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Ginzburg–Landau Equation and Motion by Mean Curvature, II: Development of the Initial Interface

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ABSTRACT. In this paper, we study the short time behavior of the solutions of a sequence of Ginzburg–Landau equations indexed by ϵ. We prove that under appropriate assumptions on the initial data, solutions converge to ± 1 in short time and behave like the one-dimensional traveling wave across the interface. In particular, energy remains uniformly bounded in ϵ.

1. Introduction

In an earlier paper [12], I have studied the asymptotic behavior of the Ginzburg–Landau equation,

$$u^\epsilon_t - \Delta u^\epsilon + \frac{1}{\epsilon^2} f(u^\epsilon) = 0, \qquad (0, \infty) \times \mathcal{R}^d, \tag{1.1}$$

$$u^\epsilon(0, x) = u^\epsilon_0(x), \qquad x \in \mathcal{R}^d. \tag{1.2}$$

The nonlinearity f is the derivative of a bi-stable potential W:

$$W(u) = \frac{1}{2}(u^2 - 1)^2, \qquad f(u) = W'(u) = 2u(u^2 - 1). \tag{1.3}$$

In [12], I proved that there are two open, disjoint subsets $\mathcal{P}$, $\mathcal{N}$ of $(0, \infty) \times \mathcal{R}^d$ and a subsequence ϵ_n satisfying

(a) $u^{\epsilon_n} \to 1$, uniformly on bounded subsets of $\mathcal{P}$,

(b) $u^{\epsilon_n} \to -1$, uniformly on bounded subsets of $\mathcal{N}$,

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(c) Γ = complement of $(P \cup \mathcal{N})$ has Hausdorff dimension d and it moves by mean curvature in the sense defined in [12], [1].

This convergence result generalizes the previous results of Rubinstein, Steinberg, and Keller [10], DeMottoni and Schatzman [8], Chen [2], Evans, Soner, and Souganidis [4], Barles, Soner, and Souganidis [1], and Ilmanen [7]. For more information on the Ginzburg–Landau equation, the weak theories for the mean curvature flow and other related topics we refer the reader to the introduction of the companion paper [12] and the references therein.

The above result was proved under the assumption (cf. (2.6) in [12]) that for every $\delta > 0$ there are positive constants K_δ and η such that for every continuous function φ,

$$\text{(A)} \qquad \sup\left\{\int |\varphi(x)|\mu^\epsilon(dx;t) \ : \ \epsilon \in (0,1), t \in \left[\delta, \frac{1}{\delta}\right]\right\}, \\ \leq K_\delta \sup\{|\varphi(x)|e^{\eta|x|}: x \in \mathcal{R}^d\}$$

where

$$\mu^\epsilon(dx;t) = \left[\frac{\epsilon}{2}|Du^\epsilon(t,x)|^2 + \frac{1}{\epsilon}W(u^\epsilon(t,x))\right]dx. \tag{1.4}$$

The main purpose of this paper is to verify (A) under some reasonable conditions on the initial data u_0^ϵ. This analysis requires a detailed description of $u^\epsilon(t,x)$ near the initial interface. Such an analysis have already been carried out by DeMottoni and Schatzman [9] and by Chen [2]. However, the condition (A) cannot be directly obtained from the results of [2], [9].

There are two key estimates in the proof of (A). The first is a detailed description of $u^\epsilon(t,x)$ near the initial interface, Theorem 4.1 below. This result is a sharper version of a result of DeMottoni and Schatzman [9] and its proof is similar to Lemma 4.1 in [5]. The description obtained in Theorem 4.1 is of independent interest. The second key step in the proof of (A) is a gradient estimate, Theorem 5.1 below.

The paper is organized as follows. In the next section the main result of this paper is described. In Section 3, a result of DeMottoni and Schatzman is recalled and an easy gradient bound is proved. The behavior of $u^\epsilon(t,x)$ near the initial interface is analyzed in Section 4 and a second gradient estimate is obtained in Section 5. A proof of the main theorem is given in the last section.

2. Main Result

Multiply (1.1) by ϵu_t^ϵ, integrate and use integration by parts to obtain

$$E^\epsilon(t_1) - E^\epsilon(t_2) = -\epsilon\int_{t_1}^{t_2}\int_{\mathcal{R}^d}(u_t^\epsilon)^2\,dx\,dt, \qquad t_1 > t_2, \tag{2.1}$$

where

$$E^\epsilon(t) = \mu^\epsilon(\mathcal{R}^d; t) = \int_{\mathcal{R}^d} \left[\frac{\epsilon}{2} |Du^\epsilon(t,x)|^2 + \frac{1}{\epsilon} W(u^\epsilon(t,x)) \right] dx.$$

Hence (A) holds with $\eta = 0$ provided that $E^\epsilon(0)$ is bounded in ϵ. In particular, an elementary computation shows that $E^\epsilon(0)$ is bounded in ϵ, if there are a function z_0^ϵ, a constant $\lambda \geq 1$, and a bounded open set Ω of finite perimeter (cf. [3], [6]) satisfying

$$u_0^\epsilon(x) = q\left(\frac{z_0^\epsilon(x)}{\epsilon}\right), \qquad q(r) = \tanh(r),$$

$$|Dz_0^\epsilon| \leq \lambda, \qquad \frac{1}{\lambda} d(x) \leq z_0^\epsilon(x) \leq \lambda d(x),$$

where $d(x)$ is the signed distance between x and the boundary of Ω.

When u_0^ϵ is independent of ϵ, we generally do not expect $E^\epsilon(0)$ to be bounded in ϵ. Indeed, let $u_0^\epsilon \equiv \beta$ for some constant $\beta \neq \pm 1$. Then $u^\epsilon(t,x) = w^\epsilon(t)$ and $E^\epsilon(t) = +\infty$ for every $t \geq 0$ and $\epsilon > 0$. However, condition (A) holds with any $\eta > 0$.

In the remainder of this paper, we assume that

$$u_0^\epsilon \text{ is independent of } \epsilon\text{, i.e., } u_0^\epsilon = u_0, \tag{2.2a}$$

$$u_0 \in C_b^3(\mathcal{R}^d),\ |u_0(x)| < 1, \tag{2.2b}$$

$$\Gamma_0 = \{x \in \mathcal{R}^d \colon u_0(x) = 0\} \text{ is bounded}, \tag{2.2c}$$

$$\inf\nolimits_{\Gamma_0} |Du_0| > 0, \tag{2.2d}$$

$$\limsup\nolimits_{R\to 0}\ \inf\nolimits_{|x|\geq R} |u_0(x)| > 0, \tag{2.2e}$$

where $C_b^3(\mathcal{R}^d)$ is the set of all bounded functions that are thrice continuously differentiable with bounded derivatives. Observe that (2.2b,c,d) imply that Γ_0 is a C^2 manifold. The main goal of this paper is to prove (A) under the above hypotheses; see Theorem 6.1 below.

3. Preliminaries

Let $d_0(x)$ be the signed distance between x and Γ_0. Choose $\lambda > 0$ such that

$$d_0 \in C^2(\Omega_\lambda), \qquad \Omega_\lambda = \{x \in \mathcal{R}^d \colon |d_0(x)| < 2\lambda\}. \tag{3.1}$$

We now recall a result of DeMottoni and Schatzman [9, Theorem 5].

Theorem 3.1. *For every $\delta, m > 0$ there are $C_1, C_2 > 0$ such that for every*

$$t \in I_\epsilon := \left[C_1 \epsilon^2 \ln\left(\frac{1}{\epsilon}\right) \ , \ C_2 \epsilon^{\frac{1}{3}} \right], \tag{3.2}$$

we have

$$\left| u^\epsilon(t,x) - q\left(\frac{d_0(x)}{\epsilon}\right) \right| \le \delta, \qquad \text{if } |d_0(x)| \le \lambda, \tag{3.3}$$

$$|u^\epsilon(t,x) - \text{sign}[u_0(x)]| \le \epsilon^m, \qquad \text{if } |d_0(x)| \ge \lambda. \tag{3.4}$$

Recall that $q(r) = \tanh(r)$. In the remainder of this paper C_1, C_2 denote the constants constructed in Theorem 3.1 with $m = 2$ and $\delta = 1/8$. Also set

$$C_3 = q^{-1}(7/8). \tag{3.5}$$

Fix $t \in I_\epsilon$. Then whenever $d(x) \in [\epsilon C_3, \lambda]$, (3.3) yields

$$u^\epsilon(t,x) \ge q^{-1}\left(\frac{d(x)}{\epsilon}\right) - \delta \ge \frac{3}{4}.$$

Also if $d(x) \ge \lambda$, (3.4) implies the above inequality, provided that $\epsilon^2 < 1/4$. Hence

$$u^\epsilon(t,x) \ge 3/4, \quad \forall \epsilon \le 1/2, \ \ t \in I_\epsilon, \quad d(x) \ge \epsilon C_3. \tag{3.6}$$

Similarly,

$$u^\epsilon(t,x) \le -3/4, \quad \forall \epsilon \le 1/2, \ \ t \in I_\epsilon, \quad d(x) \le -\epsilon C_3. \tag{3.7}$$

We close this section with a simple gradient estimate.

Lemma 3.1. *There is a constant K, independent of ϵ, satisfying*

$$|Du^\epsilon(t,x)| \le \frac{K}{\epsilon}. \tag{3.8}$$

Proof. Since $|u_0| \le 1$, $|u^\epsilon(t,x)| \le 1$ for all (t,x). Set

$$g(t,x) = \frac{1}{\epsilon^2} f(u^\epsilon(t,x)).$$

Then for all $0 \le \tau \le t$,

$$u^\epsilon(t,x) = [G(t-\tau,\cdot) * u^\epsilon(\tau,\cdot)](x) + \int_\tau^t [G(t-s-\tau,\cdot) * g(s,\cdot)](x)\,ds, \tag{3.9}$$

where * denotes the convolution and G is the heat kernel, i.e.,

$$G(\tau, y) = (4\pi\tau)^{-\frac{d}{2}} \exp\left(-\frac{|y|^2}{4\tau}\right).$$

Now, differentiate (9) with respect x_j and use the properties of the convolution and the heat kernel to obtain

$$|u^\epsilon_{x_j}(t,x)| \le \|D_j G(t-\tau,\cdot)\|_{L^1} \quad \|u^\epsilon(\tau,\cdot)\|_{L^\infty} + \int_r^t \|D_j(t-s-\tau,\cdot)\|_{L^1}\|g\|_{L^\infty}\,dx,$$
$$\le \frac{C}{\sqrt{t-\tau}} + \frac{C}{\epsilon^2}\sqrt{t-\tau}\ ,$$

where C is an appropriate constant. Choose $\tau = t - \epsilon^2$ to obtain (3.8). □

4. Behavior near the interface

In this section we prove a sharper version of (3.3), (3.4). Our approach is very similar to [5, Lemma 4.1]. Let λ be as in (3.1) and set

$$t_1 = C_1\epsilon^2 \ln\left(\frac{1}{\epsilon}\right). \tag{4.1}$$

Theorem 4.1. *There are μ, $K > 0$ such that for sufficiently small $\epsilon > 0$,*

$$u^\epsilon(t,x) \ge W(t-t_1, d_0(x)), \qquad \forall t \in I_\epsilon, d_0(x) \in [\epsilon C_3, \lambda], \tag{4.2}$$

$$u^\epsilon(t,x) \le -W(t-t_1, |d_0(x)|), \qquad \forall t \in I_\epsilon, d_0(x) \in [-\lambda, -\epsilon C_3], \tag{4.3}$$

where

$$W(t,d) = \max\left\{q\left(\frac{d-Kt}{\epsilon} - K\right) - K\epsilon - \frac{1}{4}\exp\left(-\frac{\mu t}{\epsilon}\right), \frac{3}{4}\right\}.$$

Proof. We will prove only (4.2). The proof of (4.3) is similar.

(1) In view of (3.1) there is $d \in C_b^2(\mathcal{R}^d)$ satisfying

$$d(x) = d_0(x), \quad \text{if } |d_0(x)| \le \lambda, \tag{4.4}$$

$$|d(x)| \ge \lambda, \quad \text{if } |d_0(x)| \ge \lambda, \tag{4.5}$$

$$|Dd(x)| \le 1, \quad \forall x. \tag{4.6}$$

For $\xi(t)$, $p(t) \ge 0$ (to be determined later) define

$$v(t,x) = q\left(\frac{d(x) - \epsilon C_3 - \xi(\frac{t}{\epsilon})}{\epsilon}\right) - p\left(\frac{t}{\epsilon}\right),$$

where C_3 is as in (3.5).

We will show that for appropriately chosen $\xi(\cdot)$, $p(\cdot)$, and a sufficiently small $\epsilon > 0$, v is a subsolution of (1.1) on $\{v \ge 0\}$. Indeed, a direct computation shows that

$$\begin{aligned} I &:= v_t - \Delta v + \frac{1}{\epsilon^2} f(v), \\ &= \frac{1}{\epsilon} q'(\cdots)\left[-\frac{1}{\epsilon}\xi'\left(\frac{t}{\epsilon}\right) - \Delta d(x)\right] - \frac{1}{\epsilon} p'\left(\frac{t}{\epsilon}\right), \\ &\quad + \frac{1}{\epsilon^2}[f(v) - q''(\cdots)|Dd|^2], \end{aligned}$$

where $(\cdots)$ denotes $[d(x) - \epsilon C_3 - \xi(\frac{t}{\epsilon})]/\epsilon$.

(2) Since $q(\cdots) = v + p$ and $p \ge 0$, $q(\cdots) \ge 0$ whenever $v(t,x) \ge 0$. Therefore on $\{v \ge 0\}$, $q''(\cdots) \le 0$ and (6) yields

$$q''(\cdots)|Dd|^2 \ge q''(\cdots) = f(q(\cdots)).$$

So on $\{v \ge 0\}$ we have

$$I \le -\frac{1}{\epsilon^2} q'(\cdots)\xi'\left(\frac{t}{\epsilon}\right) - \frac{1}{\epsilon} p'\left(\frac{t}{\epsilon}\right) + \frac{1}{\epsilon^2}[f(v) - f(q(\cdots))] + \frac{\beta}{\epsilon}, \tag{4.7}$$

where $\beta := \|q'\|_\infty \| \Delta d\|_\infty$.

(3) Set

$$\mu = f'\left(\frac{5}{8}\right) = \min\left\{f'(u) \colon u \ge \frac{5}{8}\right\} > 0, \tag{4.8}$$

and

$$p(\tau) = \frac{\epsilon\beta}{\mu} + \left(\frac{1}{4} - \frac{\epsilon\beta}{\mu}\right) \exp\left(-\frac{\mu\tau}{\epsilon}\right), \ \tau \geq 0. \tag{4.9}$$

We will choose $\xi \geq 0$ in step 5 satisfying

$$\xi' \geq 0. \tag{4.10}$$

(4) Suppose that

$$q(\cdots) \in \left[\frac{7}{8}, 1\right]. \tag{4.11}$$

The case $q(\cdots) \leq \frac{7}{8}$ will be analyzed in the next step. Since $|p(\tau)| \leq \frac{1}{4}$, (4.11) implies that

$$v(t, x) = q(\cdots) - p\left(\frac{t}{\epsilon}\right) \geq \frac{5}{8}.$$

Since $v = q(\cdots) - p \leq q(\cdots)$, (4.8) yields

$$f(v(t, x)) - f(q(\cdots)) \leq -\mu p\left(\frac{t}{\epsilon}\right).$$

Use (4.9), (4.10) and the above inequality in (4.7) to obtain

$$I \leq \frac{\beta}{\epsilon} - \frac{1}{\epsilon}\, p'\left(\frac{t}{\epsilon}\right) - \frac{\mu}{\epsilon^2} p\left(\frac{t}{\epsilon}\right) = 0\,,$$

on $\{v \geq 0\}$.

(5) Suppose that (4.11) does not hold, i.e.,

$$q(\cdots) \leq \frac{7}{8}.$$

Then on $\{v \geq 0\}$, $q(\cdots) \in [0, \frac{7}{8}]$ and

$$q'(\cdots) = (1 - q(\cdots)^2) \geq \left(1 - \left(\frac{7}{8}\right)^2\right) := \gamma. \tag{4.12}$$

Set

$$\alpha := \max\{|f'(u)|:\ u \in [0, 1]\}.$$

Since $v \leq 1$, on $\{v \geq 0\}$ we have

$$f(v) - f(q(\cdots)) \leq \alpha |v(t,x) - q(\cdots)| = \alpha p\left(\frac{t}{\epsilon}\right).$$

Use the above inequality and (4.12) in (4.7) to obtain

$$I \leq -\frac{\gamma}{\epsilon^2}\xi'\left(\frac{t}{\epsilon}\right) - \frac{1}{\epsilon}p'\left(\frac{t}{\epsilon}\right) + \frac{\alpha}{\epsilon^2}p\left(\frac{t}{\epsilon}\right) + \frac{\beta}{\epsilon}.$$

We now choose $\xi(\cdot)$ satisfying $\xi(0) = 0$ and

$$\xi'(\tau) = \frac{1}{\gamma}\{\beta\epsilon + \alpha p(\tau) - \epsilon p'(\tau)\} = \frac{\alpha + \mu}{\gamma}p(\tau), \quad \tau \geq 0.$$

Using (4.9) we integrate the above equation:

$$\xi(\tau) = \frac{\epsilon}{\gamma}\left(1 + \frac{\alpha}{\mu}\right)\left[\beta\tau + \left(\frac{1}{4} - \frac{\epsilon\beta}{\mu}\right)\left(1 - \exp\left(-\frac{\mu\tau}{\epsilon}\right)\right)\right].$$

Observe that this choice of ξ satisfies (4.10).

(6) By the previous two steps,

$$I \leq \quad \text{on } \{v \geq 0\}.$$

Also by (3.6)

$$u^\epsilon(t,x) \geq \frac{3}{4} \qquad \forall t \in I_\epsilon, \quad d_0(x) \geq \epsilon C_3.$$

In particular,

$$v(0,x) = q(\cdots) - \frac{1}{4} \leq \frac{3}{4} \leq u^\epsilon(t_1,x), \qquad \forall d_0(x) \geq \epsilon C_3,$$

and since $p, \xi > 0$,

$$v(t - t_1, x) \leq q(0) = 0 \leq u^\epsilon(t,x), \quad \forall t \in I_\epsilon, \quad \forall d_0(x) = \epsilon C_3.$$

Since $u^\epsilon(t,x) \geq 0$ for all $t \in I_\epsilon$ and $d_0(x) \geq \epsilon C_3$, the maximum principle yields

$$u^\epsilon(t,x) \geq v(t - t_1, x), \qquad \forall t \in I_\epsilon, \; d_0(x) \geq \epsilon C_3. \tag{4.13}$$

Now (4.2) follows from (4.13), (4), (3.6), and the definitions of p and ξ. □

5. A gradient estimate

In this section we obtain an upper bound for $|Du^\epsilon|$ away from the interface. Let t_1 be as in (4.1), $d \in C_b^2(\mathcal{R}^d)$ be an extension of d_0 satisfying (4), (5), (6), and C_3 be as in (3.5).

Theorem 5.1. *There are constants $K, \delta, \alpha > 0$ satisfying*

$$|Du^\epsilon(t,x)|^2 \le \frac{K^2}{\epsilon^2}\left[\exp\left(-\frac{\delta}{\epsilon^2}(t-t_1)\right) + \exp\left[-\frac{\alpha}{\epsilon}(|d(x)| - \epsilon C_3)\right]\right], \tag{5.1}$$

for all sufficiently small $\epsilon > 0$ and $t \in I_\epsilon$, $|d_0(x)| \ge \epsilon C_3$.

Proof. Set

$$\Omega = \{(t,x) \colon t \in I_\epsilon,\ \ |d_0(x)| > \epsilon\, C_3\},$$

$$\varphi(t,x) = |Du^\epsilon(t,x)|^2.$$

(1) Differentiate (1.1)and then multiply by $2Du^\epsilon$ to obtain

$$\varphi_t - \Delta\varphi + \frac{2}{\epsilon^2} f'(u^\epsilon)\varphi = -2\|D^2u^\epsilon\|^2 \le 0.$$

By (3.6) and (3.7),

$$|u^\epsilon(t,x)| \ge \frac{3}{4}, \qquad \forall (t,x) \in \Omega.$$

Set

$$\delta = 2f'\left(\frac{3}{4}\right) = \min\left\{f'(u) \colon |u| \ge \frac{3}{4}\right\} > 0.$$

Then

$$\varphi_t - \Delta\varphi + \frac{\delta}{\epsilon^2}\varphi \le 0 \ \text{ on } \ \Omega. \tag{5.2}$$

(2) Set

$$\Psi(t,x) = \frac{K^2}{\epsilon^2}\left\{\exp\left(-\frac{\delta}{\epsilon^2}(t-t_1)\right) + g\left(\frac{|d(x)| - \epsilon C_3}{\epsilon}\right)\right\},$$

where $K > 0$ is as in (3.8) and $g(\cdot)$ is the unique, bounded solution of

$$-g_{rr}(r) + \|\Delta d\|_\infty g_r(r) + \delta g(r) = 0, \quad r > 0, \tag{5.3}$$

satisfying $g(0) = 1$. Then

$$g(r) = e^{-\alpha r}, \qquad \alpha = \frac{1}{2}\{-\| \Delta d\|_\infty + \sqrt{\| \Delta d\|_\infty^2 + 4\delta}\}.$$

(3) We claim that Ψ is a supersolution of (5.2) on Ω. Indeed,

$$\Psi_t - \Delta\Psi + \frac{\delta}{\epsilon^2}\Psi = \frac{K^2}{\epsilon^4}\left\{-g_{rr}(\cdots)|Dd|^2 - \epsilon\frac{d}{|d|}\Delta d\, g_r(\cdots) + \delta g(\cdots)\right\},$$

where $(\cdots) = (|d(x)| - \epsilon C_3)/\epsilon$. Since $g_r \le 0 \le g_{rr}$ and $|Dd| \le 1$ (cf. (6)), (5.3) implies that for $\epsilon \le 1$,

$$\Psi_t - \Delta\Psi + \frac{\delta}{\epsilon^2}\Psi \ge 0, \qquad \text{on } \Omega.$$

(4) By (3.8),

$$\Psi(t_1, x) \ge \frac{K^2}{\epsilon^2} \ge \varphi(t_1, x), \qquad \forall\ |d_0(x)| \ge \epsilon C_3,$$

and since $g(0) = 1$,

$$\Psi(t, x) \ge \frac{K^2}{\epsilon^2} \ge \varphi(t, x), \qquad \forall\ |d_0(x)| = \epsilon C_3.$$

(5) Now an application of the maximum principle yields $\Psi \ge \varphi$ on Ω. □

6. Conclusion

Theorem 6.1. *Assume* (2.2). *Then* (A) *holds.*

Proof. Let A be a Borel subset of $\mathcal{R}^d$ with a finite Lebesgue measure. Set

$$\Omega_1 = \{x \in \mathcal{R}^d\colon |d_0(x)| \le \epsilon C_3\},$$

$$\Omega_2 = \{x \in \mathcal{R}^d\colon |d_0(x)| \in [\epsilon C_3, \lambda]\},$$

$$\Omega_3 = \{x \in \mathcal{R}^d\colon |d_0(x)| \ge \lambda\},$$

$$A_i = A \cap \Omega_i,\ i = 1, 2, 3,$$

$$I_i(t) = \int_{A_i} \frac{\epsilon}{2} |Du^\epsilon(t,x)|^2 \, dx, \quad i = 1,2,3, \quad t \ge 0,$$

$$J_i(t) = \int_{A_i} \frac{1}{\epsilon} W(u^\epsilon(t,x)) \, dx, \quad i = 1,2,3, \quad t \ge 0,$$

where λ, C_3 are as in (3.1) and (3.5), respectively. In the following steps we will estimate I_i and J_i's separately.

(1) By Lemma 3.1,

$$I_1(t) + J_1(t) \le \int_{A_1} \frac{\epsilon}{2} \frac{K^2}{\epsilon^2} + \frac{1}{\epsilon} = \left(\frac{K^2}{2} + 1 \right) \frac{|\Omega_1|}{\epsilon}.$$

Since Γ_0 is smooth and bounded, for sufficiently small $\epsilon > 0$, $|\Omega_1| \le \epsilon \hat{C}$ for an appropriate constant $\hat{C}$. Hence

$$I_1(t) + J_1(t) \le \hat{C} \left(1 + \frac{K^2}{2} \right), \qquad \forall t \ge 0.$$

(2) Set

$$C_4 = C_1 + \frac{1}{\delta}, \qquad t_4 = C_4 \epsilon^2 \ln\left(\frac{1}{\epsilon}\right),$$

where $\delta > 0$ is the constant appearing in (5.1) and C_1 is as in Theorem 3.1. Then for all $t \in I_\epsilon \cap [t_4, \infty)$, by (5.1) we have

$$|Du^\epsilon(t,x)|^2 \le \frac{K^2}{\epsilon^2} \left[\epsilon + \exp\left(-\frac{\alpha}{\epsilon} (|d(x)| - \epsilon C_3) \right) \right].$$

Therefore,

$$I_2(t) \le \frac{K^2}{2} |A_2| + \frac{K^2}{2\epsilon} \int_{\Omega_2} \exp\left(-\frac{\alpha}{\epsilon} (|d(x)| - \epsilon C_3) \right) dx.$$

By (4), $d_0 = d$ on A_2. In the above integral we use local orthogonal coordinates w, with $w_1 = d_0(x)$. Since d_0 is smooth in Ω_2, there is a constant C, depending on the $(d-1)$-dimensional measure of Γ_0, such that

$$\begin{aligned} I_2(t) &\le \frac{K^2}{2} |A_2| + \frac{K^2}{2\epsilon} C \int_{\epsilon C_3}^{\lambda} e^{-\frac{\alpha}{\epsilon}(w_1 - \epsilon C_3)} \, dw_1 \\ &\le \frac{K^2}{2} (|A_2| + \hat{C}), \quad \forall t \in I_\epsilon \cap [t_4, \infty), \end{aligned}$$

where $\hat{C}$ is an appropriate constant, possibly different than the constant appearing in the first step.

(3) For $t \in I_\epsilon \cap [t_4, \infty)$ and $|d_0(x) \geq \lambda$, (5.1) and (5) yield

$$|Du^\epsilon(t,x)|^2 \leq \frac{K^2}{\epsilon^2}[\epsilon + e^{-\frac{\alpha}{\epsilon}(\lambda - \epsilon C_3)}].$$

Therefore for sufficiently small $\epsilon \geq 0$,

$$I_3(t) \leq \frac{K^2}{2}(|A_3| + \hat{C}), \qquad \forall t \in I_\epsilon \cap [t_4, \infty),$$

for an appropriate constant $\hat{C}$, again possibly different than the constant appearing in the previous steps.

(4) Recall that we have chosen C_1, C_2 satisfying (3.4) with $m = 2$. Hence for all $|d_0(x)| \geq \lambda$, and $t \in I_3$,

$$\begin{aligned} W(u^\epsilon) &= \frac{1}{2}(1 - u^\epsilon)^2(1 + u^\epsilon)^2 \\ &\leq 2(u^\epsilon - \text{sign}(u_0))^2 \leq 2\epsilon^4. \end{aligned}$$

Therefore,

$$J_3(t) \leq 2\epsilon^3 |A_3|, \qquad \forall t \in I_\epsilon.$$

(5) Set

$$C_5 = C_1 + \frac{1}{\mu}, \qquad t_5 = C_5 \epsilon^2 \ln\left(\frac{1}{\epsilon}\right),$$

where μ is the constant appearing in Theorem 4.1 and C_1 is as in Theorem 3.1. Then (4.2) and (4.3) imply that for all $t \in I_\epsilon \cap [t_5, \infty)$, $|d_0(x)| \in [\epsilon C_3, \lambda]$

$$|u^\epsilon(t,x)| \geq \left[q\left(\frac{|d_0(x)| - Kt}{\epsilon} - K\right) - K\epsilon - \frac{1}{4}\epsilon\right]^+,$$

where $(a)^+ = \max\{a, 0\}$. Since $|W'(u)| \leq 1$ for $|u| \leq 1$, for sufficiently small $\epsilon > 0$ we have

$$\begin{aligned} J_2(t) &\leq \int_{A_2} \frac{1}{\epsilon} W\left(\left[q\left(\frac{|d_0(x)| - Kt}{\epsilon} - K\right) - \epsilon\left(K + \frac{1}{4}\right)\right]^+\right) dx \\ &\leq \int_{A_2} \frac{1}{\epsilon} W\left(\left[q\left(\frac{|d_0(x)| - Kt}{\epsilon} - K\right)\right]^+\right) dx + \left(K + \frac{1}{4}\right)|A_2| \\ &\leq \int_{\Omega_2} \frac{1}{\epsilon} W\left(\left[q\left(\frac{|d_0(x)| - 2K\epsilon}{\epsilon}\right)\right]^+\right) dx + \left(K + \frac{1}{4}\right)|A_2|, \end{aligned}$$

for all $t \in I_\epsilon \cap [t_5, \epsilon]$. Now using the same change of variables used in step 2 we obtain

$$\begin{aligned} J_2(t) &\leq \frac{C}{\epsilon}\int_{\epsilon C_3}^{\lambda} W\left(\left[q\left(\frac{w_1 - 2K}{\epsilon}\right)\right]^+\right) dw_1 + \left(K + \frac{1}{4}\right)|A_2| \\ &\leq \frac{C}{\epsilon}\int_{\epsilon C_3}^{2\epsilon K} W(0)\, dw_1 + \left(K + \frac{1}{4}\right)|A_2| \\ &\quad + C\int_0^{\lambda/\epsilon - 2K} W(q(r))\, dr. \end{aligned}$$

Since

$$W(q(r)) = \frac{(q'(r))^2}{2} = \frac{8e^{4r}}{(e^{2r}+1)^4},$$

$$J_2(t) \leq \hat{C}(|A_2| + 1).$$

(6) Combining the previous steps we conclude that

$$\begin{aligned} \mu^\epsilon(A;t) &= \sum_{i=1}^{3}(I_i(t) + J_i(t)) \\ &\leq \hat{C}(|A| + 1), \end{aligned} \tag{6.1}$$

for all $t \geq 0$ satisfying

$$t \in I_\epsilon, \quad t \geq t_4, \quad t \geq t_5, \quad t \leq \epsilon, \tag{6.2}$$

and sufficiently small $\epsilon \geq 0$.

(7) Let Ψ be a smooth positive function decaying exponentially as $|x| \to \infty$. Then using 1.1 we obtain

$$\begin{aligned} \frac{d}{dt}\int \Psi(x)\mu^\epsilon(dx;t) &= -\epsilon\int \Psi\left(-\Delta u^\epsilon + \frac{1}{\epsilon^2}f(u^\epsilon)\right)^2 dx \\ &\quad + \epsilon\int D\Psi \cdot Du^\epsilon\left(-\Delta u^\epsilon + \frac{1}{\epsilon^2}f(u^\epsilon)\right) dx \\ &\leq -\epsilon\int \Psi\left(-\Delta u^\epsilon + \frac{1}{\epsilon^2}f(u^\epsilon) - \frac{D\Psi\cdot Du^\epsilon}{2\Psi}\right)^2 \\ &\quad + \epsilon\int |Du^\epsilon|^2\,\frac{|D\Psi|^2}{4\Psi}\, dx \\ &\leq \epsilon\int |Du^\epsilon|^2\frac{|D\Psi|^2}{4\Psi}\, dx. \end{aligned}$$

Let

$$\hat{\Psi}(x) = \exp\left(-\sqrt{1+|x|^2}\right).$$

Then $|D\hat{\Psi}| \le \hat{\Psi}$ and

$$\begin{aligned}\frac{d}{dt}\int \hat{\Psi}(x)\mu^\epsilon(dx;t) &\le \frac{1}{2}\int \frac{\epsilon}{2}|Du^\epsilon|^2\hat{\Psi}\,dx\\ &\le \frac{1}{2}\int \hat{\Psi}(x)\mu^\epsilon(dx;t).\end{aligned}$$

Therefore for any $t \ge t_0 \ge 0$,

$$\int \hat{\Psi}(x)\mu^\epsilon(dx;t) \le \int \hat{\Psi}(x)\mu^\epsilon(dx;t_0)\ \ e^{\frac{t-t_0}{2}}. \tag{6.3}$$

(8) Let t_0 be a point satisfying (6.2). Then (1) yields

$$\begin{aligned}\int \hat{\Psi}(x)\mu^\epsilon(dx;t_0) &\le \sum_{i=1}^{\infty} e^{-i}\mu^\epsilon(\{|x| \in [i-1,i)\};t_0)\\ &\le \hat{C}w_d\sum_{i=1}^{\infty} e^{-i}(1+(i)^d-(i-1)^d)\\ &\le \tilde{C},\end{aligned}$$

where w_d is the volume of the unit space in $\mathcal{R}^d$ and $\tilde{C}$ is an appropriate constant. Then by (6.3)

$$\int \hat{\Psi}(x)\mu^\epsilon(dx;t) \le \tilde{C}e^{\frac{t}{2}},$$

for every sufficiently small ϵ and

$$t \ge \max\{C_1, C_4, C_5\}\epsilon^2 \ln\left(\frac{1}{\epsilon}\right). \tag{6.4}$$

(9) Now let ϕ be any continuous function satisfying

$$\Lambda := \sup\{|\phi(x)|e^{\sqrt{2}(1+|x|)}\colon x \in \mathcal{R}^d\} < \infty.$$

Then $|\phi(x)| \le \Lambda\hat{\Psi}(x)$, and

$$\int |\phi(x)|\mu^\epsilon(dx;t) \le \tilde{C}\ \Lambda\ e^{\frac{t}{2}}, \tag{6.5}$$

for all t satisfying (6.4), and sufficiently small $\epsilon > 0$. Since for every $\epsilon > 0$, by (9)

$$\mu^\epsilon(dx; t) \leq \frac{1}{\epsilon}\left(\frac{K^2}{2} + 1\right) dx.$$

Hence for every $t \geq 0$,

$$\int |\phi(x)|\mu^\epsilon(dx; t) \leq \frac{\Lambda}{\epsilon}\left(\frac{K^2}{2} + 1\right) \int \hat{\Psi}(x)\, dx. \tag{6.6}$$

Now (A) follows from (6.5) and (6.6) with $\eta = \sqrt{2}$. □

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