SIAM J. CONTROL AND OPTIMIZATION
Vol. 24, No. 6, November 1986

002

OPTIMAL CONTROL WITH STATE-SPACE CONSTRAINT II*

HALİL METE SONER†

Abstract. Optimal control of piecewise deterministic processes with state space constraint is studied. Under appropriate assumptions, it is shown that the optimal value function is the only viscosity solution on the open domain which is also a supersolution on the closed domain. Finally, the uniform continuity of the value function is obtained under a condition on the deterministic drift.

Key words. viscosity solutions, stochastic control, state-space constraint, piecewise deterministic processes

AMS(MOS) subject classifications. 93E20, 35J65, 35K60, 60J60

Introduction. We are interested in the optimal control of jump processes with a state-space constraint. By that we mean the trajectories of the controlled process have to stay within a given subset θ of $\mathbb{R}^n$. These kinds of processes arise naturally in some applications [5], [9], [10]. The deterministic counterpart of this problem is studied in [11] and the optimal value function is characterized as the viscosity solution of the corresponding Hamilton-Jacobi-Bellman (HJB) equation. Also the concept of viscosity solutions, introduced by M. G. Crandall and P.-L. Lions [2], was used to identify the boundary conditions satisfied by the optimal value function. For more information about viscosity solutions see [1], [3], [7], [8] and references therein.

In this paper we generalize the results mentioned above to a certain class of jump processes, namely piecewise deterministic processes. These kinds of processes are introduced by M. Davis [4] and used by D. Vermes in [12]. Let us summarize the construction of the piecewise deterministic processes.

Let u be a Borel measurable map of $\bar{\theta}=[0,\infty)$ into a compact, separable metric space U and $y_0(x, s; t, u)$ be the solution of

$$\frac{d}{dt}y_0(x, s, t, u)=b(y_0(x, s, t, u), u(x, t-s)) \quad \text{for } t \geqq s \tag{0.1}$$

with initial data $y(x, s, s, u)=x$. Pick the first jump time T_1 so that the jump rate is $\lambda(y_0(x, 0, t, u))$. Then construct the post-jump location Y_1 such that $Q(y_0(x, 0, \tau, u), u(x, \tau), \cdot)$ is its conditional distribution given $T_1=\tau$. Starting from Y_1 at time T_1 select the inter-jump time T_2-T_1 and the second post-jump location Y_2 similarly. Set $T_0=0$, $Y_0=x$ and iterate the procedure above to obtain $\{(T_n, Y_n)\colon n=0, 1, \cdots\}$. Between the jumps T_n and T_{n+1} the process $y(x, t, u)$ follows the deterministic trajectory passing through (Y_n, T_n), i.e.

$$y(x, t, u)=y_0(Y_n, T_n, t, u) \quad \text{if } t\in[T_n, T_{n+1}). \tag{0.2}$$

Moreover, $\{(Y_n, T_n)\}$ satisfies

$$\begin{aligned} &P(T_{n+1}-T_n \geqq t \mid T_1, Y_1, \cdots, T_n, Y_n) \\ &\qquad =\exp\left\{-\int_{T_n}^{t+T_n} \lambda(y(x, s, u), u(Y_n, s-T_n))\, ds\right\}, \end{aligned} \tag{0.3}$$

* Received by the editors March 11, 1985, and in revised form August 6, 1985. This research was supported by the National Science Foundation under grant MCS 8121940.

† Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912.

$$\begin{aligned}(0.4)\qquad & P(Y_{n+1}\in A|T_1, Y_1, \cdots, Y_n, T_{n+1}) \\ &= Q(y_0(Y_n, T_n, T_{n+1}, u), u(Y_n, T_{n+1}-T_n), A) \quad \text{for all } A\subset\bar\theta.\end{aligned}$$

The process $y(x,\cdot,u)$ is a strong Markov process and the following version of Ito's lemma is proved in [4]. Set $y(t)=y(x,t,u)$ and $u_n(t)=u(Y_n, t-T_n)$. Then for any $\psi\in C^1(\bar\theta\times[0,T])$ we have

$$\begin{aligned}(0.5)\qquad & E\psi(y(T),T) \\ &= \psi(x,0)+E\left\{\sum_{n=0}^{\infty}\int_{T_n\wedge T}^{T_{n+1}\wedge T}\left[b(y(t),u_n(t))\nabla\psi(y(t),t) + \frac{\partial}{\partial t}\psi(y(t),t)+\lambda(y(t),u_n(t)) \cdot\int_{\bar\theta}[\psi(z,t)-\psi(y(t),t)]Q(y(t),u_n(t),dz)\right]dt\right\}.\end{aligned}$$

We assume that the post-jump locations are in $\bar\theta$. Then one can define the set of admissible strategies $\mathcal{A}_{ad}$ as:

$$(0.6)\qquad \mathcal{A}_{ad}:=\left\{\begin{matrix}u:\bar\theta\times[0,\infty)\to U, \text{ Borel measurable and} \\ P(y(x,t,u)\in\bar\theta \text{ for all } t\geqq 0)=1, \text{ for all } x\in\bar\theta\end{matrix}\right\}.$$

The optimal value is given by

$$(0.7)\qquad v(x):=\inf_{u\in\mathcal{A}_{ad}} E\left\{\sum_{n=0}^{\infty}\int_{T_n}^{T_{n+1}} e^{-t}f(y(x,t,u),u(Y_n,t-T_n))\,dt\right\}.$$

It is shown, in § 2, that v is the only viscosity solution of the corresponding HJB equation, satisfying the same boundary condition as in the deterministic case [11]. This result holds if the optimal value is in $BUC(\bar\theta)$ and the dynamic programming relation (0.8) is satisfied.

$$(0.8)\qquad v(x)=\inf_{u\in\mathcal{A}_{ad}} E\left\{\int_0^{T\wedge T_1} e^{-t}f(y(x,t,u),u(x,t))\,dt+e^{-T\wedge T_1}v(y(x,T\wedge T_1,u))\right\} \quad \text{for all } T\geqq 0 \text{ and } x\in\bar\theta.$$

Finally, in §3 we show that under assumptions (A2)-(A4) v is in $BUC(\bar\theta)$ and satisfies the dynamic programming relation (0.8). Note that these assumptions yield that the optimal value of the corresponding deterministic problem is in $BUC(\bar\theta)$. By an induction argument one can extend this result to piecewise deterministic processes with finitely many jumps. We eventually pass to the limit to conclude.

1. Main result. Let θ be an open subset of $\mathbb{R}^n$ with connected boundary satisfying:

(A.1) There are positive constants h, r and $\mathbb{R}^n$-valued bounded-uniformly continuous map η of $\bar\theta$ such that

$$B(x+t\eta(x), tr)\subset\bar\theta \quad \text{for all } x\in\bar\theta \text{ and } t\in(0,h].$$

Here $B(x,R)$ denotes the ball with center x and radius R.

Remark 1.1. If θ is bounded and $\partial\theta$ is C^1, then (A.1) is satisfied. Also boundaries with corners may satisfy (A.1), for example, $\theta=\{(x,y)\in R^2: x>0, y>0\}$.

The strategies take values in U which is a compact, separable metric space. Also, we assume the following throughout the paper. Let x and y be in $\bar{\theta}$.

$$(1.1) \qquad \sup_{\alpha\in U} |\gamma(x,\alpha)-\gamma(y,\alpha)| \leqq L(\gamma)|x-y|, \qquad \gamma=b, f \text{ or } \lambda,$$

$$(1.2) \qquad \sup_{\substack{\alpha\in U\\ x\in\bar{\theta}}} |\gamma(x,\alpha)| \leqq K(\gamma), \qquad \gamma=b, f \text{ or } \lambda.$$

(1.3) For each bounded, continuous function h on $\bar{\theta}$, there is a continuous function W_h with $W_h(0)=0$ such that

$$\sup_{\alpha\in U}\left|\int_{\bar{\theta}} h(z)Q(x,\alpha,dz)-\int_{\bar{\theta}} h(z)Q(y,\alpha,dz)\right| \leqq W_h(|x-y|).$$

(1.4) $Q(x,\alpha,\bar{\theta})=1$ for all $x\in\bar{\theta}$ and $\alpha\in U$.

(1.5) $\lambda(x,\alpha)\geqq 0$ for all $x\in\bar{\theta}$ and $\alpha\in U$.

The corresponding Hamiltonian H is a continuous map of $\bar{\theta}\times\mathbb{R}^n\times BUC(\bar{\theta})$ given as:

$$(1.6) \qquad H(x,p,\psi)=\sup_{\alpha\in U}\left\{-b(x,\alpha)\cdot p-f(x,\alpha)-\lambda(x,\alpha)\int_{\bar{\theta}}[\psi(z)-\psi(x)]Q(x,\alpha,dy)\right\}.$$

This Hamiltonian is a nonlocal operator but still one can define a notion of viscosity solutions.

DEFINITION. Let K be a subset of $\mathbb{R}^n$ and $v\in BUC(\bar{K})$.

(i) We say v is a *viscosity subsolution* of $v(x)+H(x,Dv(x),v)=0$ on K if $v(x_0)+H(x_0,\nabla\psi(x_0),v)\leqq 0$ whenever $\psi\in C^1(N_{x_0})$ and $(v-\psi)$ has a global maximum, relative to K, at $x_0\in K$, where N_{x_0} is a neighborhood of x_0.

(ii) We say v is a *viscosity supersolution* of $v(x)+H(x,Dv(x),v)=0$ on K if $v(x_0)+H(x_0,\nabla\psi(x_0),v)\geqq 0$ whenever $\psi\in C^1(N_{x_0})$ and $(v-\psi)$ has a global minimum, relative to K, at $x_0\in K$, where N_{x_0} is a neighborhood of x_0.

Remark 1.2. This is an obvious generalization of the original notion introduced by M. G. Crandall and P.-L. Lions [2]. The definition we used above is analogous to one of the definitions introduced in [1].

We are interested in the following notion of viscosity solutions.

DEFINITION. $v\in BUC(\bar{\theta})$ is said to be a *constrained viscosity solution* of $v(x)+H(x,Dv(x),v)=0$ on $\bar{\theta}$ if it is a subsolution on θ and supersolution on $\bar{\theta}$.

Remark 1.3. The fact that v is a supersolution on the closed domain imposes a certain boundary condition. Suppose that v is smooth and a constrained viscosity solution. Then $H(x,\nabla v(x)+\alpha v(x),v)\geqq H(x,\nabla v(x),v)$ for all $x\in\partial\theta$ and $\alpha\geqq 0$ ($v(x)$ is the exterior normal vector). This effect is discussed in [11].

THEOREM 1.1. *Suppose* (A.1), (1.1)-(1.5) *hold. Then there is at most one constrained viscosity solution of $v(x)+H(x,Dv(x),v)=0$ on θ. Moreover if $v\in BUC(\bar{\theta})$ and dynamic programming relation* (0.8) *holds, then the optimal value function v is a constrained viscosity solution.*

2. Proof of the main theorem. We need the following lemma:

LEMMA 2.1. *$v\in BUC(\bar{\theta})$ is a viscosity subsolution of $v(x)+H(x,Dv(x),v)=0$ on θ (or supersolution on $\bar{\theta}$) if and only if*

$$v(x_0)+H(x_0,\nabla\psi(x_0),\psi)\leqq 0$$

(or $\geqq 0$) whenever $\psi\in C^1(N_{x_0})$ and $v-\psi$ has a global maximum relative to $\bar{\theta}$ at $x_0\in\theta$ (or minimum at $x_0\in\bar{\theta}$ respectively), where N_{x_0} is a neighborhood of x_0.

Proof. We will prove the statement for subsolutions only, the other statement is proved exactly the same way.

Necessity. Suppose $v \in BUC(\bar{\theta})$ is a viscosity subsolution and ψ, x_0 are as above, i.e.

$$v(x_0) - \psi(x_0) = \max_{x \in \theta} v(x) - \psi(x).$$

Then we have $v(x_0) - v(z) \geqq \psi(x_0) - \psi(z)$ for all $z \in \bar{\theta}$. Then (1.5) yields:

$$H(x_0, \nabla\psi(x_0), \psi) \leqq H(x_0, \nabla\psi(x_0, v).$$

Hence the viscosity property of v gives the result.

Sufficiency. Let $\psi \in C^1(N_{x_0})$ and $(v-\psi)(x_0) = \max_{x\in\theta}\{(v-\psi)(x)\} = 0$. For each $\varepsilon > 0$ we define Φ^ε as follows:

$$\Phi^\varepsilon(x) = \psi(x)\chi^\varepsilon(x) + v(x)(1-\chi^\varepsilon(x)) \quad \text{for } x \in \bar{\theta} \tag{2.1}$$

where χ^ε is a smooth function satisfying

$$\begin{aligned} &0 \leqq \chi^\varepsilon \leqq 1, \\ &\chi^\varepsilon(x) = 1 \quad \text{if } x \in B(x_0, \varepsilon), \\ &\chi^\varepsilon(x) = 0 \quad \text{if } x \in R^n \backslash B(x_0, 2\varepsilon). \end{aligned} \tag{2.2}$$

Observe $v(x_0) - \Phi^\varepsilon(x_0) = 0$ and $v(x) - \Phi^\varepsilon(x) = (v(x) - \psi(x))\chi^\varepsilon(x) \leqq 0$. Hence

$$v(x_0) - \Phi^\varepsilon(x_0) = \max_{x \in \bar{\theta}}\{v(x) - \Phi^\varepsilon(x)\}. \tag{2.3}$$

Thus the hypothesis of the lemma and $\nabla\Phi^\varepsilon(x_0) = \nabla\psi(x_0)$ yields

$$v(x_0) + H(x_0, \nabla\psi(x_0), \Phi^\varepsilon) \leqq 0. \tag{2.4}$$

The following estimate follows (1.5):

$$\begin{aligned} &|H(x_0, \nabla\psi(x_0), \Phi^\varepsilon) - H(x_0, \nabla\psi(x_0), v)| \\ &\leqq \sup_{\alpha \in U}\left\{\lambda(x_0, \alpha)\int |\Phi^\varepsilon(x_0) - \Phi^\varepsilon(y) - v(x_0) + v(y)|Q(x_0, \alpha, dy)\right\}. \end{aligned} \tag{2.5}$$

Observe that $\Phi^\varepsilon(x_0) = v(x_0)$ and $\Phi^\varepsilon(x_0+y) = v(x_0+y)$ for $y \notin B(0, 2\varepsilon)$. Also for $y \in B(0, 2\varepsilon)$

$$\begin{aligned} |\Phi^\varepsilon(x_0+y) - v(x_0+y)| &= |\psi(x_0+y) - v(x_0+y)|\chi^\varepsilon(x_0+y) \\ &\leqq |\psi(x_0+y) - \psi(x_0)| + |v(x_0) - v(x_0+y)| \\ &\leqq \|\nabla\psi\|_\infty |y| + \omega_v(|y|) \\ &\leqq 2\|\nabla\psi\|_\infty \varepsilon + \omega_v(2\varepsilon). \end{aligned} \tag{2.6}$$

Here ω_v is the modulus of continuity of v and we used $\psi(x_0) = v(x_0)$ in the second inequality. Combine (2.4)-(2.6) to conclude that v is a viscosity subsolution. □

Remark 2.1. It is easy to prove that in Lemma 2.1 we may replace $\psi \in C^1(N_{x_0})$ by $\psi \in C^1(\bar{\theta})$ (see [1]).

Proof of Theorem 1.1. Suppose v_1 and v_2 are two solutions in $BUC(\bar{\theta})$. For $i = 1, 2$ define f_i and $\bar{H}_i$ as follows

$$f_i(x, \alpha) = f(x, \alpha) + \lambda(x, \alpha)\int_{\bar{\theta}} (v_i(y) - v_i(x))Q(x, \alpha, dy) \quad \text{for } x \in \bar{\theta}, \alpha \in U, \tag{2.7}$$

$$\bar{H}_i(x, p) = \sup_{\alpha \in U}\{-b(x, \alpha)\cdot p - f_i(x, \alpha)\} \quad \text{for } x \in \bar{\theta}, p \in \mathbb{R}^n. \tag{2.8}$$

Notice that $\bar{H}_i$ is the Hamiltonian of the corresponding deterministic problem with running cost f_i. Using (1.1)-(1.4), one can show that the f_i's are uniformly continuous in x uniformly with respect to α. Pick $z_\delta \in \bar{\theta}$ such that

$$v_1(x) - v_2(x) \leqq v_1(z_\delta) - v_2(z_\delta) + \delta \quad \text{for all } x \in \bar{\theta}.$$

Then Corollary 2.3 in [11] yields

$$v_1(z_\delta) - v_2(z_\delta) \leqq c\delta + \omega_{f_1}(c\delta) + \omega_{f_2}(c\delta) + \sup_{\alpha \in U} \{f_1(z_\delta, \alpha) - f_2(z_\delta, \alpha)\}. \tag{2.9}$$

But $f_1(z_\delta, \alpha) - f_2(z_\delta, \alpha) \leqq \delta\lambda(z_\delta, \alpha) \leqq \delta K(\lambda)$ for every α. Substitute this into (2.9) and send δ to zero, to prove the uniqueness.

Let $\psi \in C^1(\bar{\theta})$ and $x_0 \in \theta$ such that $(v - \psi)(x_0) = \max\{(v - \psi)(x); x \in \bar{\theta}\} = 0$, where v is the optimal value. For any $u \in \mathcal{A}_{ad}$ the dynamic programming relation (0.8) yields

$$\psi(x_0) = v(x_0) \leqq E\left\{\int_0^{t \wedge T_1} e^{-s} f(y(x_0, s, u), u(x_0, s))\, ds + e^{-t \wedge T_1} v(y(x_0, t \wedge T_1, u))\right\}.$$

Set $y(s) = y(x_0, s, u)$ and $u(s) = u(x_0, s)$. Use $v \leqq \psi$ to obtain:

$$\psi(x_0) \leqq E\left\{\int_0^{t \wedge T_1} e^{-s} f(y(s), u(s))\, ds + e^{-t \wedge T_1} \psi(y(t \wedge T_1))\right\}. \tag{2.10}$$

Ito's formula (0.5) on $e^{-s\psi(y(s))}$ yields

$$E \int_0^{t \wedge T_1} e^{-s}[\psi(y(s)) - b(y(s), u(s)) \cdot \nabla\psi(y(s)) - \bar{f}(y(s), u(s))]\, ds \leqq 0 \tag{2.11}$$

where

$$\bar{f}(x, \alpha) = f(x, \alpha) + \lambda(x, \alpha) \int_{\bar{\theta}} (\psi(z) - \psi(x)) Q(x, \alpha, dz).$$

Observe that on $[0, t \wedge T_1)$ $y(\cdot)$ is a deterministic trajectory. Thus standard estimates on $y(\cdot)$ and (1.1)-(1.4) yield

$$\frac{1}{t} E \int_0^{t \wedge T_1} [\psi(x_0) - b(x_0, u(s)) \cdot \nabla\psi(x_0) - \bar{f}(x_0, u(s))]\, ds \leqq h(t) \tag{2.12}$$

where h is a continuous function with $h(0) = 0$. Since $x_0 \in \theta$, for any $\alpha \in U$ there is a strategy $u \in \mathcal{A}_{ad}$ such that $u(x_0, t) = \alpha$ for all $t \leqq \operatorname{dist}(x_0, \partial\theta)/K(b)$. Use this strategy in (2.12) to obtain

$$[\psi(x_0) - b(x_0, \alpha) \cdot \nabla\psi(x_0) - \bar{f}(x_0, \alpha)] E[(t \wedge T_1)/t] \leqq h(t).$$

Hence $\psi(x_0) + H(x_0, \nabla\psi(x_0), \psi) \leqq 0$. So Lemma 2.1 and Remark 2.1 imply that v is a subsolution on θ.

Now let $\psi \in C^1(\bar{\theta})$ and $(v - \psi)(x_0) = \min\{(v - \psi)(x): x \in \bar{\theta}\} = 0$ for some $x_0 \in \bar{\theta}$. The dynamic programming relation and $v \geqq \psi$ yield

$$v(x_0) = \psi(x_0) \geqq \inf_{u \in \mathcal{A}_{ad}} E\left[\int_0^{t \wedge T_1} e^{-s} f(y(s), u(s))\, ds + e^{-t \wedge T_1} \psi(y(t \wedge T_1))\right]. \tag{2.13}$$

For $t = 1/m$ one can pick $u_m \in \mathcal{A}_{ad}$ such that

$$\psi(x_0) + \left(\frac{1}{m}\right)^2 \geqq E\left[\int_0^{1/m \wedge T_1} e^{-s} f(y(s), u_m(s))\, ds + e^{-T_1 \wedge 1/m} \psi\left(y\left(T_1 \wedge \frac{1}{m}\right)\right)\right].$$

First, use Ito's lemma (0.5) on $e^{-s}\psi(y(s))$ then (1.1)-(1.4), as in (2.10)-(2.12) above to obtain

$$mE\int_0^{T_1\wedge 1/m}[\psi(x_0)-b(x_0,u_m(s))\cdot\nabla\psi(x_0)-\bar f(x_0,u_m(s))]\,ds\geqq K(m) \tag{2.14}$$

where $K(m)$ is converging to zero as m tends to infinity. Rewrite (2.14) as

$$[\psi(x_0)-B(m)\cdot\nabla\psi(x_0)-F(m)]E(mT_1\wedge 1)\geqq K(m) \tag{2.15}$$

where

$$B(m)=\left(E\left(T_1\wedge\frac{1}{m}\right)\right)^{-1}E\int_0^{T_1\wedge 1/m}b(x_0,u_m(s))\,ds,$$

$$F(m)=\left(E\left(T_1\wedge\frac{1}{m}\right)\right)^{-1}E\int_0^{T_1\wedge 1/m}\bar f(x_0,u_m(s))\,ds.$$

Observe that $(B(m),F(m))\in\overline{\text{co}}\{(b(x_0,\alpha),\bar f(x_0,\alpha)):\alpha\in U\}:=\overline{\text{co}}[BF(x_0)]$. Hence there is $(B,F)\in\overline{\text{co}}[BF(x_0)]$ such that $(B(m),F(m))$ converges to (B,F) on a subsequence, denoted by m again. Pass to the limit in (2.15) to obtain

$$\psi(x_0)+\sup\{-B\cdot\nabla\psi(x_0)-F:(B,F)\in\overline{\text{co}}[BF(x_0)]\}\geqq 0.$$

Also

$$\sup\{-B\cdot\nabla\psi(x_0)-F:(B,F)\in\overline{\text{co}}[BF(x_0)]\}=H(x_0,\nabla\psi(x_0),\psi).$$

Thus, Lemma 2.1 and Remark 2.1 imply that v is a viscosity supersolution on $\bar\theta$. □

3. Uniform continuity and dynamic programming. We assume the following.

(A.2) There is a Borel measurable map α of $\partial\theta$ into U and β positive satisfying $b(x,\alpha(x))\cdot\nu(x)\leqq-\beta<0$, where ν is the exterior normal vector.

(A.3) The boundary of θ is of class C^2.

(A.4) If $\partial\theta$ is not compact, there are constants ρ and l such that for any $x\in\partial\theta$ there is a $C^2(B(x,\rho))$ function T with $C^1(B(x,\rho))$ inverse T^{-1} satisfying

$$\begin{aligned}&\text{(i)}\quad T(B(x,\rho)\cap\theta)\subset\{y\in R^n: y_n>0\},\\&\text{(ii)}\quad T(B(x,\rho)\cap\partial\theta)\subset\{y\in R^n: y_n=0\},\\&\text{(iii)}\quad \|T\|_{C^2(B(x,\rho))}+\|T^{-1}\|_{C^1(B(x,\rho))}\leqq l.\end{aligned} \tag{3.1}$$

The subscript n denotes the nth component.

Remark 3.1. The assumption (A.2) holds with some $\beta>0$ if

$$\sup_{x\in\partial\theta}\min_{\alpha\in U}b(x,\alpha)\cdot\nu(x)<0.$$

Note that we do not assume that $b(x,\alpha)$ points inwards the domain θ for all α and $x\in\partial\theta$. Thus there may be controls that allow the deterministic process to reach the boundary.

Remark 3.2. The assumptions (A.2)-(A.4) are used to obtain the uniform continuity of the corresponding deterministic problem [11]. In particular see [11, Lemma 3.2].

Let v_0 be the optimal value of the deterministic problem and define v^N as follows

$$v^N(x)=\inf_{u\in\mathcal{A}_{ad}}J^N(x,u)$$

where

$$J^N(x,u)=E\left[\int_0^{T_1}e^{-t}f(y(x,t,u),u(x,t))\,dt+e^{-T_1}v^{N-1}(Y_1)\right]. \tag{3.2}$$

LEMMA 3.1. *Let $v^{N-1}\in BUC(\bar\theta)$, then the dynamic programming relation holds for v^N, i.e., for all $T>0$*

$$v^N(x)=\inf_{u\in\mathscr{A}_{ad}} E\left\{\int_0^{T\wedge T_1} e^{-T} f(y(x,t,u),u(x,t))\,dt + e^{-T_1}v^{N-1}(Y_1)\chi_{[0,T]}(T_1)+e^{-T}v^N(y(x,T,u))\chi_{(T,\infty)}(T_1)\right\} \tag{3.3}$$

where χ_A is the indicator function of set A.

Proof. Fix $x\in\bar\theta$ and T positive. Let $I^N(x,u)$ be the right-hand side of (3.3) before taking the infimum. To simplify the notation, put $y(t)=y_0(x,0;t,u)$, $u(t)=u(x,t)$, $\lambda(t)=\lambda(y(t),u(t))$ and $\Lambda(t)=\exp\{-\int_0^t\lambda(s)\,ds\}$. Recall that y_0 is the corresponding deterministic trajectory given by (0.1). In terms of these quantities $I^N(x,u)$ is given by

$$\begin{aligned} I^N(x,u)&=\int_0^\infty \lambda(t)\Lambda(t)\left[\int_0^{T\wedge t} e^{-s}f(y(s),u(s))\,ds + e^{-t}\int_{\bar\theta} v^{N-1}(z)Q(y(t),u(t),dz)\chi_{[0,T]}(t) + e^{-T}v^N(y(T))\chi_{(T,\infty)}(t)\right]dt\\ &\quad+\Lambda(\infty)\left[\int_0^T e^{-t}f(y(t),u(t))\,dt+e^{-T}v^N(y(T))\right]\\ &=\int_0^T \lambda(t)\Lambda(t)\left[\int_0^t e^{-s}f(y(s),u(s))\,ds+e^{-t}\int_{\bar\theta}v^{N-1}(z)Q(y(t),u(t),dz)\right]dt\\ &\quad+\Lambda(T)\int_0^T e^{-s}f(y(s),u(s))\,ds+\Lambda(T)\,e^{-T}v^N(y(T)). \end{aligned}\tag{3.4}$$

Since $y(T)$ is a deterministic quantity determined by x, T and u, one can pick $u^*\in\mathscr{A}_{ad}$ such that $v^N(y(T))\geqq J^N(y(T),u^*)-\delta$. Now we define $\bar u$ as follows:

$$\bar u(z,t)=u(z,t)\chi_{[0,T]}(t)+u^*(y_0(z,0,T,u),t-T)\chi_{[T,\infty)}(t). \tag{3.5}$$

Define $\bar y(t)=\bar y(x,0;t,\bar u)$ and $\bar\lambda$, $\bar\Lambda$ similarly. Then we have

$$\begin{aligned} J^N(y(T),u^*)=(\Lambda(T))^{-1}\Bigg\{&\int_0^\infty \bar\lambda(t+T)\bar\Lambda(t+T)\cdot\left[\int_0^t e^{-s}f(\bar y(s+T),\bar u(x,s+T))\,ds + e^{-t}\int_{\bar\theta}v^{N-1}(z)Q(\bar y(t+T),\bar u(x,t+T),dz)\right]dt\\ &+\bar\Lambda(\infty)\int_0^\infty e^{-s}f(\bar y(s+T),\bar u(x,s+T))\,ds\Bigg\}. \end{aligned}\tag{3.6}$$

Change variables in (3.6) and use $v^N(y(T)) \geqq J^N(y(T), u^*) - \delta$ to obtain

$$e^{-T}\Lambda(T)v^N(y(T)) \geqq \int_T^\infty \bar{\lambda}(t)\bar{\Lambda}(t)\left[\int_T^t e^{-s}f(\bar{y}(s), \bar{u}(x,s))\,dx + e^{-t}\int_{\bar{\theta}} v^{N-1}(z)Q(\bar{y}(t), \bar{u}(x,t), dz)\right]dt + \bar{\Lambda}(\infty)\int_T^\infty e^{-s}f(\bar{y}(s), \bar{u}(x,s))\,ds - e^{-T}\Lambda(T)\delta.$$

The fact that $\int_T^\infty \bar{\lambda}(t)\bar{\Lambda}(t)\,dt = \Lambda(T) - \bar{\Lambda}(\infty)$ and the above inequality yield

$$\Lambda(T)\int_0^T e^{-s}f(y(s), u(s))\,ds + \Lambda(T)\,e^{-T}v^N(y(T))$$
$$\geqq \int_T^\infty \bar{\lambda}(t)\bar{\Lambda}(t)\left[\int_0^t e^{-s}f(\bar{y}(s), \bar{u}(x,s))\,ds + e^{-t}\int_{\bar{\theta}} v^{N-1}(z)Q(\bar{y}(t), \bar{u}(x,t), dz)\right]dt$$
$$+ \bar{\Lambda}(\infty)\int_0^\infty e^{-s}f(\bar{y}(s), \bar{u}(x,s))\,ds - \delta.$$

Substitute the above inequality into (3.4) and use the fact $\lambda(t) = \bar{\lambda}(t)$, $\Lambda(t) = \bar{\Lambda}(t)$ for $t \in [0, T]$ to obtain

$$\begin{aligned} I^N(x,u) \geqq & \int_0^\infty \bar{\lambda}(t)\bar{\Lambda}(t)\left[\int_0^t e^{-s}f(\bar{y}(s), \bar{u}(x,s))\,ds + e^{-t}\int_{\bar{\theta}} v^{N-1}(z)Q(\bar{y}(t), \bar{u}(x,t), dz)\right]dt \\ & + \bar{\Lambda}(\infty)\int_0^\infty e^{-s}f(\bar{y}(s), \bar{u}(x,s))\,ds - \delta \\ = & J^N(x, \bar{u}) - \delta \geqq v^N(x) - \delta. \end{aligned} \tag{3.7}$$

Thus, $v^N(x) \leqq \inf_{u \in \mathcal{A}_{ad}} I^N(x,u)$. One can prove the other inequality similarly. □

The following lemma is an analogue of Lemma 3.2 in [11].

LEMMA 3.2. *Let $v^{N-1} \in BUC(\bar{\theta})$ and (1.1)-(1.4), (A.2)-(A.4) hold. Then for any T positive, there is a positive function h_T and a projection $\mathcal{P}_T(u)$ of any Borel measurable map u of $\bar{\theta} \times [0, \infty)$ into U such that $\mathcal{P}_T(u) \in \mathcal{A}_{ad}$ and*

$$|J_T^N(x,u) - J_T^N(x, \mathcal{P}_T(u))| \leqq h_T(\sup\{\operatorname{dist}(y_0(x,0,t,u), \bar{\theta}) : t \in [0,T]\}) \tag{3.8}$$

where h_T is a continuous function with $h_T(0) = 0$ and

$$J_T^N(x,u) = E\left\{\int_0^{T\wedge T_1} e^{-s}f(y(x,s,u), u(x,s))\,ds + e^{-T_1}v^{N-1}(Y_1)\chi_{[0,T]}(T_1)\right\}. \tag{3.9}$$

Proof. Define $t_0(x,u) = \inf\{t \geqq 0 : y_0(x,0,t,u) \in \partial\theta\}$ or infinity. Let t^* and k be as in Lemma 3.2 of [11], i.e.

$$\begin{aligned} t^* &= \min\{\rho/K(\bar{b}), l\beta/L(\bar{b})K(\bar{b}), \ln(1 + \beta l/4K(\bar{b}))/L(\bar{b})\}, \\ k &= 2/l\rho \end{aligned} \tag{3.10}$$

where $K(\bar{b}) = lK(b)$ and $L(\bar{b}) = l^2K(b) + l^2L(b)$. We now construct u^1 as in Lemma 3.2 of [11]

$$u^1(x,t) = \begin{cases} u(x,t) & \text{if } t \leqq t_0(x,u) \quad \text{or } t \geqq t_0(x,u) + k\varepsilon_0(x,u), \\ \alpha(y_0(x,0,t_0(x,u),u)) & \text{if } t_0(x,u) < t < t_0(x,u) + k\varepsilon_0(x,u) \end{cases} \tag{3.11}$$

where α as in (A.2) and $\varepsilon_0(x, u) = \sup\{\operatorname{dist}(y_0(x, 0, t, u), \bar{\theta}) : t \in [0, t^*]\}$. Since $x \to t_0(x, u)$ is a Borel measurable map u^1 is measurable. Also in Lemma 3.2 of [11] it is proved that

$$y_0(x, 0, t, u^1) \in \bar{\theta} \quad \text{for all } t \in [0, t^*]. \tag{3.12}$$

Now construct a sequence of strategies $\{u^n : n = 1, 2, \cdots\}$ by the following recursive formula

$$t_n(x, u^n) = \inf\{t \geqq nt^* : y_0(x, 0, t, u^n) \in \partial\theta\} \quad \text{or infinity,}$$

$$\varepsilon_n(x, u^n) = \sup_{t \in [0, (n+1)t^*]} [\operatorname{dist}(y_0(x, 0, t, u^n), \bar{\theta})],$$

then

$$u^{n+1}(x, t) = \begin{cases} u^n(x, t) & \text{if } t \leqq t_n(x, u^n) \quad \text{or } t \geqq t_n(x, u^n) + k\varepsilon_n(x, u^n), \\ \alpha(y_0(x, 0, t_n(x, u^n), u^n)) & \text{if } t_n(x, u^n) < t < t_n(x, u^n) + k\varepsilon_n(x, u^n). \end{cases}$$

Iterate (3.12) to get $y_0(x, 0, t, u^n) \in \bar{\theta}$ for $t \in [0, nt^*]$. We have to estimate the Lebesgue measure of the following set

$$M^n(x) = \{t \in [0, nt^*] : u^n(x, t) \neq u(x, t)\}. \tag{3.13}$$

For every t we have

$$|y_0(x, 0, t, u^n) - y_0(x, 0, t, u)|$$
$$\leqq \int_{[0,t] \cap M^n(x)} 2K(b) + \int_{[0,t] \setminus M^n(x)} L(b) |y_0(x, 0, s, u) - y_0(x, 0, s, u^n)| \, ds;$$

thus the Gronwall's inequality implies

$$\varepsilon_n(x, u^n) \leqq \varepsilon_n(x, u) + 2K(b) \, e^{(n+1)L(b)t^*} \operatorname{meas} M^n(x).$$

The construction of u^n yields

$$\begin{aligned} \operatorname{meas} M^n(x) &\leqq k \sum_{i=0}^{n-1} \varepsilon_i(x, u^i) \\ &\leqq nk\varepsilon_{n-1}(x, u) + k2K(b) \, e^{nL(b)t^*} \sum_{i=0}^{n-1} \operatorname{meas} M^i(x). \end{aligned} \tag{3.14}$$

Iterate this inequality to obtain $C(n)$, depending only on n, $K(b)$ and $L(b)$, such that

$$\operatorname{meas} M^n(x) \leqq C(n) \varepsilon_{n-1}(x, u). \tag{3.15}$$

Now, for given T, choose $\bar{n}$ so that $T \leqq \bar{n}t^*$. Then define $\mathcal{P}_T u$ as

$$\mathcal{P}_T u(x, t) = \begin{cases} u^{\bar{n}}(x, t) & \text{for } t \leqq T, \\ \tilde{u}(y_0(x, 0, T, u^{\bar{n}}), t - T) & \text{for } t > T \end{cases}$$

where $\tilde{u}$ is any strategy in $\mathcal{A}_{ad}$.

Observe that for any u

$$J_T^N(x, u) = \int_0^T \lambda(t)\Lambda(t) \left[\int_0^t e^{-s} f(y(s), u(s)) \, ds + e^{-t} \int_{\bar{\theta}} v^{N-1}(z) Q(y(t), u(t), dz) \right] dt$$
$$+ \Lambda(T) \int_0^T e^{-s} f(y(s), u(s)) \, ds.$$

By using (1.1)-(1.4) one can show

(3.16) $|J^N_T(x, u) - J^N_T(x, \mathcal{P}_T(u))| \leqq CT[d_T(x) + W_{v^{N-1}}(d_T(x)) + \text{meas } M^{\bar{n}}(x)]$

where C is a positive constant and

$$d_T(x) = \sup_{t\in[0,T]} \{|y_0(x, 0, t, u) - y_0(x, 0, t, \mathcal{P}_T(u))|\}$$
$$\leqq 2K(b)\, e^{L(b)T} \text{ meas } M^{\bar{n}}(x).$$

Plug this into (3.16) together with (3.15) to conclude Lemma 3.2. □

LEMMA 3.3. *Let* (A.2)-(A.4) *and* (1.1)-(1.5) *hold; then* $v^N \in BUC(\bar{\theta})$ *and there is a* δ*-optimal strategy* u^* *such that*

$$J^N(x, u^*) \leqq v^N(x) + \delta \quad \textit{for all } x \in \bar{\theta}.$$

Proof. It is proved that $v^0 \in BUC(\bar{\theta})$ [11, Thm. 3.3]. Now suppose $v^{N-1} \in BUC(\bar{\theta})$ and define

$$\omega(r) = \sup\{|v^N(x) - v^N(y)|: x, y \in \bar{\theta} \text{ and } |x - y| < r\} \quad \text{for } r > 0.$$

At the origin $\omega(0) = \lim_{r\downarrow 0} \omega(r)$. Using Lemmas 3.1 and 3.2, we can conclude, as in Theorem 3.3 of [11], that for some t positive,

$$\omega(r) \leqq h_t(Cr) + e^{-t}\omega(\tilde{C}r) + Ctr \tag{3.17}$$

where $\tilde{C} > 1$, $C > 0$ and h_t is as in Lemma 3.2. Iterate (3.17) to obtain

$$\omega(\tilde{C}^{-n}) \leqq e^{-nt}\omega(1) + \sum_{m=0}^{n-1} e^{-mt}[h_t(C\tilde{C}^{(m-n)}) + C\tilde{C}^{(m-n)}t].$$

Use dominated convergence theorem to get $\lim_{n\to\infty} \omega(\tilde{C}^{-n}) = 0$. Hence $v^N \in BUC(\bar{\theta})$.

Pick $\{x_m: n = 1, 2, \cdots\} \subset \bar{\theta}$ such that $\bar{\theta} \subset \bigcup_m B(x_m, r)$ where r to be chosen. Then select $\{u_n: n = 1, 2, \cdots\} \subset \mathcal{A}_{ad}$

$$J^N(x_m, u_m) \leqq v^N(x_m) + \frac{\delta}{2}. \tag{3.18}$$

Now define u as follows

$$u(x, t) = u_m(x_m, t) \quad \text{if } x \in \theta_m = \bigcup_{n=1}^{m} B(x_n, r) \Big\backslash \bigcup_{n=1}^{m-1} B(x_n, r).$$

Let $u^* = \mathcal{P}_T(u)$ where T to be chosen. Note that u^* depends both on r and T but this dependence is suppressed in the notation. For every $x \in \theta_m$ we have

$$\begin{aligned} |J^N(x, u^*) - J^N(x_m, u)| &\leqq |J^N_T(x, u^*) - J^N_T(x, u)| + |J^N_T(x, u) - J^N_T(x_m, u)| \\ &\quad + 2 \sup_{\substack{x\in\theta \\ u\in\mathcal{A}_{ad}}} |J^N(x, u) - J^N_T(x, u)| := I_1 + I_2 + I_3. \end{aligned} \tag{3.19}$$

The construction of u and standard ODE estimates yield

$$\sup_{\substack{x\in\theta_m \\ t\in[0,T]}} |y_0(x, 0; t, u) - y_0(x_m, 0; t, u)| \leqq C(T)r \tag{3.20}$$

where $C(T)$ is a positive constant. Since $y_0(x_m, 0; t, u) \in \bar{\theta}$ for all $t \geqq 0$ above inequality implies that

$$\sup_{\substack{x\in\bar{\theta} \\ t\in[0,T]}} d(y_0(x, 0; t, u), \bar{\theta}) \leqq C(T)r. \tag{3.21}$$

Now, in the case of I_1 use (3.8), (3.21) and in the case of I_2 use (3.20), (1.1)-(1.4) and the continuity of v^{N-1} to obtain

$$I_1+I_2 \leqq h^N(T, r) \tag{3.22}$$

where $h^N \in C([0,\infty)\times[0,\infty))$ with $\lim_{r\downarrow 0} h^N(T, r)=0$ for every T and N. Also (3.2) and (3.9) imply

$$\begin{aligned}|J^N(x,u)-J_T^N(x,u)| &= \left| E\left[e^{-T_1}v^{N-1}(Y_1)\chi_{(T,\infty)}(T_1)+\int_{T\wedge T_1}^{T_1} e^{-s}f(y(x,s,u),u(x,s))\,ds\right]\right| \\ &\leqq e^{-T}[\|v^{N-1}\|_{L^\infty(\bar\theta)}+K(f)].\end{aligned}$$

Recall that $K(f)$ is the sup-norm of f and it is easy to show that v^N is bounded by $K(f)$ for every N. Hence we have

$$I_3 \leqq 4K(f)\,e^{-T}. \tag{3.23}$$

Substitute (3.22)-(3.23) into (3.19) to get for all $x_m \in \theta_m$

$$|J^N(x,u^*)-J^N(x_m,u)| \leqq h^N(T,r)+4K(f)\,e^{-T}. \tag{3.24}$$

The continuity of v^N, (3.18) and the above inequality imply

$$J^N(x,u^*) \leqq v^N(x)+\frac{\delta}{2}+\omega_{v^N}(r)+h^N(T,r)+4K(f)\,e^{-T}.$$

Recall that $\lim_{r\downarrow 0} h^N(T,r)=0$; thus we can choose T and r so that $J^N(x,u^*) \leqq v^N(x)+\delta$. □

THEOREM 3.4. *If* (A.2)-(A.4), (1.1)-(1.5) *hold, then* $v \in BUC(\bar\theta)$ *and the dynamic programming relation* (0.8) *holds.*

Proof. Iterating the second assertion of the previous lemma, one gets

$$\begin{aligned} v^N(x) = \inf_{\{u_1,\cdots,u_N\}\subset \mathcal{A}_{ad}} E\Bigg\{ &\sum_{n=0}^{N-1}\int_{T_n}^{T_{n+1}} e^{-t}f(y_0(Y_n,T_n,t,u_n),u_n(Y_n,t-T_n))\,dt \\ &+\int_{T_N}^{\infty} e^{-t}f(y_0(Y_N,T_N,t,u_N)u_N(Y_N,t-T_N))\Bigg\}. \end{aligned} \tag{3.25}$$

Now define v^∞ by

$$v^\infty(x)=\inf_{\{u_n:\, n=1,2,\cdots\}\subset\mathcal{A}_{ad}} E\left\{\sum_{n=0}^{\infty}\int_{T_n}^{T_{n+1}} e^{-t}f(y_0(Y_n,T_n,t,u_n),u_n(Y_n,t-T_n))\,dt\right\}. \tag{3.26}$$

Hence we have

$$\sup_{x\in\bar\theta}|v^N(x)-v^\infty(x)| \leqq 2K(f) \sup_{\{u_1,\cdots,u_N\}\subset\mathcal{A}_{ad}} E(e^{-T_N}).$$

To prove that v^N converges to v^∞ it suffices to show that $E(e^{-T_N})$ is decreasing to zero independent of control.

Let $\{u_n: n=1,\cdots\}\subset\mathcal{A}_{ad}$ and set $\lambda_n(t)=\lambda(y_0(Y_{n-1},0,t,u_n),u_n(Y_{n-1},t))$.

$$\begin{aligned} E(e^{-T_n+T_{n-1}}|Y_1,\cdots,Y_{n-1},T_{n-1}) &= \int_0^\infty e^{-t}\lambda_n(t)\exp\left\{-\int_0^t \lambda_n(s)\,ds\right\}dt \\ &= 1-\int_0^\infty e^{-t}\exp\left\{-\int_0^t\lambda_n(s)\,ds\right\}dt \\ &\leqq 1-\int_0^\infty e^{(-(1+K(\lambda))t)}\,dt := \gamma < 1. \end{aligned}$$

Observe that γ is independent of control and strictly less than one. Hence

$$\sup_{\{u_1,\cdots,u_N\}\subset \mathcal{A}_{ad}} E(e^{-T_N}) \leqq \gamma^N. \tag{3.27}$$

Therefore v^N converges to v^∞ uniformly on $\bar{\theta}$. So $v^\infty \in BUC(\bar{\theta})$ and v^∞ satisfies (0.8).

We now proceed to show that $v^\infty = v$. First observe that $v^\infty \leqq v$ because of the definitions of v^∞ and v. Also we can construct u^* as in the previous lemma such that for all $x \in \bar{\theta}$

$$E\left\{\int_0^{T_1} e^{-t} f(y_0(x, 0, t, u^*), u^*(x, t))\, dt + e^{-T_1} v^\infty(Y_1)\right\} \leqq v^\infty(x) + \delta. \tag{3.28}$$

Apply (3.28) at $x = Y_1$ to obtain

$$v^\infty(Y_1) \geqq -\delta + E\left\{\int_0^{T_2-T_1} e^{-t} f(y_0(Y_1, 0, t, u^*), u^*(Y_1, t))\, dt + e^{-(T_2-T_1)} v^\infty(Y_2) \middle| Y_1, T_1\right\}.$$

The above inequality and (3.28) yields

$$\begin{aligned} &E\left\{\sum_{n=0}^{1} \int_{T_n}^{T_{n+1}} e^{-t} f(y_0(Y_n, T_n, t, u^*), u^*(Y_n, t - T_n))\, dt + e^{-T_2} v^\infty(Y_2)\right\} \\ &\quad \leqq v^\infty(x) + \delta E \sum_{n=0}^{1} e^{-T_n}. \end{aligned} \tag{3.29}$$

Iterate this procedure to obtain

$$J(x, u^*) \leqq v^\infty(x) + \delta E\left(\sum_{n=0}^{\infty} e^{-T_n}\right) \leqq v^\infty(x) + \delta(1-\gamma)^{-1}$$

where γ is as in (3.27). □

Acknowledgments. This paper comprises a part of the author's thesis written under the direction of Professor W. H. Fleming at Brown University. The author would like to express thanks to Professor W. H. Fleming for suggesting the problem, helpful conversations, good advice and careful reading of an earlier manuscript, which led to substantial changes.

REFERENCES

[1] M. G. CRANDALL, L. C. EVANS AND P.-L. LIONS, *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 282 (1984), pp. 487-502.

[2] M. G. CRANDALL AND P.-L. LIONS, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 277 (1983), pp. 1-42.

[3] M. G. CRANDALL AND P. E. SOUGANIDIS, *Developments in the theory of nonlinear first-order partial differential equations*, Proc. International Symposium on Differential Equations, Birmingham, AL, Knowles and Lewis, eds., North-Holland, Amsterdam, 1983.

[4] M. H. A. DAVIS, *Piecewise deterministic Markov processes: a general class of non-diffusion stochastic models*, J. Roy. Statist. Soc. (B), to appear.

[5] S. D. DESKMUKH AND S. R. PLISKA, *Optimal consumption and exploration of non-renewable resources under uncertainty*, Econometrica, 48 (1980), pp. 177-200.

[6] W. H. FLEMING AND R. RISHEL, *Deterministic and Stochastic Optimal Control*, Springer, Berlin, 1975.

[7] P.-L. LIONS, *Generalized Solutions of Hamilton-Jacobi Equations*, Research Notes in Mathematics 69, Pitman, London, 1982.

[8] P.-L. LIONS, *Optimal control and viscosity solutions*, Proc. Rome meeting, 1984, to appear in Springer Lecture Notes in Mathematics.

[9] S. R. PLISKA, *On a functional differential equation that arises in a Markov control problem*, J. Differential Equations, 28 (1978), pp. 390-405.

[10] H. M. SONER, *Optimal control of a one-dimensional storage process*, Appl. Math. Optim., 13 (1985), pp. 175-191.

[11] ———, *Optimal control with state space constraint*, I, this Journal, 24 (1986), pp. 552-561.

[12] D. VERMES, *Optimal control of piecewise deterministic processes*, Stochastics, to appear.