* $\leq$, a far weaker assumption. As the preferences of agents are monotone with respect to the common order, we assume that market participants share a common view of when one contract is better than another[7].

Note that this commonly shared pre-order is incomplete, usually. One example is the pointwise order of contracts; indeed, as we assume that our order $\leq$ is consistent with the pointwise order, it follows that agents' preferences are monotone with respect to the pointwise order. If more is commonly known about the environment, one might want to impose stronger assumptions on the pre-order. For example, the common order could be generated by the expected value of payoffs under a common prior $\mathbb{P}$, a situation that corresponds to the "risk–neutral" world as we show below. Or we can use the almost–sure ordering under that prior, the standard (implicit) assumption in finance models. In the situation of a *common set of priors*, the so–called quasi–sure ordering is a natural choice that

[7] In a multiple prior setting, one is naturally led to the distinction of objective and subjective rationality discussed by Gilboa, Maccheroni, Marinacci, and Schmeidler (2010); in their paper, the common order is given by Bewley's incomplete expected utility model whereas the single agent has a complete multiple prior utility function.

is induced by the family of (potentially non–equivalent) priors. More examples are discussed below in Section 4.

It might be interesting to note that it is possible to *derive* the common pre-order $\leq$ from a given set of admissible agents by using the *uniform* order derived from a set of preference relations $\hat{\mathcal{A}}$ which are convex and τ-lower semi-continuous. Let

$$\mathcal{Z}_{\preceq} := \{Z \in \mathcal{H} : X \preceq Z + X \preceq X,\ \forall\ X \in \mathcal{H}\},$$

be the set of negligible (or null) contracts for the preference relation $\preceq \in \hat{\mathcal{A}}$. We call $\mathcal{Z}_{uni} := \bigcap_{\preceq \in \hat{\mathcal{A}}} \mathcal{Z}_{\preceq}$ the set of *unanimously negligible* contracts. We define the uniform pre-order $\leq_{uni}$ on $\mathcal{H}$ by setting $X \leq_{uni} Y$ if and only if there exists $Z \in \mathcal{Z}_{uni}$ such that $X(\omega) \leq Y(\omega) + Z(\omega)$ for all $\omega \in \Omega$. $(\mathcal{H}, \leq_{uni})$ is then a pre-ordered vector space[8], and agents' preferences are monotone with respect to the uniform pre-order.

Relevant Claims

Our notion of arbitrage uses the concept of relevant claims, a subset of the set of positive claims. The interpretation is that a claim is relevant if some agent views it as a desirable gain without any downside risk. For most models, it is perfectly natural to identify the set of relevant claims with the set of positive claims, and the reader is invited to make this identification at first reading. In fact, if we think of the traded claims as consumption bundles, then for each contingent consumption plan R that is non-zero, there will be an agent who strictly prefers R to zero. This class of contracts describes the direction of strict monotonicity that are identified by the market participants.

In some finance applications, it makes sense to work with a smaller set of relevant claims. For example, if some positive claims cannot be liquidated without cost, agents would not consider them as free lunches. When turning complex derivatives into cash involves high transaction costs, it is reasonable to consider as relevant only a restricted class of positive claims, possibly only cash.

Agents

We aim to clarify the relation between arbitrage–free financial markets and equilibrium. For a given arbitrage–free market, we ask if it can be reasonably thought of as an equilibrium outcome in some economy, and vice-versa. Following Harrison and Kreps (1979), we think of agents about whom some things are known, without assuming that we know exactly their preferences or their number. We thus impose a number of properties on preferences that are standard in economics. In particular, the preferences are monotone with respect to

[8] In the same spirit, one could define a pre-order $X \leq_u' Y \Leftrightarrow X \preceq Y$ for all $\preceq \in \mathcal{A}$. In general this will not define a pre-ordered vector space $(\mathcal{H}, \leq')$. The analysis of the paper carries over with minor modifications.

the common weak pre-order; this assumption is a mere tautology if we interpret the pre-order as a *common* pre-order. Moreover, we impose some weak form of continuity with respect to some topology. Convexity reflects a preference for diversification.

The Financial Market

We model the financial market in a rather reduced form with the help of the convex cone $\mathcal{I}$. This abstract approach is sufficient for our purpose of discussing the relation of arbitrage and viability. In the next example, we show how the usual models of static and dynamic trading are embedded.

EXAMPLE 3.1 We consider four markets with increasing complexity.

1. In a one period setting with finitely many states $\Omega = \{1, \dots, N\}$, a financial market with $J+1$ securities can be described by its initial prices $x_j \geq 0, j = 0, \dots, J$ and a $(J+1) \times N$–payoff matrix F, compare LeRoy and Werner (2014). A portfolio $\bar{H} = (H_0, \dots, H_J) \in \mathbb{R}^{J+1}$ has the payoff $\bar{H}F = \left(\sum_{j=0}^{J} H_j F_{j\omega}\right)_{\omega=1,\dots,N}$; its initial cost satisfies $H \cdot x = \sum_{j=0}^{J} H_j x_j$. If the zeroth asset is riskless with a price $x_0 = 1$ and pays off 1 in all states of the world, then a net trade with zero initial cost can be expressed in terms of the portfolio of risky assets $H = (H_1, \dots, H_J) \in \mathbb{R}^J$ and the return matrix $\hat{F} = (F_{j\omega} - x_j)_{j=1,\dots,J,\omega=1,\dots,N}$. $\mathcal{I}$ is given by the image of the $J \times N$ return matrix $\hat{F}$, i.e.

$$\mathcal{I} = \{H\hat{F} : H \in \mathbb{R}^J\}.$$

2. Our model includes the case of finitely many trading periods. Let $\mathbb{F} := (\mathcal{F}_t)_{t=0}^T$ be a filtration on $(\Omega, \mathcal{F})$ and $S = (S_t)_{t=0}^T$ be an adapted stochastic process with values in $\mathbb{R}_+^J$ for some $J \geq 1$; S models the uncertain assets. We assume that a riskless bond with interest rate zero is also given. Then, the set of net trades can be described by the gains from trade processes: $\ell \in \mathcal{H}$ is in $\mathcal{I}$ provided that there exists predictable integrands $H_t \in \left(\mathcal{L}^0(\Omega, \mathcal{F}_{t-1})\right)^J$ for $t = 1, \dots, T$ such that,

$$\ell = (H \cdot S)_T := \sum_{t=1}^{T} H_t \cdot \Delta S_t, \quad \text{where} \quad \Delta S_t := (S_t - S_{t-1}).$$

In the frictionless case, the set of net trades is a subspace of $\mathcal{H}$. In general, one might impose restrictions on the set of admissible trading strategies. For example, one might exclude short-selling of risky assets. More generally, trading strategies might belong to a suitably defined convex cone; in these cases, the marketed subspace $\mathcal{I}$ is a convex cone, too.

3. In Harrison and Kreps (1979), the market is described by a marketed space $M \subset \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ and a (continuous) linear functional π on M. In this case, $\mathcal{I}$ is

the kernel of the price system, i.e.

$$\mathcal{I} = \{X \in M : \pi(X) = 0\}.$$

4. In continuous time, the set of net trades consists of stochastic integrals of the form

$$\mathcal{I} = \left\{ \int_0^T \theta_u \cdot dS_u \ : \ \theta \in \mathcal{A}_{adm} \right\},$$

for a suitable set of *admissible* strategies $\mathcal{A}_{adm}$. There are several possible choices of such a set. When the stock price process S is a semi-martingale one example of $\mathcal{A}_{adm}$ is the set of all S-integrable, predictable processes whose integral is bounded from below.[9] Other natural choices for $\mathcal{A}_{adm}$ would consist of simple integrands only; when S is a continuous process and $\mathcal{A}_{adm}$ is the set of process with finite variation then the above integral can be defined through integration by parts (see Dolinsky and Soner (2014a, 2015)).

Viability

Knightian uncertainty requires a careful adaptation of the notion of economic viability. Harrison and Kreps (1979) and Kreps (1981) show that the absence of arbitrage is equivalent to a *representative agent* equilibrium. Under Knightian uncertainty, such a single agent construction is generally impossible as the next example shows[10].

EXAMPLE 3.2 Suppose that we have a situation of *full Knightian uncertainty*[11]. Take $\Omega = [0, 1]$, $\mathcal{F}$ the Borel sets, let $\mathcal{H}$ be the set of all bounded, measurable functions on Ω. Let the common pre-order be given by the pointwise order. Take the relevant contracts $\mathcal{R} = \mathcal{P}^+$. A bounded measurable function is thus relevant if it is nonnegative everywhere and is strictly positive for at least one $\omega \in \Omega$. Suppose that the set of net trades is given by multiples of $\ell(\omega) = 1_{(0,1]}(\omega)$. Note that the contract ℓ itself is both relevant and achievable with zero wealth (a net trade), hence is an arbitrage. Consequently, by Theorem 2.1, this market is not viable in the sense of Section 2.

[9] In continuous time, to avoid doubling strategies a lower bound (maybe more general than above) has to be imposed on the stochastic integrals. In such cases, the set $\mathcal{I}$ is not a linear space.

[10] Compare Example 3 in Kreps (1981).

[11] The chosen example is only for illustrative purposes. The arguments that follow carry over to more sophisticated models of Knightian uncertainty that are described by a non-dominated set of priors $\mathcal{M}$, as in Epstein and Ji (2013), Vorbrink (2014), or Beissner and Denis (2018). In that setting, Ω is the set of continuous functions on $[0, \infty)$ that represent the possible trajectories of financial prices. In the market, there is uncertainty about the true volatility of the price process, yet there is a unanimous agreement that it lies in a certain interval $[\underline{\sigma}, \overline{\sigma}]$. The class of probability measures $\{P_\sigma\}_{\sigma \in [\underline{\sigma}, \overline{\sigma}]}$ such that the price process has volatility σ under P_σ defines a non-dominated set of priors.

Consider the Gilboa–Schmeidler utility function

$$U(X) = \inf_{\omega \in \Omega} u(\omega + X(\omega))$$

for some strictly monotone, strictly concave function $u : \mathbb{R} \to \mathbb{R}$. This particular agent weakly prefers the zero trade to any multiple of ℓ. Indeed, we have $U(0) = u(0) \geq U(\lambda \ell)$ for all $\lambda \in \mathbb{R}$. For positive λ, the agent cares only about the worst state $\omega = 0$ in which the claim $\lambda \ell$ has a payoff of zero. The agent does not want to short-sell the claim either because he would then lose money in each state of the world except at $\omega = 0$.

For this single agent economy, the equilibrium condition (2.1) is satisfied as the zero trade is optimal; however, the relevance condition (2.2) does not hold true because the agent is indifferent between 0 and the relevant contract $1_{(0,1]}(\omega)$. Condition (2.2) excludes situations in which the agents of the economy do not desire relevant payoffs. Note that the above agent is not really "representing" the market $(\mathcal{H}, \tau, \leq, \mathcal{I}, \mathcal{R})$ because he does not desire the relevant contract ℓ.

The notion of economic equilibrium does not require the existence of a representative agent, of course. It is natural, and – in fact – closer to reality, to allow for a sufficiently rich set of heterogeneous agents in an economy. Our above definition of viability thus allows for an economy populated by heterogeneous agents.

As the above example has shown, under Knightian uncertainty, reasonable preferences need not be strictly monotone with respect to every positive (or relevant) trade nor rule out arbitrage. One might just aim to replace the above Gilboa–Schmeidler utility function by a *strictly monotone* utility. While this approach is feasible in a probabilistic setting, it does not work under Knightian uncertainty because strictly monotone utility functions do not exist, in general[12]. In our definition of viability, we thus generalize the definition by Harrison and Kreps by replacing the assumption of strict monotonicity with the property (2.2) that requires that each relevant contract is desirable for *some* agent in equilibrium[13].

Our notion of equilibrium does not model endowments explicitly as we assume that the zero trade is optimal for each agent. Let us explain why this reduced approach comes without loss of generality. In general, an agent is given by a

[12]On the mathematical side, this is related to the absence of strictly positive linear functionals. For example, it is well known that there is no linear functional on the space of bounded measurable functions on $[0, 1]$ that assigns a strictly positive value to $\mathbf{1}_{\{\omega\}}$ for every $\omega \in \Omega$ (Aliprantis and Border (1999)). It is thus impossible to construct a probability measure that assigns a positive mass to a continuum of singletons. This fact carries over to more complex models of Knightian uncertainty as the uncertain volatility model.

[13]Note that the definition with $\mathcal{R} = \mathcal{P}^+$ is equivalent to the one given by Harrison and Kreps in the probabilistic setup. At the same time, it allows to overcome the problem identified in the previous Example 3.2 because the single agent economy of the example violates (2.2). Our definition is equivalent to the definition by Harrison and Kreps in the probabilistic setup because strictly monotone preferences satisfy (2.2).

preference relation $\succeq \in \mathcal{A}$ and an endowment $e \in \mathcal{H}$. Given the set of net trades, the agent chooses $\ell^* \in \mathcal{I}$ such that $e + \ell^* \succeq e + \ell$ for all $\ell \in \mathcal{I}$. By suitably modifying the preference relation, this can be reduced to the optimality of the zero trade at the zero endowment for a suitably modified preference relation. Let $X \succeq' Y$ if and only if $X + e + \ell^* \succeq Y + e + \ell^*$. It is easy to check that $\succeq'$ is also an admissible preference relation. For the new preference relation $\succeq'$, we then have $0 \succeq' \ell$ if and only if $e + \ell^* \succeq e + \ell^* + \ell$. As $\mathcal{I}$ is a cone, $\ell + \ell^* \in \mathcal{I}$, and we conclude that we have indeed $0 \succeq' \ell$ for all $\ell \in \mathcal{I}$.

Sublinear Expectations

Our fundamental theorem of asset pricing characterizes the absence of arbitrage with the help of a non–additive expectation $\mathcal{E}$. In decision theory, non–additive probabilities have a long history; Schmeidler (1989) introduces an extension of expected utility theory based on non–additive probabilities. The widely used max-min expected utility model of Gilboa and Schmeidler (1989) is another instance. If we define the subjective expectation of a payoff to be the minimal expected payoff over a class of priors, then the resulting notion of expectation has the common properties of an expectation like monotonicity, preservation of constants, but is no longer additive.

In our case, the non–additive expectation has a more objective than subjective flavor because it describes the pricing functional of the market. Whereas an additive probability measure is sufficient to characterize viable markets in models with a common prior, in general, such a construction is no longer feasible. Indeed, Harrison and Kreps (1979) prove that viability implies that the linear market pricing functional can be extended from the marketed subspace to a *strictly positive* linear functional on the whole space of contingent claims. Under Knightian uncertainty, however, the fact that strictly positive linear functionals *do not exist* is the rule rather than the exception (compare Footnote 12). We thus rely on a non–additive notion of expectation[14].

The pricing functional assigns a nonpositive value to all net trades; in this sense, net trades have the (super)martingale property under this expectation. If we assume for the sake of the discussion that the set of net trades is a linear subspace, then the pricing functional has to be additive over that subspace. As a consequence, the value of all net trades under the sublinear pricing expectation is zero. For contingent claims that lie outside the marketed subspace, the pricing operation of the market is sub–additive.

The following two examples illustrate the issue. We start with the simple case of complete financial markets within finite state spaces. Here, an additive probability is sufficient to characterise the absence of arbitrage, as is well known.

[14] Beissner and Riedel (2019) develop a general equilibrium model based on such non–additive pricing functionals.

EXAMPLE 3.3 (The atom of finance and complete markets) The basic one–step binomial model, that we like to call the atom of finance, consists of two states of the world, $\Omega = \{1, 2\}$. An element $X \in \mathcal{H}$ can be identified with a vector in $\mathbb{R}^2$. Let $\leq$ be the usual partial order of $\mathbb{R}^2$. Then $\mathcal{Z} = \{0\}$ and $X \in \mathcal{P}$ if and only if $X \geq_\Omega 0$. The relevant contracts are the positive ones, $\mathcal{R} = \mathcal{P}^+$.

There is a riskless asset B and a risky asset S. At time zero, both assets have value $B_0 = S_0 = 1$. The riskless asset yields $B_1 = 1 + r$ for an interest rate $r > -1$ at time one, whereas the risky asset takes the values u in state 1 and respectively d in state 2 with $u > d$.

We use the riskless asset B as numéraire. The discounted net return on the risky asset is $\hat{\ell} := S_1/(1 + r) - 1$. $\mathcal{I}$ is the linear space spanned by $\hat{\ell}$. There is no arbitrage if and only if the unique candidate for a full support martingale probability of state one

$$p^* = \frac{1 + r - d}{u - d}$$

belongs to $(0, 1)$ which is equivalent to $u > 1 + r > d$. p^* induces the unique martingale measure $\mathbb{P}^*$ with expectation

$$\mathbb{E}^*[X] = p^* X(1) + (1 - p^*) X_1(2)\,.$$

$\mathbb{P}^*$ is a linear measure with full support. The market is viable with $A = \{\preceq^*\}$, the preference relation given by the linear expectation $\mathbb{P}^*$, i.e. $X \preceq^* Y$ if and only if $\mathbb{E}^*[X] \leq \mathbb{E}^*[Y]$. Indeed, under this preference $\ell \sim^* 0$ for any $\ell \in \mathcal{I}$ and $X \prec^* X + R$ for any $X \in \mathcal{H}$ and $R \in \mathcal{P}^+$. In particular, any $\ell \in \mathcal{I}$ is an optimal portfolio and the market is viable.

The preceding analysis carries over to all finite Ω and complete financial markets.

We now turn to a somewhat artificial one period model with uncountably many states. It serves well the purpose to illustrate the need for sublinear expectations and thus stands in an exemplary way for more complex models involving continuous time and uncertain violatility, e.g.

EXAMPLE 3.4 (Highly incomplete one-period models) This example shows that sublinear expectations are necessary to characterize the absence of arbitrage under Knightian uncertainty and with incomplete markets.

Let $\Omega = [0, 1]$ and $\leq$ be the usual pointwise partial order. Payoffs X are bounded Borel measurable functions on Ω. As in the previous example we have $\mathcal{Z} = \{0\}$ and $X \in \mathcal{P}$ if and only if $X \geq_\Omega 0$. Let the relevant contracts be again $\mathcal{R} = \mathcal{P}^+$. Assume that there is a riskless asset with interest rate $r \geq 0$. Let the risky asset have the price $S_0 = 1$ at time 0 and assume it pays off $S_1(\omega) = 2\omega$ at time 1. As in the previous example, $\mathcal{I}$ is spanned by the net return $\hat{\ell} := S_1/(1 + r) - 1$.

There exist uncountably many martingale measures because any probability measure $\mathbb{Q}$ satisfying $\int_\Omega 2\omega\, \mathbb{Q}(d\omega) = 1 + r$ is a martingale measure. Denote by $\mathcal{Q}_{ac}$ the set of all martingale measures.

No single martingale measure is sufficient to characterize the absence of arbitrage because there is no single linear martingale probability measure with full support. Indeed, such a measure would have to assign a non-zero value to every point in Ω, an impossibility for uncountable Ω. Hence, the equivalence "no arbitrage" to "there is a martingale measure with some monotonicity property" does not hold true if one insists on having a linear martingale measure. Instead, one needs to work with the nonlinear expectation

$$\mathcal{E}(X) := \sup_{\mathbb{Q} \in \mathcal{Q}_{ac}} \mathbb{E}_{\mathbb{Q}}[X]$$

for $X \in \mathcal{H}$. We claim that $\mathcal{E}$ has full support and characterizes the absence of arbitrage in the sense of Theorem 2.1 and Theorem 2.2.

To see that $\mathcal{E}$ has full support, note that $R \in \mathcal{P}^+$ if and only if $R \geq 0$ and there is $\omega^* \in \Omega$ so that $R(\omega^*) > 0$. Define $\mathbb{Q}^*$ by

$$\mathbb{Q}^* := \frac{1}{2}\left(\delta_{\{\omega^*\}} + \delta_{\{1-\omega^*\}}\right).$$

Then $\mathbb{Q}^* \in \mathcal{Q}_{ac}$, and we have

$$\mathcal{E}(R) \geq \mathbb{E}_{\mathbb{Q}^*}[R] = \frac{1}{2}R(\omega^*) + \frac{1}{2}R(1-\omega^*) > 0 = \mathcal{E}(0).$$

This example shows that the heterogeneity of agents needed in equilibrium to support an arbitrage-free financial market and the necessity to allow for sublinear expectations are two complementary faces of the same issue.

4. THE EFFICIENT MARKET HYPOTHESIS AND ROBUST FINANCE

The Efficient Market Hypothesis (EMH) plays a fundamental role in the history of Financial Economics. Fama (1970) suggests that expected returns of all securities are equal to the safe return of a suitable bond. This conjecture of the financial market's being a "fair game" dates back to Bachelier (1900) and was rediscovered by Paul Samuelson (1965; 1973).

Our framework allows for a discussion of the various forms of the EMH from a general point of view. We show that the EMH is a result of the strength of assumptions one is willing to make on the common order of the market. If we are convinced that agents' preferences are monotone with respect to expected payoffs (respectively, the almost sure ordering) under a common prior, we obtain the strong (respectively, weak) form of the EMH. If we are not willing to make such a strong assumption on agents' probabilistic sophistication, weaker Knightian analogs of the EMH result.

Throughout this section, let us assume that we have a frictionless one–period or discrete–time multiple period financial market as in Example 3.1, 1. and 2. In particular, the set of net trades $\mathcal{I}$ is a subspace of $\mathcal{H}$.

4.1. *Strong Efficient Market Hypothesis under Risk*

In its original version, the efficient market hypothesis postulates that the "real world probability" or historical measure $\mathbb{P}$ is itself a martingale measure. We can reach this conclusion if the common order is given by the expectation under a common prior.

Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$. Set $\mathcal{H} = \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Let the common order by given by $X \leq Y$ if and only if the expected payoffs under the common prior $\mathbb{P}$ satisfy

$$\mathbb{E}_{\mathbb{P}}[X] \leq \mathbb{E}_{\mathbb{P}}[Y]. \tag{4.1}$$

In this case, negligible contracts coincide with the contracts with mean zero under $\mathbb{P}$. Moreover, $X \in \mathcal{P}$ if $\mathbb{E}_{\mathbb{P}}[X] \geq 0$. We take $\mathcal{R} = \mathcal{P}_+$.

PROPOSITION 4.1 *Under the assumptions of this subsection, the financial market is viable if and only if the common prior $\mathbb{P}$ is a martingale measure. In this case, $\mathbb{P}$ is the unique martingale measure.*

PROOF: Note that the common order as given by (4.1) is complete. If $\mathbb{P}$ is a martingale measure, the common order $\leq$ itself defines a linear preference relation under which the market is viable with $A = \{\leq\}$.

On the other hand if the market is viable, Theorem 2.2 ensure that there exists a sublinear martingale expectation with full support. By the Riesz duality theorem, a martingale functional $\phi \in \mathcal{Q}_{ac}$ can be identified with a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$. It is absolutely continuous (in our sense defined above) if and only if it assigns the value 0 to all negligible claims. As a consequence, we have $E_{\mathbb{Q}}[X] = 0$ whenever $E_{\mathbb{P}}[X] = 0$. Then $\mathbb{Q} = \mathbb{P}$ follows[15]. *Q.E.D.*

Hence the only absolutely continuous martingale measure is the common prior itself. As a consequence, all traded assets have zero net expected return under the common prior. A financial market is thus viable if and only if the strong form of the expectations hypothesis holds true.

4.2. *Weak Efficient Market Hypothesis under Risk*

In its weak form, the efficient market hypothesis states that expected returns are equal under some (pricing) probability measure $\mathbb{P}^*$ that is equivalent to the common prior (or "real world" probability) $\mathbb{P}$. This hypothesis can be derived in our framework as follows.

Let $\mathbb{P}$ be a probability on $(\Omega, \mathcal{F})$ and $\mathcal{H} = \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. In this example, the common order is given by the almost surely order under the common prior $\mathbb{P}$, i.e.,

$$X \leq Y \quad \Leftrightarrow \quad \mathbb{P}(X \leq Y) = 1.$$

[15] If $\mathbb{Q} \neq \mathbb{P}$, there is an event $A \in \mathcal{F}$ with $\mathbb{Q}(A) < \mathbb{P}(A)$. Set $X = 1_A - \mathbb{P}(A)$. Then $0 = E_{\mathbb{P}}[X] > \mathbb{Q}(A) - \mathbb{P}(A) = E_{\mathbb{Q}}[X]$.

A payoff is negligible if it vanishes $\mathbb{P}$–almost surely and is positive if it is $\mathbb{P}$–almost surely nonnegative. The typical choice for relevant contracts $\mathcal{R}$ are the $\mathbb{P}$–almost surely nonnegative payoffs that are strictly positive with positive $\mathbb{P}$–probability[16]

$$\mathcal{R} = \left\{ R \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})_+ \ : \ \mathbb{P}(R > 0) > 0 \right\}.$$

A functional $\phi \in \mathcal{H}'_+$ is an absolutely continuous martingale functional if and only if it can be identified with a probability measure $\mathbb{Q}$ that is absolutely continuous with respect to $\mathbb{P}$ and if all net trades have expectation zero under ϕ. In other words, discounted asset prices are $\mathbb{Q}$-martingales. We thus obtain a version of the Fundamental Theorem of Asset Pricing under risk, similar to Harrison and Kreps (1979) and Dalang, Morton, and Willinger (1990).

PROPOSITION 4.2 *Under the assumptions of this subsection, the financial market is viable if and only if there is a martingale measure $\mathbb{Q}$ that has a bounded density with respect to $\mathbb{P}$.*

PROOF: If $\mathbb{P}^*$ is a martingale measure equivalent to $\mathbb{P}$, define $X \preceq^* Y$ if and only if $\mathbb{E}^{\mathbb{P}^*}[X] \leq \mathbb{E}^{\mathbb{P}^*}[Y]$. Then the market is viable with $A = \{\preceq^*\}$.

If the market is viable, Theorem 2.2 ensures that there exists a sublinear martingale expectation with full support. By the Riesz duality theorem, a martingale functional $\phi \in \mathcal{Q}_{ac}$ can be identified with a probability measure $\mathbb{Q}_\phi$ that is absolutely continuous with respect to $\mathbb{P}$, has a bounded density with respect to $\mathbb{P}$, and all net trades have zero expectation zero under $\mathbb{Q}_\phi$. In other words, discounted asset prices are $\mathbb{Q}_\phi$-martingales. From the full support property, the family $\{\mathbb{Q}_\phi\}_{\phi \in \mathcal{Q}_{ac}}$ has the same null sets as $\mathbb{P}$. By the Halmos-Savage Theorem, there exists an equivalent martingale measure $\mathbb{P}^*$. *Q.E.D.*

4.3. *The EMH under Knightian Uncertainty*

We turn our attention to the EMH under Knightian uncertainty. We consider first the case when the common order is derived from a common set of priors, inspired by the multiple prior approach in decision theory (Bewley (2002); Gilboa and Schmeidler (1989)). We then discuss a second-order Bayesian approach that is inspired by the smooth ambiguity model (Klibanoff, Marinacci, and Mukerji (2005)).

[16] Note that it is possible to derive these sets of positive and relevant contracts from the assumption that preferences in $\mathcal{A}$ consist of risk averse von Neumann-Morgenstern expected utility maximizers with strictly increasing Bernoulli utility function. Define $X \preceq Y$ if and only if $\mathbb{E}^{\mathbb{P}}[U(X)] \leq \mathbb{E}^{\mathbb{P}}[U(Y)]$ for all strictly increasing and concave real functions U. It is well known that this order coincides with second order stochastic dominance under $\mathbb{P}$. A random variable Y dominates 0 in the sense of second order stochastic dominance if and only if it is $\mathbb{P}$–almost surely nonnegative. Moreover, the set of relevant contracts $\mathcal{R}$ corresponds to the set of contracts that are uniformly desirable for all agents.

4.3.1. *Strong Efficient Market Hypothesis under Knightian Uncertainty*

We consider a generalization of the original EMH to Knightian uncertainty that shares a certain analogy with Bewley's incomplete expected utility model (Bewley (2002)) and Gilboa and Schmeidler's maxmin expected utility (Gilboa and Schmeidler (1989))[17]. Agents might have different subjective perceptions, but they share a common set of priors $\mathcal{M}$. Their preferences are weakly monotone with respect to the uniform order induced by expectations over the set of priors.

More formally, let Ω be a metric space and $\mathcal{M}$ be a convex, weak*-closed set of common priors on $(\Omega, \mathcal{F})$. Define a semi-norm

$$\|X\|_{\mathcal{M}} := \sup_{\mathbb{P}\in\mathcal{M}} \mathbb{E}_{\mathbb{P}}|X|.$$

Let $\mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$ be the closure of continuous and bounded functions on Ω under the semi-norm $\|\cdot\|_{\mathcal{M}}$. If we identify the functions which are $\mathbb{P}$-almost surely equal for every $\mathbb{P} \in \mathcal{M}$, then $\mathcal{H} = \mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$ is a Banach space. Furthermore, the topological dual of $\mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$ can be identified with probability measures that admit a bounded density with respect to some measure in $\mathcal{M}$, compare Bion-Nadal, Kervarec, et al. (2012); Beissner and Denis (2018). Therefore, any absolutely continuous martingale functional $\mathbb{Q} \in \mathcal{Q}_{ac}$ is a probability measure and $\mathcal{M}$ is closed in the weak* topology induced by $\mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$.

Consider the uniform order induced by expectations over $\mathcal{M}$,

$$X \leq Y \quad \Leftrightarrow \quad \forall \mathbb{P} \in \mathcal{M} \quad \mathbb{E}_{\mathbb{P}}[X] \leq \mathbb{E}_{\mathbb{P}}[Y]\,.$$

Then, $Z \in \mathcal{Z}$ if $\mathbb{E}_{\mathbb{P}}[Z] = 0$ for every $\mathbb{P} \in \mathcal{M}$. A contract X is positive if $\mathbb{E}_{\mathbb{P}}[X] \geq 0$ for every $\mathbb{P} \in \mathcal{M}$. A natural choice for the relevant contracts consists of nonnegative contracts with a positive return under some prior belief, i.e.

$$\mathcal{R} = \{R \in \mathcal{H} : 0 \leq \inf_{\mathbb{P}\in\mathcal{M}} \mathbb{E}_{\mathbb{P}}[R] \text{ and } 0 < \sup_{\mathbb{P}\in\mathcal{M}} \mathbb{E}_{\mathbb{P}}[R]\}\,.$$

PROPOSITION 4.3 *Under the assumptions of this subsection, if the financial market is viable, then the set of absolutely continuous martingale functionals $\mathcal{Q}_{ac}$ is a subset of the set of priors $\mathcal{M}$.*

PROOF: Set $\mathcal{E}_{\mathcal{M}}(X) := \sup_{\mathbb{P}\in\mathcal{M}} \mathbb{E}_{\mathbb{P}}[X]$. Then, $Y \leq 0$ if and only if $\mathcal{E}_{\mathcal{M}}(Y) \leq 0$. Fix $\mathbb{Q} \in \mathcal{Q}_{ac}$ with the preference relation given by $X \preceq_{\mathbb{Q}} Y$ if $\mathbb{E}_{\mathbb{Q}}[X - Y] \leq 0$.

Let us assume that $\mathbb{Q} \notin \mathcal{M}$. Since $\mathcal{M}$ is a weak*-closed and convex subset of the topological dual of $\mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$, there exists $X^* \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$ with $\mathcal{E}_{\mathcal{M}}(X^*) < 0 < \mathbb{E}_{\mathbb{Q}}[X^*]$ by the Hahn-Banach theorem. In particular, $X^* \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$ and $X^* \leq 0$. Since $\preceq_{\mathbb{Q}}$ is weakly monotone with respect to $\leq$, $X^* \preceq_{\mathbb{Q}} 0$. Hence, $\mathbb{E}_{\mathbb{Q}}[X^*] \leq 0$ contradicting the choice of X^*. Therefore, $\mathcal{Q}_{ac} \subset \mathcal{M}$.

Q.E.D.

[17]For the relation between the two approaches, compare also the discussion of objective and subjective ambiguity in Gilboa, Maccheroni, Marinacci, and Schmeidler (2010).

Expected returns of traded securities are thus not necessarily the same under all $\mathbb{P} \in \mathcal{M}$. However, for a smaller class $\mathcal{Q}_{ac}$ of $\mathcal{M}$ they remain the same and thus the strong form of EMH holds for this subset of the priors.

Let $\mathcal{H}_{\mathcal{M}}$ be the subspace of claims that have no ambiguity in the mean in the sense that $\mathbb{E}_{\mathbb{P}}[X]$ is the same constant for all $\mathbb{P} \in \mathcal{M}$. Consider the submarket $(\mathcal{H}_{\mathcal{M}}, \tau, \leq, \mathcal{I}_{\mathcal{M}}, \mathcal{R}_{\mathcal{M}})$ with $\mathcal{I}_{\mathcal{M}} := \mathcal{I} \cap \mathcal{H}_{\mathcal{M}}$ and $\mathcal{R}_{\mathcal{M}} := \mathcal{R} \cap \mathcal{H}_{\mathcal{M}}$. Restricted to this market, the measures $\mathcal{Q}_{ac}$ and $\mathcal{M}$ are identical and the strong EMH holds true.

The following simple example illustrates these points.

Example 4.4 Let $\Omega = \{0,1\}^2$, $\mathcal{H}$ be all functions on Ω. Then, $\mathcal{H} = \mathbb{R}^4$ and we write $X = (x, y, v, w)$ for any $X \in \mathcal{H}$. Let $\mathcal{I} = \{(x, y, 0, 0) : x + y = 0\}$. Consider the priors given by

$$\mathcal{M} := \left\{ \left(p, \frac{1}{2} - p, \frac{1}{4}, \frac{1}{4}\right) \ : \ p \in \left[\frac{1}{6}, \frac{1}{3}\right] \right\}.$$

There is Knightian uncertainty about the first two states, yet no Knightian uncertainty about the last two states. One directly verifies that $\mathcal{Q}_{ac} = \{\mathbb{Q}^*\} = \{(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})\}$. Notice that $\mathbb{Q}^* \in \mathcal{M}$.

In this case, $\mathcal{H}_{\mathcal{M}} = \{X = (x, y, v, w) \in \mathcal{H} \ : \ x = y\}$. In particular, all priors in $\mathcal{M}$ coincide with $\mathbb{Q}^*$ when restricted to $\mathcal{H}_{\mathcal{M}}$. Hence, for the claims that are mean-ambiguity-free, the strong efficient market hypothesis holds true.

4.3.2. *Weak Efficient Market Hypothesis under Knightian Uncertainty*

Let $\mathcal{M}$ be a common set of priors on $(\Omega, \mathcal{F})$. Set $\mathcal{H} := \mathcal{B}_b$. Let the common order be given by the quasi-sure ordering under the common set of priors $\mathcal{M}$, i.e.

$$X \leq Y \quad \Leftrightarrow \quad \mathbb{P}(X \leq Y) = 1, \ \ \forall \, \mathbb{P} \in \mathcal{M}.$$

In this case, a contract X is negligible if it vanishes $\mathcal{M}$–*quasi surely*, i.e. with probability one for all $\mathbb{P} \in \mathcal{M}$. An indicator function 1_A is thus negligible if the set A is *polar*, i.e. a null set with respect to every probability in $\mathcal{M}$. Take the set of relevant contracts to be[18]

$$\mathcal{R} = \{ R \in \mathcal{P} \ : \ \exists \, \mathbb{P} \in \mathcal{M} \quad \text{such that} \quad \mathbb{P}(R > 0) > 0 \, \}.$$

Proposition 4.5 *Under the assumptions of this subsection, the financial market is viable if and only if there is a set of finitely additive measures $\mathcal{Q}$ that are martingale measures and that has the same polar sets as the common set of priors $\mathcal{P}$.*

[18] These sets of positive and relevant contracts can be derived from Gilboa–Schmeidler utilities. Define $X \preceq Y$ if and only if $\mathcal{E}_{\mathcal{M}}[U(X)] := \inf_{\mathbb{P} \in \mathcal{M}} E_{\mathbb{P}}[U(X)] \leq \mathcal{E}_{\mathcal{M}}[U(Y)]$ for all strictly increasing and concave real functions U. The $0 \preceq Y$ is equivalent to Y dominating the zero contract in the sense of second order stochastic dominance under all $\mathbb{P} \in \mathcal{M}$. Hence, Y is nonnegative almost surely for all $\mathbb{P} \in \mathcal{M}$.

PROOF: Suppose that the market is viable. We show that the class $\mathcal{Q}_{ac}$ from Theorem 2.2 satisfies the desired properties. The martingale property follows by definition and from the fact that $\mathcal{I}$ is a linear space. Suppose that A is polar. Then, 1_A is negligible and from the absolute continuity property, it follows $\phi(A) = 0$ for any $\phi \in \mathcal{Q}_{ac}$. On the other hand, if A is not polar, $1_A \in \mathcal{R}$ and from the full support property, it follows that there exists $\phi_A \in \mathcal{Q}_{ac}$ such that $\phi_A(A) > 0$. Thus, A is not $\mathcal{Q}_{ac}$-polar. We conclude that $\mathcal{M}$ and $\mathcal{Q}_{ac}$ share the same polar sets. For the converse implication, define $\mathcal{E}(\cdot) := \sup_{\phi\in\mathcal{Q}} \mathbb{E}_\phi[\cdot]$. Using the same argument as above, $\mathcal{E}$ is a sublinear martingale expectation with full support. From Theorem 2.2 the market is viable. *Q.E.D.*

Under Knightian uncertainty, there is indeterminacy in arbitrage–free prices as there is always a range of economically justifiable arbitrage–free prices. Such indeterminacy has been observed in full general equilibrium analysis as well (Rigotti and Shannon (2005); Dana and Riedel (2013); Beissner and Riedel (2019)). In this sense, Knightian uncertainty shares a similarity with incomplete markets and other frictions like transaction costs, but the economic reason for the indeterminacy is different.

4.3.3. *A second-order Bayesian version of the EMH*

Let us next turn to the case that arises when the order is modeled by corresponds to a second–order Bayesian approach, in the spirit of the smooth ambiguity model (Klibanoff, Marinacci, and Mukerji (2005)),

Let $\mathcal{F}$ be a sigma algebra on Ω and $\mathfrak{P} = \mathfrak{P}(\Omega)$ the set of all probability measures on $(\Omega, \mathcal{F})$. Let μ be a second order prior, i.e. a probability measure[19] on $\mathfrak{P}$. We define a common prior in this setting as follows. The set function $\hat{\mathbb{P}} : \mathcal{F} \to [0,1]$ defined as $\hat{\mathbb{P}}(A) = \int_{\mathfrak{P}} \mathbb{P}(A)\mu(d\mathbb{P})$ is a probability measure on $(\Omega, \mathcal{F})$. Let $\mathcal{H} = \mathcal{L}^1(\Omega, \mathcal{F}, \hat{\mathbb{P}})$.

The common order is given by

$$X \leq Y \quad \Leftrightarrow \quad \mu\left(\{\mathbb{P} \in \mathfrak{P} \ : \ \mathbb{P}(X \leq Y) = 1 \ \}\right) = 1 \,.$$

A contract is positive if it is $\mathbb{P}$–almost surely nonnegative for all priors in the support of the second order prior μ. A natural choice for relevant contracts is

$$\mathcal{R} = \{R \in \mathcal{P} \ : \mu\left[\mathbb{P} \in \mathfrak{P} : \mathbb{P}(R > 0) > 0\right] > 0\} \,,$$

i.e. the set of beliefs $\mathbb{P}$ under which the contract is strictly positive with positive probability is not negligible according to the second order prior[20]

[19] From Theorem 15.18 of Aliprantis and Border (1999), the space of probability measure is a Borel space if and only if Ω is a Borel space. This allows to define second order priors.

[20] These sets of positive and relevant contracts can be derived from smooth ambiguity utility

PROPOSITION 4.6 *Under the assumptions of this subsection, the financial market is viable if and only if there is a martingale measure* $\mathbb{Q}$ *that has the form*

$$\mathbb{Q}(A) = \int_{\mathfrak{P}} \int_A D \, d\mathbb{P} \, \mu(d\mathbb{P})$$

for some state price density D*.*

PROOF: The set function $\hat{\mathbb{P}} : \mathcal{F} \to [0,1]$ defined as $\hat{\mathbb{P}}(A) = \int_{\mathfrak{P}} \mathbb{P}(A)\mu(d\mathbb{P})$ is a probability measure on $(\Omega, \mathcal{F})$. The induced $\hat{\mathbb{P}}$-a.s. order coincides with $\leq$ of this subsection. The result thus follows from Proposition 4.2. *Q.E.D.*

The smooth ambiguity model thus leads to a second–order Bayesian approach for asset returns. All asset returns are equal to the safe return for some second order martingale measure; the expectation is the average expected return corresponding to a risk–neutral second order prior $\mathbb{Q}$.

4.3.4. *On Recent Results in Mathematical Finance*

We conclude this section by relating our work to recent results in Mathematical Finance. Our approach gives a microeconomic foundation to the characterization of absence of arbitrage in “robust” or “model–free” finance.

In this subsection, Ω is a metric space. We say $X \leq Y$ if

(4.2) $\qquad X \leq_\Omega Y,$

which implies $\mathcal{Z} = \{0\}$.

In the finance literature, this approach is called *model-independent* as it does not rely on any probability measure. There is still a model, of course, given by Ω.

A contract is nonnegative, $X \in \mathcal{P}$, if $X(\omega) \geq 0$ for every $\omega \in \Omega$ and $R \in \mathcal{P}^+$ if $R \in \mathcal{P}$ and there exists $\omega_0 \in \Omega$ such that $R(\omega_0) > 0$.

In the literature several different notions of arbitrage have been used. Our framework allows to unify these different approaches under one framework with the help of the notion of relevant contracts. It is our view that all these different definitions simply depend on the agents perception of relevance[21].

We start with the following large set of relevant contracts

$$\mathcal{R}_{op} := \mathcal{P}^+ = \{ R \in \mathcal{P} \; : \; \exists \omega_0 \in \Omega \text{ such that } R(\omega_0) > 0 \; \}.$$

functions. Define $X \preceq Y$ if and only if

$$\int_{\mathfrak{P}} \psi\left(\mathbb{E}_{\mathbb{P}}[U(X)]\right) \mu(d\mathbb{P}) \leq \int_{\mathfrak{P}} \psi\left(\mathbb{E}_{\mathbb{P}}[U(X)]\right) \mu(d\mathbb{P})$$

for all strictly increasing and concave real functions U and ψ. Recall that ψ reflects uncertainty aversion.

The $0 \preceq Y$ is equivalent to Y dominating the zero contract in the sense of second order stochastic dominance for μ–almost all $\mathbb{P} \in \mathfrak{P}$, i.e. when $Y \geq 0$ in the sense defined above.

[21] One might also compare the similar approach in Burzoni, Frittelli, and Maggis (2016).

With this notion of relevance, an investment opportunity ℓ is an arbitrage if $\ell(\omega) \geq 0$ for every ω with a strict inequality for some ω, corresponding to the notion of *one point arbitrage* considered in Riedel (2015). In this setting, no arbitrage is equivalent to the existence a set of martingale measures $\mathcal{Q}_{op}$ so that for each point there exists $\mathbb{Q} \in \mathcal{Q}_{op}$ putting positive mass to that point.

In a second example, one requires the relevant contracts to be continuous, i.e.,

$$\mathcal{R}_{open} := \{ R \in C_b(\Omega) \cap \mathcal{P} \ : \ \exists \omega_0 \in \Omega \text{ such that } R(\omega_0) > 0 \ \} .$$

It is clear that when $R \in \mathcal{R}$ then it is non-zero on an open set. Hence, in this example the empty set is the only small set and the large sets are the ones that contain a non-empty open set.

Then, $\ell \in \mathcal{I}$ is an arbitrage opportunity if it is nonnegative and is strictly positive on an open set, corresponding to the notion of *open arbitrage* that appears in Burzoni, Frittelli, and Maggis (2016); Riedel (2015); Dolinsky and Soner (2014b).

Acciaio, Beiglböck, Penkner, and Schachermayer (2016) defines a contract to be an arbitrage when it is positive everywhere. In our context, this defines the relevant contracts as those that are positive everywhere, i.e.,

$$\mathcal{R}_+ := \{ R \in \mathcal{P} \ : \ \ R(\omega) > 0, \ \forall \ \omega \in \Omega \ \} .$$

Bartl, Cheridito, Kupper, and Tangpi (2017) consider a slightly stronger notion of relevant contracts. Their choice is

$$\mathcal{R}_u = \{ R \in \mathcal{P} \ : \ \exists c \in (0, \infty) \text{ such that } R \equiv c \ \} . \tag{4.3}$$

Hence, $\ell \in \mathcal{I}$ is an arbitrage if is uniformly positive, which is sometimes called *uniform arbitrage*. Notice that with the choice $\mathcal{R}_u$, the notions of arbitrage and free lunch with vanishing risk are equivalent.

The no arbitrage condition with $\mathcal{R}_u$ is the weakest while the one with $\mathcal{R}_{op}$ is the strongest. The first one is equivalent to the existence of one sublinear martingale expectation. The latter one is equivalent to the existence of a sublinear expectation that puts positive measure to all points.

In general, the no-arbitrage condition based on $\mathcal{R}_+$ is not equivalent to the absence of uniform arbitrage. However, absence of uniform arbitrage implies the existence of a linear bounded functional that is consistent with the market. In particular, risk neutral functionals are positive on $\mathcal{R}_u$. Moreover, if the set $\mathcal{I}$ is "large" enough then one can show that the risk neutral functionals give rise to countably additive measures. In Acciaio, Beiglböck, Penkner, and Schachermayer (2016), this conclusion is achieved by using the so-called "power-option" placed in the set $\mathcal{I}$ as a static hedging possibility, compare also Bartl, Cheridito, Kupper, and Tangpi (2017).

5. PROOF OF THE THEOREMS

Let $(\mathcal{H}, \tau, \leq \mathcal{I}, \mathcal{R})$ be a given financial market. Recall that $(\mathcal{H}, \tau)$ is a metrizable topological vector space; we write $\mathcal{H}'$ for its topological dual. We let $\mathcal{H}'_+$ be the set of all positive functionals, i.e., $\varphi \in \mathcal{H}'_+$ provided that $\varphi(X) \geq 0$ for every $X \geq 0$ and $X \in \mathcal{H}$.

The following functional generalizes the notion of super-replication functional from the probabilistic to our order-theoretic framework. It plays a central role in our analysis. For $X \in \mathcal{H}$, let

$$\mathcal{D}(X) := \inf\{\, c \in \mathbb{R} \;:\; \exists \{\ell_n\}_{n=1}^{\infty} \subset \mathcal{I},\ \{e_n\}_{n=1}^{\infty} \subset \mathcal{H}_+,\ e_n \stackrel{\tau}{\to} 0, \text{ such that } \quad c + e_n + \ell_n \geq X\} \,. \tag{5.1}$$

Following the standard convention, we set $\mathcal{D}(X)$ to plus infinity, when the above set is empty. Note that $\mathcal{D}$ is extended real valued. In particular, it takes the value $+\infty$ when there are no super-replicating portfolios. It might also take the value $-\infty$ if there is no lower bound.

We observe first that the absence of free lunches with vanishing risk can be equivalently described by the statement that the super-replication functional $\mathcal{D}$ assigns a strictly positive value to all relevant contracts.

PROPOSITION 5.1 *The financial market is strongly free of arbitrage if and only if $\mathcal{D}(R) > 0$ for every $R \in \mathcal{R}$.*

PROOF: Suppose $\{\ell_n\}_{n=1}^{\infty} \subset \mathcal{I}$ is a free lunch with vanishing risk. Then, there is $R^* \in \mathcal{R}$ and $\{e_n\}_{n=1}^{\infty} \subset \mathcal{H}_+$ with $e_n \stackrel{\tau}{\to} 0$ so that $e_n + \ell_n \geq R^*$. In view of the definition, we obtain $\mathcal{D}(R^*) \leq 0$.

To prove the converse, suppose that $\mathcal{D}(R^*) \leq 0$ for some $R^* \in \mathcal{R}$. Then, the definition of $\mathcal{D}(R^*)$ implies that there is a sequence of real numbers $\{c_k\}_{k=1}^{\infty}$ with $c_k \downarrow \mathcal{D}(R^*)$, net trades $\{\ell_{k,n}\}_{n=1}^{\infty} \subset \mathcal{I}$, and $\{e_{k,n}\}_{n=1}^{\infty} \subset \mathcal{H}_+$ with $e_{k,n} \stackrel{\tau}{\to} 0$ for $n \to \infty$ such that

$$c_k + e_{k,n} + \ell_{k,n} \geq R^*, \qquad \forall\, n, k \in \mathbb{N}.$$

Let $B_r(0)$ be the ball with radius r centered at zero with the metric compatible with τ. For every k, choose $n = n(k)$ such that $e_{k,n} \in B_{\frac{1}{k}}(0)$. Set $\tilde{\ell}_k := \ell_{k,n(k)}$ and $\tilde{e}_k := e_{k,n(k)} + (c_k \vee 0)$. Then, $\tilde{e}_k + \tilde{\ell}_k \geq R^*$ for every k. Since $\tilde{e}_k \stackrel{\tau}{\to} 0$, $\{\tilde{\ell}_k\}_{k=1}^{\infty}$ is a free lunch with vanishing risk. *Q.E.D.*

It is clear that $\mathcal{D}$ is convex and we now use the tools of convex duality to characterize this functional in more detail. Recall the set of absolutely continuous martingale functionals $\mathcal{Q}_{ac}$ defined in Section 2.

PROPOSITION 5.2 *Assume that the financial market is strongly free of arbitrage. Then, the super-replication functional $\mathcal{D}$ defined in (5.1) is a lower semi-continuous, sublinear martingale expectation with full support. Moreover,*

$$\mathcal{D}(X) = \sup_{\varphi \in \mathcal{Q}_{ac}} \varphi(X), \quad X \in \mathcal{H}.$$

The technical proof of this statement can be found in Appendix A. The important insight is that the super-replication functional can be described by a family of linear functionals. In the probabilistic setup, they correspond to the family of (absolutely continuous) martingale measures. With the help of this duality, we are now able to carry out the proof of our first main theorem.

PROOF OF THEOREM 2.1: Suppose first that the market is viable and for some $R^* \in \mathcal{R}$, there are sequences $\{e_n\}_{n=1}^{\infty} \subset \mathcal{H}_+$ and $\{\ell_n\}_{n=1}^{\infty} \subset \mathcal{I}$ with $e_n \stackrel{\tau}{\to} 0$, and $e_n + \ell_n \geq R^*$. By viability, there is a family of agents $\{\preceq_a\}_{a \in A} \subset \mathcal{A}$ such that $\ell_a^* = 0$ is optimal for each agent $a \in A$ and for some $a \in A$ we have $R^* \succ_a 0$. Since $\leq$ is a pre-order compatible with the vector space operations, we have $-e_n + R^* \leq \ell_n$. As $\preceq_a \in \mathcal{A}$ is monotone with respect to $\leq$, we have $-e_n + R^* \preceq_a \ell_n$. Since $\ell_a^* = 0$ is optimal, we get $-e_n + R^* \preceq_a 0$. By lower semi–continuity of $\preceq$, we conclude that $R^* \preceq_a 0$, a contradiction.

Suppose now that the market is strongly free of arbitrage. By Proposition 5.1, $\mathcal{D}(R) > 0$, for every $R \in \mathcal{R}$. In particular, this implies that the family $\mathcal{Q}_{ac}$ is nonempty, as otherwise the supremum over $\mathcal{Q}_{ac}$ would be $-\infty$. For each $\varphi \in \mathcal{Q}_{ac}$, define $\preceq_\varphi$ by,

$$X \preceq_\varphi Y, \quad \Leftrightarrow \quad \varphi(X) \leq \varphi(Y).$$

One directly verifies that $\preceq_\varphi \in \mathcal{A}$. Moreover, $\varphi(\ell) \leq \varphi(0) = 0$ for any $\ell \in \mathcal{I}$ implies that $\ell_\varphi^* = 0$ is optimal for $\preceq_\varphi$ and (2.1) is satisfied. Finally, Proposition 5.1 and Proposition 5.2 imply that for any $R \in \mathcal{R}$, there exists $\varphi \in \mathcal{Q}_{ac}$ such that $\varphi(R) > 0$; thus, (2.2) is satisfied. We deduce that $\{\preceq_\varphi\}_{\varphi \in \mathcal{Q}_{ac}}$ supports the financial market $(\mathcal{H}, \tau, \leq, \mathcal{I}, \mathcal{R})$. *Q.E.D.*

The previous arguments also imply our version of the fundamental theorem of asset pricing. In fact, with absence of arbitrage, the super-replication function is a lower semi-continuous sublinear martingale expectation with full support. Convex duality allows to prove the converse.

PROOF OF THEOREM 2.2: Suppose the market is viable. From Theorem 2.1, it is strongly free of arbitrage. From Proposition 5.2, the super-replication functional is the desired lower semi-continuous sublinear martingale expectation with full support.

Suppose now that $\mathcal{E}$ is a lower semi-continuous sublinear martingale expectation with full support. In particular, $\mathcal{E}$ is a convex, lower semi-continuous, proper functional. Then, by the Fenchel-Moreau theorem,

$$\mathcal{E}(X) = \sup_{\varphi \in dom(\mathcal{D}^*)} \varphi(X),$$

where $dom(\mathcal{D}^*) = \{\varphi \in \mathcal{H}' \, : \, \varphi(X) \le \mathcal{E}(X), \; \forall \; X \in \mathcal{H} \; \}$. We now proceed as in the proof of Theorem 2.1, to verify the viability of $(\mathcal{H}, \tau, \le, \mathcal{I}, \mathcal{R})$ using the preference relations $\{\preceq_\varphi\}_{\varphi \in \mathcal{D}^*}$. *Q.E.D.*

6. CONCLUSION

This paper studies the economic viability of a given financial market without assuming a common prior of the state space. We show that it is possible to understand viability and the absence of arbitrage based on a common notion of "more" that is shared by all potential agents of the economy. A given financial market is viable if and only if a *sublinear* pricing functional exists that is consistent with the given asset prices.

Our paper also shows how the properties of the common order are reflected in expected equilibrium returns. When the common order is given by the expected value under some common prior, expected returns under that prior have to be equal in equilibrium, and thus, Fama's Efficient Market Hypothesis results. If the common order is determined by the almost sure order under some common prior, we obtain the weak form of the efficient market hypothesis that states that expected returns are equal under some (martingale) measure that shares the same null sets as the common prior.

In situations of Knightian uncertainty, it might be too demanding to impose a common prior for all agents. When Knightian uncertainty is described by a class of priors, it is necessary to replace the linear (martingale) expectation by a sublinear expectation. It is then no longer possible to reach the conclusion that expected returns are equal under some probability measure. Knightian uncertainty might thus be an explanation for empirical violations of the Efficient Market Hypothesis. In particular, there is always a range of economically justifiable arbitrage–free prices. In this sense, Knightian uncertainty shares similarities with markets with friction or that are incomplete, but the economic reason for the price indeterminacy is different.

APPENDIX A: PROOF OF PROPOSITION 5.2

We separate the proof in several steps. Recall that the super-replication functional $\mathcal{D}$ is defined in (5.1).

LEMMA A.1 *Assume that the financial market is strongly free of arbitrage. Then, $\mathcal{D}$ is convex, lower semi–continuous and $\mathcal{D}(X) > -\infty$ for every $X \in \mathcal{H}$.*

PROOF: The convexity of $\mathcal{D}$ follows immediately from the definitions. To prove lower semi-continuity, consider a sequence $X_k \stackrel{\tau}{\to} X$ with $\mathcal{D}(X_k) \le c$. Then, by definition, for every k there exists a sequence $\{e_{k,n}\}_{n=1}^\infty \subset \mathcal{H}_+$ with $e_{k,n} \stackrel{\tau}{\to} 0$ for $n \to \infty$ and a sequence $\{\ell_{k,n}\}_{n=1}^\infty \subset \mathcal{I}$ such that $c + \frac{1}{k} + e_{k,n} + \ell_{k,n} \ge X_k$, for every k, n. Let $B_r(0)$ be the ball of radius r centered around zero in the metric compatible with τ. Choose $n = n(k)$ such that $e_{k,n} \in B_{\frac{1}{k}}$ and set $\tilde{e}_k := e_{k,n(k)}$, $\tilde{\ell}_k := \ell_{k,n(k)}$. Then, $c + \frac{1}{k} + \tilde{e}_k + (X - X_k) + \tilde{\ell}_k \ge X$ and $\frac{1}{k} + \tilde{e}_k + (X - X_k) \stackrel{\tau}{\to} 0$ as $k \to \infty$. Hence, $\mathcal{D}(X) \le c$. This proves that $\mathcal{D}$ is lower semi-continuous.

The constant claim 1 is relevant and by Proposition 5.1, $\mathcal{D}(1) \in (0,1]$; in particular, it is finite. Towards a counter-position, suppose that there exists $X \in \mathcal{H}$ such that $\mathcal{D}(X) = -\infty$. For $\lambda \in [0,1]$, set $X_\lambda := X + \lambda(1-X)$. The convexity of $\mathcal{D}$ implies that $\mathcal{D}(X_\lambda) = -\infty$ for every $\lambda \in [0,1)$. Since $\mathcal{D}$ is lower semi-continuous, $0 < \mathcal{D}(1) \le \lim_{\lambda\to 1} \mathcal{D}(X_\lambda) = -\infty$, a contradiction. *Q.E.D.*

LEMMA A.2 *Assume that the financial market is strongly free of arbitrage. The super-replication functional $\mathcal{D}$ is a sublinear expectation with full-support. Moreover, $\mathcal{D}(c) = c$ for every $c \in \mathbb{R}$, and*

$$\mathcal{D}(X+\ell) \le \mathcal{D}(X), \quad \forall\, \ell \in \mathcal{I},\ X \in \mathcal{H}. \tag{A.1}$$

In particular, $\mathcal{D}$ has the martingale property.

PROOF: We prove this result in two steps.

Step 1. In this step we prove that $\mathcal{D}$ is a sublinear expectation. Let $X, Y \in \mathcal{H}$ such that $X \le Y$. Suppose that there are $c \in \mathbb{R}$, $\{\ell_n\}_{n=1}^\infty \subset \mathcal{I}$ and $\{e_n\}_{n=1}^\infty \subset \mathcal{H}_+$ with $e_n \xrightarrow{\tau} 0$ satisfying, $Y \le c + e_n + \ell_n$. Then, from the transitivity of $\le$, we also have $X \le c + e_n + \ell_n$. Hence, $\mathcal{D}(X) \le \mathcal{D}(Y)$ and consequently $\mathcal{D}$ is monotone with respect to $\le$.

Translation-invariance, $\mathcal{D}(c+g) = c + \mathcal{D}(g)$, follows directly from the definitions.

We next show that $\mathcal{D}$ is sub-additive. Fix $X, Y \in \mathcal{H}$. If either $\mathcal{D}(X) = \infty$ or $\mathcal{D}(Y) = \infty$. Then, since by Lemma A.1 $\mathcal{D} > -\infty$, we have $\mathcal{D}(X) + \mathcal{D}(Y) = \infty$ and the sub-additivity follows directly. Now we consider the case $\mathcal{D}(X), \mathcal{D}(Y) < \infty$. Hence, there are $c_X, c_Y \in \mathbb{R}$, $\{\ell_n^X\}_{n=1}^\infty, \{\ell_n^Y\}_{n=1}^\infty \subset \mathcal{I}$ and $\{e_n^X\}_{n=1}^\infty, \{e_n^Y\}_{n=1}^\infty \subset \mathcal{H}_+$ with $e_n^X, e_n^Y \xrightarrow{\tau} 0$ satisfying,

$$c_X + \ell_n^X + e_n^X \ge X, \quad c_Y + \ell_n^Y + e_n^Y \ge Y.$$

Set $\bar{c} := c_X + c_Y$, $\bar{\ell}_n := \ell_n^X + \ell_n^Y$, $\bar{e}_n := \ell_n^X + \ell_n^Y$. Since $\mathcal{I}$, $\mathcal{P}$ are positive cones, $\{\bar{\ell}_n\}_{n=1}^\infty \subset \mathcal{I}$, $\bar{e}_n \xrightarrow{\tau} 0$ and

$$\bar{c} + \bar{e}_n + \bar{\ell}_n \ge X + Y \quad \Rightarrow \quad \mathcal{D}(X+Y) \le \bar{c}.$$

Since this holds for any such c_X, c_Y, we conclude that

$$\mathcal{D}(X+Y) \le \mathcal{D}(X) + \mathcal{D}(Y).$$

Finally we show that $\mathcal{D}$ is positively homogeneous of degree one. Suppose that $c + e_n + \ell_n \ge X$ for some constant c, $\{\ell_n\}_{n=1}^\infty \subset \mathcal{I}$ and $\{e_n\}_{n=1}^\infty \subset \mathcal{H}_+$ with $e_n \xrightarrow{\tau} 0$. Then, for any $\lambda > 0$ and for any $n \in \mathbb{N}$, $\lambda c + \lambda e_n + \lambda \ell_n \ge \lambda X$. Since $\lambda \ell_n \in \mathcal{I}$ and $\lambda e_n \xrightarrow{\tau} 0$, this implies that

$$\mathcal{D}(\lambda X) \le \lambda\, \mathcal{D}(X), \quad \lambda > 0,\ X \in \mathcal{H}. \tag{A.2}$$

Notice that above holds trivially when $\mathcal{D}(X) = +\infty$. Conversely, if $\mathcal{D}(\lambda X) = +\infty$ we are done. Otherwise, we use (A.2) with λX and $1/\lambda$,

$$\mathcal{D}(X) = \mathcal{D}\left(\frac{1}{\lambda}\lambda X\right) \le \frac{1}{\lambda}\mathcal{D}(\lambda X), \quad \Rightarrow \quad \lambda \mathcal{D}(X) \le \mathcal{D}(\lambda X).$$

Hence, $\mathcal{D}$ positively homogeneous and it is a sublinear expectation.

Step 2. In this step, we assume that the financial market is strongly free of arbitrages. Since $0 \in \mathcal{I}$, we have $\mathcal{D}(0) \le 0$. If the inequality is strict we obviously have a free lunch with vanishing risk, hence $\mathcal{D}(0) = 0$ and from translation-invariance the same applies to every $c \in \mathbb{R}$. Moreover, by Proposition 5.1, $\mathcal{D}$ has full support. Thus, we only need to prove (A.1).

Suppose that $X \in \mathcal{H}, \ell \in \mathcal{I}$ and $c + e_n + \ell_n^X \ge X$. Since $\mathcal{I}$ is a convex cone, $\ell_n^X + \ell \in \mathcal{I}$ and $c + e_n + (\ell + \ell_n^X) \ge X + \ell$. Therefore, $\mathcal{D}(X+\ell) \le c$. Since this holds for all such constants, we conclude that $\mathcal{D}(X+\ell) \le \mathcal{D}(X)$ for all $X \in \mathcal{H}$. In particular $\mathcal{D}(\ell) \le 0$ and the martingale property is satisfied. *Q.E.D.*

REMARK A.3 Note that for $\mathcal{H} = (\mathcal{B}_b, \|\cdot\|_\infty)$, the definition of $\mathcal{D}$ reduces to the classical one:

$$\mathcal{D}(X) := \inf \{ \ c \in \mathbb{R} \ : \ \exists \ \ell \in \mathcal{I}, \text{ such that } c + \ell \geq X \ \} . \tag{A.3}$$

Indeed, if $c + \ell \geq X$ for some c and ℓ, one can use the constant sequences $\ell_n \equiv \ell$ and $e_n \equiv 0$ to get that $\mathcal{D}$ in (5.1) is less or equal than the one in (A.3). For the converse inequality observe that if $c + e_n + \ell_n \geq X$ for some c, ℓ_n and e_n with $\|e_n\|_\infty \to 0$, then the infimum in (A.3) is less or equal than c. The thesis follows. Lemma A.1 is in line with the well known fact that the classical super-replication functional in $\mathcal{B}_b$ is Lipschitz continuous with respect to the sup-norm topology.

The results of Lemma A.2 and Lemma A.1 imply that the super-replication functional defined in (5.1) is a regular convex function in the language of convex analysis, compare, e.g., Rockafellar (2015). By the classical Fenchel-Moreau theorem, we have the following dual representation of $\mathcal{D}$,

$$\mathcal{D}(X) = \sup_{\varphi \in \mathcal{H}'} \ \{\varphi(X) - \mathcal{D}^*(\varphi)\} , \quad X \in \mathcal{H}, \quad \text{where}$$
$$\mathcal{D}^*(\varphi) = \sup_{Y \in \mathcal{H}} \ \{\varphi(Y) - \mathcal{D}(Y)\} , \quad \varphi \in \mathcal{H}'.$$

Since $\varphi(0) = \mathcal{D}(0) = 0$, $\mathcal{D}^*(\varphi) \geq \varphi(0) - \mathcal{D}(0) = 0$ for every $\varphi \in \mathcal{H}'$. However, it may take the value plus infinity. Set,

$$dom(\mathcal{D}^*) := \left\{ \ \varphi \in \mathcal{H}' \ : \ \mathcal{D}^*(\varphi) < \infty \right\} .$$

LEMMA A.4 *We have*

$$dom(\mathcal{D}^*) = \left\{ \ \varphi \in \mathcal{H}'_+ \ : \ \mathcal{D}^*(\varphi) = 0 \right\} = \left\{ \ \varphi \in \mathcal{H}'_+ \ : \ \varphi(X) \leq \mathcal{D}(X), \ \ \forall \ X \in \mathcal{H} \right\} . \tag{A.4}$$

In particular,

$$\mathcal{D}(X) = \sup_{\varphi \in dom(\mathcal{D}^*)} \ \varphi(X), \quad X \in \mathcal{H}.$$

Furthermore, there are free lunches with vanishing risk in the financial market, whenever $dom(\mathcal{D}^*)$ is empty.

PROOF: Clearly the two sets on the right of (A.4) are equal and included in $dom(\mathcal{D}^*)$. The definition of $\mathcal{D}^*$ implies that

$$\varphi(X) \leq \mathcal{D}(X) + \mathcal{D}^*(\varphi), \quad \forall \ X \in \mathcal{H}, \ \varphi \in \mathcal{H}'.$$

By homogeneity,

$$\varphi(\lambda X) \leq \mathcal{D}(\lambda X) + \mathcal{D}^*(\varphi), \quad \Rightarrow \quad \varphi(X) \leq \mathcal{D}(X) + \frac{1}{\lambda}\mathcal{D}^*(\varphi),$$

for every $\lambda > 0$ and $X \in \mathcal{H}$. Suppose that $\varphi \in dom(\mathcal{D}^*)$. We then let λ go to infinity to arrive at $\varphi(X) \leq \mathcal{D}(X)$ for all $X \in \mathcal{B}_b$. Hence, $\mathcal{D}^*(\varphi) = 0$.

Fix $X \in \mathcal{H}_+$. Since $\leq$ is monotone with respect to $\leq_\Omega$, $-X \leq 0$. Then, by the monotonicity of $\mathcal{D}$, $\varphi(-X) \leq \mathcal{D}(-X) \leq \mathcal{D}(0) \leq 0$. Hence, $\varphi \in \mathcal{H}'_+$.

Now suppose that $dom(\mathcal{D}^*)$ is empty or, equivalently, $\mathcal{D}^* \equiv \infty$. Then, the dual representation implies that $\mathcal{D} \equiv -\infty$. In view of Proposition 5.1, there are free lunches with vanishing risk in the financial market. *Q.E.D.*

We next show that, under the assumption of absence of free lunch with vanishing risk with respect to any $\mathcal{R}$, the set $dom(\mathcal{D}^*)$ is equal to $\mathcal{Q}_{ac}$ defined in Section 2. Since any relevant set $\mathcal{R}$ by hypothesis contains $\mathcal{R}_u$ defined in (4.3), to obtain this conclusion it would be sufficient to assume the absence of free lunch with vanishing risk with respect to any $\mathcal{R}_u$.

LEMMA A.5 *Suppose the financial market is strongly free of arbitrage with respect to $\mathcal{R}$. Then, $dom(\mathcal{D}^*)$ is equal to the set of absolutely continuous martingale functionals $\mathcal{Q}_{ac}$.*

PROOF: The fact that $dom(\mathcal{D}^*)$ is non-empty follows from Lemma A.2 and Lemma A.4. Fix an arbitrary $\varphi \in dom(\mathcal{D}^*)$. By Lemma A.2, $\mathcal{D}(c) = c$ for every constant $c \in \mathbb{R}$. In view of the dual representation of Lemma A.4,

$$c\varphi(1) = \varphi(c) \leq \mathcal{D}(c) = c, \quad \forall c \in \mathbb{R}.$$

Hence, $\varphi(1) = 1$.

We continue by proving the monotonicity property. Suppose that $X \in \mathcal{P}$. Since $0 \in \mathcal{I}$, we obviously have $\mathcal{D}(-X) \leq 0$. The dual representation implies that $\varphi(-X) \leq \mathcal{D}(-X) \leq 0$. Thus, $\varphi(X) \geq 0$.

We now prove the supermartingale property. Let $\ell \in \mathcal{I}$. Obviously $\mathcal{D}(\ell) \leq 0$. By the dual representation, $\varphi(\ell) \leq \mathcal{D}(\ell) \leq 0$. Hence φ is a martingale functional. The absolute continuity follows as in Lemma E.3. Hence, $\varphi \in \mathcal{Q}_{ac}$.

To prove the converse, fix an arbitrary $\varphi \in \mathcal{Q}_{ac}$. Suppose that $X \in \mathcal{H}$, $c \in \mathbb{R}$, $\{\ell_n\}_{n=1}^{\infty} \subset \mathcal{I}$ and $\{e_n\}_{n=1}^{\infty} \subset \mathcal{H}_+$ with $e_n \xrightarrow{\tau} 0$ satisfy, $c + e_n + \ell_n \geq X$. From the properties of φ,

$$0 \leq \varphi(c + e_n + \ell_n - X) = \varphi(c + e_n - X) + \varphi(\ell_n) \leq c - \varphi(X - e_n).$$

Since $e_n \xrightarrow{\tau} 0$ and φ is continuous, $\varphi(X) \leq \mathcal{D}(X)$ for every $X \in \mathcal{H}$. Therefore, $\varphi \in dom(\mathcal{D}^*)$. *Q.E.D.*

PROOF OF PROPOSITION 5.2: It follows directly from Lemma A.4 and Lemma A.5. *Q.E.D.*

We have the following immediate corollary, which states that is the first part of the Fundamental Theorem of Asset Pricing in this context.

COROLLARY A.6 *The financial market is strongly free of arbitrage if and only $\mathcal{Q}_{ac} \neq \emptyset$ and for any $R \in \mathcal{R}$, there exists $\varphi_R \in \mathcal{Q}_{ac}$ such that $\varphi_R(R) > 0$.*

PROOF: By contradiction, suppose that there exists R^* such that $e_n + \ell_n \geq R^*$ with $e_n \xrightarrow{\tau} 0$. Take φ_{R^*} such that $\varphi_{R^*}(R^*) > 0$ and observe that $0 < \varphi_{R^*}(R^*) \leq \varphi(e_n + \ell_n) \leq \varphi(e_n)$. Since $\varphi \in \mathcal{H}'_+$, $\varphi(e_n) \to 0$ as $n \to \infty$, which is a contradiction.

In the other direction, assume that the financial market is strongly free of arbitrage. By Lemma A.5, $dom(\mathcal{D}^*) = \mathcal{Q}_{ac}$. Let $R \in \mathcal{R}$ and note that, by Proposition 5.1, $\mathcal{D}(R) > 0$. It follows that there exists $\varphi_R \in dom(\mathcal{D}^*) = \mathcal{Q}_{ac}$ satisfying $\varphi_R(R) > 0$. *Q.E.D.*

REMARK A.7 The set of positive functionals $\mathcal{Q}_{ac} \subset \mathcal{H}'_+$ is the analogue of the set of local martingale measures of the classical setting. Indeed, all elements of $\varphi \in \mathcal{Q}_{ac}$ can be regarded as supermartingale "measures", since $\varphi(\ell) \leq 0$ for every $\ell \in \mathcal{I}$. Moreover, the property $\varphi(Z) = 0$ for every $Z \in \mathcal{Z}$ can be regarded as absolute continuity with respect to null sets. The full support property is our analog to the converse absolute continuity. However, the full-support property cannot be achieved by a single element of $\mathcal{Q}_{ac}$.

Bouchard and Nutz (2015) study arbitrage for a set of priors $\mathcal{M}$. The absolute continuity and the full support properties then translate to the statement that "$\mathcal{M}$ and $\mathcal{Q}$ have the same polar sets". In the paper by Burzoni, Frittelli, and Maggis (2016), a class of relevant sets $\mathcal{S}$ is given and the two properties can summarised by the statement "the set $\mathcal{S}$ is not contained in the polar sets of $\mathcal{Q}$".

Also, when $\mathcal{H} = \mathcal{B}_b$, $\mathcal{H}'$ is the class of bounded additive measures ba. It is a classical question whether one can restrict $\mathcal{Q}$ to the set of countable additive measures $ca_r(\Omega)$. In several of the examples described in Section 3 and 4 this is proved. However, there are examples for which this is not true.

APPENDIX B: LINEARLY GROWING CLAIMS

Let $\mathcal{B}(\Omega, \mathcal{F})$ be the set of all $\mathcal{F}$ measurable real-valued functions on Ω. Any Banach space contained in $\mathcal{B}(\Omega, \mathcal{F})$ satisfies the requirements for $\mathcal{H}$. In our examples, we used the spaces $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$ (defined in the subsection 4.3.1) and $\mathcal{B}_b(\Omega, \mathcal{F})$, the set of all bounded functions in $\mathcal{B}(\Omega, \mathcal{F})$, with the supremum norm. In the latter case, the super-hedging functional enjoys several properties as discussed in Remark A.3.

Since we require that $\mathcal{I} \subset \mathcal{H}$ (see Section 2), in the case of $\mathcal{H} = \mathcal{B}_b(\Omega, \mathcal{F})$ this means that all the trading instruments are bounded. This could be restrictive in some applications and we now provide another example that overcomes this difficulty. To define this set, fix $L^* \in \mathcal{B}(\Omega, \mathcal{F})$ with $L^* \geq_\Omega 1$. Consider the linear space

$$\mathcal{B}_\ell := \left\{ X \in \mathcal{B}(\Omega, \mathcal{F}) \ : \ \exists \ \alpha \in \mathbb{R}^+ \quad \text{such that} \quad |X| \leq_\Omega \alpha L^* \right\}$$

equipped with the norm,

$$\|X\|_\ell := \inf\{\alpha \in \mathbb{R}^+ \ : \ |X| \leq_\Omega \alpha L^* \ \} = \left\| \frac{X}{L^*} \right\|_\infty .$$

We denote the topology induced by this norm by τ_ℓ. Then, $\mathcal{B}_\ell(\Omega, \mathcal{F})$ with τ_ℓ is a Banach space and satisfies our assumptions. Note that if $L^* = 1$, then $\mathcal{B}_\ell(\Omega, \mathcal{F}) = \mathcal{B}_b(\Omega, \mathcal{F})$.

Now, suppose that

$$L^*(\omega) := c^* + \hat{\ell}(\omega), \qquad \omega \in \Omega, \tag{B.1}$$

for some $c^* > 0$, $\hat{\ell} \in \mathcal{I}$. Then, one can define the super-replication functional as in (A.3).

APPENDIX C: NO ARBITRAGE VERSUS NO FREE-LUNCH-WITH-VANISHING-RISK

We recall the definition of arbitrage. Let $(\mathcal{H}, \tau, \leq, \mathcal{I}, \mathcal{R})$ be a financial market. We say that an achievable contract $\ell \in \mathcal{I}$, is an *arbitrage* if there exists a relevant contract $R^* \in \mathcal{R}$ with $\ell \geq R^*$.

When $\mathcal{R} = \mathcal{P}^+$ the definitions above become simpler. In this case $\ell \in \mathcal{I}$ is an arbitrage if and only if $\ell \in \mathcal{P}^+$. From Definition **??** it is clear that an arbitrage opportunity is always a free lunch with vanishing risk. The purpose of this section is to investigate when these two notions are equivalent.

C.1. *Attainment*

We first show that the attainment property is useful in discussing the connection between two different notions of arbitrage. We start with a definition.

Definition C.1 We say that a financial market has the *attainment property*, if for every $X \in \mathcal{H}$ there exists a minimizer in (5.1), i.e., there exists $\ell_X \in \mathcal{I}$ satisfying,

$$\mathcal{D}(X) + \ell_X \geq X.$$

Proposition C.2 *Suppose that a financial market has the attainment property. Then, it is strongly free of arbitrage if and only if it has no arbitrages.*

Proof: Let $R^* \in \mathcal{R}$. By hypothesis, there exist $\ell \in \mathcal{I}^*$ so that $\mathcal{D}(R^*) + \ell^* \geq R^*$. If the market has no arbitrage, then we conclude that $\mathcal{D}(R^*) > 0$. In view of Proposition 5.1, this proves that the financial market is also strongly free of arbitrage. Since no arbitrage is weaker condition, they are equivalent. *Q.E.D.*

C.2. *Finite discrete time markets*

In this subsection and in the next section, we restrict ourselves to arbitrage considerations in finite discrete-time markets.

We start by introducing a discrete filtration $\mathbb{F} := (\mathcal{F}_t)_{t=0}^T$ on subsets of Ω. Let $S = (S_t)_{t=0}^T$ be an adapted stochastic process[22,23] with values in $\mathbb{R}_+^M$ for some M. For every $\ell \in \mathcal{I}$ there exist predictable integrands $H_t \in \mathcal{B}_b(\Omega, \mathcal{F}_{t-1})$ for all $t = 1, \dots, T$ such that,

$$\ell = (H \cdot S)_T := \sum_{t=1}^{T} H_t \cdot \Delta S_t, \quad \text{where} \quad \Delta S_t := (S_t - S_{t-1}).$$

Denote by $\ell_t := (H \cdot S)_t$ for $t \in \mathcal{I}$ and $\ell := \ell_T$.

Set $\hat{\ell} = \sum_{k,i} S_k^i - S_0^i$. Then, one can directly show that with an appropriate c^*, we have $L^* := c^* + \hat{\ell} \geq 1$. Define $\mathcal{B}_\ell$ using $\hat{\ell}$, set $\mathcal{H} = \mathcal{B}_\ell$ and denote by $\mathcal{I}_\ell$ the subset of $\mathcal{I}$ with H_t *bounded* for every $t = 1, \dots, T$.

We next prescribe the equivalence relation and the relevant sets. Our starting point is the set of negligible sets $\mathcal{Z}$ which we assume is given. We also make the following structural assumption.

Assumption C.3 Assume that the trading is allowed only at finite time points labeled through $1, 2, \dots, T$. Let $\mathcal{I}$ be given as above and let $\mathcal{Z}$ be a lattice which is closed with respect to pointwise convergence.

We also assume that $\mathcal{R} = \mathcal{P}^+$ and the pre-order is given by,

$$X \leq Y \quad \Leftrightarrow \quad \exists Z \in \mathcal{Z} \quad \text{such that} \quad X \leq_\Omega Y + Z.$$

In particular, $X \in \mathcal{P}$ if and only if there exists $Z \in \mathcal{Z}$ such that $Z \leq_\Omega X$.

An example of the above structure is the Example 4.3.2. In that example, $\mathcal{Z}$ is polar sets of a given class $\mathcal{M}$ of probabilities. Then, in this context all inequalities should be understood as $\mathcal{M}$ quasi-surely. Also note also that the assumptions on $\mathcal{Z}$ are trivially satisfied when $\mathcal{Z} = \{0\}$. In this latter case, inequalities are pointwise.

Observe that in view of the definition of $\leq$ and the fact $\mathcal{R} = \mathcal{P}^+$, $\ell \in \mathcal{I}$ is an arbitrage if and only if there is $R^* \in \mathcal{P}^+$ and $Z^* \in \mathcal{Z}$, so that $\ell \geq_\Omega R^* + Z^*$. Hence, $\ell \in \mathcal{I}$ is an arbitrage if and only is $\ell \in \mathcal{P}^+$. We continue by showing the equivalence of the existence of an arbitrage to the existence of a one-step arbitrage.

Lemma C.4 *Suppose that Assumption C.3 holds. Then, there exists arbitrage if and only if there exists $t \in \{1, \dots, T\}$, $h \in \mathcal{B}_b(\Omega, \mathcal{F}_{t-1})$ such that $\ell := h \cdot \Delta S_t$ is an arbitrage.*

Proof: The sufficiency is clear. To prove the necessity, suppose that $\ell \in \mathcal{I}$ is an arbitrage. Then, there is a predictable process H so that $\ell = (H \cdot S)_T$. Also $\ell \in \mathcal{P}^+$, hence, $\ell \notin \mathcal{Z}$ and there exists $Z \in \mathcal{Z}$ such that $\ell \geq Z$. Define

$$\hat{t} := \min\{t \in \{1, \dots, T\} \ : \ (H \cdot S)_t \in \mathcal{P}^+ \ \} \leq T.$$

First we study the case where $\ell_{\hat{t}-1} \in \mathcal{Z}$. Define

$$\ell^* := H_{\hat{t}} \cdot \Delta S_{\hat{t}},$$

and observe that $\ell_{\hat{t}} = \ell_{\hat{t}-1} + \ell^*$. Since $\ell_{\hat{t}-1} \in \mathcal{Z}$, we have that $\ell^* \in \mathcal{P}^+$ iff $\ell_{\hat{t}} \in \mathcal{P}^+$ and consequently the lemma is proved.

[22] When working with N stocks, a canonical choice for Ω would be

$$\Omega = \{\omega = (\omega_0, \dots, \omega_T) \ : \ \omega_i \in [0, \infty)^N, \ i = 0, \dots, T \ \}.$$

Then, one may take $S_t(\omega) = \omega_t$ and $\mathbb{F}$ to be the filtration generated by S.

[23] Note that we do not specify any probability measure.

Suppose now $\ell_{\hat{t}-1} \notin \mathcal{Z}$. If $\ell_{\hat{t}-1} \geq_\Omega 0$, then $\ell_{\hat{t}-1} \in \mathcal{P}$ and, thus, also in $\mathcal{P}^+$, which is not possible from the minimality of $\hat{t}$. Hence the set $A := \{\ell_{\hat{t}-1} <_\Omega 0\}$ is non empty and $\mathcal{F}_{\hat{t}-1}$-measurable. Define, $h := H_{\hat{t}}\chi_A$ and $\ell^* := h \cdot \Delta S_{\hat{t}}$. Note that,

$$\ell^* = \chi_A(\ell_{\hat{t}} - \ell_{\hat{t}-1}) \geq_\Omega \chi_A \ell_{\hat{t}} \geq_\Omega \chi_A Z \in \mathcal{Z}.$$

This implies $\ell^* \in \mathcal{P}$. Towards a contradiction, suppose that $\ell^* \in \mathcal{Z}$. Then,

$$\ell_{\hat{t}-1} \geq_\Omega \chi_A \ell_{\hat{t}-1} \geq \chi_A (Z - \ell^*) \in \mathcal{Z},$$

Since, by assumption, $\ell_{\hat{t}-1} \notin \mathcal{Z}$ we have $\ell_{\hat{t}-1} \in \mathcal{P}^+$ from which $\hat{t}$ is not minimal. *Q.E.D.*

The following is the main result of this section. For the proof we follow the approach of Kabanov and Stricker (2001) which is also used in Bouchard and Nutz (2015). We consider the financial market $\Theta_* = (\mathcal{B}_\ell, \|\cdot\|_\ell, \leq_\Omega, \mathcal{I}, \mathcal{P}^+)$ described above.

THEOREM C.5 *In a finite discrete time financial market satifying the Assumption C.3, the following are equivalent:*

1. *The financial market Θ_* has no arbitrages.*
2. *The attainment property holds and Θ_* is free of arbitrage.*
3. *The financial market Θ_* is strongly free of arbitrages.*

PROOF: In view of Proposition C.2 we only need to prove the implication 1 ⇒ 2.

For $X \in \mathcal{H}$ such that $\mathcal{D}(X)$ is finite we have that

$$c_n + \mathcal{D}(H) + \ell_n \geq_\Omega X + Z_n,$$

for some $c_n \downarrow 0$, $\ell_n \in \mathcal{I}$ and $Z_n \in \mathcal{Z}$. Note that since $\mathcal{Z}$ is a lattice we assume, without loss of generality, that $Z_n = (Z_n)^-$ and denote by $\mathcal{Z}^- := \{Z^- \mid Z \in \mathcal{Z}\}$.

We show that $\mathcal{C} := \mathcal{I} - (\mathcal{L}^0_+(\Omega, \mathcal{F}) + \mathcal{Z}^-)$ is closed under pointwise convergence where $\mathcal{L}^0_+(\Omega, \mathcal{F})$ denotes the class of pointwise nonnegative random variables. Once this result is shown, by observing that $X - c_n - \mathcal{D}(X) = W_n \in \mathcal{C}$ converges pointwise to $X - \mathcal{D}(X)$ we obtain the attainment property.

We proceed by induction on the number of time steps. Suppose first $T = 1$. Let

$$W_n = \ell_n - K_n - Z_n \to W, \tag{C.1}$$

where $\ell_n \in \mathcal{I}$, $K_n \geq_\Omega 0$ and $Z_n \in \mathcal{Z}^-$. We need to show $W \in \mathcal{C}$. Note that any ℓ_n can be represented as $\ell_n = H_1^n \cdot \Delta S_1$ with $H_1^n \in \mathcal{L}^0(\Omega, \mathcal{F}_0)$.

Let $\Omega_1 := \{\omega \in \Omega \mid \liminf |H_1^n| < \infty\}$. From Lemma 2 in Kabanov and Stricker (2001) there exist a sequence $\{\tilde{H}_1^k\}$ such that $\{\tilde{H}_1^k(\omega)\}$ is a convergent subsequence of $\{H_1^k(\omega)\}$ for every $\omega \in \Omega_1$. Let $H_1 := \liminf H_1^n \, \chi_{\Omega^1}$ and $\ell := H_1 \cdot \Delta S_1$.

Note now that $Z_n \leq_\Omega 0$, hence, if $\liminf |Z_n| = \infty$ we have $\liminf Z_n = -\infty$. We show that we can choose $\tilde{Z}_n \in \mathcal{Z}^-$, $\tilde{K}_n \geq_\Omega 0$ such that $\tilde{W}_n := \ell_n - \tilde{K}_n - \tilde{Z}_n \to W$ and $\liminf \tilde{Z}_n$ is finite on Ω_1. On $\{\ell_n \geq_\Omega W\}$ set $\tilde{Z}_n = 0$ and $\tilde{K}_n = \ell_n - W$. On $\{\ell_n <_\Omega W\}$ set

$$\tilde{Z}_n = Z_n \vee (\ell_n - W), \quad \tilde{K}_n = K_n \chi_{\{Z_n = \tilde{Z}_n\}}.$$

It is clear that $Z_n \leq_\Omega \tilde{Z}_n \leq_\Omega 0$. From Lemma E.1 we have $\tilde{Z}_n \in \mathcal{Z}$. Moreover, it is easily checked that $\tilde{W}_n := \ell_n - \tilde{K}_n - \tilde{Z}_n \to W$. Nevertheless, from the convergence of ℓ_n on Ω_1 and $\tilde{Z}_n \geq_\Omega -(W - \ell_n)^+$, we obtain $\{\omega \in \Omega_1 \mid \liminf \tilde{Z}_n > -\infty\} = \Omega_1$. As a consequence also $\liminf \tilde{K}_n$ is finite on Ω_1, otherwise we could not have that $\tilde{W}_n \to W$. Thus, by setting $\tilde{Z} := \liminf \tilde{Z}_n$ and $\tilde{K} := \liminf \tilde{K}_n$, we have $W = \ell - \tilde{K} - \tilde{Z} \in \mathcal{C}$.

On Ω_1^C we may take $G_1^n := H_1^n/|H_1^n|$ and let $G_1 := \liminf G_1^n \chi_{\Omega_1^C}$. Define, $\ell_G := G_1 \cdot \Delta S_1$. We now observe that,

$$\{\omega \in \Omega_1^C \mid \ell_G(\omega) \leq 0\} \subseteq \{\omega \in \Omega_1^C \mid \liminf Z_n(\omega) = -\infty\}.$$

Indeed, if $\omega \in \Omega_1^C$ is such that $\liminf Z_n(\omega) > -\infty$, applying again Lemma 2 in Kabanov and Stricker (2001), we have that

$$\liminf_{n\to\infty} \frac{X(\omega) + Z_n(\omega)}{|H_1^n(\omega)|} = 0,$$

implying $\ell_G(\omega)$ is nonnegative. Set now

$$\tilde{Z}_n := Z_n \vee -(\ell_G)^-.$$

From $Z_n \leq_\Omega \tilde{Z}_n \leq_\Omega 0$, again by Lemma E.1, $\tilde{Z}_n \in \mathcal{Z}$. By taking the limit for $n \to \infty$ we obtain $(\ell_G)^- \in \mathcal{Z}$ and thus, $\ell_G \in \mathcal{P}$. Since the financial market has no arbitrages $G_1 \cdot \Delta S_1 = Z \in \mathcal{Z}$ and hence one asset is redundant. Consider a partition Ω_2^i of Ω_1^C on which $G_1^i \neq 0$. Since $\mathcal{Z}$ is stable under multiplication (Lemma E.2), for any $\ell^* \in \mathcal{I}$, there exists $Z^* \in \mathcal{Z}$ and $H^* \in \mathcal{L}^0(\Omega_2^i, \mathcal{F}_0)$ with $(H^*)^i = 0$, such that $\ell^* = H^* \cdot \Delta S_1 + Z^*$ on Ω_2^i. Therefore, the term ℓ_n in (C.1) is composed of trading strategies involving only $d-1$ assets. Iterating the procedure up to d-steps we have the conclusion.

Assuming now that C.1 holds for markets with $T-1$ periods, with the same argument we show that we can extend to markets with T periods. Set again $\Omega_1 := \{\omega \in \Omega \mid \liminf |H_1^n| < \infty\}$. Since on Ω_1 we have that,

$$W_n - H_1^n \cdot \Delta S_1 = \sum_{t=2}^{T} H_t^n \cdot \Delta S_t - K_n - Z_n \to W - H_1 \cdot \Delta S_1.$$

The induction hypothesis allows to conclude that $W - H_1 \cdot S_1 \in \mathcal{C}$ and therefore $W \in \mathcal{C}$. On Ω_1^C we may take $G_1^n := H_1^n/|H_1^n|$ and let $G_1 := \liminf G_1^n \chi_{\Omega_1^C}$. Note that $W_n/|H_1^n| \to 0$ and hence

$$\sum_{t=2}^{T} \frac{H_t^n}{|H_1^n|} \cdot \Delta S_t - \frac{K_n}{|H_1^n|} - \frac{Z_n}{|H_1^n|} \to -G_1 \cdot \Delta S_1.$$

Since $\mathcal{Z}$ is stable under multiplication $\frac{Z_n}{|H_1^n|} \in \mathcal{Z}$ and hence, by inductive hypothesis, there exists $\tilde{H}_t$ for $t = 2, \ldots, T$ and $\tilde{Z} \in \mathcal{Z}$ such that

$$\tilde{\ell} := G_1 \cdot \Delta S_1 + \sum_{t=2}^{T} \tilde{H}_t \cdot \Delta S_t \geq_\Omega \tilde{Z} \in \mathcal{Z}.$$

The No Arbitrage condition implies that $\tilde{\ell} \in \mathcal{Z}$. Once again, this means that one asset is redundant and, by considering a partition Ω_2^i of Ω_1^C on which $G_1^i \neq 0$, we can rewrite the term ℓ_n in (C.1) with $d-1$ assets. Iterating the procedure up to d-steps we have the conclusion. *Q.E.D.*

The above result is consistent with the fact that in classical "probabilistic" model for finite discrete-time markets only the no-arbitrage condition and not the no-free lunch condition has been utilized.

APPENDIX D: COUNTABLY ADDITIVE MEASURES

In this section, we show that in general finite discrete time markets, it is possible to characterize viability through countably additive functionals. We prove this result by combining some results from Burzoni, Frittelli, Hou, Maggis, and Obłój (2017) which we collect in Appendix

E.2. We refer to that paper for the precise technical requirements for $(\Omega, \mathbb{F}, S)$, we only point out that, in addition to the previous setting, Ω needs to be a Polish space.

We let $\mathcal{Q}^{ca}$ be the set of countably additive positive probability measures $\mathbb{Q}$, with finite support, such that S is a $\mathbb{Q}$-martingale and $\mathcal{Z}^- := \{-Z^- \mid Z \in \mathcal{Z}\}$. For $X \in \mathcal{H}$, set

$$\mathcal{Z}(X) := \left\{ Z \in \mathcal{Z}^- \ : \ \exists \ell \in \mathcal{I} \quad \text{such that} \quad \mathcal{D}(X) + \ell \geq_\Omega X + Z \ \right\},$$

which is always non-empty when $\mathcal{D}(X)$, e.g. $\forall X \in \mathcal{B}_b$. By the lattice property of $\mathcal{Z}$, if $\mathcal{D}(X) + \ell \geq_\Omega X + Z$ the same is true if we take $Z = Z^-$. From Theorem C.5 we know that, under no arbitrage, the attainment property holds and, hence, $\mathcal{Z}(X)$ is non-empty for every $X \in \mathcal{H}$. For $A \in \mathcal{F}$, we define

$$\mathcal{D}_A(X) := \inf \{c \in \mathbb{R} \ : \exists \ell \in \mathcal{I} \quad \text{such that} \quad c + \ell(\omega) \geq X(\omega), \ \forall \omega \in A \ \}$$
$$\mathcal{Q}^{ca}_A := \{\mathbb{Q} \in \mathcal{Q}^{ca} \ : \ \mathbb{Q}(A) = 1 \ \}.$$

We need the following technical result in the proof of the main Theorem.

PROPOSITION D.1 *Suppose Assumption C.3 holds and the financial market has no arbitrages. Then, for every $X \in \mathcal{H}$ and $Z \in \mathcal{Z}(X)$, there exists $A_{X,Z}$ such that*

$$A_{X,Z} \subset \{ \ \omega \in \Omega \ : \ Z(\omega) = 0 \ \}, \tag{D.1}$$

and

$$\mathcal{D}(X) = \mathcal{D}_{A_{X,Z}}(X) = \sup_{\mathbb{Q} \in \mathcal{Q}^{ca}_{A_{X,Z}}} \mathbb{E}_{\mathbb{Q}}[X].$$

Before proving this result, we state the main result of this section.

THEOREM D.2 *Suppose Assumption C.3 holds. Then, the financial market has no arbitrages if and only if for every $(Z, R) \in \mathcal{Z}^- \times \mathcal{P}^+$ there exists $\mathbb{Q}_{Z,R} \in \mathcal{Q}^{ca}$ satisfying*

$$\mathbb{E}_{\mathbb{Q}_{Z,R}}[R] > 0 \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}_{Z,R}}[Z] = 0. \tag{D.2}$$

PROOF: Suppose that the financial market has no arbitrages. Fix $(Z, R) \in \mathcal{Z}^- \times \mathcal{P}^+$ and $Z_R \in \mathcal{Z}(R)$. Set $Z^* := Z_R + Z \in \mathcal{Z}(R)$. By Proposition D.1, there exists $A_* := A_{R,Z^*}$ satisfying the properties listed there. In particular,

$$0 < \mathcal{D}(R) = \sup_{\mathbb{Q} \in \mathcal{Q}^{ca}_{A_*}} \mathbb{E}_{\mathbb{Q}}[R].$$

Hence, there is $\mathbb{Q}^* \in \mathcal{Q}^{ca}_{A_*}$ so that $\mathbb{E}_{\mathbb{Q}^*}[R] > 0$. Moreover, since $Z_R, Z \in \mathcal{Z}^-$,

$$A_* \subset \{Z^* = 0\} = \{Z_R = 0\} \cap \{Z = 0\}.$$

In particular, $\mathbb{E}_{\mathbb{Q}^*}[Z] = 0$.

To prove the opposite implication, suppose that there exists $R \in \mathcal{P}^+$, $\ell \in \mathcal{I}$ and $Z \in \mathcal{Z}$ such that $\ell \geq_\Omega R + Z$. Then, it is clear that $\ell \geq_\Omega R - Z^-$. Let $\mathbb{Q}^* := \mathbb{Q}_{-Z^-,R} \in \mathcal{Q}^{ca}$ satisfying (D.2). By integrating both sides against $\mathbb{Q}^*$, we obtain

$$0 = \mathbb{E}_{\mathbb{Q}^*}[\ell] \geq \mathbb{E}_{\mathbb{Q}^*}[R - Z^-] = \mathbb{E}_{\mathbb{Q}^*}[R] > 0.$$

which is a contradiction. Thus, there are no arbitrages. *Q.E.D.*

We continue with the proof of Proposition D.1.

Proof of Proposition D.1: Since there are no arbitrages, by Theorem C.5 we have the attainment property. Hence, for a given $X \in \mathcal{H}$, the set $\mathcal{Z}(X)$ is non-empty.

Step 1. We show that, for any $Z \in \mathcal{Z}(X)$, $\mathcal{D}(X) = \mathcal{D}_{\{Z=0\}}(X)$.
Note that, since $\mathcal{D}(X) + \ell \geq_\Omega X + Z$, for some $\ell \in \mathcal{I}$, the inequality $\mathcal{D}_{\{Z=0\}}(X) \leq \mathcal{D}(X)$ is always true. Towards a contradiction, suppose that the inequality is strict, namely, there exist $c < \mathcal{D}(X)$ and $\tilde{\ell} \in \mathcal{I}$ such that $c + \tilde{\ell}(\omega) \geq X(\omega)$ for any $\omega \in \{Z = 0\}$. We show that

$$\tilde{Z} := (c + \tilde{\ell} - X)^- \chi_{\{Z<0\}} \in \mathcal{Z}.$$

This together with $c + \tilde{\ell} \geq_\Omega X + \tilde{Z}$ yields a contradiction. Recall that $\mathcal{Z}$ is a linear space so that $nZ \in \mathcal{Z}$ for any $n \in \mathbb{N}$. From $nZ \leq_\Omega \tilde{Z} \vee (nZ) \leq_\Omega 0$, we also have $\tilde{Z}_n := \tilde{Z} \vee (nZ) \in \mathcal{Z}$, by Lemma E.1. By noting that $\{\tilde{Z} < 0\} \subset \{Z < 0\}$ we have that $\tilde{Z}_n(\omega) \to \tilde{Z}(\omega)$ for every $\omega \in \Omega$. From the closure of $\mathcal{Z}$ under pointwise convergence, we conclude that $\tilde{Z} \in \mathcal{Z}$.

Step 2. For a given set $A \in \mathcal{F}_T$, we let $A^* \subset A$ be the set of scenarios visited by martingale measures (see (E.2) in the Appendix for more details). We show that, for any $Z \in \mathcal{Z}(X)$, $\mathcal{D}(X) = \mathcal{D}_{\{Z=0\}^*}(X)$.

Suppose that $\{Z = 0\}^*$ is a proper subset of $\{Z = 0\}$ otherwise, from Step 1, there is nothing to show. From Lemma E.6 there is a strategy $\tilde{\ell} \in \mathcal{I}$ such that $\tilde{\ell} \geq 0$ on $\{Z = 0\}$[24]. Lemma E.5 (and in particular (E.4)) yield a finite number of strategies $\ell_1^t, \dots \ell_{\beta_t}^t$ with $t = 1, \dots T$, such that

$$\text{(D.3)} \qquad \{\hat{Z} = 0\} = \{Z = 0\}^* \qquad \text{where} \qquad \hat{Z} := Z - \sum_{t=1}^{T} \sum_{i=1}^{\beta_t} \chi_{\{Z=0\}} (\ell_i^t)^+ \ .$$

Moreover, for any $\omega \in \{Z = 0\} \setminus \{Z = 0\}^*$, there exists (i, t) such that $\ell_i^t(\omega) > 0$. We are going to show that, under the no arbitrage hypothesis, $\ell_i^t \in \mathcal{Z}$ for any $i = 1, \dots \beta_t$, $t = 1, \dots T$. In particular, from the lattice property of the linear space $\mathcal{Z}$, we have $\hat{Z} \in \mathcal{Z}$.

We illustrate the reason for $t = T$, by repeating the same argument up to $t = 1$ we have the thesis. We proceed by induction on i. Start with $i = 1$. From Lemma E.5 we have that $\ell_i^T \geq 0$ on $\{Z = 0\}$ and, therefore, $\{\ell_1^T < 0\} \subseteq \{Z < 0\}$. Define $\tilde{Z} := -(\ell_1^T)^- \leq_\Omega 0$. By using the same argument as in Step 1, we observe that $nZ \leq_\Omega \tilde{Z} \vee (nZ) \leq_\Omega 0$ with $nZ \in \mathcal{Z}$ for any $n \in \mathbb{N}$. From $\{\ell_1^T < 0\} \subseteq \{Z < 0\}$ and the closure of $\mathcal{Z}$ under pointwise convergence, we conclude that $\tilde{Z} \in \mathcal{Z}$. From no arbitrage, we must have $\ell_1^T \in \mathcal{Z}$.
Suppose now that $\ell_j^T \in \mathcal{Z}$ for every $1 \leq j \leq i - 1$. From Lemma E.5, we have that $\ell_i^T \geq 0$ on $\{Z - \sum_{j=1}^{i-1} \ell_i^T = 0\}$ and, therefore, $\{\ell_i^T < 0\} \subseteq \{Z - \sum_{j=1}^{i-1} \ell_i^T < 0\}$. The argument of Step 1 allows to conclude that $\ell_i^T \in \mathcal{Z}$.

We are now able to show the claim. The inequality $\mathcal{D}_{\{Z=0\}^*}(X) \leq \mathcal{D}_{\{Z=0\}}(X) = \mathcal{D}(X)$ is always true. Towards a contradiction, suppose that the inequality is strict, namely, there exist $c < \mathcal{D}(X)$ and $\tilde{\ell} \in \mathcal{I}$ such that $c + \tilde{\ell}(\omega) \geq X(\omega)$ for any $\omega \in \{Z = 0\}^*$. We show that

$$\tilde{Z} := (c + \tilde{\ell} - X)^- \chi_{\Omega \setminus \{Z=0\}^*} \in \mathcal{Z}.$$

This together with $c + \tilde{\ell} \geq_\Omega X + \tilde{Z}$, yields a contradiction. To see this recall that, from the above argument, $\hat{Z} \in \mathcal{Z}$ with $\hat{Z}$ as in (D.3). Moreover, again by (D.3), we have $\{\tilde{Z} < 0\} \subset \{\hat{Z} < 0\}$. The argument of Step 1 allows to conclude that $\tilde{Z} \in \mathcal{Z}$.

Step 3. We are now able to conclude the proof. Fix $Z \in \mathcal{Z}(X)$ and set $A_{X,Z} := \{Z = 0\}^*$. Then,

$$\mathcal{D}(X) = \mathcal{D}_{\{Z=0\}}(X) = \mathcal{D}_{(A_{X,Z})^*}(X) = \sup_{Q \in \mathcal{Q}^{ca}_{A_{X,Z}}} \mathbb{E}_Q[X],$$

where the first two equalities follow from Step 1 and Step 2 and the last equality follows from Proposition E.7. *Q.E.D.*

[24]Note that restricted to $\{Z = 0\}$ this strategy yields no risk and possibly positive gains, in other words, this is a good candidate for being an arbitrage.

APPENDIX E: SOME TECHNICAL TOOLS

E.1. *Preferences*

We start with a simple but a useful condition for negligibility.

LEMMA E.1 *Consider two negligible contracts $\hat{Z}, \tilde{Z} \in \mathcal{Z}$. Then, any contract $Z \in \mathcal{H}$ satisfying $\hat{Z} \le Z \le \tilde{Z}$ is negligible as well.*

PROOF: By definitions, we have,

$$X \preceq X + \hat{Z} \preceq X + Z \preceq X + \tilde{Z} \preceq X \quad \Rightarrow \quad X \sim X + Z.$$

Thus, $Z \in \mathcal{Z}$. *Q.E.D.*

LEMMA E.2 *Suppose that $\mathcal{Z}$ is closed under pointwise convergence. Then, $\mathcal{Z}$ is stable under multiplication, i.e., $ZH \in \mathcal{Z}$ for any $H \in \mathcal{H}$.*

PROOF: Note first that $Z_n := Z((H \wedge n) \vee -n) \in \mathcal{Z}$. This follows from by Lemma E.1 and the fact that $\mathcal{Z}$ is a cone. By taking the limit for $n \to \infty$, the result follows. *Q.E.D.*

We next prove that $\mathcal{E}(Z) = 0$ for every $Z \in \mathcal{Z}$.

LEMMA E.3 *Let $\mathcal{E}$ be a sublinear expectation. Then,*

$$\begin{aligned} \mathcal{E}(c + \lambda[X+Y]) &= c + \mathcal{E}(\lambda[X+Y]) = c + \lambda\mathcal{E}(X+Y) \\ &\le c + \lambda\,[\,-(-\mathcal{E}(X) - \mathcal{E}(Y))]\,, \end{aligned} \tag{E.1}$$

for every $c \in$, $\lambda \ge 0$, $X, Y \in \mathcal{H}$. In particular,

$$\mathcal{E}(Z) = 0, \quad \forall\ Z \in \mathcal{Z}.$$

PROOF: Let $X, Y \in \mathcal{H}$. The sub-additivity of $U_{\mathcal{E}}$ implies that

$$U_{\mathcal{E}}(X') + U_{\mathcal{E}}(Y') \le U_{\mathcal{E}}(X' + Y'), \quad \forall\ X', Y' \in \mathcal{H},$$

even when they take values $\pm\infty$. The definition of $U_{\mathcal{E}}$ now yields,

$$\mathcal{E}(X+Y) = -U_{\mathcal{E}}(-X-Y) \le -\left[U_{\mathcal{E}}(-X) + U_{\mathcal{E}}(-Y)\right] = -\left(-\mathcal{E}(X) - \mathcal{E}(Y)\right).$$

Then, (E.1) follows directly from the definitions.

Let $Z \in \mathcal{Z}$. Then, $-Z, Z \in \mathcal{P}$ and $\mathcal{E}(Z), \mathcal{E}(-Z) \ge 0$. Since $-Z \in \mathcal{P}$, the monotonicity of $\mathcal{E}$ implies that $\mathcal{E}(X - Z) \ge \mathcal{E}(X)$ for any $X \in \mathcal{H}$. Choose $X = Z$ to arrive at

$$0 = \mathcal{E}(0) = \mathcal{E}(Z - Z) \ge \mathcal{E}(Z) \ge 0.$$

Hence, $\mathcal{E}(Z)$ is equal to zero. *Q.E.D.*

E.2. *Finite Time Markets*

We here recall some results from Burzoni, Frittelli, Hou, Maggis, and Obłój (2017) (see Section 2 therein for the precise specification of the framework). We are given a filtered space $(\Omega, \mathbb{F}, \mathcal{F})$ with Ω a Polish space and $\mathbb{F}$ containing the natural filtration of a Borel-measurable process S. We denote by $\mathcal{Q}$ the set of martingale measures for the process S, whose support is a finite number of points. For a given set $A \in \mathcal{F}$, $\mathcal{Q}_A = \{Q \in \mathcal{Q} \mid Q(A) = 1\}$. We define the set of scenarios charged by martingale measures as

$$A^* := \{\omega \in \Omega \mid \exists Q \in \mathcal{Q}_A \text{ s.t. } Q(\omega) > 0\} = \bigcup_{Q \in \mathcal{Q}_A} supp(Q). \tag{E.2}$$

DEFINITION E.4 We say that $\ell \in \mathcal{I}$ is a one-step strategy if $\ell = H_t \cdot (S_t - S_{t-1})$ with $H_t \in \mathcal{L}(X, \mathcal{F}_{t-1})$ for some $t \in \{1, \ldots, T\}$. We say that $a \in \mathcal{I}$ is a one-point Arbitrage on A iff $a(\omega) \geq 0 \; \forall \omega \in A$ and $a(\omega) > 0$ for some $\omega \in A$.

The following Lemma is crucial for the characterization of the set A^* in terms of arbitrage considerations.

LEMMA E.5 *Fix any $t \in \{1, \ldots, T\}$ and $\Gamma \in \mathcal{F}$. There exist an index $\beta \in \{0, \ldots, d\}$, one-step strategies $\ell^1, \ldots, \ell^\beta \in \mathcal{I}$ and $B^0, \ldots, B^\beta$, a partition of Γ, satisfying:*

1. *if $\beta = 0$ then $B^0 = \Gamma$ and there are No one-point Arbitrages, i.e.,*

$$\ell(\omega) \geq 0 \; \forall \omega \in B^0 \Rightarrow \ell(\omega) = 0 \; \forall \omega \in B^0.$$

2. *if $\beta > 0$ and $i = 1, \ldots, \beta$ then:*
   - ▷ $B^i \neq \emptyset$;
   - ▷ $\ell^i(\omega) > 0$ *for all* $\omega \in B^i$
   - ▷ $\ell^i(\omega) \geq 0$ *for all* $\omega \in \cup_{j=i}^{\beta} B^j \cup B^0$.

We are now using the previous result, which is for some fixed t, to identify A^*. Define

$$A_T := A$$

$$A_{t-1} := A_t \setminus \bigcup_{i=1}^{\beta_t} B_t^i, \quad t \in \{1, \ldots, T\}, \tag{E.3}$$

where $B_t^i := B_t^{i,\Gamma}$, $\beta_t := \beta_t^\Gamma$ are the sets and index constructed in Lemma E.5 with $\Gamma = A_t$, for $1 \leq t \leq T$. Note that, for the corresponding strategies ℓ_i^t we have

$$A_0 = \bigcap_{t=1}^{T} \bigcap_{i=1}^{\beta_t} \{\ell_i^t = 0\}. \tag{E.4}$$

LEMMA E.6 *A_0 as constructed in (E.3) satisfies $A_0 = A^*$. Moreover, No one-point Arbitrage on $A \Leftrightarrow A^* = A$.*

PROPOSITION E.7 *Let $A \in \mathcal{F}$. We have that for any $\mathcal{F}$-measurable random variable g,*

$$\pi_{A^*}(g) = \sup_{Q \in \mathcal{Q}_A} \mathbb{E}_Q[g]. \tag{E.5}$$

with $\pi_{A^*}(g) = \inf\{x \in \mathbb{R} \mid \exists a \in \mathcal{I}$ such that $x + a_T(\omega) \geq g(\omega) \; \forall \omega \in A^*\}$. In particular, the left hand side of (E.5) is attained by some strategy $a \in \mathcal{I}$.

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