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Asymptotic Expansions for Markov Processes with Lévy Generators*

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Abstract. This paper considers a deterministic flow in n-dimensional space, perturbed by a Markov jump process with small variance. Asymptotic expansions are obtained for certain functionals of Feynman-Kac type, in powers of a small parameter representing a noise intensity. The methods are analytical rather than probabilistic.

1. Introduction

In this paper we consider a deterministic flow on R^n perturbed by a Markov jump process, with a small variance. Our purpose is to obtain asymptotic expansions for certain quantities, as the size of the noise intensity tends to zero. We start with a description of the random process.

For a small constant $\varepsilon > 0$, define an operator $\mathscr{L}^\varepsilon$ by

$$\mathscr{L}^\varepsilon \varphi(x) = b(x)\cdot\nabla\varphi(x) + \frac{1}{\varepsilon}\int_{R^n\setminus\{0\}} [\varphi(x+\varepsilon y) - \varphi(x) - \varepsilon y\cdot\nabla\varphi(x)]m(x, dy), \tag{1.1}$$

where φ is a smooth function on R^n, $b(x)\in R^n$, and $m(x, dy)$ is a positive Borel measure on $R^n\setminus\{0\}$. Further, we assume that there are $c(x, z)\in R^n$, $f(x, z)\geq 0$

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satisfying

$$m(x, A) = \int_{\{z\in R^n\setminus\{0\}: c(x,z)\in A\}} f(x, z)\,\frac{dz}{|z|^{n+1}}, \tag{1.2}$$

$$\sup_x \int |c(x, z)|^2 f(x, z)\,\frac{dz}{|z|^{n+1}} < \infty \tag{1.3}$$

for every Borel subset A of $R^n\setminus\{0\}$ (see the Appendix for a discussion of this condition). Using the substitution $y = c(x, z)$ the integral (1.1) can be written as an integral in z. This form for $\mathcal{L}^\varepsilon\varphi$ is used by some authors. We consider the Markov process $y^\varepsilon(t)$ generated by $\mathcal{L}^\varepsilon$. Such a process can be obtained as the solution of a stochastic integral equation driven by a Cauchy process (as in [14]), or by solving the associated martingale problem (as in [16]). The parameter $\varepsilon > 0$ measures the variance of $y^\varepsilon(\cdot)$, and as ε tends to zero $y^\varepsilon(t)$ converges weakly to the solution of

$$\frac{d}{dt}\,y^0(t) = b(y^0(t)).$$

The quantity we are interested in is a Feynman–Kac-type functional of $y^\varepsilon(\cdot)$. Namely,

$$u^\varepsilon(x, t) = -\varepsilon \ln E_{x,t}\, e^{-(1/\varepsilon)g(y^\varepsilon(T))}, \qquad x\in R^n, \quad t\in[0, T], \tag{1.4}$$

where $E_{x,t}$ is the mathematical expectation conditioned on $y^\varepsilon(t) = x$, and $g(x)\in R$.

The convergence of $u^\varepsilon(x, t)$ as ε tends to zero, or equivalently the first term in the asymptotic expansion of $u^\varepsilon(x, t)$, is a special case of the theory of large deviations. The reader may refer to Freidlin and Wentzel [7], Stroock [17], and Varadhan [18].

This problem arises in a number of applications, such as macroscopic chemical kinetics, queueing theory, production planning, and large traffic networks. We refer to Parekh and Walrand [13], Weiss [19], and Knessl *et al.* [8]–[12]. In [8]–[10] formal expansions for quantities like $u^\varepsilon(x, t)$ and exit times were obtained.

The function $u^\varepsilon(x, t)$ satisfies the following nonlinear integrodifferential equation:

$$-\frac{\partial}{\partial t}\,u^\varepsilon(x, t) + \mathcal{H}^\varepsilon(x, \nabla u^\varepsilon(x, t), u^\varepsilon(\cdot, t)) = 0, \qquad x\in R^n, \quad t\in[0, T), \tag{1.5$^\varepsilon$}$$

$$u^\varepsilon(x, T) = g(x), \qquad x\in R^n, \tag{1.6}$$

where, for $x, p\in R^n$ and $\varphi\in C_b^2$; the space to twice continuously differentiable functions on R^n, which are bounded along with their derivatives of order up to two,

$$\mathcal{H}^\varepsilon(x, p, \varphi) = -b(x)\cdot p + \int_{R^n\setminus\{0\}} [e^{-(1/\varepsilon)[\varphi(x+\varepsilon y)-\varphi(x)]} - 1 + y\cdot\nabla\varphi(x)]\,m(x, dy).$$

In Section 3 we show that, under certain assumptions on the coefficients (Lemma 3.1),

$$\sup_{\varepsilon}\|u^{\varepsilon}\|_{W^{1,\infty}(R^n\times(0,T))}\le K, \tag{1.7}$$

$$u^{\varepsilon}(x+y,t)+u^{\varepsilon}(x-y,t)-2u^{\varepsilon}(x,t)\le K|x-y|^2, \tag{1.8}$$

where $W^{1,\infty}(R^n\times(0,T))$ denotes the Sobolev space of Lipschitz continuous, bounded functions. The inequality (1.7) together with the theory of viscosity solutions imply that $u^{\varepsilon}(x,t)$ tends to a limit $u^0(x,t)$, uniformly on bounded subsets of $R^n\times[0,T]$, as ε approaches zero. Moreover, $u^0(x,t)$ is the unique solution of the following equation:

$$-\frac{\partial}{\partial t}u^0(x,t)+\mathscr{H}(x,\nabla u^0(x,t))=0, \qquad (x,t)\in R^n\times[0,T], \tag{1.9}$$

with (1.6). Here,

$$\mathscr{H}(x,p)=-b(x)\cdot p+\int_{R^n\setminus\{0\}}[e^{-y\cdot p}-1+y\cdot p]m(x,dy). \tag{1.10}$$

Under the assumptions of Section 2, $\mathscr{H}(x,p)$ is a C^{∞} function. Still, the limit $u^0(x,t)$, in general, is not of class C^1. However, there is an open, dense subset N of $R^n\times[0,T]$ on which $u^0(x,t)$ is C^{∞}-smooth. And, due to (1.8) $\nabla u^{\varepsilon}(x,t)$ tends to $\nabla u^0(x,t)$ on N.

We can now state the main result of this paper:

Theorem 5.1. *Suppose* (A1)–(A6) *hold. Then, for any positive integer m,*

$$u^{\varepsilon}(x,t)=u^0(x,t)+\varepsilon u^1(x,t)+\cdots+\varepsilon^m u^m(x,t)+o(\varepsilon^m) \tag{1.11}$$

uniformly in any compact subset of N.

The coefficients $u^m(x)$ are smooth functions, and each u^m is given by a functional of $u^0,\dots,u^{m-1}$. A more precise statement can be found in Section 5.

The nonlinearity in equation $(1.5)^{\varepsilon}$ for u^{ε} has the special form (1.7). This is because u^{ε} is obtained via a logarithmic transformation (1.4) from a solution I^{ε} to a linear evolution equation (see (2.2) below). We have results similar to Theorem 5.1 when the nonlinearity $\mathscr{H}^{\varepsilon}$ takes a more general form, of the kind appearing in the dynamic programming equation for controlled Markov processes with generators of the sort appearing in (1.1). For certain technical reasons we have considered only $(1.5)^{\varepsilon}$ with the Cauchy data (1.6). It would be of interest to extend the results to consider $(1.5)^{\varepsilon}$ in a cylinder $D\times[0,T)$, with u^{ε} given on $(\partial D\times[0,T))\cup(D\times\{T\})$. This corresponds to stopping the process y^{ε} at time $\min(\tau^{\varepsilon},T)$, where τ^{ε} is the exit time from D. It should be remarked that the asymptotic series expansion for solutions u^{ε} to this boundary problem are not well understood in the case of nearly deterministic diffusions. For nearly deterministic diffusions, the operator in (1.1) takes the form

$$\mathscr{L}^{\varepsilon}\varphi(x)=b(x)\cdot\nabla\varphi(x)+\varepsilon\mathscr{M}(\varphi),$$

where $\mathcal{M}(\varphi)$ is a second-order elliptic partial differential operator. Recently Fleming and Souganidis [6] obtained an asymptotic expansion for solutions to some equations of the form

$$0=\varepsilon\Delta u^{\varepsilon}+H(x,\nabla u^{\varepsilon}),$$

arising in stochastic control. Our methods are in part adapted from [6].

The asymptotic series (1.11) is equivalent to a WKB-type expansion for the function $I^{\varepsilon}=\exp[-\varepsilon^{-1}u^{\varepsilon}]$:

$$I^{\varepsilon}=\exp[-\varepsilon^{-1}u^{0}](v_0+\varepsilon v_1+\cdots+\varepsilon^{m}v_m+o(\varepsilon^{m})),$$

with leading term $v_0=\exp(-u^1)\neq 0$. For the case of nearly deterministic diffusions, Azencott [1] obtained asymptotic series expansions of WKB type by probabilistic methods. In these expansions, the leading term in general depends on ε.

The paper is organized as follows: assumptions are stated in the next section, in Section 3 estimates (1.7) and (1.8) are obtained, Section 4 is devoted to a sequence of "almost" linear equations related to the higher-order terms in the expansion; finally the main result is proved in Section 5. Also, a brief discussion of condition (1.2) is given in the Appendix.

2. Assumptions

In whatever follows, C_b^{∞} denotes the space of infinitely differentiable functions, which are bounded along with their derivatives, $\pi(dz)=dz|z|^{-(n+1)}$, for $\psi\in C_b^{\infty}(R^n)$ and any multi-index $\alpha=(\alpha_1,\ldots,\alpha_n)$, $D^{\alpha}\psi(x)=(\partial^{|\alpha|}/\partial x_2^{\alpha_1}\cdots\partial x_n^{\alpha_n})\psi(x)$, $|\alpha|=\sum\alpha_i$. We assume:

(A1) $b(\cdot),\ g(\cdot),\ c(\cdot,z),\ f(\cdot,z)\in C_b^{\infty}(R^n)$,

(A2) $\displaystyle\sup_{x\in R^n}\int_{R^n\setminus\{0\}}|D^{\alpha}c(x,z)|f(x,z)\pi(dz)\le K,\qquad \forall|\alpha|\ge 1,$

(A3) $\displaystyle\int_{R^n\setminus\{0\}}\sup_{x\in R^n}|D^{\alpha}f(x,z)|\pi(dz)\le K,\qquad \forall|\alpha|\ge 1,$

(A4) $\displaystyle\lim_{\delta\downarrow 0}\sup_{x\in R^n}\int_{0<|z|\le\delta}|c(x,z)|^2f(x,z)\pi(dz)=0,$

(A5) $\displaystyle\sup_{z,x\in R^n}[|D^{\alpha}f(x,z)|+|D^{\alpha}c(x,z)|]\le K,\qquad \forall|\alpha|\ge 0.$

Lemma 2.1. *Suppose* (A1)–(A5) *hold. Then there exists a unique solution* $u^{\varepsilon}(x,t)$ *of* $(1.5)^{\varepsilon}$ *and* (1.6). *Moreover,* $u^{\varepsilon}\in C_b^{\infty}(R^n\times(0,T))$.

Outline of the Proof. Consider the function

$$w(x,t)=e^{-(1/\varepsilon)u^{\varepsilon}(x,t)}. \tag{2.1}$$

Then, $w(x, t)$ solves

$$-\frac{\partial}{\partial t} w(x, t) - \mathscr{L}^{\varepsilon} w(x, t) = 0, \qquad (x, t) \in R^n \times [0, T), \tag{2.2}$$

$$w(x, T) = e^{-(1/\varepsilon) g(x)}, \qquad x \in R^n. \tag{2.3}$$

Approximate equation (2.2) by

$$-\frac{\partial}{\partial t} w(x, t) - \mathscr{L}^{\varepsilon}_{\delta} w(x, t) = 0, \qquad (x, t) \in R^n \times [0, T), \tag{2.2$^\delta$}$$

where $\delta > 0$, and for a smooth function φ

$$\begin{aligned}\mathscr{L}^{\varepsilon}_{\delta}\varphi(x) &= b(x)\cdot\nabla\varphi(x) \\ &\quad + \int_{|z|\ge\delta} \frac{1}{\varepsilon}[\varphi(x+\varepsilon c(x, z)) - \varphi(x) - \varepsilon\nabla\varphi(x)\cdot c(x, z)] f(x, z)\pi(dz) \\ &= \nabla\varphi(x)\cdot\left[b(x) - \int_{|z|\ge\delta} c(x, z) f(x, z)\pi(dz)\right] \\ &\quad + \frac{1}{\varepsilon}\int_{|z|\ge\delta}[\varphi(x+\varepsilon c(x, z)) - \varphi(x)] f(x, z)\pi(dz).\end{aligned} \tag{2.4}$$

Since $(2.2)^\delta$ is a linear equation and the last term in (2.4) is bounded, it is easy to show the existence of a C^∞-smooth solution w^δ of $(2.2)^\delta$ and (2.3).

We obtain next sup-norm estimates of the derivatives of w^δ. The maximum principle yields

$$w^\delta(x, t) \ge \exp\left(-\frac{1}{\varepsilon}\|g\|_\infty\right) \tag{2.5}$$

and

$$w^\delta(x, t) \le \exp\left(\frac{1}{\varepsilon}\|g\|_\infty\right).$$

Also, $w_i(x, t) = (\partial/\partial x_i) w^\delta(x, t)$ satisfies the following equation:

$$\begin{aligned}&-\frac{\partial}{\partial t} w_i(x, t) - \mathscr{L}^{\varepsilon}_{\delta} w_i(x, t) - \sum_{j=1}^{n} w_j(x, t)\frac{\partial}{\partial x_i} b_j(x, t) \\ &\quad - \sum_{j=1}^{n}\int_{|z|\ge\delta}(w_j(x+\varepsilon c(x, z), t) - w_j(x, t))\frac{\partial}{\partial x_i} c_j(x, t) f(x, z)\pi(dz) \\ &\quad - \frac{1}{\varepsilon}\int_{|z|\ge\delta}(w(x+\varepsilon c(x, z), t) - w(x, t) - \varepsilon\nabla w(x, t) c(x, t)) \\ &\quad \times\frac{\partial}{\partial x_i} f(x, z)\pi(dz) = 0\end{aligned} \tag{2.6^i}$$

with terminal data

$$w_i(x, T) = -\frac{1}{\varepsilon}\frac{\partial}{\partial x_i} g(x)\, e^{-(1/\varepsilon) g(x)}, \qquad x \in R^n. \tag{2.7^i}$$

Suppose that there are $(x_0, t_0) \in R^n \times [0, T]$ and $i_0 \in \{1, \ldots, n\}$ satisfying

$$e^{\rho t_0} w_{i_0}(x_0, t_0) = \sup\{e^{\rho t}|w_i(x, t)| : (x, t) \in R^n \times [0, T], i = 1, \ldots, n\}, \tag{2.8}$$

where $\rho > 0$ shall be chosen in the proof. We may assume that $t_0 < T$. Then, by using (2.8) and equation $(2.6)^{i_0}$ we obtain

$$\begin{aligned}
\rho w_{i_0}(x_0, t_0) &\leq -\frac{\partial}{\partial t} w_{i_0}(x_0, t_0) \\
&= \mathcal{L}^\varepsilon_\delta w_{i_0}(x_0, t_0) + \sum_j w_j(x_0, t_0) \frac{\partial}{\partial x_{i_0}} b_j(x_0, t_0) \\
&\quad + \sum_j \int_{|z| \geq \delta} [w_j(x_0 + \varepsilon c(x_0, z), t_0) - w_j(x_0, t_0)] \\
&\quad \times \frac{\partial}{\partial x_{i_0}} c_j(x_0, t) f(x_0, z) \pi(dz) \\
&\quad - \sum_j \int_{|z| \geq \delta} w_j(x_0, t_0) c_j(x_0, z) \frac{\partial}{\partial x_{i_0}} f(x_0, z) \pi(dz) \\
&\quad + \int_{|z| \geq \delta} \frac{1}{\varepsilon} [w(x_0 + \varepsilon c(x_0, z), t_0) - w(x_0, t_0)] \frac{\partial}{\partial x_{i_0}} f(x_0, z) \pi(dz).
\end{aligned}$$

Since $w_{i_0}(x_0, t_0) \geq |w_j(x, t_0)|$ for all x, and j,

$$\begin{aligned}
\rho w_{i_0}(x_0, t_0) \leq w_{i_0}(x_0, t_0) \Big[& n \|D^2 b\| + 2 \int |Dc(x_0, z)| f(x_0, z) \pi(dz) \\
& + \int |c(x_0, z)| |Df(x_0, z)| \pi(dz) \Big] \\
+ \frac{2}{\varepsilon} \|w\|_\infty & \int |Df(x_0, t)| \pi(dz).
\end{aligned}$$

In view of (A2), (A3), and (A5), there is $K > 0$, independent of δ, such that

$$\rho w_{i_0}(x_0, t_0) \leq K w_{i_0}(x_0, t_0) + \frac{K}{\varepsilon}.$$

An appropriate choice of ρ yields that $Dw^\delta(x, t)$ is bounded uniformly in δ, if such a point (x_0, t_0) exists. But we can always obtain such a point by using the difference quotients rather than w_i, and then perturbing them around appropriate points.

The higher-order derivative estimates can be proved similarly. Therefore, on a subsequence $w^\delta(x, t)$ converges to a C^∞-smooth function $w(x, t)$, uniformly on bounded subsets of $R^n \times [0, T]$. Furthermore, $w(x, t)$ solves (2.2) and (2.3). Finally, the function $u^\varepsilon(x, t) = -\varepsilon \ln w(x, t)$ is a solution of $(1.5)^\varepsilon$ and (1.6), and, due to the positivity of $w(x, t)$, (2.5), $u^\varepsilon \in C^\infty_b(R^n \times (0, T))$.

The uniqueness of $u^\varepsilon(x, t)$ follows from the maximum principle. We refer to Lemma 3.1 in [15] for a similar result. □

Our last assumption is a nondegeneracy condition. It is a direct analogue of the uniform ellipticity assumption, used in the diffusion case.

$$\text{(A6)}\quad \begin{cases} \text{There are } \rho \in (0,1) \text{ and } \delta > 0 \text{ satisfying} \\ \inf_{x,y\in R^n} m(x, K(y,\delta,\rho)) = c_0 > 0, \end{cases}$$

where $m(x,\cdot)$ is as in (1.2), and $K(y,\delta,\rho)$ is given by

$$K(y,\delta,\rho) = \{\nu \in R^n\colon |\nu| \geq \delta \text{ and } \nu\cdot y \leq -p|y||\nu|\}.$$

An immediate consequence of (A6) is the following. (Compare this result with (A_3) in [6].)

Lemma 2.2. *Suppose* (A6) *holds. Then, for any* $\nu \in R^n$,

$$\inf_{x\in R^n} \sum_{i,j=1} \nu_i\nu_j \frac{\partial^2}{\partial p_i\,\partial p_j} \mathscr{H}(x,p) \geq K|\nu|^2,$$

where $K > 0$ *is a suitable constant.*

3. Estimates

In this section we prove that the family $\{u^\varepsilon\}$ is uniformly Lipschitz continuous, and satisfy a uniform one-sided second-order estimate.

Lemma 3.1. *Suppose that* (A1)-(A6) *hold. Then* $u^\varepsilon(x,t)$ *satisfies* (1.7) *and* (1.8) *with a constant* $K > 0$ *independent of* ε.

Before we give the proof of Lemma 3.1, we indicate some consequences of it. The estimate (1.7) implies the precompactness of the set $\{u^\varepsilon\}$ in the weak topology of $W^{1,\infty}_{\text{loc}}$. Moreover, the theory of viscosity solutions [3, Theorem 1.4] yields that any limit point u^0 of $\{u^\varepsilon\}$, as ε tends to zero, is a viscosity solution to (1.9). Due to the uniqueness of solutions to (1.9) with terminal condition (1.6) (see [3] and [4]), we obtain that as ε approaches to zero, $u^\varepsilon(x,t)$ converges to $u^0(x,t)$ uniformly on compact subsets of $R^n\times[0,T]$.

Also, it is known that the estimate (1.8) yields information about the convergence of $\nabla u^\varepsilon(x,t)$ to $\nabla u^0(x,t)$. More precisely, we have the following result.

Theorem 3.2. *Let* $u^0(x,t)$ *be the unique viscosity solution of* (1.9) *and* (1.6). *Then* $u^\varepsilon(x,t)$ *converges to* $u^0(x,t)$ *uniformly on compact subsets of* $R^n\times[0,T]$. *Moreover, if* $u^0(x,t)$ *is differentiable at* (x,t), *then*

$$\lim_{\substack{(y,s)\to(x,t)\\ \varepsilon\downarrow 0}} \nabla u^\varepsilon(y,s) = \nabla u^0(x,t). \tag{3.1}$$

Proof. On account of (1.8),

$$u^\varepsilon(y+\eta,s) - u^\varepsilon(y,s) - \nabla u^\varepsilon(y,s)\cdot\eta \leq K|\eta|^2$$

for any $\eta \in R^n$. We refer to Lemma 4.2 of [2] for an elementary proof of this fact. Now pass to the limit in the above inequality as $\varepsilon \downarrow 0$ and $(y, s) \to (x, t)$. Then, any limit point $\bar{p}$ of $\nabla u^\varepsilon(y, s)$ satisfies the following:

$$u^0(x+\eta, t) - u^0(x, t) - \bar{p} \cdot \eta \leq K|\eta|^2.$$

Since $u^0(x, t)$ is differentiable at (x, t), and it satisfies (1.8), $\bar{p} = \nabla u^0(x, t)$ (see Corollary 4.12 in [2]). □

Proof of Lemma 3.1. Set $w^\varepsilon(x, t) = (\partial/\partial t)u^\varepsilon(x, t)$. Differentiating $(1.5)^\varepsilon$ with respect to t, we obtain

$$\begin{cases} -\dfrac{\partial}{\partial t} w^\varepsilon(x, t) + \displaystyle\int \left\{ \exp\left(-\frac{1}{\varepsilon}[u^\varepsilon(x+\varepsilon y, t) - u^\varepsilon(x, t)] \right) \right. \\ \qquad\qquad \times \left[\dfrac{1}{\varepsilon}(w^\varepsilon(x, t) - w^\varepsilon(x+\varepsilon y, t)) \right] \\ \qquad\qquad \left. + y \cdot \nabla w^\varepsilon(x, t) \right\} m(x, dy) = 0, \\ w^\varepsilon(x, t) = \mathcal{H}^\varepsilon(x, \nabla g(x), g(\cdot)). \end{cases} \tag{3.2$^\varepsilon$}$$

Since the constant functions $\phi^+ \equiv \sup \mathcal{H}^\varepsilon(x, \nabla g(x), g)$ and $\phi^- \equiv \inf \mathcal{H}^\varepsilon(x, \nabla g(x), g)$ are super and subsolutions of $(3.2)^\varepsilon$, the comparison principle yields

$$\|w^\varepsilon\|_\infty \leq \sup_x |\mathcal{H}^\varepsilon(x, \nabla g(x), g)| \leq K. \tag{3.3}$$

Let

$$c(1, \varepsilon) = \|\nabla u^\varepsilon\|_{L^\infty(R^n \times (0,T))}, \tag{3.4}$$

$$c(2, \varepsilon) = \sup_{x,y,t} [D^2 u^\varepsilon(x, t) y \cdot y] |y|^{-2}. \tag{3.5}$$

Using equation $(1.5)^\varepsilon$, and the inequality $1 - e^r \leq -r$, we obtain

$$\begin{aligned} -\frac{\partial}{\partial t} & u^\varepsilon(x, t) + \mathcal{H}(x, \nabla u^\varepsilon(x, t)) \\ &= \mathcal{H}(x, \nabla u^\varepsilon(x, t)) - \mathcal{H}^\varepsilon(x, \nabla u^\varepsilon(x, t), u^\varepsilon(\cdot, t)) \\ &= \int_{R^n \setminus \{0\}} e^{-y \cdot \nabla u^\varepsilon(x,t)} \\ &\qquad \times \left[1 - \exp\left(-\frac{1}{\varepsilon}(u^\varepsilon(x+\varepsilon y, t) - u^\varepsilon(x, t) - \varepsilon y \cdot \nabla u^\varepsilon(x, t)) \right) \right] m(x, dy) \\ &\leq \int_{R^n \setminus \{0\}} e^{-y \cdot \nabla u^\varepsilon(x,t)} \frac{1}{\varepsilon} [u^\varepsilon(x+\varepsilon y, t) - u^\varepsilon(x, t) - \varepsilon y \cdot \nabla u^\varepsilon(x, t)] m(x, dy). \end{aligned}$$

Also, the definition of $c(2, \varepsilon)$ yields

$$u^\varepsilon(x+\varepsilon y, t) - u^\varepsilon(x, t) - \varepsilon y \cdot \nabla u^\varepsilon(x, t) \leq \varepsilon^2 c(2, \varepsilon)|y|^2.$$

The above inequalities, (A.5), (1.3), and (3.3) yield

$$\begin{aligned}\mathcal{H}(x, \nabla u^\varepsilon(x,t)) &\le \frac{\partial}{\partial t} u^\varepsilon(x,t) + \varepsilon K c(2,\varepsilon)\, e^{Kc(1,\varepsilon)} \\ &\le K + \varepsilon K c(2,\varepsilon)\, e^{Kc(1,\varepsilon)}.\end{aligned}$$

Due to Lemma 2.2, there is a strictly increasing function $\gamma(r) > 0$ satisfying

$$\{p \in R^n : \sup_x \mathcal{H}(x,p) \le r\} \subset \{p \in R^n : |p| \le \gamma(r)\}, \qquad \forall r > 0.$$

Hence

$$c(1,\varepsilon) \le \gamma(K + \varepsilon K c(2,\varepsilon)\, e^{Kc(1,\varepsilon)}). \tag{3.6}$$

Next, we estimate $c(2, \varepsilon)$ in terms of $c(1, \varepsilon)$; see (3.9). Assume that there are $\bar{x}, \bar{y} \in R^n$, $\bar{t} \in [0, T]$ satisfying

$$\begin{aligned}&e^{\rho \bar{t}}[u^\varepsilon(\bar{x},\bar{t}) + u^\varepsilon(\bar{y},\bar{t}) - 2u^\varepsilon(\tfrac{1}{2}(\bar{x}+\bar{y}),\bar{t})] - c_0|\bar{x}-\bar{y}|^2 \\ &\quad = \max_{x,y,t} \{e^{\rho t}[u^\varepsilon(x,t) + u^\varepsilon(y,t) - 2u^\varepsilon(\tfrac{1}{2}(x+y),t)] - c_0|x-y|^2\},\end{aligned}$$

where $c_0 \ge e^{\rho t}\|D^2 g\|_\infty$, and $\rho > 0$ shall be chosen in the proof. We can always obtain such points $\bar{x}$, $\bar{y}$, $\bar{t}$, by slightly perturbing u^ε. We refer to the proof of Theorem 4.1 of [3] for details. The next step in the proof is to obtain the following estimate:

$$\rho[u^\varepsilon(\bar{x},\bar{t}) + u^\varepsilon(\bar{y},\bar{t}) - 2u^\varepsilon(\tfrac{1}{2}(\bar{x}+\bar{y}),\bar{t})] \le e^{Kc(1,\varepsilon)}[K + c_0\, e^{-\rho \bar{t}}]|\bar{x}-\bar{y}|^2$$

with a constant $K > 0$, independent of ε. Once the technical and long proof of the above inequality is completed, we establish (1.7) and (1.8) by using the standard arguments.

Set $\bar{\eta} = \frac{1}{2}(\bar{x}+\bar{y})$. Since, $D^2 g$ is bounded, we may assume that $\bar{t} \ne T$. Then

$$\begin{aligned}&\rho[u^\varepsilon(\bar{x},\bar{t}) + u^\varepsilon(\bar{y},\bar{t}) - 2u^\varepsilon(\bar{\eta},\bar{t})] \\ &\quad \le -\frac{\partial}{\partial t} u^\varepsilon(\bar{x},\bar{t}) - \frac{\partial}{\partial t} u^\varepsilon(\bar{y},\bar{t}) + 2\frac{\partial}{\partial t} u^\varepsilon(\bar{\eta},\bar{t}) \\ &\quad = -\mathcal{H}^\varepsilon(\bar{x}, p^\varepsilon + 2c_0(\bar{x}-\bar{y})\, e^{-\rho \bar{t}}, u^\varepsilon(\cdot,\bar{t})) \\ &\qquad - \mathcal{H}^\varepsilon(\bar{y}, p^\varepsilon - 2c_0(\bar{x}-\bar{y})\, e^{-\rho \bar{t}}, u^\varepsilon(\cdot,\bar{t})) \\ &\qquad + 2\mathcal{H}^\varepsilon(\bar{\eta}, p^\varepsilon, u^\varepsilon(\cdot,\bar{t})),\end{aligned}$$

where $p^\varepsilon = \nabla u^\varepsilon(\bar{\eta},\bar{t})$. Recalling the identity (1.2) and the definition of $\mathcal{H}^\varepsilon$, we obtain

$$\begin{aligned}&\rho[u^\varepsilon(\bar{x},\bar{t}) + u^\varepsilon(\bar{y},\bar{t}) - 2u^\varepsilon(\bar{\eta},\bar{t})] \\ &\quad \le \int_{R^n\setminus\{0\}} [-e^{I_1(z)} f(\bar{x},z) - e^{I_2(z)} f(\bar{y},z) + 2\, e^{I_3(z)} f(\bar{\eta},z)]\,\pi(dz) \\ &\qquad + \int_{R^n\setminus\{0\}} [-(c(\bar{x},z)\cdot p^\varepsilon - 1) f(\bar{x},z) - (c(\bar{y},z)\cdot p^\varepsilon - 1) f(\bar{y},z) \\ &\qquad\qquad + 2(c(\bar{\eta},z)\cdot p^\varepsilon - 1) f(\bar{\eta},z)]\,\pi(dz)\end{aligned}$$

$$+e^{-\rho\bar{t}}(b(\bar{x})-b(\bar{y}))\cdot 2c_0(\bar{x}-\bar{y})+(b(\bar{x})+b(\bar{y})-2b(\bar{\eta}))\cdot p^\varepsilon$$

$$+e^{-\rho\bar{t}}c_0\int_{R^n\setminus\{0\}}(\bar{x}-\bar{y})\cdot(c(\bar{x},z)-c(\bar{y},z))\pi(dz),$$

where

$$I_1(z)=-\frac{1}{\varepsilon}[u^\varepsilon(\bar{x}+\varepsilon c(\bar{x},z),\bar{t})-u^\varepsilon(\bar{x},\bar{t})],$$

$$I_2(z)=-\frac{1}{\varepsilon}[u^\varepsilon(\bar{y}+\varepsilon c(\bar{y},z),\bar{t})-u^\varepsilon(\bar{y},\bar{t})],$$

$$I_3(z)=-\frac{1}{\varepsilon}[u^\varepsilon(\bar{\eta}+\varepsilon c(\bar{\eta},z),\bar{t})-u^\varepsilon(\bar{\eta},\bar{t})].$$

Now, using (A1)-(A3), we obtain

$$\rho[u^\varepsilon(\bar{x},\bar{t})+u^\varepsilon(\bar{y},\bar{t})-2u^\varepsilon(\bar{\eta},\bar{t})]$$

$$\leq\int_{R^n\setminus\{0\}}(-e^{I_1(z)}f(\bar{x},z)-e^{I_2(z)}f(\bar{y},z)+e^{I_3(z)}[f(\bar{x},z)+f(\bar{y},z)])\pi(dz)$$

$$+K\,e^{\|I_3\|_\infty}|\bar{x}-\bar{y}|^2+K|p^\varepsilon||\bar{x}-\bar{y}|^2+Kc_0|\bar{x}-\bar{y}|^2\,e^{-\rho\bar{t}}.$$

Observe that $\|I_3\|_\infty\leq Kc(1,\varepsilon)$, and $|p^\varepsilon|\leq c(1,\varepsilon)$. Hence,

$$\rho[u^\varepsilon(\bar{x},\bar{t})+u^\varepsilon(\bar{y},\bar{t})-2u^\varepsilon(\bar{\eta},\bar{t})]$$

$$\leq[K\,e^{Kc(1,\varepsilon)}+K\,e^{-\rho\bar{t}}c_0]|\bar{x}-\bar{y}|^2$$

$$+\int_{R^n\setminus\{0\}}[-e^{I_1(z)}f(\bar{x},z)-e^{I_2(z)}f(\bar{y},z)+e^{I_3(z)}(f(\bar{x},z)+f(\bar{y},z))]\pi(dz). \tag{3.7}$$

The choice of $\bar{x}$, $\bar{y}$, $\bar{t}$ yields, for every $c_1, c_2, c_3\in R^n$,

$$u^\varepsilon(\bar{x}+\varepsilon c_1,\bar{t})+u^\varepsilon(\bar{y}+\varepsilon c_2,\bar{t})-2u^\varepsilon(\bar{\eta}+\varepsilon c_3,\bar{t})$$

$$\leq u^\varepsilon(\bar{x}+\varepsilon c_1,\bar{t})+u^\varepsilon(\bar{y}+\varepsilon c_2,\bar{t})-2u^\varepsilon\left(\bar{\eta}+\frac{\varepsilon}{2}(c_1+c_2),\bar{t}\right)$$

$$+c(1,\varepsilon)\varepsilon|c_1+c_2-2c_3|$$

$$\leq u^\varepsilon(\bar{x},\bar{t})+u^\varepsilon(\bar{y},\bar{t})-2u^\varepsilon(\bar{\eta},\bar{t})+c(1,\varepsilon)\varepsilon|c_1+c_2-2c_3|$$

$$+e^{-\rho\bar{t}}c_0[|\bar{x}+\varepsilon c_1-\bar{y}-\varepsilon c_2|^2-|\bar{x}-\bar{y}|^2].$$

Letting $c_1=c(\bar{x},z)$, $c_2=c(\bar{y},z)$, and $c_3=c(\bar{\eta},z)$ in the above inequality, we arrive at

$$2I_3\leq I_1+I_2+c_0\,e^{-\rho\bar{t}}[\varepsilon|c(\bar{x},z)-c(\bar{y},z)|^2+2(\bar{x}-\bar{y})\cdot(c(\bar{x},z)-c(\bar{y},z))]$$

$$+2c(1,\varepsilon)|c(\bar{x},z)+c(\bar{y},z)-2c(\bar{\eta},z)|.$$

Since $e^{I_i(z)} \le e^{Kc(1,\varepsilon)}$ for $i=1,2,3$, the above inequality implies that

$$\begin{aligned} &e^{I_3(z)} - e^{(1/2)(I_1(z)+I_2(z))} \\ &\quad \le \max[e^{I_3(z)}, e^{(1/2)(I_1(z)+I_2(z))}] \max[0, I_3 - \tfrac{1}{2}(I_1(z)+I_2(z))] \\ &\quad \le e^{Kc(1,\varepsilon)} c_0\, e^{-\rho\bar{t}} |c(\bar{x}, z) - c(\bar{y}, z)| [\varepsilon |c(\bar{x}, z) - c(\bar{y}, z)| + 2|\bar{x} - \bar{y}|] \\ &\qquad + 2c(1, \varepsilon)\, e^{Kc(1,\varepsilon)} |c(\bar{x}, z) + c(\bar{y}, z) - 2c(\bar{\eta}, z)|. \end{aligned}$$

Now substitute the above estimate into (3.7):

$$\begin{aligned} &\rho[u^\varepsilon(\bar{x}, \bar{t}) + u^\varepsilon(\bar{y}, \bar{t}) - 2u^\varepsilon(\bar{\eta}, \bar{t})] \\ &\quad \le [K\, e^{Kc(1,\varepsilon)} + Kc_0\, e^{-\rho\bar{t}}] |\bar{x} - \bar{y}|^2 \\ &\qquad + \int_{R^n \setminus \{0\}} [-e^{I_1(z)} f(\bar{x}, z) - e^{I_2(z)} f(\bar{y}, z) \\ &\qquad\qquad + e^{(1/2)(I_1(z)+I_2(z))} (f(\bar{x}, z) + f(\bar{y}, z))] \pi(dz) \\ &\qquad + \int_{R^n \setminus \{0\}} e^{Kc(1,\varepsilon)} c_0\, e^{-\rho\bar{t}} |c(\bar{x}, z) - c(\bar{y}, z)| \\ &\qquad \times [\varepsilon |c(\bar{x}, z) - c(\bar{y}, z)| + 2|\bar{x} - \bar{y}|] \pi(dz) \\ &\qquad + \int_{R^n \setminus \{0\}} 2c(1, \varepsilon)\, e^{Kc(1,\varepsilon)} |c(\bar{x}, z) + c(\bar{y}, z) - 2c(\bar{\eta}, z)| \pi(dz) \\ &\quad \le e^{Kc(1,\varepsilon)} [K + c_0\, e^{-\rho\bar{t}}] |\bar{x} - \bar{y}|^2 \\ &\qquad + \int_{R^n \setminus \{0\}} [-e^{I_1(z)} f(\bar{x}, z) - e^{I_2(z)} f(\bar{y}, z) \\ &\qquad\qquad + e^{(1/2)(I_1(z)+I_2(z))} (f(\bar{x}, z) + f(\bar{y}, z))] \pi(dz). \end{aligned} \tag{3.8}$$

The last inequality follows from (A2), and a straightforward algebraic manipulation. Since $f(x, z) \ge 0$ and $f(\cdot, z) \in C_b^\infty(R^n)$, for every $x, z \in R^n$

$$|\nabla f(x, z)| \le 2 \| D^2 f \|_\infty^{1/2} [f(x, z)]^{1/2}.$$

Using the above inequality, we obtain

$$\begin{aligned} |f(\bar{x}, z) - f(\bar{y}, z)| &\le \int_0^1 |\nabla f(\bar{x} + \tau(\bar{y} - \bar{x}), z)|\, d\tau |\bar{x} - \bar{y}| \\ &\le \int_0^1 [|\nabla f(\bar{x}, z)| + \| D^2 f \|_\infty \tau |\bar{x} - \bar{y}|]\, d\tau |\bar{x} - \bar{y}| \\ &\le 2 \| D^2 f \|_\infty^{1/2} (f(\bar{x}, z))^{1/2} |\bar{x} - \bar{y}| + \tfrac{1}{2} \| D^2 f \|_\infty |\bar{x} - \bar{y}|^2 \\ &\le 2 \| D^2 f \|_\infty^{1/2} (f(\bar{x}, z) + f(\bar{y}, z))^{1/2} |\bar{x} - \bar{y}| + \tfrac{1}{2} \| D^2 f \|_\infty |\bar{x} - \bar{y}|^2. \end{aligned}$$

Hence

$$J=\int_{R^n\backslash\{0\}}[-e^{I_1(z)}f(\bar{x},z)-e^{I_2(z)}f(\bar{y},z)$$
$$+e^{(1/2)(I_1(z)+I_2(z))}(f(\bar{x},z)+f(\bar{y},z))]\pi(dz)$$
$$\leq\int_{R^n\backslash\{0\}}[-\tfrac{1}{2}(f(\bar{x},z)+f(\bar{y},z))(e^{(1/2)I_1(z)}-e^{(1/2)I_2(z)})^2$$
$$+\tfrac{1}{2}|f(\bar{x},z)-f(\bar{y},z)||e^{I_1(z)}-e^{I_2(z)}|]\pi(dz)$$
$$\leq\int_{R^n\backslash\{0\}}[-\tfrac{1}{2}(f(\bar{x},z)+f(\bar{y},z))(e^{(1/2)I_1(z)}-e^{(1/2)I_2(z)})^2$$
$$+\|D^2f\|_\infty^{1/2}(f(\bar{x},z)+f(\bar{y},z))^{1/2}|\bar{x}-\bar{y}|$$
$$\times|e^{(1/2)I_1(z)}-e^{(1/2)I_2(z)}||e^{(1/2)I_1(z)}+e^{(1/2)I_2(z)}|$$
$$+\tfrac{1}{4}\|D^2f\|_\infty|\bar{x}-\bar{y}|^2\sup(e^{I_1(z)}+e^{I_2(z)})]\pi(dz).$$

Recall that $e^{I_i(z)}\leq e^{Kc(1,\varepsilon)}$ for $i=1,2$. This observation, together with the integrability of $\|D^2f\|_\infty$, (A3), yields

$$J\leq\int_{R^n\backslash\{0\}}[-\tfrac{1}{2}(f(\bar{x},z)+f(\bar{y},z))|e^{(1/2)I_1(z)}-e^{(1/2)I_2(z)}|^2$$
$$+\|D^2f\|_\infty^{1/2}\,e^{Kc(1,\varepsilon)}|\bar{x}-\bar{y}|(f(\bar{x},z)+f(\bar{y},z))^{1/2}$$
$$\times|\,e^{(1/2)I_1(z)}-e^{(1/2)I_2(z)}|]\pi(dz)$$
$$+K\,e^{Kc(1,\varepsilon)}|\bar{x}-\bar{y}|^2.$$

Finally, by using the inequality $ab\leq\frac{1}{2}(a^2+b^2)$, we obtain

$$J\leq\int_{R^n\backslash\{0\}}\tfrac{1}{2}\|D^2f\|_\infty\,e^{2Kc(1,\varepsilon)}|\bar{x}-\bar{y}|^2\pi(dz)+K\,e^{Kc(1,\varepsilon)}|\bar{x}-\bar{y}|^2$$
$$\leq K\,e^{Kc(1,\varepsilon)}|\bar{x}-\bar{y}|^2.$$

Substitute this estimate back into (3.8):

$$\rho[u^\varepsilon(\bar{x},\bar{t})+u^\varepsilon(\bar{y},\bar{t})-2u^\varepsilon(\bar{\eta},\bar{t})]\leq e^{Kc(1,\varepsilon)}[K+c_0\,e^{-\rho\bar{t}}]|\bar{x}-\bar{y}|^2.$$

Also, for every $x,y\in R^n$, and $t\in[0,T]$,

$$e^{\rho t}[u^\varepsilon(x,t)+u^\varepsilon(y,t)-2u^\varepsilon(\tfrac{1}{2}(x+y),t)]-c_0|x-y|^2$$
$$\leq e^{\rho\bar{t}}[u^\varepsilon(\bar{x},\bar{t})+u^\varepsilon(\bar{y},\bar{t})-2u^\varepsilon(\bar{\eta},\bar{t})]-c_0|\bar{x}-\bar{y}|^2$$
$$\leq\left[e^{\rho\bar{t}}\,e^{Kc(1,\varepsilon)}(K+e^{-\rho\bar{t}}c_0)\frac{1}{\rho}-c_0\right]|\bar{x}-\bar{y}|^2.$$

Hence for $\rho=\rho(c(1,\varepsilon))$ and $c_0=c_0(c(1,\varepsilon))$ sufficiently large, the last expression is negative and

$$u^\varepsilon(x,t)+u^\varepsilon(y,t)-2u^\varepsilon(\tfrac{1}{2}(x+y),t)\leq c_0(c(1,\varepsilon))\,e^{-\rho(c(1,\varepsilon))t}|x-y|^2.$$

Consequently,

$$c_2(2, \varepsilon) \le c_0(c(1, \varepsilon)). \tag{3.9}$$

Now, substitute (3.9) into (3.6):

$$\begin{aligned} c(1, \varepsilon) &\le \gamma(K + \varepsilon K c_0(c(1, \varepsilon))\, e^{Kc(1,\varepsilon)}) \\ &= \gamma(K + \varepsilon F(c(1, \varepsilon))), \end{aligned} \tag{3.10}$$

where $F(x) = Kc_0(x)\exp(Kx)$. Since $\gamma(r)$ is continuous and increasing, for sufficiently small $\varepsilon > 0$, there are $0 < A(\varepsilon) < B(\varepsilon) \le \infty$ satisfying

$$0 \le x \le \gamma(K + \varepsilon F(x)) \quad \Rightarrow \quad x \in [0, A(\varepsilon)] \cup [B(\varepsilon), \infty), \tag{3.11a}$$

$$\sup_{\varepsilon} A(\varepsilon) \le K. \tag{3.11b}$$

Therefore, $c(1, \varepsilon) \in [0, A(\varepsilon)] \cup [B(\varepsilon), \infty)$. We claim that $c(1, \varepsilon)$, in fact, belongs to the first of these intervals. Indeed, for $\tau \in [0, 1]$ let $u^{\varepsilon,\tau}(x, t)$ denote the solution of $(1.5)^{\varepsilon}$ with terminal data $u^{\varepsilon,\tau}(x, \tau) = \tau g(x)$. Define $c_\tau(1, \varepsilon)$ to be the $W^{1,\infty}$-norm of $u^{\varepsilon,\tau}$. Then, for all $\tau \in [0, 1]$,

$$c_\tau(1, \varepsilon) \in [0, A(\varepsilon)] \cup [B(\varepsilon), \infty).$$

Moreover, the map $\tau \mapsto c_\tau(1, \varepsilon)$ is continuous on $[0, 1]$ for each $\varepsilon > 0$. Since $c_0(1, \varepsilon) = 0$ and $A(\varepsilon) < B(\varepsilon)$, we conclude that $c_\tau(1, \varepsilon) \in [0, A(\varepsilon)]$ for all $\tau \in [0, 1]$. The proof of the lemma is now complete, in view of (3.9) and (3.11b). □

4. An "Almost Linear" Equation

In this section we obtain L^∞-estimates for the solutions to a sequence of equations. These estimates are used in the proof of Theorem 5.1.

We continue by deriving the first of this sequence of equations. Let $u^0(x, t)$ be the unique viscosity solution to (1.9) and (1.6). In Theorem 3.2 we have shown the convergence of $u^\varepsilon(x, t)$ to $u^0(x, t)$, as ε tends to zero. Hence, $u^0(x, t)$ is the leading term in the asymptotic expansion of $u^\varepsilon(x, t)$. To derive an equation for the next term in the expansion, consider the function

$$u^{\varepsilon,1}(x, t) = \frac{1}{\varepsilon}[u^\varepsilon(x, t) - u^0(x, t)].$$

Then $u^{\varepsilon,1}(x, t)$ solves the following equation:

$$-\frac{\partial}{\partial t} u^{\varepsilon,1}(x, t) + \frac{1}{\varepsilon}[\mathscr{H}^\varepsilon(x, \nabla u^\varepsilon(x, t), u^\varepsilon(\cdot, t)) - \mathscr{H}(x, \nabla u^0(x, t))] = 0. \tag{4.1}$$

Introduce a function $\mathbb{H}^\varepsilon$ on $R^n \times R^n \times C_b^2(R^n)$:

$$\begin{aligned} \mathbb{H}^\varepsilon(x, p, \psi) &= -b(x) \cdot p \\ &\quad + \int_{R^n \setminus \{0\}} \left[\exp\left(-\frac{1}{\varepsilon}[\psi(x + \varepsilon y) - \psi(x) - \varepsilon y \cdot (\nabla \psi(x) - p)]\right) - 1 + y \cdot p\right] \\ &\quad \times m(x, dy). \end{aligned} \tag{4.2}$$

Observe that

$$\mathscr{H}^\varepsilon(x, \nabla\psi(x), \psi) = \mathbb{H}^\varepsilon(x, \nabla\psi(x), \psi).$$

Using $\mathbb{H}^\varepsilon$, rearrange the terms in (4.1) as follows:

$$\begin{aligned} -\frac{\partial}{\partial t} u^{\varepsilon,1}(x,t) &+ \frac{1}{\varepsilon}[\mathbb{H}^\varepsilon(x, \nabla u^\varepsilon(x,t), u^\varepsilon(\cdot,t)) - \mathbb{H}^\varepsilon(x, \nabla u^\varepsilon(x,t), u^0(\cdot,t))] \\ &+ \frac{1}{\varepsilon}[\mathbb{H}^\varepsilon(x, \nabla u^\varepsilon(x,t), u^0(\cdot,t)) - \mathbb{H}^\varepsilon(x, \nabla u^0(x,t), u^0(\cdot,t))] \\ &+ \frac{1}{\varepsilon}[\mathbb{H}^\varepsilon(x, \nabla u^0(x,t), u^0(\cdot,t)) - \mathscr{H}(x, \nabla u^0(x,t))] = 0. \end{aligned} \tag{4.3}$$

Let

$$A_1^\varepsilon(x,t) = \frac{1}{\varepsilon}[\mathbb{H}^\varepsilon(x, \nabla u^0(x,t), u^0(\cdot,t)) - \mathscr{H}(x, \nabla u^0(x,t))]. \tag{4.4}$$

If $u^0(x,t)$ is smooth in a neighborhood of a point, then $A_1^\varepsilon(x,t)$ converges to a limit as ε tends to zero. Next, consider

$$\begin{aligned} &\frac{1}{\varepsilon}[\mathbb{H}^\varepsilon(x, \nabla u^\varepsilon(x,t), u^0(x,t)) - \mathbb{H}^\varepsilon(x, \nabla u^0(x,t), u^0(x,t))] \\ &\quad = \int_0^1 \mathbb{H}_p^\varepsilon(x, \nabla u^0(x,t) + \tau(\nabla u^\varepsilon(x,t) - \nabla u^0(x,t)), u^0(x,t))\, d\tau \\ &\qquad \times \frac{1}{\varepsilon}(\nabla u^\varepsilon(x,t) - \nabla u^0(x,t)) \\ &\quad = b_1^\varepsilon(x,t) \cdot \nabla u^{\varepsilon,1}(x,t), \end{aligned} \tag{4.5}$$

where $b_1^\varepsilon(x,t)$ is defined to be the integral term. Due to (3.1), $b_1^\varepsilon(x,t)$ converges uniformly to $\mathscr{H}_p(x, \nabla u^0(x,t))$ provided that $u^0(x,t)$ is smooth in a neighborhood of (x,t). In view of (4.4) and (4.5), equation (4.3) can be rewritten as

$$\begin{aligned} -\frac{\partial}{\partial t} u^{\varepsilon,1}(x,t) &+ b_1^\varepsilon(x,t) \cdot \nabla u^{\varepsilon,1}(x,t) + A_1^\varepsilon(x,t) \\ &+ \frac{1}{\varepsilon}[\mathbb{H}^\varepsilon(x, \nabla u^\varepsilon(x,t), \theta^{\varepsilon,1}(\cdot,t) + \varepsilon u^{\varepsilon,1}(\cdot,t)) \\ &\quad - \mathbb{H}^\varepsilon(x, \nabla u^\varepsilon(x,t), \theta^{\varepsilon,1}(\cdot,t))] = 0, \end{aligned} \tag{4.6$^{\varepsilon,1}$}$$

where $\theta^{\varepsilon,1}(x,t) = u^0(x,t)$.

Now, suppose that the expansion (1.11) holds up to $m-1$. Then the function $u^{\varepsilon,m}(x,t) = \varepsilon^{-m}[u^\varepsilon(x,t) - u^0(x,t) - \cdots - \varepsilon^{m-1}u^m(x,t)]$ satisfies an equation of the following form:

$$-\frac{\partial}{\partial t} u^{\varepsilon,m}(x,t) + b_m^\varepsilon(x,t) \cdot \nabla u^{\varepsilon,m}(x,t) + A_m^\varepsilon(x,t)$$

$$+\frac{1}{\varepsilon^m}[\mathbb{H}^\varepsilon(x, \nabla u^\varepsilon(x,t), \theta^{\varepsilon,m}(\cdot,t)+\varepsilon^m u^{\varepsilon,m}(\cdot,t))$$

$$-\mathbb{H}^\varepsilon(x, \nabla u^\varepsilon(x,t), \theta^{\varepsilon,m}(\cdot,t))]=0, \qquad (4.6)^{\varepsilon,m}$$

where $b^\varepsilon_m(x,t)$, $A^\varepsilon_m(x,t)$, and $\theta^{\varepsilon,m}(x,t)$ are determined by $u^\varepsilon(x,t)$ and the coefficients $u^0(x,t),\dots,u^{m-1}(x,t)$. The specific form of these functions is given in the proof of Theorem 5.1. In this section we only use the following property of b^ε_m, A^ε_m, and θ^ε_m:

$$\text{(H1)}\quad \begin{cases} b^\varepsilon_m, A^\varepsilon_m, \theta^\varepsilon_m \in C^\infty_b(N), \\ \|\theta^\varepsilon_m\|_{C^2} \le K, \\ \lim_{\varepsilon\downarrow 0} b^\varepsilon_m(x,t) = \mathscr{H}_p(x, \nabla u^0(x,t)), \\ \qquad \text{uniformly on compact subsets of } N, \end{cases}$$

where N is the region on which $u^0(x,t)$ is C^∞-smooth. On account of (A1)-(A6), and Lemma 2.2, N is an open and dense subset of $R^n\times[0,\tau]$ [5, Theorem 2]. Moreover, for every point $(x,t)\in N$, there is a smooth curve $\gamma^*(x,t)(\cdot)$ satisfying

$$(\gamma^*(x,t)(s), s)\in N, \qquad \forall s\in[t,T], \qquad (4.7a)$$

$$\gamma^*(x,t)(t)=x, \qquad (4.7b)$$

$$\frac{d}{ds}\gamma^*(x,t)(s) = -\mathscr{H}_p(\gamma^*(x,t)(s), \nabla u^0(\gamma^*(x,t)(s),s)), \qquad \forall s\in(t,T]. \quad (4.7c)$$

The curve $\gamma^*(x,t)$ is, in fact, the unique minimizer of the variational problem associated with the Hamilton–Jacobi equation (1.9), satisfying the initial data (4.7b). The reader may refer to [5] for the proofs of these classical results.

Next we construct a subregion $N(x_0,t_0)\subset N$ for every $(x_0,t_0)\in N$. This construction is essentially the same as the one given in [6]. First, consider the following first-order linear evolution equation:

$$\frac{\partial}{\partial t} w(x,t) - \mathscr{H}_p(x, \nabla u^0(x,t))\cdot\nabla w(x,t) = 1, \qquad (x,t)\in N,$$

and

$$w(x,T)=\varphi(x), \qquad x\in R^n,$$

where φ is a smooth function. Since the characteristics of the above equation are $\gamma^*(x,t)(\cdot)$,

$$w(x,t)=\varphi(\gamma^*(x,t)(T))-(T-t).$$

For $\gamma>0$, let $\tilde F_\gamma(x,t)$ be the solution of the above equation with terminal data $\varphi(x)=(1-(1/\gamma)|x-\gamma^*(x_0,t_0)(T)|^2)(T-t_0)+\gamma$. Then,

$$\tilde F_\gamma(x,t)=\left(1-\frac{1}{\gamma}|\gamma^*(x,t)(T)-\gamma^*(x_0,t_0)(T)|^2\right)(T-t_0)+\gamma-(T-t),$$

$$(x,t)\in N.$$

Define a subregion $N_\gamma \subset N$ by

$$N_\gamma = \{(x, t) \in N;\ \tilde{F}_\gamma(x, t) > 0\}.$$

Observe that $(x_0, t_0) \in N_\gamma$ for every $\gamma > 0$. Also, the closure $\bar{N}_\gamma$ of N_γ is a subset of N, provided that γ is sufficiently small. Let $N(x_0, t_0) = N_\gamma$ and $F = \tilde{F}_\gamma$ for this choice of γ. Then the following are satisfied by $N(x_0, t_0)$ and F:

$$F(x, t) > 0, \qquad (x, t) \in N(x_0, t_0), \tag{4.8a}$$

$$F(x, t) \le 0, \qquad (x, t) \notin N(x_0, t_0), \tag{4.8b}$$

$$\frac{\partial}{\partial t} F(x, t) - \mathscr{H}_p(x, \nabla u^0(x, t)) = 1, \qquad (x, t) \in N(x_0, t_0), \tag{4.8c}$$

$$\gamma^*(x, t)(s) \in N(x_0, t_0), \qquad (x, t) \in N(x_0, t_0), \quad s \in [t, T], \tag{4.8d}$$

$$F \in C_b^\infty(N') \qquad \text{for a suitable neighborhood } N' \text{ of } N(x_0, t_0). \tag{4.8e}$$

We are now ready to prove the main result of this section.

Proposition 4.1. *Assume that $u^{\varepsilon,m}(x, t)$ is a solution to $(4.6)^{\varepsilon,m}$ on N satisfying*

$$u^{\varepsilon,m}(x, T) = 0, \qquad (x, T) \in N,$$

$$\|u^{\varepsilon,m}\|_{L^\infty(N)} \le k\varepsilon^{-m}.$$

Further, assume that (H1) *hold. Then, for every $(x_0, t_0) \in N$, there is $\varepsilon_0 = \varepsilon_0(x_0, t_0)$ such that*

$$|u^{\varepsilon,m}(x, t)| \le k\varepsilon^{-m} \exp\left(-\frac{1}{\sqrt{\varepsilon}} F(x, t)\right) + \|A_m^\varepsilon\|_\infty (T - t) \tag{4.9}$$

for every $\varepsilon \le \varepsilon_0$, and $(x, t) \in N(x_0, t_0)$.

Proof. Set

$$z(x, t) = k\varepsilon^{-m} \exp\left(-\frac{1}{\sqrt{\varepsilon}} F(x, t)\right) + a^\varepsilon (T - t),$$

where $a^\varepsilon = \|A_m^\varepsilon\|_\infty$. We shall show that $z(x, t)$ and $-z(x, t)$ are super- and subsolutions of $(4.6)^{\varepsilon,m}$ in $N(x_0, t_0)$, respectively. Then (4.9) follows from the comparison result for equations of type $(4.6)^{\varepsilon,m}$. We refer to Lemma 3.1 of [15] for a proof of the comparison principle.

Consider

$$L^\varepsilon(x, t) = \frac{1}{\varepsilon^m} [\mathbb{H}^\varepsilon(x, \nabla u^\varepsilon(x, t), \theta^{\varepsilon,m}(\cdot, t) + \varepsilon^m z(\cdot, t))$$
$$- \mathbb{H}^\varepsilon(x, \nabla u^\varepsilon(x, t), \theta^{\varepsilon,m}(\cdot, t))].$$

Using the definition of $\mathbb{H}^\varepsilon$, estimate L^ε as follows:

$$
\begin{aligned}
|L^\varepsilon(x,t)| \le \int_{R^n\setminus\{0\}} & \exp\Big(-\frac{\varepsilon^m}{\varepsilon}[\theta^{\varepsilon,m}(x+\varepsilon y,t)-\theta^{\varepsilon,m}(x,t) \\
&\qquad -\varepsilon y\cdot(\nabla\theta^{\varepsilon,m}(x,t)-\nabla u^\varepsilon(x,t))]\Big) \\
&\times\frac{1}{\varepsilon^m}\left|\exp\Big(-\frac{\varepsilon^m}{\varepsilon}[z(x+\varepsilon y,t)-z(x,t)-\varepsilon y\cdot\nabla z(x,t)]\Big)-1\right| m(x,dy) \\
\le \int_{R^n\setminus\{0\}} & \exp[\varepsilon\|D^2\theta^{\varepsilon,m}\|_\infty|y|^2+\|\nabla u^\varepsilon\|_\infty|y|] \\
&\times\exp\left[\frac{\varepsilon^m}{\varepsilon}|z(x+\varepsilon y,t)-z(x,t)-\varepsilon y\cdot\nabla z(x,t)|\right] \\
&\times\frac{1}{\varepsilon}|z(x+\varepsilon y,t)-z(x,t)-\varepsilon y\cdot\nabla z(x,t)|m(x,dy). \qquad (4.10)
\end{aligned}
$$

The specific form of $z(x,t)$ yields, for all (x,t), $(x+\varepsilon y,t)\in N'$ (here N' is as in (4.8e)),

$$
\begin{aligned}
&\frac{1}{\varepsilon}|z(x+\varepsilon y,t)-z(x,t)-\varepsilon y\cdot\nabla z(x,t)| \\
&=\frac{k}{\varepsilon^{m+1}}\exp\Big(-\frac{1}{\sqrt{\varepsilon}}F(x,t)\Big)\Big|\exp\Big(-\frac{1}{\sqrt{\varepsilon}}[F(x+\varepsilon y,t)-F(x,t)]\Big)-1 \\
&\qquad\qquad +\sqrt{\varepsilon}\,y\cdot\nabla F(x,t)\Big| \\
&=\frac{k}{\varepsilon^{m+1}}\exp\Big(-\frac{1}{\sqrt{\varepsilon}}F(x,t)\Big)\Big|\sqrt{\varepsilon}\int_0^1\Big[y\cdot\nabla F(x,t)-y\cdot\nabla F(x+\tau\varepsilon y,t) \\
&\qquad\qquad \times\exp\Big(-\frac{1}{\sqrt{\varepsilon}}[F(x+\tau\varepsilon y,t)-F(x,t)]\Big)\Big]\,d\tau\Big| \\
&\le\frac{k}{\varepsilon^{m+1}}\exp\Big(-\frac{1}{\sqrt{\varepsilon}}F(x,t)\Big) \\
&\quad\times\Big[\|D^2F\|_\infty\varepsilon^{3/2}|y|^2+\sqrt{\varepsilon}\Big|\int_0^1 y\cdot\nabla F(x+\tau\varepsilon y,t) \\
&\qquad\times\Big[\exp\Big(-\frac{1}{\sqrt{\varepsilon}}[F(x+\tau\varepsilon y,t)-F(x,t)]\Big)-1\Big]\,d\tau\Big|\Big] \\
&\le\frac{k}{\varepsilon^{m+1}}\exp\Big(-\frac{1}{\sqrt{\varepsilon}}F(x,t)\Big) \\
&\quad\times[\|D^2F\|_\infty\varepsilon^{3/2}|y|^2+\|DF\|_\infty^2\exp(\sqrt{\varepsilon}\|DF\|_\infty|y|)\varepsilon|y|^2].
\end{aligned}
$$

Substitute the above inequality into (4.10). Then, by using the boundedness and the integrability of $c(x, z)$, we arrive at

$$|L^{\varepsilon}(x, t)| \le \int_{R^n\setminus\{0\}} k\frac{1}{\varepsilon^m}\exp\left(-\frac{1}{\sqrt{\varepsilon}}F(x, t)\right)|y|^2 m(x, dy)$$

$$\le k\frac{1}{\varepsilon^m}\exp\left(-\frac{1}{\sqrt{\varepsilon}}F(x, t)\right), \qquad (x, t)\in N(x_0, t_0), \quad \varepsilon \text{ small.} \tag{4.11}$$

Also, for $(x, t)\in N(x_0, t_0)$,

$$-\frac{\partial}{\partial t}z(x, t)+b_m^{\varepsilon}(x, t)\cdot\nabla z(x, t)+A_m^{\varepsilon}(x, t)$$

$$= \|A_m^{\varepsilon}\|_{\infty}+\frac{K}{\varepsilon^{m+1/2}}\exp\left(-\frac{1}{\sqrt{\varepsilon}}F(x, t)\right)$$

$$\times\left[\frac{\partial}{\partial t}F(x, t)-b_m^{\varepsilon}(x, t)\cdot\nabla F(x, t)\right]+A_m^{\varepsilon}(x, t)$$

$$\ge \frac{K}{\varepsilon^{m+1/2}}\exp\left(-\frac{1}{\sqrt{\varepsilon}}F(x, t)\right)$$

$$\times\left[\frac{\partial}{\partial t}F(x, t)-\mathcal{H}_p(x, \nabla u^0(x, t))\cdot\nabla F(x, t)+\|b_m^{\varepsilon}-\mathcal{H}_p\|_{\infty}\|\nabla F\|_{\infty}\right]$$

$$\ge \frac{K}{2\varepsilon^{m+1/2}}\exp\left(-\frac{1}{\sqrt{\varepsilon}}F(x, t)\right)$$

for sufficiently small ε. The final inequality is obtained by using (4.8c) and (H1). The above estimate, together with (4.11), yields that $z(x, t)$ is a supersolution to $(4.6)^{\varepsilon,m}$ in $N(x_0, t_0)$. Then the comparison principle implies that

$$u^{\varepsilon,m}(x, t)-z(x, t)\le \sup[u^{\varepsilon,m}(y, s)-z(y, s);\ (y, s)\notin N(x_0, t_0)].$$

Due to (4.8b), and the boundary behavior of $u^{\varepsilon}(x, t)$, $u^{\varepsilon}\le z$ on $\partial N(x_0, t_0)$. Consequently, $u^{\varepsilon}\le z$ on $N(x_0, t_0)$.

A similar computation shows that $-z(x, t)$ is a subsolution to $(4.6)^{\varepsilon,m}$, and therefore $u^{\varepsilon,m}\ge -z$ on $N(x_0, t_0)$. □

5. Asymptotic Expansion

Theorem 5.1. *Suppose that* (A1)-(A6) *hold. Then, for each* $m = 1, 2, \ldots,$

$$u^{\varepsilon}(x, t) = u^0(x, t)+\varepsilon u^1(x, t)+\cdots+\varepsilon^m u^m(x, t)+o(\varepsilon^m) \tag{1.11}$$

uniformly on compact subsets of N. The coefficients $u^m \in C_b^{\infty}(N)$ *and for each* $m\ge 1$ *they satisfy*

$$-\frac{\partial}{\partial t}u^m(x, t)+\mathcal{H}_p(x, \nabla u^0(x, t))\cdot\nabla u^m(x, t)+A_m(x, t)=0, \qquad (x, t)\in N, \tag{5.1^m}$$

$$u_m(x, T) = 0, \qquad (x, T) \in N, \tag{5.2}$$

where $A_m(x, t)$ is a function of $u^0, \ldots, u^{m-1}$ and their derivatives.

We obtain $(5.1)^m$ by formally differentiating $(1.5)^\varepsilon$ m times with respect to ε, and then setting $\varepsilon = 0$. For example,

$$A_1(x, t) = -\tfrac{1}{2} \int_{R^n \setminus \{0\}} e^{-y \cdot \nabla u^0(x,t)} (D^2 u^0(x, t) y \cdot y) m(x, dy),$$

$$A_2(x, t) = \int_{R^n \setminus \{0\}} e^{-y \cdot \nabla u^0(x,t)} \Big[-\tfrac{1}{3} \sum \frac{\partial^\varepsilon}{\partial x_i \, \partial x_j \, \partial x_k} u^0(x, t) y_i y_j y_k - D^2 u^1(x, t) y \cdot y$$
$$+ (\nabla u^1(x, t) \cdot y + \tfrac{1}{2} D^2 u^0(x, t) y \cdot y)^2 \Big] m(x, dy).$$

Proof. For $m = 0, 1, 2, \ldots$ define

$$w^{\varepsilon,m}(x, t) = u^0(x, t) + \cdots + \varepsilon^m u^m(x, t), \tag{5.3}$$

$$u^{\varepsilon,m+1}(x, t) = \frac{1}{\varepsilon^{m+1}} [u^\varepsilon(x, t) - w^{\varepsilon,m}(x, t)], \tag{5.4}$$

where $u^m(x, t)$ is the solution of $(5.1)^m$ with terminal data (5.2). Observe that for each m, characteristics of $(5.1)^m$ are equal to the curves $\gamma^*(x, t)$ defined in Section 4. Since $\gamma^*(x, t)(s) \in N$ for all $s \in [t, T]$, we can solve $(5.1)^m$ on N. Moreover, $u^m \in C_b^\infty(N)$.

Using equations $(1.5)^\varepsilon$, (1.9), and $(5.1)^m$, we obtain the following equation for $u^{\varepsilon,m+1}$:

$$-\frac{\partial}{\partial t} u^{\varepsilon,m+1}(x, t) + \frac{1}{\varepsilon^{m+1}} \Big[\mathbb{H}^\varepsilon(x, \nabla u^\varepsilon(x, t), u^\varepsilon(\cdot, t))$$
$$- \mathcal{H}_p(x, \nabla u^0(x, t)) \cdot \nabla w^{\varepsilon,m}(x, t) - \sum_{k=0}^{m} \varepsilon^k A_k(x, t) \Big] = 0,$$

where $A_0(x, t) = \mathcal{H}(x, \nabla u^0(x, t)) - \mathcal{H}_p(x, \nabla u^0(x, t)) \cdot \nabla u^0(x, t)$. Rearranging the terms, we can show that $u^{\varepsilon,m+1}$ solves $(4.6)^{\varepsilon,m+1}$ with the following choices for the coefficients:

$$b^\varepsilon_{m+1}(x, t) = \int_0^1 \mathbb{H}^\varepsilon_p(x, \nabla w^{\varepsilon,m} + \tau(\nabla u^\varepsilon(x, t) - \nabla w^{\varepsilon,m}(x, t)), w^{\varepsilon,m}(\cdot, t)) \, d\tau,$$

$$\theta^{\varepsilon,m+1}(x, t) = w^{\varepsilon,m}(x, t),$$

and

$$A^\varepsilon_{m+1}(x, t) = \frac{1}{\varepsilon^{m+1}} \Big[\mathbb{H}^\varepsilon(x, \nabla w^{\varepsilon,m}(x, t), w^{\varepsilon,m}(\cdot, t))$$
$$- \mathcal{H}_p(x, \nabla u^0(x, t)) \cdot \nabla w^{\varepsilon,m}(x, t) - \sum_{k=0}^{m} \varepsilon^k A_k(x, t) \Big].$$

On account of Theorem 3.2, $b_m^\varepsilon(x, t)$ converges to $\mathcal{H}_p(x, \nabla u^0(x, t))$ as ε tends to zero, uniformly on compact subsets of N. Since $w^{\varepsilon,m} \in C_b^\infty$, inductively we can show that A_{m+1}^ε converges to A_{m+1}, as ε tends to zero. Hence, condition (H1) of Section 4 holds, and Proposition 4.1 implies that for any $(x_0, t_0) \in N$

$$|u^{\varepsilon,m+1}(x, t)| \leq K\varepsilon^{-(m+1)} \exp\left(-\frac{1}{\sqrt{\varepsilon}} F(x, t)\right) + K(T-t), \qquad (x, t) \in N(x_0, t_0),$$

where $N(x_0, t_0)$ is as in Section 4. Since $F > 0$ on $N(x_0, t_0)$, (5.4) and the above estimate yields

$$\lim_{\varepsilon \downarrow 0} \sup_{(x,t) \in N(x_0,t_0)} \frac{1}{\varepsilon^m} |u^\varepsilon(x, t) - w^{\varepsilon,m}(x, t)| = 0. \qquad \square$$

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Appendix

Due to a theorem of Skorokhod [14, Lemma II, p. 77], the existence of $c(x, z)$ and $f(x, z)$ satisfying (1.2) and (1.3) is equivalent to

$$\sup_x \int \frac{|z|^2}{1+|z|^2} m(x, dz) < \infty. \tag{A.1}$$

Moreover, given a positive measure $m(x, dz)$ there is more than one pair $c(x, z)$ and $f(x, z)$ satisfying (1.2). In fact, we can always take $f(x, z) \equiv 1$. The reason for allowing $f(x, z)$ in our model is the smoothness assumptions (A1)-(A5) imposed on the coefficients.

We give the following simple example to clarify this point. Let $n = 1$, and

$$m(x, A) = \lambda(x)\chi_A(1)$$

for $x \in (-\infty, \infty)$ and Borel set A. Then the following pairs,

$$f_1(x, z) = 1, \qquad c_1(x, z) = \chi_{(1/(1+\lambda(x)),1]}(z),$$

and

$$f_2(x, z) = \lambda(x)\chi_{(1/2,1]}(z), \qquad c_2(x, z) = \chi_{(1/2,1]}(z),$$

both satisfy (1.2). But $c_1(x, z)$ does not satisfy (A1)-(A5) even if $\lambda(x)$ is a smooth function.

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