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OPTIMAL CONTROL WITH STATE-SPACE CONSTRAINT I*

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Abstract. We investigate the optimal value of a deterministic control problem with state space constraint. We show that the optimal value function is the only viscosity subsolution, on the open domain, and the viscosity supersolution, on the closed domain, of the corresponding Bellman equation. Finally, the uniform continuity of the optimal value function is obtained under an assumption on the vector field.

Key words. optimal control, viscosity solutions, Hamilton-Jacobi-Bellman equations, state-space constraint

AMS(MOS) subject classifications. 93E20, 35J65, 35K60, 60J60

Introduction. This paper is concerned with the optimal control of deterministic trajectories given a state-space constraint. The dynamics of the controlled process are (0.1) below. More precisely, let the control u be a Borel measurable map from $[0, \infty)$ into a compact, separable, metric space U and $y(x, \cdot, u)$ be the controlled process. The trajectories $y(x, t, u)$ are the solutions of

$$\frac{d}{dt} y(x, t, u) = b(y(x, t, u), u(t)) \tag{0.1}$$

with initial data $y(x, 0, u) = x$. Let θ be an open subset of R^n and $\mathscr{A}_x$ be the set of strategies under which $y(x, t, u)$ lies in $\bar{\theta}$ (bar denotes the closure). In this paper we refer to $\mathscr{A}_x$ as the set of admissible controls. The structure of $\mathscr{A}_x$ constitutes a state-space constraint. We now associate a discounted cost to every admissible control u and x in $\bar{\theta}$. Given these the optimal value function is

$$v(x) = \inf_{u \in \mathscr{A}_x} \int_0^\infty e^{-t} f(y(x, t, u), u(t))\, dt. \tag{0.2}$$

Note that v is not necessarily continuous. This is caused by the complicated structure of the set valued function $x \to \mathscr{A}_x$. However, as will be shown in § 3 the optimal value function is uniformly continuous on $\bar{\theta}$ given that at every point x on $\partial\theta$ (boundary of θ) there is an $\alpha(x)$ in U such that $b(x, \alpha(x)) \cdot \nu(x) \leqq -\beta < 0$. Here $\nu(x)$ is the exterior normal vector.

If v is uniformly continuous one can make use of the notion of weak (or so-called viscosity) solution of Hamilton-Jacobi equations introduced by M. G. Crandall and P.-L. Lions [2]. In [2] they proved the uniqueness of the viscosity solutions of Hamilton-Jacobi equations in a wide-class of cases. In [1], M. G. Crandall, L. C. Evans, P.-L. Lions provide a simpler introduction to the subject. The book by P.-L. Lions [5] and the review paper by M. G. Crandall and P. E. Souganidis [3] provide a view of the scope of the theory and the references to much of the recent literature. Finally, P.-L. Lions in [6] states results related to constrained problems and viscosity solutions. He proves that under an assumption, stronger than the one above, the optimal value function is locally Lipschitz. Then by using the "everywhere characterization"

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of Lipschitz viscosity solutions [7], he obtains an existence and uniqueness result for the Bellman equation $v+H(x, Dv)=0$.

Let $H \in C(\bar{\theta} \times R^n; R)$ be given by

$$H(x, p)=\sup_{\alpha \in U}\{-b(x, \alpha) \cdot p-f(x, \alpha)\}. \tag{0.3}$$

In § 2, the optimal value function is characterized as the only viscosity solution of $v(x)+H(x, Dv(x))=0$ on θ given the appropriate boundary conditions. Note that v is not a priori defined on $\partial\theta$. The only information at the boundary is given by the state-space constraint. To motivate the boundary condition assume that there is a continuous optimal feedback strategy $\alpha^*(x)$ and v is continuously differentiable on $\bar{\theta}$. The constraint imposes the inequality $b(x, \alpha^*(x)) \cdot \nu(x) \leqq 0$ at the boundary of θ. Also, the optimality of α^* yields $H(x, \nabla v(x))=-b(x, \alpha^*(x)) \cdot \nabla v(x)-f(x, \alpha^*(x))$. Given these, one can show

$$H(x, \nabla v(x)) \leqq H(x, \nabla v(x)+\beta\nu(x)) \quad \text{for all } \beta \geqq 0 \text{ and } x \in \partial\theta. \tag{0.4}$$

Moreover if ψ is differentiable and $v-\psi$ has a minimum on $\bar{\theta}$ at $x \in \partial\theta$, then $\nabla\psi(x)= \nabla v(x)+\beta\nu(x)$ for some positive β. In view of (0.4),

$$v(x)+H(x, \nabla v(x)) \leqq v(x)+H(x, \nabla\psi(x)). \tag{0.5}$$

Since v is smooth, it is easy to show that $v(x)+H(x, \nabla v(x))=0$ on $\bar{\theta}$. Hence $v(x)+ H(x, \nabla\psi(x)) \geqq 0$ whenever ψ is smooth and $v-\psi$ has a minimum, relative to $\bar{\theta}$, at $x \in \partial\theta$. In fact, it is proved that v is the only solution of the Bellman equation with this property (Theorem 2.2).

One can view the inequality (0.4) as a constraint on the normal derivative of v at the boundary. Suppose the Hamiltonian $H(x, p)$ is differentiable with respect to p. Then (0.4) reads as $H_p(x, \nabla v(x)) \cdot \nu(x) \geqq 0$. This implicitly imposes a constraint on $\nabla v(x)$ at the boundary. We give the following simple example to clarify this point.

Example. Let $\theta=(0,1)$, $U=[-1,1]$, $b(x, u)=u$, $f(x, u)=-u$ if $u \in[0,1]$ and $f(x, u)=0$ otherwise. The corresponding Hamiltonian $H(x, p)$ is given by

$$H(x, p)=\begin{cases}1-p & \text{if } p \leqq \frac{1}{2}, \\ p & \text{if } p>\frac{1}{2}.\end{cases}$$

At $x=1$ the condition (0.4) implies that $v_x(1) \geqq \frac{1}{2}$ and at $x=0$ we have $v_x(0) \leqq \frac{1}{2}$. For this example $v(x)=\frac{1}{2} e^{x-1}-1$ is the only solution of $v(x)+H(x, v_x(x))=0$ on $x \in(0,1)$ satisfying the inequalities $v_x(0) \leqq \frac{1}{2}$ and $v_x(1) \geqq \frac{1}{2}$.

1. Statement of the problem. Let θ be an open subset of $\mathbb{R}^n$ with a connected boundary satisfying:

(A1) There are positive constants h, r and an R^n-value bounded, uniformly continuous map η of $\bar{\theta}$ satisfying

$$B(x+t\eta(x), rt) \subset \theta \quad \text{for all } x \in \bar{\theta} \quad \text{and } t \in(0, h].$$

Here $B(x, r)$ denotes the ball with center x and radius r.

Remark. If θ is bounded and $\partial\theta$ is C^1, then it satisfies (A1). Also boundaries with isolated corners may satisfy (A1), for example, $\theta=\{(x, y) \in R^2: x>0, y>0\}$.

We assume the following throughout the paper:

(1.0) The controls take values in a compact metric space U.

For all $x, y \in R^n$, $u \in U$ the functions

$$b: R^n \times U \to R^n,$$
$$f: R^n \times U \to R$$

satisfy

$$\sup_{\alpha \in U} |b(x, \alpha) - b(v, \alpha)| \leqq L(b)|x-y| \quad \text{for all } x, y, \tag{1.1}$$

$$\sup_{\alpha \in U} |b(x, \alpha)| \leqq K(b) \quad \text{for all } x, \tag{1.2}$$

$$\sup_{\alpha \in U} |f(x, \alpha) - f(y, \alpha)| \leqq \omega_f(|x-y|) \quad \text{for all } x, y, \tag{1.3}$$

$$\sup_{\alpha \in U} |f(x, \alpha)| \leqq K(f) \quad \text{for all } x \tag{1.4}$$

where ω_f is a nondecreasing continuous function with $\omega_f(0) = 0$.

Consider $\mathscr{A}$, the set of all measurable maps of $[0, \infty)$ into U. For any $u \in \mathscr{A}$ and $x \in \bar{\theta}$ let $y(x, \cdot, u)$ be the solution of (0.1) with initial data $y(x, 0, u) = x$. The associated discounted cost $J(x, u)$ is

$$J(x, u) = \int_0^\infty e^{-t} f(y(x, t, u), u(t))\, dt. \tag{1.5}$$

We allow only the controls which leave $y(x, \cdot, u)$ in $\bar{\theta}$. To have a feasible problem, we assume that the set of admissible controls is nonempty, i.e.

$$\mathscr{A}_x = \{u \in \mathscr{A}: y(x, t, u) \in \bar{\theta} \text{ for all } t \geqq 0\} \neq \varnothing \quad \text{for all } x \in \bar{\theta}. \tag{A2}$$

Under these assumptions the optimal value function

$$v(x) = \inf_{u \in \mathscr{A}_x} J(x, u), \qquad x \in \bar{\theta} \tag{1.6}$$

is bounded.

2. Hamilton–Jacobi–Bellman equation. We begin by recalling the notion of viscosity solutions [1], [2]. Let K be a subset of R^n. We will use the notations $C^1(K)$ and $BUC(\bar{\theta})$ to mean the set of continuously differentiable functions in a neighborhood of K and the set of bounded uniformly continuous functions on $\bar{\theta}$, respectively.

DEFINITIONS 1.1. Let K be a subset of R^n and $v \in BUC(\bar{K})$.

(i) We say v is a *viscosity subsolution* of $v(x) + H(x, Dv(x)) = 0$ on K if

$$v(x_0) + H(x_0, \nabla \psi(x_0)) \leqq 0$$

whenever $\psi \in C^1(\bar{K})$ and $v - \psi$ has a maximum, relative to K, at $x_0 \in K$.

(ii) We say v is a *viscosity supersolution* of $v(x) + H(x, Dv(x)) = 0$ on K if

$$v(x_0) + H(x_0, \nabla \psi(x_0)) \geqq 0$$

whenever $\psi \in C^1(\bar{K})$ and $v - \psi$ has a minimum, relative to K, at $x_0 \in K$.

If v is both subsolution and supersolution, then v is called a viscosity solution.

Remark. A viscosity solution v satisfies the equation at every point where v is differentiable.

In order to consider the state space constraint problem, we extend the definition as follows.

DEFINITION 2.1. $v \in BUC(\bar{\theta})$ is said to be a *constrained viscosity solution* of $v(x) + H(x, Dv(x)) = 0$ on $\bar{\theta}$ if it is a subsolution on θ and a supersolution on $\bar{\theta}$.

Remark. The fact that v is a supersolution on the closed domain imposes a boundary condition. To demonstrate this, suppose $v \in C^1(\bar{\theta})$ is a constrained viscosity solution; then $v(x)+H(x, \nabla v(x))=0$ for all $x \in \bar{\theta}$. But also $v(x)+H(x, \nabla v(x)+\alpha \nu(x)) \geqq 0$, for all $x \in \partial\theta$ and α positive, because if $v-\psi$ has a minimum at $x_0 \in \partial\theta$, then $\nabla\psi(x_0)=\nabla v(x_0)+\alpha\nu(x_0)$ for some $\alpha \geqq 0$. Hence v satisfies (0.5).

THEOREM 2.1. *Suppose that* (A1), (A2), (1.0)-(1.4) *hold and that the optimal value function v is in $BUC(\bar{\theta})$. Then v is the only constrained viscosity solution of $v(x)+H(x, \nabla(x))=0$ on $\bar{\theta}$.*

Proof. First recall that the optimal value satisfies the dynamic programming principle, i.e.: for any positive T

$$v(x)=\inf_{u \in \mathscr{A}_x}\left\{\int_0^T e^{-t} f(y(x, t, u), u(t))\, dt+e^{-T} v(y(x, T, u))\right\}. \tag{2.1}$$

Let $\psi \in C^1(\bar{\theta})$, $x_0 \in \theta$ and $(v-\psi)(x_0)=\max\,[(v-\psi)(x)\colon x \in \bar{\theta}]=0$. Then, for any $u \in \mathscr{A}_{x_0}$ and t positive, the dynamic programming relation yields:

$$\psi(x_0) \leqq \int_0^t e^{-s} f(y(x_0, s, u), u(s))\, ds+e^{-t} \psi(y(x_0, t, u))$$

which implies

$$\begin{aligned}\frac{1}{t} \int_0^t & [\psi(y(x_0, s, u))-b(y(x_0, s, u), u(s)) \\ & \cdot \nabla \psi(y(x_0, s, u))-f(y(x_0, s, u), u(s))]\, e^{-s}\, ds \leqq 0.\end{aligned} \tag{2.2}$$

Use (1.1), (1.3) and the fact $|y(x_0, s, u)-x_0| \leqq K(b) s$ to obtain:

$$\psi(x_0)-\frac{1}{t} \int_0^t b(x_0, u(s))\, ds \cdot \nabla \psi(x_0)-\frac{1}{t} \int_0^t f(x_0, u(s))\, ds \leqq h(t). \tag{2.3}$$

Here $h(t)$ denotes a continuous function of $[0, \infty)$ into R with value zero at the origin. Put $t_0=\operatorname{dist}(x_0, \partial\theta)/K(b)$ and for any $\alpha \in U$ define u as follows:

$$u(t)=\alpha \chi_{[0, t_0)}(t)+\tilde{u}(t-t_0) \chi_{[t_0, \infty)}(t) \tag{2.4}$$

where $\tilde{u}$ is any control in $\mathscr{A}_{y(x_0, t_0, u)}$. Then u is in $\mathscr{A}_{x_0}$. Use u in (2.3) to get:

$$\psi(x_0)-b(x_0, \alpha) \cdot \nabla \psi(x_0)-f(x_0, \alpha) \leqq h(t) \quad \text{for all } \alpha \in U \text{ and } t \leqq t_0. \tag{2.5}$$

Send t to zero to prove v is a subsolution on θ. Now let $\psi \in C^1(\bar{\theta})$ and $(v-\psi)(x_0)=\min_{x \in \bar{\theta}}[(v-\psi)(x)]=0$ for some $x_0 \in \bar{\theta}$. Then we have

$$\psi(x_0)=\inf_{u \in \mathscr{A}_{x_0}}\left[\int_0^T f(y(x_0, t, u), u(t))\, e^{-t}\, dt+e^{-T} v(y(x_0, T, u))\right] \quad \text{for } T \geqq 0. \tag{2.6}$$

Thus there is a sequence $\{u^m\}_{m=1}^{\infty} \subset \mathscr{A}_{x_0}$ such that

$$\psi(x_0)+\frac{1}{m^2} \geqq \int_0^{1/m} e^{-t} f(y(x_0, t, u^m), u^m(t))\, dt+e^{-1/m} \psi\left(y\left(x_0, \frac{1}{m}, u^m\right)\right). \tag{2.7}$$

Use (1.1) and (1.3) and proceed as in (2.3) and (2.4) to obtain

$$\psi(x_0)-m \int_0^{1/m} b(x_0, u^m(t))\, dt \cdot \nabla \psi(x_0)-m \int_0^{1/m} f(x_0, u^m(t))\, dt \geqq -K(m) \tag{2.8}$$

where $K(m)$ denotes a sequence of numbers which converges to zero as m tends to infinity. Observe that $(b^m, f^m) = (m\int_0^{1/m} b(x_0, u^m(t))\,dt,\ m\int_0^{1/m} f(x_0, u^m(t))\,dt)$ lies in the closed convex hull of $BF(x_0) = \{(b(x_0, \alpha), f(x_0, \alpha)),\ \alpha \in U\}$ which is compact. Thus there is a subsequence denoted by m again and $(b, f) \in \overline{co}BF(x_0)$ such that (b^m, f^m) converges to (b, f). Send m to infinity in (2.8) to get

$$\psi(x_0) - b \cdot \nabla\psi(x_0) - f \geqq 0: \tag{2.9}$$

hence

$$\psi(x_0) + \sup\{-b \cdot \nabla\psi(x_0) - f: (b, f) \in \overline{co}BF(x_0)\} \geqq 0. \tag{2.10}$$

But $H(x_0, \nabla\psi(x_0)) = \sup\{-b \cdot \nabla\psi(x_0) - f: (b, f) \in \overline{co}\,BF(x_0)\}$. This proves that the optimal value v is a constrained viscosity solution. The uniqueness is an immediate consequence of the following theorem. □

Consider two running costs $\{f_i;\ i = 1, 2\}$ and the corresponding Hamiltonians H_i defined as in (0.3).

THEOREM 2.2. *Suppose v_1 is a viscosity subsolution of $v(x) + H_1(x, Dv(x)) = 0$ on θ and v_2 is a viscosity supersolution of $v(x) + H_2(x, Dv(x)) = 0$ on $\bar\theta$. Let* (A1), (1.1), (1.2) *hold and f_i satisfy* (1.3) *and* (1.4) *for $i = 1, 2$. Then*

$$\sup_{x\in\bar\theta}[v_1(x) - v_2(x)] \leqq \sup_{\substack{x\in\bar\theta\\ \alpha\in U}}[f_1(x, \alpha) - f_2(x, \alpha)]. \tag{2.11}$$

Before we give the proof, we briefly sketch the technique introduced by Crandall, Evans and Lions [1] and point out the modification we need. Let ξ be a smooth bump function ξ, for example, $\xi(r) = 1 - r^2$. Let $m = \max\{\|v_1\|_\infty, \|v_2\|_\infty\}$ and define

$$\Phi(x, y) = v_1(x) - v_2(y) + 3m\xi\left(\frac{x - y}{\varepsilon}\right). \tag{2.12}$$

Suppose Φ attains its maximum at $(x_0, y_0) \in \bar\theta \times \bar\theta$. It follows that $|x_0 - y_0| \leqq \varepsilon$. Now consider the map $x \to v_1(x) - v_2(y_0) + 3m\xi(x - y_0/\varepsilon)$. It has a maximum at x_0. If $x_0 \in \theta$, the viscosity property yields

$$v_1(x_0) + H_1(x_0, p_\varepsilon) \leqq 0, \tag{2.13}$$

$$p_\varepsilon = \frac{3m}{\varepsilon}\nabla\xi\left(\frac{x_0 - y_0}{\varepsilon}\right). \tag{2.14}$$

Similarly, consider the map $y \to v_1(x_0) - v_2(y) + 3m\xi(x_0 - y/\varepsilon)$. At $y_0 \in \bar\theta$ it has a maximum. The viscosity property implies

$$v_2(y_0) + H_2(y_0, p_\varepsilon) \geqq 0. \tag{2.15}$$

Subtract (2.15) from (2.13) and use the fact $|x_0 - y_0| \leqq \varepsilon$ to obtain

$$\begin{aligned} v_1(x_0) - v_2(y_0) &\leqq H_2(y_0, p_\varepsilon) - H_1(x_0, p_\varepsilon) \\ &\leqq 0(\varepsilon) + \sup_{x\in\theta\alpha\in U}[f_1(x, \alpha) - f_2(x, \alpha)]. \end{aligned} \tag{2.16}$$

This will give (2.11) since one can estimate $\sup_{x\in\theta}[v_1(x) - v_2(x)]$ by $v_1(x_0) - v_2(y_0)$. In general, however, x_0 may lie on $\partial\Omega$. To complete the proof of the theorem, we have to modify Φ so that x_0 lies in θ.

Proof of Theorem 2.1. Let η, r be as in (A1), pick $z_\delta \in \bar\theta$ and ρ positive such that

$$|\eta(x) - \eta(y)| \leqq \frac{r}{2} \quad \text{for all } x, y \in \bar\theta \text{ and } |x - y| < \rho, \tag{2.17}$$

$$v_1(z_\delta)-v_2(z_\delta)\geqq\sup_{x\in\bar\theta}[v_1(x)-v_2(x)]-\frac{\delta}{2}. \tag{2.18}$$

Define $\Phi^\varepsilon:\bar\theta\times\bar\theta\to R$ as follows:

$$\Phi^\varepsilon(x,y)=v_1(x)-v_2(y)-\left|\frac{x-y}{\varepsilon}-\frac{2}{r}\eta(z_\delta)\right|^2-\left|\frac{y-z_\delta}{\rho}\right|^2. \tag{2.19}$$

Note that $z_\delta+(\varepsilon 2/r)\eta(z_\delta)$ is in θ for small ε. Use these to obtain

$$\begin{aligned}\Phi^\varepsilon\left(z_\delta+\frac{2\varepsilon}{r}\eta(z_\delta),\,z_\delta\right)&\geqq v_1(z_\delta)-v_2(z_\delta)-\omega_1(c\varepsilon)\\ &\geqq\sup_{x\in\theta}[v_1(x)-v_2(x)]-\frac{\delta}{2}-\omega_1(c\varepsilon)\end{aligned} \tag{2.20}$$

where $\omega_1(r)$ is the modulus of continuity and c is a positive constant. We also have

$$\Phi^\varepsilon(x,y)\leqq v_1(x)-v_2(x)+\omega_1(|x-y|)-\left|\frac{x-y}{\varepsilon}-\frac{2}{r}\eta(z_\delta)\right|^2-\left|\frac{y-z_\delta}{\rho}\right|^2. \tag{2.21}$$

Suppose $\Phi^\varepsilon(x,y)\geqq\Phi^\varepsilon(z_\delta+(2\varepsilon/r)\eta(z_\delta),z_\delta)$. Use (2.20) and (2.21) to obtain

$$\left|\frac{y-z_\delta}{\rho}\right|^2+\left|\frac{x-y}{\varepsilon}-\frac{2}{r}\eta(z_\delta)\right|^2\leqq\omega_1(c\varepsilon)+\frac{\delta}{2}+\omega_1(|x-y|). \tag{2.22}$$

Since ω_1 is bounded it follows that $\Phi^\varepsilon(x,y)\leqq\Phi^\varepsilon(z_\delta+(2\varepsilon/r)\eta(z_\delta),z_\delta)$ for $x,y\notin B(z_\delta,K)$ for sufficiently large K. Hence Φ^ε achieves its maximum, say at x_0,y_0. Also (2.22) yields that there is m positive such that $|x_0-y_0|\leqq m\varepsilon$. We use this in (2.22) to obtain

$$\left|\frac{y_0-z_\delta}{\rho}\right|^2+\left|\frac{x_0-y_0}{\varepsilon}-\frac{2}{r}\eta(z_\delta)\right|^2\leqq\omega_1(c\varepsilon)+\frac{\delta}{2}+\omega_1(m\varepsilon). \tag{2.23}$$

Pick ε and δ so that the right-hand side of (2.23) is less than one. Hence $|y_0-z_\delta|\leqq\rho$, (2.17) implies there in e is the unit ball such that $\eta(y_0)=\eta(z_\delta)+(r/2)e$. Also, there is e' again in the unit ball such that $x_0=y_0+(\varepsilon 2/r)\eta(z_\delta)+\varepsilon e'$. Combining these yields

$$x_0=y_0+\frac{2\varepsilon}{r}\left(\eta(y_0)+r\left[-\frac{e}{2}+\frac{e'}{2}\right]\right)\in B(y_0+t\eta(y_0),tr) \tag{2.24}$$

with $t=2\varepsilon/r$. Thus, (A1) implies $x_0\in\theta$ if ε is small. Now consider the maps

$$\bar\psi(x)=v_2(y_0)+\left|\frac{x-y_0}{\varepsilon}-\frac{2}{r}\eta(z_\delta)\right|^2+\left|\frac{y_0-z_\delta}{\rho}\right|^2, \tag{2.25}$$

$$\psi(y)=v_1(x_0)-\left|\frac{x_0-y}{\varepsilon}-\frac{2}{r}\eta(z_\delta)\right|^2-\left|\frac{y-z_\delta}{\rho}\right|^2. \tag{2.26}$$

Then $v_1-\bar\psi$ has a maximum at $x_0\in\theta$ and $v_2-\psi$ has a minimum at $y_0\in\bar\theta$. The viscosity property yields

$$v_1(x_0)+H_1(x_0,p_\varepsilon)\leqq 0, \tag{2.27}$$

$$v_2(y_0)+H_2(y_0,p_\varepsilon+q_\varepsilon)\geqq 0 \tag{2.28}$$

where

$$p_\varepsilon = 2\left(\frac{x_0 - y_0}{\varepsilon} - \frac{2}{r}\eta(z_\delta)\right)\frac{1}{\varepsilon}, \qquad q_\varepsilon = -2\left(\frac{y_0 - z_\delta}{\rho^2}\right).$$

Subtract (2.28) from (2.27),

$$\begin{aligned} v_1(x_0) - v_2(y_0) \leqq {} & [H_2(y_0, p_\varepsilon + q_\varepsilon) - H_2(x_0, p_\varepsilon)] \\ & + [H_2(x_0, p_\varepsilon) - H_1(x_0, p_\varepsilon)] := I(\varepsilon) + J(\varepsilon), \end{aligned} \tag{2.29}$$

$$J(\varepsilon) \leqq \sup_{\alpha \in U} [f_1(x_0, \alpha) - f_2(x_0, \alpha)]. \tag{2.30}$$

Using (1.2) and (1.3) yields

$$I(\varepsilon) \leqq \omega_{f_2}(|x_0 - y_0|) + |p_\varepsilon| L(b) |x_0 - y_0| + K(b) |q_\varepsilon|. \tag{2.31}$$

(2.23) yields $|q_\varepsilon| \leqq h(\varepsilon) + \delta/2$, $\varepsilon |p_\varepsilon| \leqq h(\varepsilon) + \delta/2$, and $|x_0 - y_0| \varepsilon^{-1} \leqq C$ for some C independent of ε. Here $h(\varepsilon)$ denotes a continuous function of ε which has value zero at the origin. Thus, we have

$$I(\varepsilon) \leqq \omega_{f_2}(C\varepsilon) + L(b) h(\varepsilon) C + K(b) h(\varepsilon) \leqq h(\varepsilon) + C\delta. \tag{2.32}$$

Substitute (2.30), (2.32) into (2.29) to get

$$v_1(x_0) - v_2(y_0) \leqq h(\varepsilon) + \sup_{\alpha \in U} [f_1(x_0, \alpha) - f_2(x_0, \alpha)] + C\delta. \tag{2.33}$$

Also we have,

$$\begin{aligned} \max_{x \in \bar\theta} \{v_1(x) - v_2(x)\} & \leqq v_1\left(z_\delta + \frac{2\varepsilon}{r}\eta(z_\delta)\right) - v_2(z_\delta) + \delta + \omega_1(c\varepsilon) \\ & \leqq \max\{\phi^\varepsilon(x, y) \colon x, y \in \bar\theta\} + \delta + \omega_1(c\varepsilon). \end{aligned}$$

Using (2.23) and (2.33), one can show that

$$\max\{\phi^\varepsilon(x, y) \colon x, y \in \bar\theta\} \leqq h(\varepsilon) + c\delta + \sup_{x \in \bar\theta, \alpha \in U} \{f_1(x, \alpha) - f_2(y, \alpha)\}. \tag{2.34}$$

Now send first ε then δ to zero. □

In fact, using the fact that x_0 is close to z_δ in (2.33), one can improve the result as follows:

COROLLARY 2.3. *Let $z_\delta \in \bar\theta$ be as in* (2.18); *then under the hypothesis of Theorem* 2 *we have*

$$v_1(z_\delta) - v_2(z_\delta) \leqq \sup_{\alpha \in U} [f_1(z_\delta, \alpha) - f_2(z_\delta, \alpha)] + C\delta + \omega_{f_1}(C\delta) + \omega_{f_2}(C\delta)$$

where C is a positive constant depending on $K(b)$, $L(b)$.

3. Uniform continuity of the value function. In this section we prove the continuity of the value function under the following assumptions.

(A3) There is a positive constant β such that for any $x \in \partial\theta$ there is $\alpha(x) \in U$ satisfying $b(x, \alpha(x)) \cdot \nu(x) \leqq -\beta < 0$, where ν is the exterior normal vector.

(A4) The boundary $\partial\theta$ is of class C^2.

(A5) If $\partial\theta$ is not compact there are positive constants ρ and l such that, for any $x\in\partial\theta$ there is a $T\in C^2(B(x,\rho)$ with inverse $T^{-1}\in C^1(B(x,\rho))$ satisfying

$$\text{(3.1)}\qquad \begin{aligned} &\text{(i)} \quad T(B(x,\rho)\cap\theta)\subset\{y\in R^n, y_n>0\},\\ &\text{(ii)} \quad T(B(x,\rho)\cap\partial\theta)\subset\{y\in R^n, y_n=0\},\\ &\text{(iii)} \quad \|T\|_{C^2(B(x,\rho))}+\|T^{-1}\|_{C^1(B(x,\rho))}\leqq l. \end{aligned}$$

The subscript n denotes the nth component.

Remark. The condition (3.1) is satisfied locally if (A4) holds. By using a technique similar to the one indicated below, one can prove the continuity of the optimal value function under (A3) and (A4). Instead of this we use (A5) together with (A3) and (A4) to obtain a uniform modulus of continuity for the optimal value function.

LEMMA 3.1. *Suppose* (A3)–(A5) *hold. Then, for all* $x\in\partial\theta$, $\bar b(x,\alpha(x))_n\geqq l\beta$ *where*

$$\bar b(y,\alpha)=\nabla T(T^{-1}(y))\cdot b(T^{-1}(y),\alpha),\ y\in B(x,\rho)\text{ and }\alpha\in U. \tag{3.2}$$

Proof. Given $x_0\in\partial\theta$ pick T as in (A5). Observe $\partial\theta\cap B(x_0,\rho)\subset\{x\colon T_n(x)=0\}$. Hence $\nu(x_0)=-\nabla T_n(x_0)/|\nabla T_n(x_0)|$.

$$\begin{aligned} \bar b(T(x_0),\alpha(x_0))_n&=\nabla T_n(x_0)\cdot b(x_0,\alpha(x_0))\\ &=-|\nabla T_n(x_0)|\nu(x_0)\cdot b(x_0,\alpha(x_0))\geqq l\beta. \end{aligned}\tag{3.3}$$ □

Remark. The vector field $\bar b$ is the image of b under the transformation T.

Let u be an admissible control for $x_0\in\bar\theta$. Then u is not necessarily admissible at any point x, regardless how close that point is to x_0. The following lemma provides a way to project it into $\mathcal{A}_x$ by changing the cost proportionally to $|x-x_0|$.

LEMMA 3.2. *Assume that* (A3)–(A5), (1.1)–(1.4) *hold. Then there exist* $t^*>0$ *and* $L>0$ *such that for any* $x\in\bar\theta$ *and* $u\in\mathcal{A}$ *there is* $\bar u$ *in* $\mathcal{A}_x$ *satisfying*

$$|J_{t^*}(x,\bar u)-J_{t^*}(x,u)|\leqq L\sup_{t\in[0,t^*]}[\operatorname{dist}(y(x,t,u),\bar\theta)] \tag{3.4}$$

where

$$J_{t^*}(x,u)=\int_0^{t^*}e^{-t}f(y(x,t,u),u(t))\,dt. \tag{3.5}$$

Proof. In the proof we shall determine t^*, sufficiently small. Let t_0 be the first entrance to $\partial\theta$, i.e.,

$$\begin{aligned} t_0=\inf\{0<t\leqq t^*, y(x,t,u)\in\partial\theta\}\\ \text{or } t^*\text{ if } y(x,t,u)\in\theta\text{ for all } t\leqq t^*. \end{aligned}\tag{3.6}$$

Let $\varepsilon=\sup\{\text{dist.}(y(x,t,u),\bar\theta);\ t\in[0,t^*]\}$. Define $\bar u$ as follows

$$\bar u(t)=u(t)\chi_{[0,t_0)\cup(t_0+k\varepsilon,\infty)}(t)+\alpha(y(x,t_0,u))\chi_{[t_0,t_0+k\varepsilon]}(t) \tag{3.7}$$

where k is to be chosen and $\alpha(x)$ is as in (A3). We claim that $y(x,t,\bar u)\in\bar\theta$ for $t\leqq t^*$. At $y(x,t_0,u)\in\partial\theta$ there is a map T satisfying (3.1). Set $z(x,t,u)=T(y(x,t,u))$ for any u in $\mathcal{A}$, then z obeys the differential equation

$$\frac{d}{dt}z(x,t,u)=\bar b(z(x,t,u),u(t)) \tag{3.8}$$

where $\bar b$ is as in (3.2). The vector field $\bar b$ is Lipschitz continuous on $N=T(B(y(x,t_0,u),\rho))$. Moreover, on N it is bounded by $K(\bar b)=lK(b)$ and its Lipschitz constant $L(\bar{b})$ is no more than $l^2K(b)+l^2L(b)$. (Here l is as in (A5).) If we choose t^* less than $\rho(K(b))^{-1}$, then $y(x, t, \bar{u})$ lies in $B(y(x, t_0, u), \rho)$ for all $t \leqq t^*$. Hence, $z(x, t, \bar{u}) \in N$ for all $t \leqq t^*$ and without loss of generality we assume $\bar{b}$ is Lipschitz continuous with Lipschitz constant $l^2K(b)+l^2L(b)$. To prove the claim, it suffices to show $(z(x, t, \bar{u}))_n \geqq 0$ on $t \in [0, t^*]$. Consider

$$\psi(t) = (z(x, t+k\varepsilon, \bar{u}) - z(x, t, u))_n \quad \text{for } t \geqq t_0. \tag{3.9}$$

Then

$$\begin{aligned}\psi(t) &= \psi(t_0) + \int_{t_0}^{t} [\bar{b}(z(x, s+k\varepsilon, \bar{u}), u(t) - \bar{b}(z(x, s, u), u(t))]_n \, ds \\ &\geqq \psi(t_0) - L(\bar{b})|z(x, t_0+k\varepsilon, \bar{u}) - z(x, t_0, u)| \int_{t_0}^{t} e^{L(\bar{b})(s-t_0)} \, ds \\ &\geqq \psi(t_0) - K(\bar{b})k\varepsilon(e^{L(\bar{b})(t-t_0)} - 1) \\ &\geqq \psi(t_0) - K(\bar{b})k\varepsilon(e^{L(\bar{b})(t^*-t_0)} - 1) \quad \text{for } t \in [t_0, t^*].\end{aligned} \tag{3.10}$$

Now choose t^* less than $(L(\bar{b}))^{-1} \ln (1+\beta l/4K(\bar{b}))$, where β is as in (A3). We have

$$\psi(t) \geqq \psi(t_0) - k\varepsilon\beta l\tfrac{1}{4} \quad \text{for } t \in [t_0, t^*]. \tag{3.11}$$

We need an estimate for $\psi(t_0)$. To simplify the notation, let $\bar{b}_0 = \bar{b}(z(x, t_0, u)), \alpha(y(x, t_0, u))$ and $\omega(t) = z(x, t_0, u) + (t-t_0)\bar{b}_0$. Then, Lemma 3.1 yields that $(\bar{b}_0)_n \geqq \beta l$ and hence,

$$\omega(t)_n \leqq \beta l(t-t_0) \quad \text{for } t \geqq t_0. \tag{3.12}$$

Using standard O.D.E. estimates, one can obtain

$$|z(x, t, \bar{u}) - \omega(t)| \leqq \tfrac{1}{2}L(\bar{b})K(\bar{b})(t-t_0)^2 \quad \text{for } t \geqq t_0. \tag{3.13}$$

Thus

$$z(x, t_0+k\varepsilon, \bar{u})_n \geqq \beta l k\varepsilon - \tfrac{1}{2}L(\bar{b})K(\bar{b})(k\varepsilon)^2. \tag{3.14}$$

Since $y(x, t_0, u) \in \partial\theta$, we have $\psi(t_0) = z(x, t_0+k\varepsilon, \bar{u})_n$. Substitute (3.14) into (3.11) to get

$$\psi(t) \geqq k\varepsilon(\beta l\tfrac{3}{4} - \tfrac{1}{2}L(\bar{b})K(\bar{b})k\varepsilon). \tag{3.15}$$

Choose k to be the minimum of $\beta l/2\varepsilon L(\bar{b})K(\bar{b})$ and $2/\beta$. Then for $t \leqq t^*$

$$\psi(t) \geqq \varepsilon l = \sup \{[-z(x, t, u)_n], t \in [0, t^*]\}. \tag{3.16}$$

Hence, $z(x, t+k\varepsilon, \bar{u})_n = \psi(t) + z(x, t, u)_n \geqq 0$ for all $t \in [t_0, t^*]$. One can prove (3.4) by using the standard estimates. □

THEOREM 3.3. *Suppose* (A3)-(A5), (1.1), (1.2) *and* (1.4) *hold. Then the value function v is in $BUC(\bar{\theta})$.*

Proof. Without loss of generality one can assume f is Lipschitz in x uniformly with respect to α. If not, we take a sequence f^n of Lipschitz continuous functions converging to f uniformly. Let $x, y \in \bar{\theta}$ and $|x-y| < r$. For any positive δ, pick a δ-optimal control u in $\mathcal{A}_y$, i.e.

$$J_{t^*}(y, u) + e^{-t^*} v(y(y, t^*, u)) \leqq v(y) + \delta \tag{3.17}$$

where t^* is as in Lemma 3.2. Construct $\bar{u} \in \mathcal{A}_x$ as in Lemma 3.2, and set $\varepsilon = \sup [\text{dist}(y(x, t, u), \bar{\theta}), t \in [0, t^*]]$. Using standard estimates, one can get $\varepsilon \leqq Cr$ for some C positive. Thus, we have

$$|J_{t^*}(x, u) - J_{t^*}(x, \bar{u})| \leqq LCr. \tag{3.18}$$

Also the construction of $\bar{u}$ yields

$$|y(x, t^*, u) - y(x, t^*, \bar{u})| \leqq \bar{C}r \quad \text{for some } \bar{C} > 0. \tag{3.19}$$

Also

$$|y(x, t^*, \bar{u}) - y(y, t^*, u)| \leqq \bar{C}r + |y(x, t^*, u) - y(y, t^*, u)| \leqq \tilde{C}r \quad \text{for some } \tilde{C} > 1. \tag{3.20}$$

And

$$|J_{t^*}(x, \bar{u}) - J_{t^*}(y, u)| \leqq LCr + |J_{t^*}(x, u) - J_{t^*}(y, u)| \leqq Cr \quad \text{for some } C > 0. \tag{3.21}$$

Let $\omega(r) = \sup\{|v(x) - v(y)|, x, y \in \bar{\theta}, |x - y| < r\}$ for $r > 0$. At the origin $\omega(0) = \lim_{r \downarrow 0} \omega(r)$. Combine (3.17), (3.20) and (3.21) and use the dynamic programming principle to obtain

$$\begin{aligned} v(x) - v(y) &\leqq J_{t^*}(x, \bar{u}) + e^{-t^*} v(y(x, t^*, \bar{u})) \\ &\qquad - J_{t^*}(y, u) - e^{-t^*} v(y(y, t^*, u)) + \delta \\ &\leqq Cr + e^{-t^*} \omega(\tilde{C}r) + \delta \quad \text{for all } \delta > 0. \end{aligned} \tag{3.22}$$

Hence we have

$$\omega(r) \leqq Cr + e^{-t^*} \omega(\tilde{C}r) \quad \text{and} \quad \tilde{C} > 1. \tag{3.23}$$

Assume $\tilde{C} e^{-t^*} \neq 1$ and iterate (3.23) to obtain

$$\begin{aligned} \omega(0) = \lim_{n \to \infty} \omega(\tilde{C}^{-n}) &\leqq \lim_{n \to \infty} \left[C\tilde{C}^{-n} \sum_{l=0}^{n-1} [\tilde{C} e^{-t^*}]^l + e^{-nt^*} \omega(1) \right] \\ &= C \lim_{n \to \infty} \left[\frac{e^{-nt^*} - \tilde{C}^{-n}}{\tilde{C} e^{-t^*} - 1} \right] = 0. \qquad \square \end{aligned}$$

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