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Option pricing with transaction costs and a nonlinear Black-Scholes equation

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Abstract. In a market with transaction costs, generally, there is no nontrivial portfolio that dominates a contingent claim. Therefore, in such a market, preferences have to be introduced in order to evaluate the prices of options. The main goal of this article is to quantify this dependence on preferences in the specific example of a European call option. This is achieved by using the utility function approach of Hodges and Neuberger together with an asymptotic analysis of partial differential equations. We are led to a nonlinear Black-Scholes equation with an adjusted volatility which is a function of the second derivative of the price itself. In this model, our attitude towards risk is summarized in one free parameter a which appears in the nonlinear Black-Scholes equation : we provide an upper bound for the probability of missing the hedge in terms of a and the magnitude of the proportional transaction cost which shows the connections between this parameter a and the risk.

Key words: Transaction costs, options, viscosity solutions, dynamic programming

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1 Introduction

In a complete financial market without transaction costs, the celebrated Black-Scholes no-arbitrage argument [5] provides not only a rational option pricing formula but also a hedging portfolio that replicates the contingent claim. However,

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the Black-Scholes hedging portfolio requires continuous trading and therefore, in a market with proportional transaction costs, it is prohibitively expensive. In fact, in such a market, there is no portfolio that replicates the European call option and we are forced to relax the hedging condition, requiring the portfolio only to dominate rather than replicate the value of the option. With this relaxation there is always the trivial dominating hedging portfolio of holding one share of the stock on which the call is written. A straightforward arbitrage argument indicates that any viable option price should not be larger than the smallest initial capital that can support a dominating portfolio. Although this approach to option pricing has provided interesting results in markets without transaction costs but with constraints (see Cvitanič-Karatzas [12], Karatzas-Kuo [18], Broadie-Cvitanič-Soner [7]), in the presence of transaction costs, Soner-Shreve-Cvitanič [22] proved that the minimal hedging portfolio that dominates a European call option is the trivial one; thus showing the necessity of an alternate relaxation of perfect hedging in markets with transaction costs.

Several such relaxations have already been proposed. Leland [20] considers a model that allows transactions only at discrete times. By a formal δ-hedging argument, he derives an option price that is equal to a Black-Scholes price but with an adjusted volatility

$$\hat{\sigma} = \sigma \left(1 + \sqrt{\frac{2}{\pi}} \; \frac{\mu}{\sigma\sqrt{\Delta t}} \right)^{\frac{1}{2}},$$

where σ is the original volatility, μ is the proportional transaction cost and Δt is the transaction frequency. In this formula, both μ and Δt are assumed to be small while keeping the ratio $\mu/\sqrt{\Delta t}$ order one. For typical market numbers, this is indeed the case. For instance : with $\sigma = 0.2$, $\mu = 0.01$ and one transaction a week, the Leland volatility, $\hat{\sigma}$, is equal to σ times 1.13.

One crucial step in Leland's very interesting argument is the implicit use of the approximation

$$W(t + \Delta t) - W(t) \approx \sqrt{\frac{2}{\pi}} \; \sqrt{\Delta t},$$

where $W(\cdot)$ is the standard one-dimensional Brownian motion. Clearly, as $\Delta t \downarrow 0$, $W(t + \Delta t) - W(t)$ converges to zero like $\sqrt{\Delta t}$, but a convincing argument for the following approximation

$$W(t + \Delta t) - W(t) \approx c^* \sqrt{\Delta t},$$

with an arbitrary constant c^*, can also be made. Then the resulting option price has the adjusted volatility

$$\hat{\sigma}(c^*) = \sigma \left(1 + c^* \; \frac{\mu}{\sigma\sqrt{\Delta t}} \right)^{\frac{1}{2}}. \tag{1.1}$$

The "optimal" choice of c^* is an interesting question related to the risk inherent in markets with transaction costs (see also Kusuoka [19]).

Leland's derivation assumes the convexity of the resulting option price. Recently, an extension of this approach to general prices is obtained by Avellaneda and Paras [1].

In a second approach [6], Boyle and Vorst study the option pricing problem in discrete time with a binomial tree model for the value of the stock. Using a central limit theorem, they show that, as the time step Δt and the transaction cost μ tend to zero, the price of the discrete option converges to a Black-Scholes price with adjusted volatility $\hat{\sigma}(1)$. However, one should note that, here Δt is equal to the mean time length for a change in the value of the stock, not the transaction frequency. In a related paper [4], Bensaid-Lesne-Pagès-Scheinkman investigate the discrete time, dominating policies.

A completely different approach to option pricing is to introduce preferences. In [17], Hodges and Neuberger consider the difference between the maximum utility from final wealth when there is no option liability and when there is such a liability. Then, they postulate that the price of the option should be equal to the unique cash increment which offsets this difference. Remarkably, in the absence of market frictions, the option price obtained from utility maximization is equal to the Black-Scholes price. Hence, the utility maximization approach provides an extension of the Black-Scholes option pricing theory. In the presence of transaction costs, this theory is further developed by Davis-Panas-Zariphopoulou [14].

Clearly the price defined this way depends on the particular utility function, on the initial wealth and the portfolio of the investor, and on the mean return rate of the stock. Constantinides and Zariphopoulou [9] modified the original definition and obtained universal bounds independent of the utility function.

In this paper, we will use the utility maximization definition and asymptotic analysis to derive an option pricing formula. We will also provide an upper estimate on the probability of missing the hedge by a given amount. This latter result, might be used to choose the utility function necessary in the approach of Hodges and Neuberger.

In our analysis, we use the exponential utility function

$$U^\epsilon(\xi) := 1 - \exp(-\frac{\xi}{\epsilon}), \qquad \xi \in \mathscr{R}^1,$$

with a parameter $\epsilon > 0$, where $1/\epsilon$ is equal to the product of the risk-aversion factor and the number of options to be sold (a brief discussion of this is given in Sect. 2.1 below). We let μ be the proportional transaction cost, p be the stock price at time t, and $\Psi^{\epsilon,\mu}(p,t)$ be the option price with utility function U^ϵ and, then, study the behavior of $\Psi^{\epsilon,\mu}$ as

$$\epsilon \downarrow 0, \qquad \mu \downarrow 0, \qquad \frac{\mu}{\sqrt{\epsilon}} = a,$$

where a is any constant. In Theorem 3.1, we show that the limiting price $\Psi(p,t : a)$ solves a nonlinear Black-Scholes equation

$$\Psi_t + rp\Psi_p + \frac{1}{2}\sigma^2 p^2 \Psi_{pp} \left[1 + S\left(e^{r(T-t)} a^2 p^2 \Psi_{pp}\right)\right] = r\Psi, \tag{1.2}$$

with the usual terminal condition

$$\Psi(p, T : a) = (p - q)^+, \tag{1.3}$$

where $S(\cdot)$ is a nonlinear function defined in Sect. 3, r is the constant interest rate, σ is the constant volatility, and, respectively, T and q are the maturity and the strike price of the European call option. So formally, Ψ is equal to a Black-Scholes price with variable volatility

$$\sigma(p, t) = \sigma \left[1 + S\left(e^{r(T-t)} a^2 p^2 \Psi_{pp}(p, t : a)\right)\right]^{\frac{1}{2}}.$$

In contrast to (1.1), this volatility adjustment depends on the second derivative of the price. Since as in the Black-Scholes theory, the optimal hedge is nearly equal to Ψ_p, we expect to transact more in regions with high Ψ_{pp} and therefore, this dependence of the volatility adjustment on Ψ_{pp} is natural.

In the foregoing discussion the parameter a is given by,

$$a = \frac{\mu}{\sqrt{\epsilon}} = \mu\sqrt{\gamma N},$$

where γ is the risk aversion factor, N is the number of options to be sold, and μ is the proportional transaction cost (see Sect. 2.1). Therefore, the choice of a depends on how much risk we are willing to take. In our second main result, Theorem 3.2, we quantify this statement. Let $w(\cdot)$ and $P(\cdot)$ be, respectively, the wealth and stock price processes. We show that, for $t < T$ and $k > 0$,

$$\begin{aligned} \min \mathbb{P}(w(T) &\leq -k \mid w(t) = \Psi(p, t : a), P(t) = p) \\ &\leq \exp\left(-\frac{a^2}{\mu^2}\left[k + O\left(\frac{\mu^2}{a^2}\right)\right]\right), \end{aligned}$$

where $O(r)$ denotes any function of one variable satisfying $O(r) \to 0$, as $r \downarrow 0$, and the minimum is taken over all portfolios. (A precise statement and the definition of the wealth process are given in Sect. 3; $w(T)$ is the wealth after the option liability is paid off).

In a recent paper [23], Whalley and Wilmott study the limit of $\Psi^{\epsilon,\mu}$, as $\mu \downarrow 0$, while keeping ϵ fixed. Using formal, matched asymptotics, they obtain detailed information about the dependence of $\Psi^{\epsilon,\mu}$ and the optimal hedging strategy on the parameter μ. Their results are formal and are quite different from ours.

The paper is organized as follows. The model is described in the next section and the main results are summarized in Sect. 3. A formal derivation of (1.2) and a discussion of fixed μ is also given in that section. Section 4 is devoted to the proof of the convergence result and we close the paper with a discussion of a formal hedging policy and applications to other contingent claims.

2 The model

We consider a financial market which consists of one money market and one stock, the price of which evolves according to

$$dP(s) = P(s)\,[\alpha ds + \sigma dW(s)]\,, \qquad s \in [t,T], \tag{2.1}$$

with initial data $P(t) = p$. Here $W(\cdot)$ is a standard one-dimensional Brownian motion, α is the constant mean return rate and σ is the constant volatility. For simplicity we set the interest rate, r, to zero and discuss the non-zero interest rate case in Sect. 3.1 below.

Following Constantinides [8], we let $X(\cdot)$ and $Y(\cdot)$, respectively, be the processes of dollar holdings in the money market and the shares of stocks owned. A *trading strategy* is a pair $(L(\cdot), M(\cdot))$ of adapted, left continuous, nondecreasing processes with $L(t) = M(t) = 0$, which are interpreted as, respectively, the cumulative transfers, measured in shares of the stock, from money market to stock and vice versa. Given a proportional transaction cost $\mu \in (0,1)$ and initial values x, y, the corresponding portfolio $X(s) = X(s;t,x,y,L(\cdot),M(\cdot))$ and $Y(s) = Y(s;t,x,y,L(\cdot),M(\cdot))$ evolves according to

$$\begin{aligned} X(s) &= x - \int_t^s P(\tau)\,(1+\mu)dL(\tau) \\ &\quad + \int_t^s P(\tau)\,(1-\mu)dM(\tau)\,,\ s \in [t,T], \qquad (2.2) \\ Y(s) &= y + L(s) - M(s), \qquad s \in [t,T]. \qquad (2.3) \end{aligned}$$

The utility maximization approach of Hodges and Neuberger to pricing a European call option with maturity T and strike price q is the following. Let U be a utility function, i.e., a concave nondecreasing function on the real line. First consider the optimization problem of maximizing the expected utility from final wealth when there are no option liabilities. The resulting value function is given by,

$$V^f(x,y,p,t) := \sup_{L(\cdot),M(\cdot)} E\{\ U\,(X(T) + Y(T)P(T))\ \}. \tag{2.4}$$

In the second problem, we suppose that we have sold N European call options. Then our final wealth will be

$$X(T) + Y(T)P(T) - N(P(T) - q)^+\ ,$$

and the value function is given by,

$$V(x,y,p,t)\ :=\ \sup_{L(\cdot),M(\cdot)} E\{\ U\left(X(T) + Y(T)P(T) - N(P(T) - q)^+\right)\ \}. \tag{2.5}$$

Hodges and Neuberger postulate that the price of each option is equal to the maximal solution Λ of the algebraic equation (in fact, the unique solution in most cases)

$$V(x + N\Lambda, y, p, t) = V^f(x, y, p, t).$$

Clearly Λ depends on the initial data (x, y, p, t), and also on N and the utility function $U(\cdot)$.

In the foregoing formulation, we neglect the transaction cost of settling the option in cash. However, this difference is not important as the resulting error is proportional to the option price times the transaction cost μ. A brief discussion of this point is given in Sect. 5.2.

Optimal control problems of the above type have been studied extensively: we refer the reader, in particular, to Davis-Norman [13], Davis-Panas-Zariphopoulou [14], Fleming-Soner [16], Shreve-Soner [21], Zariphopoulou [24], [25].

2.1 Exponential utility and scaling

Following Hodges-Neuberger [17] and Davis-Panas-Zariphopoulou [14], we restrict our attention to exponential utility functions

$$U_\gamma(\xi) := 1 - e^{-\gamma\xi}, \qquad \xi \in \mathscr{R}^1,$$

where the parameter $\gamma > 0$ is the risk-aversion factor. Then the option price Λ is a function of the initial data (x, y, p, t) and γ, N. By the linearity of the state equations (2.2) and (2.3),

$$\Lambda(Nx, Ny, p, t; \gamma, N) = \Lambda(x, y, p, t; \gamma N, 1).$$

Hence selling N options with risk-aversion factor of γ yields the same price as selling only one option with risk-aversion factor of γN. (We will show, in Proposition 2.1 below, that Λ is independent of x).

2.2 Asymptotic analysis

The foregoing scaling argument leads us to consider the asymptotic analysis as γN tends to infinity. So we set

$$\epsilon = \frac{1}{\gamma N} \ ,$$

$$U^\epsilon(\xi) = 1 - e^{-\xi/\epsilon} \ .$$

Then the two optimization problems of Hodges and Neuberger take the form:

$$v^{\epsilon,f}(x, y, p, t) = 1 - \inf_{L(\cdot), M(\cdot)} E \ \exp\left(-\frac{1}{\epsilon}[X(T) + Y(T)P(T)]\right)$$

$$v^\epsilon(x, y, p, t) = 1 - \inf_{L(\cdot), M(\cdot)} E \ \exp\left(-\frac{1}{\epsilon}\left[X(T) + Y(T)P(T) - (P(T) - q)^+\right]\right) \ .$$

The superscript f in $v^{\epsilon,f}$ indicates that the first optimization problem is *free* from the option liability. To simplify the analysis, we define z^ϵ and $z^{\epsilon,f}$ by,

$$v^{\epsilon,f}(x,y,p,t) \;=\; 1 - \;\exp\left(-\frac{1}{\epsilon}[x + yp - z^{\epsilon,f}(x,y,p,t)]\right) \;,$$

$$v^{\epsilon}(x,y,p,t) \;=\; 1 - \;\exp\left(-\frac{1}{\epsilon}[x + yp - z^{\epsilon}(x,y,p,t)]\right) \;.$$

It is clear that

$$z^{\epsilon,f}(x,y,p,T) = 0, \qquad z^{\epsilon}(x,y,p,T) = (p-q)^{+} \;, \tag{2.6}$$

and the option price Λ is given by,

$$\Lambda(x,y,p,t;\frac{1}{\epsilon},1) = z^{\epsilon}(x,y,p,t) - z^{\epsilon,f}(x,y,p,t).$$

We gather several properties of z^{ϵ} and $z^{\epsilon,f}$ into the following proposition.

Proposition 2.1. *For any $\epsilon > 0$, z^{ϵ} and $z^{\epsilon,f}$ are independent of x and they are continuous viscosity solutions of*

$$\max\{-z_t - \frac{1}{2}\sigma^2 p^2 z_{pp} - \frac{1}{2\epsilon}\sigma^2 p^2 (z_p - y)^2 \; - \alpha p(z_p - y) \;;\; |z_y| - \mu p \;\} = 0, \tag{2.7}$$

in $\mathscr{R}^1 \times (0,\infty) \times (0,T)$. Moreover,

$$-\;\frac{\epsilon\;\alpha^2}{2\sigma^2}\;(T-t) \le z^{\epsilon,f}(y,p,t) \le z^{\epsilon}(y,p,t) \;,$$

$$z^{\epsilon,f}(y,p,t) \le \mu p|y|, \qquad z^{\epsilon}(y,p,t) \le p + \mu p|y-1|,$$

and

$$\varphi(p,t) - \;\frac{\epsilon\;\alpha^2}{2\sigma^2}\;(T-t) \le z^{\epsilon}(y,p,t) \;,$$

where φ is the Black-Scholes price.

Proof. 1. Let v^{ϵ} and $v^{\epsilon,f}$ be the value functions defined above. Then, by the theory of stochastic optimal control, v^{ϵ} and $v^{\epsilon,f}$ are the unique continuous viscosity solutions of the dynamic programming equation

$$\begin{aligned} \min\{ -v_t - \frac{1}{2}\sigma^2 p^2 v_{pp} - \alpha p v_p \;;\; -v_y + p(1+\mu)v_x \;; \\ v_y - p(1-\mu)v_x \;\} = 0 \;. \end{aligned} \tag{2.8}$$

See, for instance Fleming and Soner [16], for a proof of these facts. We now derive (2.7), by using the definitions of z^{ϵ} and $z^{\epsilon,f}$ and calculus. By uniqueness, z^{ϵ} and $z^{\epsilon,f}$ defined from v^{ϵ} and $v^{\epsilon,f}$, are uniquely characterized as the unique continuous viscosity solutions of (2.7).

Note that the coefficients of (2.7) and the terminal data (2.6) are independent of the x variable. Hence there is a unique continuous viscosity solution of (2.7) and the terminal data (2.6) which is independent of x, and therefore, by uniqueness, z^{ϵ} and $z^{\epsilon,f}$ are independent of x as well. A more intuitive proof of this fact is given in [14].

2. Set $z(y,p,t) := -\dfrac{\epsilon\,\alpha^2}{2\sigma^2}\,(T-t)$. We directly calculate that

$$-z_t - \frac{1}{2}\sigma^2 p^2 z_{pp} - \frac{1}{2\epsilon}\sigma^2 p^2 (z_p - y)^2 \;-\alpha p(z_p - y)$$

$$= -\frac{\epsilon\,\alpha^2}{2\sigma^2} - \frac{1}{2\epsilon}\sigma^2 p^2 y^2 \;+\alpha p y = -\frac{\sigma^2}{2\epsilon}\left(py - \frac{\alpha\epsilon}{\sigma^2}\right)^2 \le 0.$$

This calculation shows that

$$V(x,y,p,t) := U^{\epsilon}(x + yp - z(y,p,t))$$

is a smooth supersolution of the dynamic programming equation (2.8) and, by a classical verification theorem (see, for instance [16]), $v^{\epsilon,f} \le V$. This inequality yields $z^{\epsilon} \ge z$.

3. Suppose that $y = 1$. Choose $\hat{L} \equiv \hat{M} \equiv 0$. Then $(\hat{X}(s),\hat{Y}(s)) = (x,1)$ solves (2.2), (2.3) and, therefore,

$$\begin{aligned} v^{\epsilon}(x,1,p,t) &= 1 - \exp\left(-\frac{1}{\epsilon}[x + p - z^{\epsilon}(1,p,t)]\right) \\ &= \sup E\ \left\{U^{\epsilon}(X(T) + Y(T)P(T) - (P(T)-q)^+)\right\} \\ &\ge E\ \left\{U^{\epsilon}(\hat{X}(T) + \hat{Y}(T)P(T) - (P(T)-q)^+\ )\right\} \\ &= E\ \left\{U^{\epsilon}(x + P(T) - (P(T)-q)^+)\right\} \ge U^{\epsilon}(x) = 1 - \exp\left(-\frac{x}{\epsilon}\right). \end{aligned}$$

Hence $z^{\epsilon}(1,p,t) \le p$.

Now suppose that $y > 1$. Choose $\hat{L} \equiv 0$ and $\hat{M}(s) = y - 1$ for all $s > t$ so that the solution of (2.2), (2.3) is given by, $(\hat{X}(s),\hat{Y}(s)) = (x + p(1-\mu)(y-1), 1)$ for all $s > t$. Then, a similar argument shows that $z^{\epsilon}(y,p,t) \le p + p\mu(y-1)$. When $y < 1$, we choose $\hat{M} \equiv 0$ and $\hat{L}(s) = 1 - y$ and argue as before to complete the proof of the upper bound for z^{ϵ}. The upper bound for $z^{\epsilon,f}$ is proved similarly, after observing that $z^{\epsilon,f}(0,p,t) \le 0$.

4. Set

$$z(y,p,t) := \varphi(p,t) - \frac{\epsilon\,\alpha^2}{2\sigma^2}\,(T-t).$$

We proceed as in Step 2 using the fact that φ satisfies the linear Black-Scholes equation. This calculation shows that

$$V(x,y,p,t) := U^{\epsilon}(x + yp - z(y,p,t)) \le v^{\epsilon}(x,y,p,t),$$

and therefore, $z^{\epsilon} \ge z$. □

We close this section by a technical lower bound on z^ϵ. This estimate will be used in the proof of Lemma 4.2, and its proof is given in Appendix B.

Lemma 2.2. *For any $\mu \le 1/2$ and $0 < \eta \le T$, there exists a constant $K(\eta)$ such that*

$$z^\epsilon(y, p, t) \ge \mu p|y| - K(\eta)T\epsilon\ ,$$

for any $0 \le t \le T - \eta$, $p \in (0, \infty)$ and $y \in \mathscr{R}^1$.

3 The main theorem

In this section, we state the main convergence result. Its proof is given in Appendix B.

Theorem 3.1. *Suppose that $\mu = a\sqrt{\epsilon}$ for some constant $a > 0$. As $\epsilon \downarrow 0$,*

$$z^{\epsilon,f} \to 0, \qquad z^\epsilon \to \Psi,$$

where Ψ is the unique solution of the nonlinear Black-Scholes equation (1.2) *with the terminal data* (1.3) *which satisfies*

$$\lim_{p\to+\infty} \frac{\Psi(p, s)}{p} = 1\ , \tag{3.1}$$

uniformly for $s \in [0, T]$.

The nonlinear volatility correction $S(A)$ is the unique solution of

$$\frac{d}{dA}\ [S(A)] = \frac{S(A) + 1}{2\sqrt{AS(A)}\ - A}, \qquad \forall\ A \neq 0, \tag{3.2}$$

with $S(0) = 0$.

In what follows, when the dependence on a is important, we will use the notation $\Psi(p, s : a)$, in all other cases, we will employ the notation $\Psi(p, s)$.

In the Appendix, it is shown that the function $A \mapsto A(1 + S(A))$ is non-decreasing in $\mathscr{R}^1$. This implies that the nonlinear Black-Scholes equation (1.2) is a degenerate parabolic equation and the theory of viscosity solutions applies to this nonlinear equation.

In Theorem 3.1, the convergence of $z^{\epsilon,f}$ to zero is an immediate consequence of Proposition 2.1. The behavior of $z^{\epsilon,f}$, as $\epsilon \to 0$ with a fixed μ, is also interesting and partially studied in [22].

A straightforward analysis of the ordinary differential equation (3.2) implies that

$$\lim_{A\to\infty} \frac{S(A)}{A} = 1, \qquad \lim_{A\to-\infty} S(A) = -1.$$

Since none of the above properties will be used in our analysis, we omit their proofs.

Remark. The nonlinear extension of the Leland's equation obtained by Avellaneda and Paras [1] has the same from as (1.2) with a nonlinear function

$$\hat{S}(A) = \sqrt{\frac{2}{\pi}}\;\frac{\mu}{\sigma\sqrt{\Delta t}}\frac{A}{|A|},$$

where, as in the Introduction, Δt is transaction frequency.

An immediate corollary of Theorem 3.1 is an upper bound for the probability of missing the hedge by a given constant k.

Theorem 3.2. *For given constants $a, k > 0$, and initial data $X(t) = x$, $Y(t) = y$, $P(t) = p$,*

$$\inf_{L(\cdot),M(\cdot)} \mathbb{P}\left(X(T) + Y(T)P(T) - (P(T) - q)^+ \le -k\right)$$

$$\le \exp\left(-\frac{a^2}{\mu^2}\;\left[k + x + yp - \Psi(p,t:a) + O(\frac{\mu^2}{a^2})\right]\right), \tag{3.3}$$

where $O(r)$ denotes any function of one variable satisfying $O(r) \to 0$, as $r \downarrow 0$.

Proof. This is a simple consequence of Chebyshev's inequality. Set

$$\epsilon = \frac{\mu^2}{a^2}, \quad Z(T) = X(T) + Y(T)P(T) - (P(T) - q)^+, \quad F(\xi) = e^{-\xi/\epsilon}.$$

Then

$$\inf_{L(\cdot),M(\cdot)} \mathbb{P}(Z(T) \le -k) = \inf_{L(\cdot),M(\cdot)} E\left(\mathbb{1}_{\{Z(T)\le -k\}}\right) \le \inf_{L(\cdot),M(\cdot)} E\left(\frac{F(Z)}{F(-k)}\right)$$

$$= (1 - v^\epsilon(x,y,p,t))\,e^{-k/\epsilon} = \exp(-\frac{1}{\epsilon}[k + x + yp - z^\epsilon(y,p,t)])$$

$$= \exp\left(-\frac{a^2}{\mu^2}\;\left[k + x + yp - \Psi(p,t:a) + O(\frac{\mu^2}{a^2})\right]\right).$$

In the last step, we used the asymptotic behavior of z^ϵ as described in Theorem 3.1. □

Note that, if there exists an optimal policy $(L^*(\cdot), M^*(\cdot))$, then the above theorem can be applied to this policy to obtain an estimate without the infimum in (3.3).

3.1 Non-zero interest rate

In this case, the state equation (2.2) has the form:

$$\frac{d}{ds}X(s) = rX(s) - P(s)(1+\mu)dL(s) + P(s)(1-\mu)dM(s), \qquad s \in [t,T].$$

Set

$$\tilde{X}(s) := e^{r(T-s)}X(s), \qquad \tilde{Y}(s) := Y(s), \qquad \tilde{P}(s) := e^{r(T-s)}P(s),$$

so that the triplet $(\tilde{X}(s), \tilde{Y}(s), \tilde{P}(s))$ solves the equations (2.1), (2.2), (2.3) with mean return rate $\tilde{\alpha} = \alpha - r$ and initial data

$$(\tilde{X}(t), \tilde{Y}(t), \tilde{P}(t)) = (\tilde{x}, \tilde{y}, \tilde{p}) := (e^{r(T-t)}x, y, e^{r(T-t)}p).$$

Define

$$\tilde{v}^{\epsilon,f}(\tilde{x}, \tilde{y}, \tilde{p}, t) := 1 - \inf_{L(\cdot),M(\cdot)} E \ \exp\left(-\frac{1}{\epsilon}\left[\tilde{X}(T) + \tilde{Y}(T)\tilde{P}(T)\right] \right),$$

$$\tilde{v}^{\epsilon}(\tilde{x}, \tilde{y}, \tilde{p}, t) := 1 - \inf_{L(\cdot),M(\cdot)} E \ \exp\left(-\frac{1}{\epsilon}\left[\tilde{X}(T) + \tilde{Y}(T)\tilde{P}(T) - N(\tilde{P}(T) - q)^+\right] \right).$$

Since $(\tilde{X}(T), \tilde{Y}(T), \tilde{P}(T)) = (X(T), Y(T), P(T))$, it is clear that

$$v^{\epsilon}(x, y, p, t) = \tilde{v}^{\epsilon}(e^{r(T-t)}x, y, e^{r(T-t)}p, t),$$

$$v^{\epsilon,f}(x, y, p, t) = \tilde{v}^{\epsilon,f}(e^{r(T-t)}x, y, e^{r(T-t)}p, t).$$

We define $\tilde{z}^{\epsilon}$ and $\tilde{z}^{\epsilon,f}$ as in Sect. 2, so that, by Theorem 3.1,

$$\tilde{\Psi}(\tilde{p}, t : a) := \lim_{\epsilon \downarrow 0} \left(\tilde{z}^{\epsilon}(\tilde{y}, \tilde{p}, t) - \tilde{z}^{\epsilon,f}(\tilde{y}, \tilde{p}, t) \right)$$

solves the nonlinear Black-Scholes equation (1.2) with $r = 0$.

Recall that the option price $\Lambda^{\epsilon}(y, p, t)$ is defined to be the unique solution of the algebraic equation:

$$v^{\epsilon}(x + \Lambda^{\epsilon}, y, p, y) = v^{\epsilon,f}(x, y, p, t).$$

Hence,

$$\Lambda^{\epsilon}(y, p, t) = e^{-r(T-t)}\left[\tilde{z}^{\epsilon}(y, e^{r(T-t)}p, t) - \tilde{z}^{\epsilon,f}(y, e^{r(T-t)}p, t)\right]$$

and

$$\begin{aligned} \Psi(p, t : a) &:= \lim_{\epsilon \downarrow 0} \Lambda^{\epsilon}(y, p, t) \\ &= e^{-r(T-t)} \lim_{\epsilon \downarrow 0} \left[\tilde{z}^{\epsilon}(y, e^{r(T-t)}p, t) - \tilde{z}^{\epsilon,f}(y, e^{r(T-t)}p, t)\right] \\ &= e^{-r(T-t)}\tilde{\Psi}(e^{r(T-t)}p, t : a). \end{aligned}$$

Since $\tilde{\Psi}$ solves (1.2) with $r = 0$, it is straightforward to show that Ψ satisfies (1.2) with the non-zero interest rate r.

3.2 Formal derivation of (1.2)

In this subsection, we give a formal derivation of the nonlinear Black-Scholes equation (1.2). A rigorous proof of convergence will be given in the Appendix.

We start our formal derivation by assuming that z^ϵ has the form

$$z^\epsilon(y,p,t) \approx \Psi(p,t) + \epsilon C(r^\epsilon, A), \tag{3.4}$$

where Ψ and C are two functions to be determined, and

$$r^\epsilon \;=\; r^\epsilon(y,p,t) := ap\frac{\Psi_p(p,t) - y}{\sqrt{\epsilon}}$$

$$A \;=\; A^\epsilon(p,t) := a^2p^2\Psi_{pp}(p,t).$$

We will use (2.7) to derive the equations satisfied by Ψ and C. So we start by obtaining approximate expressions for the derivatives of z^ϵ, by differentiating (3.4). In the following computations, we assume that r^ϵ is order one and keep only the terms that are order one. But, since the coefficient of the z_p^ϵ term in (2.7) is ϵ^{-1}, we keep the terms of order $\sqrt{\epsilon}$ in the expansion of z_p^ϵ. After recalling that $\mu = a\sqrt{\epsilon}$, the resulting expressions are:

$$z_t^\epsilon \;\approx\; \Psi_t \qquad\qquad z_y^\epsilon \;\approx\; \mu p C_r$$

$$z_p^\epsilon \;\approx\; \Psi_p + \epsilon r_p^\epsilon C_r = \Psi_p + \epsilon\left(\frac{r^\epsilon}{p} + \frac{ap\Psi_{pp}}{\sqrt{\epsilon}}\right)C_r \;\approx\; \Psi_p + \sqrt{\epsilon}\left(\frac{A}{ap}\right)C_r$$

$$z_{pp}^\epsilon \;\approx\; \Psi_{pp} + \epsilon(r_p^\epsilon)^2 C_{rr} \;\approx\; \Psi_{pp} + \left(\frac{A}{ap}\right)^2 C_{rr}\ .$$

Then, the gradient constraint, $|z_y^\epsilon| \le \mu p$, in (2.7) is equivalent to

$$|C_r| \le 1.$$

Set

$$I^\epsilon := -z_t^\epsilon - \frac{1}{2}\sigma^2 p^2 z_{pp}^\epsilon - \frac{1}{2\epsilon}\sigma^2p^2(z_p^\epsilon - y)^2 \;-\alpha p(z_p^\epsilon - y)$$

so that

$$\begin{aligned} I^\epsilon \;\approx\; & -\Psi_t - \frac{1}{2}\sigma^2p^2\left(\Psi_{pp} + \frac{A^2}{a^2p^2}C_{rr}\right) \\ & -\frac{1}{2\epsilon}\sigma^2p^2\left(\Psi_p - y + \sqrt{\epsilon}\frac{A}{ap}C_r\right)^2 - \alpha p(\Psi_p - y) \\ =\; & -\Psi_t - \frac{1}{2}\sigma^2p^2\Psi_{pp} - \frac{\sigma^2}{2a^2}A^2C_{rr} - \frac{\sigma^2}{2a^2}\left(r^\epsilon + AC_r\right)^2 - \alpha\sqrt{\epsilon}\frac{r^\epsilon}{a} \\ \approx\; & -\Psi_t - \frac{1}{2}\sigma^2p^2\Psi_{pp} - \frac{\sigma^2}{2a^2}\left[A^2C_{rr} + (r^\epsilon + AC_r)^2\right]. \end{aligned}$$

Then, (2.7) is equivalent to

$$\max\left\{-\Psi_t - \frac{1}{2}\sigma^2 p^2 \Psi_{pp} - \frac{\sigma^2}{2a^2}\left[A^2 C_{rr} + (r^\epsilon + AC_r)^2\right] \;;\; |C_r| - 1\right\} = 0,$$

for all p, t and r^ϵ. Since this equation holds for all y and therefore for all r, we conclude that there is a function H of A, so that

$$-\Psi_t(p,t) - \frac{1}{2}\sigma^2 p^2 \Psi_{pp}(p,t) = H(A(p,t)),$$

and

$$\max\left\{H(A) - \frac{\sigma^2}{2a^2}\left[A^2 C_{rr} + (r + AC_r)^2\right]; |C_r| - 1\right\} = 0.$$

Set

$$S(A) := \frac{2a^2}{A\sigma^2} H(A),$$

and recall the definition of $A(p,t)$. This implies that Ψ is a solution of (1.2).

Without any additional conditions, the equation

$$\max\left\{-A^2 C_{rr}(r;A) - (r + AC_r(r;A))^2 + AS(A) \;;\; |C_r(r;A)| - 1\right\} = 0, \tag{3.5}$$

has more than one solution. For instance, $S(A) = 0$ with $C(r;A) = r$ or $C(r;A) = -r$ are two solutions different than the solution constructed in the Appendix. To characterize the latter as the unique solution, we observe that, in view of Proposition 2.1 and Lemma 2.2, z^ϵ behaves like $\mu p|y|$ for sufficiently large $|y|$. Therefore, in order to match this behavior, $C(r;A)$ should satisfy

$$\lim_{|r|\to\infty} \frac{C(r;A)}{|r|} = 1\ . \tag{3.6}$$

In Appendix A, we show that, for any $A \neq 0$, there exists a unique pair $(C(r;A), S(A))$ such that $C(r;A)$ is a smooth solution of (3.5) which satisfies (3.6) and $C(0;A) = C_r(0,A) = 0$, and we explicitly construct this solution. We need to impose these last conditions to have a unique solution for the variational inequality (3.5) (notice in particular that C is defined only up to a constant). Our choice is motivated by the fact that, at $p = 0$, z^ϵ, z_y^ϵ and Ψ vanish. Finally, the case $A = 0$ is a degenerate case but it will not be used in the proof of Theorem 3.1.

4 Concluding remarks

4.1 Optimal hedge

The theory of singular optimal control [16, Sect. VIII], provides us with a general strategy of constructing optimal controls (also, see [21]). In this particular problem, the optimal state process $(X^*(\cdot), Y^*(\cdot), P(\cdot))$ has to stay within the so-called continuation region

$$\mathscr{C}(t) := \{\ (x, y, p) \in \mathscr{R}^1 \times \mathscr{R}^1 \times [0, \infty)\ :\ \ |z_y^\epsilon(y, p, t)| < \mu p\ \}.$$

Since

$$z^\epsilon(y, p, t) \approx \Psi(p, t : a) + \epsilon C(ap \frac{\Psi_p(p, t) - y}{\sqrt{\epsilon}}; a^2 p^2 \Psi_{pp}(p, t)),$$

$$z_y^\epsilon(y, p, t) \approx -\mu p C_r(\cdots)\ ,$$

by (A.4),

$$\mathscr{C}(t)\ \approx\ \{\ (x, y, p)\ :\ \ |C_r(ap \frac{\Psi_p(p, t : a) - y}{\sqrt{\epsilon}}; a^2 p^2 \Psi_{pp}(p, t : a))| < 1\ \}.$$

$$=\ \{\ (x, y, p)\ :\ \ |\Psi_p(p, t : a) - y| < \frac{\sqrt{\epsilon}}{ap} g(a^2 p^2 \Psi_{pp}(p, t))\ \}.$$

In summary, the optimal y^* is approximately equal to $\Psi_p(p, t : a)$ and the optimal strategy is to keep $Y^*(s)$ in the interval

$$[\Psi_p(P(s), s : a) - \Gamma(P(s), s), \Psi_p(P(s), s : a) + \Gamma(P(s), s)],$$

for all $s \in [t, T]$. Here

$$\Gamma(p, t) = \frac{\sqrt{\epsilon}}{ap}\ g(a^2 p^2 \Psi_{pp}(p, t)),$$

where g is as in (A.4)

4.2 Higher order correction

In view of the formal argument given in Section 3.3, partially justified by the perturbed test function argument introduced in the proof of Theorem 3.1, we expect that

$$z^\epsilon(y, p, t) \approx \Psi(p, t : a) + \epsilon C(ap \frac{\Psi_p(p, t) - y}{\sqrt{\epsilon}}; a^2 p^2 \Psi_{pp}(p, t)).$$

Since $\mu = a\sqrt{\epsilon}$ and $C(r; A) \approx |r|$ for large $|r|$, we simplify the above approximation as follows:

$$z^\epsilon(y, p, t) \approx \Psi(p, t : a) + \mu p |\Psi_p(p, t) - y|\ . \tag{4.1}$$

Although in practice μ is small, this additional correction term might be significant, as indicated in the following example:

Example. Consider a European call option of one year maturity with strike price $q = \$40$ and market parameters

$$\sigma = 0.2, \qquad \mu = 0.01, \qquad r = 0.$$

Then, for most $t > 0$, $\Psi_p(40, t)$ is greater than a half and, at $p = 40$, $y = 0$, the correction term is at least \$0.2 while the Black-Scholes price at $t = 1$ is

$$\varphi(40, 1) = \$3.17,$$

and

$$\Psi(40, 1 : 0.02) = \$3.86, \qquad \Psi(40, 1 : 0.03) = \$4.13.$$

We computed Ψ by using a simple, explicit, finite difference scheme. Since the diffusion coefficient is very large when p is near the strike price and t is close to the maturity, we used a very small time step near the maturity. For the a value, we just simply used several integer multiples of μ.

Since the optimal hedge y^* is approximately equal to Ψ_p, the correction $\mu p|\Psi_p - y|$ is simply equal to the initial cost of moving our stock holdings from the initial value y to its optimal value $y^* = \Psi_p$.

4.3 Other contingent claims

In this paper, we have developed a pricing technique which depends on asymptotic analysis and utility maximization. This methodology equally applies to other call or put options. One specific example is the cash-settled European call option. In that example, our final wealth is given by,

$$w(T) := X(T) + Y(T)P(T) - \mu|Y(T)P(T)| - (P(T) - q)^+ \ .$$

Let $v^{\epsilon,f}$ and $z^{\epsilon,f}$ be as in Sect. 2 and define

$$\begin{aligned}\hat{v}^\epsilon(x, y, p, t) \ &:= \sup_{L(\cdot),M(\cdot)} E\ U^\epsilon(w(T)) \\ &:= \ 1 - \ \exp\left(-\frac{1}{\epsilon}[x + yp - \hat{z}^\epsilon(x, y, p, t)]\right) \ .\end{aligned}$$

Set $\hat{\Lambda}^\epsilon$ be the corresponding option price. Then,

$$\hat{\Lambda}^\epsilon = \hat{z}^\epsilon - z^{\epsilon,f} \geq z^\epsilon - z^{\epsilon,f} = \Lambda^\epsilon$$

and a minor modification of our proof shows that, as $\epsilon \downarrow 0$, $\hat{\Lambda}^\epsilon$ converges to Ψ as well.

4.4 Numerical experiments

In this subsection, we summarize the results of few numerical experiments we have done with the nonlinear Black-Scholes equation (1.2). For comparison, we have also computed the call prices with the Leland correction. As in Leland's paper [20], we used

$$\sigma = 0.2, \qquad r = 10\%, \qquad T = 1\,\text{year},$$

Table 1. Leland correction with weekly transactions

Strike	$\mu = 0.0$ Call Price	$\mu = 0.0025$ Call Price	$\mu = 0.01$ Call Price	$\mu = 0.04$ Call Price
80	27.97	28.00	28.10	28.58
90	19.93	20.01	20.22	21.12
100	13.27	13.35	13.64	14.92
110	8.09	8.22	8.62	10.10
120	4.67	4.78	5.14	6.58
Adjusted volatility $\hat{\sigma}$	0.2	0.2034	0.2134	0.2492

Table 2. Leland correction with monthly transactions

Strike	$\mu = 0.0$ Call Price	$\mu = 0.0025$ Call Price	$\mu = 0.01$ Call Price	$\mu = 0.04$ Call Price
80	27.97	28.05	28.25	29.42
90	19.93	20.12	20.53	22.42
100	13.27	13.49	14.09	16.60
110	8.09	8.43	9.16	11.98
120	4.67	4.95	5.66	8.48
Adjusted volatility $\hat{\sigma}$	0.2	0.2070	0.2265	0.2918

in all our computations. We have computed the call price when the current price is \$ 100 and varied the strike from \$80 to \$ 120.

Table 1 summarizes the call prices with Leland correction when the transaction frequency is once a week. Results with monthly transactions are tabulated in Table 2. Call prices computed by the nonlinear equation (1.2) are summarized in Table 3. The only additional parameter needed for (1.2) is the a value. In particular, the transaction cost value μ does not appear explicitly in these computations. However, the value of μ is clearly very important for the the upper bound of the "missed-hedge" probability (3.3). To indicate this dependence, in the last two rows of Table 3, we tabulate the quantity

Table 3. Nonlinear Black-Scholes

Strike	$a = 0.0$ Call Price	$a = 0.005$ Call Price	$a = 0.01$ Call Price	$a = 0.015$ Call Price	$a = 0.02$ Call Price
80	27.96	28.17	28.33	28.48	28.64
90	19.92	20.44	20.79	21.11	21.40
100	13.23	14.03	14.56	15.02	15.44
110	8.08	9.13	9.76	10.30	10.79
120	4.62	5.64	6.28	6.82	7.31
β_1, $\mu = 0.0025$	0.0	0.0018	10^{-7}	2.4×10^{-16}	1.6×10^{-28}
β_1, $\mu = 0.01$	0.0	0.78	0.37	0.11	0.018

$$\beta_k := \exp\left(- \frac{a^2}{\mu^2}\, k \right),$$

with $k = 1$. These numbers are the asymptotics upper bounds for the "missed-hedge" probability

$$I\!\!P\left[X(T) + Y(T)P(T) - (P(T) - q)^+ \leq -k \mid w(t) = \Psi(p, t : a) \right],$$

where $w(t) = X(t) + Y(t)P(t)$ is the initial wealth.

Numerical methods we use to compute the call prices with linear volatility and the nonlinear volatility are slightly different. This accounts for the small discrepancy observed for the call values with $\mu = 0.0$ in Tables 1 and 2, and the call values in Table 3 with a=0.0. In both computations, we have used an explicit finite difference scheme. However, since the nonlinear volatility is very large near the maturity and the strike price, we used a smaller time step near the maturity and then increased the time step for larger time values. For the lateral boundary conditions, we used a Dirichlet data at $S = 250$: we impose the Call Price at \$ 250 and at time t to be $(250 - \text{Strike}) \times e^{-r(T-t)}$.

Appendix A

In this section, we solve the variational inequality

(A.1) $$\max \left\{ -A^2 C_{rr}(r) - (r + AC_r)^2 + AS(A) \; ; \; |C_r| - 1 \right\} = 0, \forall\, r \in \mathscr{R}^1,$$

with the conditions

(A.2) $$\lim_{|r| \to \infty} \frac{C(r; A)}{|r|} = 1 \quad \text{and} \quad C(0; A) = C_r(0; A) = 0\,,$$

where A is a given parameter and the unknowns are the scalar function $C(\cdot; A)$ and the constant $S(A)$.

Note that the parameter r in this section, is not the interest rate but rather an independent variable.

We first construct a solution. Fix $A \in \mathscr{R}^1$, $A \neq 0$ and set

$$W(r) := C_r(r; A)$$

so that W solves

$$\max \left\{ -A^2 W_r(r) - (r + AW)^2 + AS(A) \; ; \; |W| - 1 \right\} = 0, \quad \forall\, r \in \mathscr{R}^1.$$

We expect W to be an nondecreasing, odd function. Therefore, we look for constants $S(A) > 0$, $g > 0$ and a continuously differentiable function

$$W : [0, \infty) \to [0, \infty)$$

satisfying

(A.3) $$A^2 W_r(r) + (r + AW(r))^2 = AS(A), \qquad r \in (0, g),$$

$$W(r) = 1, \qquad r \in [g, \infty),$$

$$W(r) \leq 1, \qquad A^2 W_r(r) + (r + AW(r))^2 \geq AS(A), \qquad r \in (0, \infty),$$

with boundary data $W(0) = 0$. The smoothness of W implies that $W_r(g) = 0$ and, by (A.3),

$$\sqrt{AS(A)} = g + A.$$

We analyze three cases separately.

Case I: $A > 0$. Set

$$\lambda = (AS(A) + A)^{\frac{1}{2}},$$

so that (A.3) yields

$$W(r) = \frac{1}{A} \left[\lambda \tanh\left(\frac{\lambda r}{A}\right) - r \right], \qquad r \in [0, g].$$

Then, we solve for $S(A)$ by using the identity $W(g) = 1$. This yields the following algebraic equation for $S(A) > 0$:

$$0 = F(S(A), A) := \tanh^{-1}\left(\sqrt{\frac{S(A)}{1 + S(A)}}\right) - \sqrt{1 + S(A)} \left(\sqrt{S(A)} - \sqrt{A}\right).$$

By differentiating the identity $F(S(A), A) = 0$ with respect to A, we derive the following differential equation for $S(A)$:

$$S'(A) = -\frac{F_A(S(A), A)}{F_S(S(A), A)} = \frac{1 + S(A)}{2\sqrt{S(A)A} - A}.$$

Case II: $A < 0$. Set

$$\lambda = (-AS(A) - A)^{\frac{1}{2}}.$$

Then, by (A.3),

$$W(r) = \frac{1}{A}\left[\lambda \tanh\left(-\frac{\lambda r}{A}\right) - r\right], \qquad r \in [0, g].$$

and the identity $W(g) = 1$ yields the following algebraic equation for $S(A) < 0$:

$$0 = F(S(A), A) := \tanh^{-1}\left(\sqrt{\frac{-S}{1+S}}\right) - \sqrt{1+S}\left(\sqrt{-S} + \sqrt{-A}\right).$$

In this case, the differential equation for $S(A)$ is:

$$S'(A) = -\frac{F_A(S(A), A)}{F_S(S(A), A)} = \frac{1 + S(A)}{2\sqrt{S(A)A} - A}.$$

Case III: $A = 0$. This is a degenerate case. A lengthy omputation shows that $S(A) \to 0$ when $A \to 0$ and we set $S(0) = g(0) = 0$, $W \equiv 1$.

For any A, set

$$C(r; A) := \int_0^{|r|} W(\xi)\, d\xi, \qquad r \in \mathscr{R}^1.$$

Then, C solves (A.1), even in the case $A = 0$. We also note that, by the construction of C,

(A.4) $$|C_r(r; A)| < 1 \quad \Leftrightarrow \quad |r| < g(A),$$

(A.5) $$rC_r(r; A) \geq 0,$$

for any r and A and that, for every $a > 0$,

(A.6) $$\inf_{r, |A| \leq a} \{ C(r; A) - |r| \} \geq -\infty.$$

Finally, for $A \neq 0$, we compute

$$\frac{d}{dA}[A(1 + S(A))] = (1 + S(A))\frac{2\sqrt{AS(A)}}{2\sqrt{AS(A)} - A};$$

but an easy analysis shows that

$$1 + S(A) \geq 0 \quad \text{and} \quad 2\sqrt{AS(A)} - A \geq 0 \quad \text{for any } A \neq 0,$$

and therefore

$$\frac{d}{dA}[A(1 + S(A))] \geq 0 \quad \text{if } A \neq 0.$$

This implies that $A(1+S(A))$ is a nondecreasing function of A since it is continuous at $A = 0$.

We conclude this Appendix by showing the uniqueness of the pair $(C(r; A), S(A))$. To this end, we consider two solutions $(C(r; A), S(A))$ and $(C'(r; A), S'(A))$ of (A.1)-(A.2). For $\mu < 1$, close to 1, we introduce the function

$$r \mapsto \mu C(r;A) - C'(r;A)\ .$$

Because of (A.2), this function achieves its maximum on $\mathscr{R}^1$ at some point $\overline{r}$. Since C and C' are smooth,

$$\mu C_r(\overline{r};A) = C_r'(\overline{r};A) \quad \text{and} \quad \mu C_{rr}(\overline{r};A) \le C_{rr}'(\overline{r};A)\ .$$

But, since $|C_r(\overline{r};A)| \le 1$ and $\mu < 1$, this implies that $|C_r'(\overline{r};A)| < 1$ and, therefore,

$$-A^2 C_{rr}'(\overline{r};A) - (\overline{r} + AC_r'(\overline{r};A))^2 + AS'(A) = 0\ .$$

Multiply the equation by μ and then subtract it from the preceding equality. The result is:

$$(\overline{r} + AC_r'(\overline{r};A))^2 - \mu(\overline{r} + AC_r(\overline{r};A))^2 + \mu AS(A) - AS'(A) \le 0\ .$$

An algebraic computation, using $\mu < 1$, $|C_r(\overline{r};A)| < 1$, yields

$$\begin{aligned}\mu AS(A) - AS'(A) &\le (\mu - 1)\overline{r}^2 + \mu(1-\mu)A^2[C_r(\overline{r};A)]^2 \\ &\le \mu(1-\mu)A^2\ .\end{aligned}$$

Letting μ to 1, we first obtain that $S(A) = S'(A)$ and then the fact that $C(r;A) = C'(r;A)$ follows from an ODE argument.

Appendix B

In this section, we give a proof of our convergence result. The chief tool of our analysis is the theory of viscosity solutions of Crandall and Lions [11]. In particular, we will use the weak viscosity limits of Barles and Perthame [3] and the perturbed test function method of Evans [15]. For information on the theory of viscosity solutions, we refer the reader to the “User’s Guide” of Crandall et al. [10], and to Fleming and Soner [16] for its applications to optimal stochastic control.

Following Barles and Perthame [2], for $(y,p,t) \in \mathscr{R}^1 \times [0,\infty) \times [0,T]$, we define

$$z^*(y,p,t) := \limsup_{\rho\downarrow 0}\ \limsup_{\epsilon\downarrow 0}\ Z^+(y,p,t;\epsilon,\rho),$$

where

$$Z^+(y,p,t;\epsilon,\rho) := \sup\{\ z^\epsilon(\hat{y},\hat{p},\hat{t})\ :\ |y-\hat{y}| + |p-\hat{p}| + |t-\hat{t}| \le \rho\ \},$$

and

$$z_*(y,p,t) := \liminf_{\rho\downarrow 0}\ \liminf_{\epsilon\downarrow 0}\ Z^-(y,p,t;\epsilon,\rho),$$

where

$$Z^-(y,p,t;\epsilon,\rho) := \inf\{\ z^\epsilon(\hat{y},\hat{p},\hat{t})\ :\ \hat{y} \in \mathscr{R}^1,\ |p-\hat{p}| + |t-\hat{t}| \le \rho\ \}.$$

In view of Proposition 2.1, z_* and z^* are well-defined, and, by its definition, z_* is independent of y. Moreover, since $|z_y^\epsilon| \le \mu p$,

$$z^\epsilon(y_1, p_1, t_1) - z^\epsilon(y_2, p_1, t_1) \le \mu p_1 |y_1 - y_2| \ ,$$

for any $y_1, y_2 \in \mathscr{R}^1$, $p_1 \in [0, \infty]$ and $t_1 \in [0, T]$ and, therefore, z^* is also independent of y.

By Proposition 2.1 and the definitions of z^* and z_*,

$$z^*(0, t) = z_*(0, t) = 0 \qquad \forall t \in [0, T], \tag{B.1}$$

and

$$\varphi(p, t) \le z_*(p, t) \le z^*(p, t) \le p, \tag{B.2}$$

for any $p \in [0, \infty)$ and $t \in [0, T]$. Therefore, since the property (3.1) is satisfied by the Black and Scholes price φ, it is also satisfied by z^* and z_*.

Our method of proof, which is standard in the theory of viscosity solutions, is this: we will first show that z^* and z_* are, respectively, a subsolution and a supersolution of (1.2)-(1.3). We will then use a comparison theorem to conclude that z^* and z_* are both equal to the unique continuous solution Ψ of the nonlinear Black-Scholes equation (1.2)-(1.3) satisfying the growth condition (3.1). In the theory of viscosity solutions, it is often the case that the proofs of the subsolution and the supersolution properties are very similar to each other. Interestingly, this is not the case here, partly because the definitions of z^*, z_* and the estimates on z^ϵ are not symmetric.

We start our analysis by proving that z^* is a viscosity subsolution of (1.2)-(1.3). Here we claim that the terminal condition (1.3) is achieved only in the viscosity sense. For a discussion of generalized boundary conditions, see [16, Sect. II.13]. The definition of this generalized viscosity property is also given in the proof of the following lemma.

Lemma B.1. z^* *is a viscosity subsolution of* (1.2)-(1.3) *with* $r = 0$.

Proof. Let $w(p, t)$ be a smooth test function and assume that $(p_0, t_0) \in (0, \infty) \times [0, T]$ is a *strict local* maximizer of the difference $z^* - w$ on $[0, \infty) \times [0, T]$. By adding a small quadratic term, if necessary, we may assume that

$$w_{pp}(p_0, t_0) \ne 0. \tag{B.3}$$

In order to verify that z^* is a viscosity subsolution, we need to prove the following: if $t_0 < T$, then

$$-w_t - \frac{1}{2}\sigma^2 p^2 w_{pp} \left[1 + S\left(a^2 p^2 w_{pp}\right)\right] \le 0, \tag{B.4}$$

at (p_0, t_0), and if $t_0 = T$, then we have to show that either (B.4) holds at (p_0, t_0) or $z^*(p_0, t_0) \le (p_0 - q)^+$.

1. Now suppose that $t_0 = T$. If $z^*(p_0, T) \le (p_0 - q)^+$, then there is nothing to prove. So we assume that either $t_0 < T$ or, $t_0 = T$ and $z^*(p_0, T) > (p_0 - q)^+$.

2. Recall that $\mu = a\sqrt{\epsilon}$. For ϵ and $0 < \delta \ll 1$, set

$$A := A^{\delta}(p,t) = a^2\ p^2\ (1+\delta)^2\ w_{pp}(p,t),$$

and

$$r^{\epsilon,\delta}(y,p,t)\ :=\ (1+\delta)ap\ \frac{(w_p(p,t)-y)}{\sqrt{\epsilon}},$$

$$w^{\epsilon,\delta}(y,p,t)\ :=\ w(p,t)+\epsilon C(r^{\epsilon,\delta}(p,t);A^{\delta}(p_0,t_0)\ ),$$

where $C(\cdot;A)$ is a smooth solution of

$$\max\ \left\{\ -A^2C_{rr}(r)-(r+AC_r)^2+AS(A)\ ;\ |C_r|-1\ \right\}=0,\quad \forall\ r\in\mathscr{R}^1.$$

This solution is constructed in Appendix A.

To simplify the notation, set $A_0 := A^{\delta}(p_0,t_0)$. Note that, since $p_0 > 0$ and $w_{pp}(p_0,t_0) \neq 0$, $A_0 \neq 0$.

3. Fix δ. We claim that there is a sequence $\epsilon_n \downarrow 0$ and local maximizers $(y_n,p_n,t_n) \in \mathscr{R}^1 \times (0,\infty) \times [0,T)$ of the function

$$(y,p,t) \mapsto z^{\epsilon_n}(y,p,t) - w^{\epsilon_n,\delta}(y,p,t) - |y - w_p(p,t)|^4$$

satisfying

(B.5) $(p_n,t_n) \to (p_0,t_0)$, $\quad z^{\epsilon_n}(y_n,p_n,t_n) \to z^*(p_0,t_0)$ and $y_n \to w_p(p_0,t_0)$.

Indeed, $(w_p(p_0,t_0),p_0,t_0)$ is a strict local maximizer of the function

$$(y,p,t) \mapsto z^*(p,t) - w(p,t) - |y - w_p(p_0,t_0)|^4\ ,$$

and

$$z^*(p,t) = \limsup_{\rho\downarrow 0}\ \limsup_{\epsilon\downarrow 0}\ Z^+(y,p,t;\epsilon,\rho)$$

for any $y \in \mathscr{R}^1$, $p \in [0,\infty)$ and $t \in [0,T]$. Moreover, $w^{\epsilon,\delta}$ converges locally uniformly to w in $\mathscr{R}^1 \times [0,\infty) \times [0,T]$. Then, the existence of such sequences ϵ_n and (y_n,p_n,t_n) is proved in the Appendix of Barles and Perthame [2].

We claim that $t_n < T$ for all sufficiently large n. Indeed, if $t_0 < T$, then this claim follows from the convergence of t_n to t_0. So we may assume that $t_0 = T$ and $z^*(p_0,T) > (p_0-q)^+$. Suppose that $t_n = T$. Then,

$$(p_0-q)^+ < z^*(p_0,T) = \lim z^{\epsilon_n}(y_n,p_n,T) = \lim(p_n-q)^+ = (p_0-q)^+\ .$$

Hence, $t_n < T$ for all sufficiently large n.

4. By calculus, at (y_n,p_n,t_n),

$$
\begin{aligned}
z_p^{\epsilon_n} &= w_p^{\epsilon_n,\delta} = w_p + \frac{\sqrt{\epsilon_n}}{a(1+\delta)p_n}\ A\ C_r(\cdots) + E_1,\\
z_{pp}^{\epsilon_n} &\le w_{pp}^{\epsilon_n,\delta} = w_{pp} + \left(\frac{A}{a(1+\delta)p_n}\right)^2\ C_{rr}(\cdots) + E_2\\
z_t^{\epsilon_n} &= w_t^{\epsilon_n,\delta} = w_t + E_3,
\end{aligned}
$$

where $(\cdots) = (r^{\epsilon_n,\delta}(y_n, p_n, t_n); A_0)$,

$$
\begin{aligned}
E_1 &:= \epsilon_n \frac{r^{\epsilon_n,\delta}}{p_n} C_r(\cdots) + 4(w_p - y_n)^3 w_{pp}\\
E_2 &:= C_{rr}(\cdots)[\epsilon_n \frac{(r^{\epsilon_n,\delta})^2}{p_n^2} + 2\sqrt{\epsilon_n}\, a(1+\delta) r^{\epsilon_n,\delta} w_{pp}]\\
&\quad + a(1+\delta)\sqrt{\epsilon_n}\, C_r(\cdots)[2w_{pp} + p_n w_{ppp}]\\
&\quad + 4(w_p - y_n)^3 w_{ppp} + 12(w_p - y_n)^2 (w_{pp})^2\\
E_3 &:= a(1+\delta)p_n \sqrt{\epsilon_n}\, w_{pt} C_r(\cdots) + 4(w_p - y_n)^3 w_{pt}.
\end{aligned}
$$

By Proposition 2.1, at (y_n, p_n, t_n),

$$
\begin{aligned}
0 &\ge -z_t^{\epsilon_n} - \frac{1}{2}\sigma^2 p_n^2 z_{pp}^{\epsilon_n} - \frac{1}{2\epsilon_n}\sigma^2 p_n^2 (z_p^{\epsilon_n} - y_n)^2 - \alpha p_n (z_p^{\epsilon_n} - y_n)\\
&= -z_t^{\epsilon_n} - \frac{1}{2}\sigma^2 p_n^2 z_{pp}^{\epsilon_n} - \frac{1}{2\epsilon_n}\sigma^2 \left[p_n (z_p^{\epsilon_n} - y_n) + \frac{\alpha\epsilon_n}{\sigma^2}\right]^2 + \frac{\alpha^2\epsilon_n}{2\sigma^2}\\
&\ge -w_t - \frac{1}{2}\sigma^2 p_n^2 \left[w_{pp} + \frac{A^2}{a^2(1+\delta)^2 p_n^2} C_{rr}(\cdots)\right] + F_1\\
&\quad - \frac{\sigma^2\ p_n^2}{2\epsilon_n}\left[w_p - y_n + \frac{\sqrt{\epsilon_n}}{a(1+\delta)p_n}\ (AC_r(\cdots) + F_2)\right]^2,
\end{aligned}
$$

where

$$
\begin{aligned}
F_1 &:= -\frac{1}{2}\sigma^2 p_n^2 E_2 - E_3 + \frac{\alpha^2\epsilon_n}{2\sigma^2}\\
F_2 &:= \frac{a(1+\delta)}{\sqrt{\epsilon_n}} (p_n E_1 + \frac{\alpha\epsilon_n}{\sigma^2})\\
&= a(1+\delta)\ \sqrt{\epsilon_n}\,(r^{\epsilon_n,\delta} C_r(\cdots) + \frac{\alpha}{\sigma^2}).
\end{aligned}
$$

We rewrite this inequality as follows:

$$\text{(B.6)} \qquad 0 \geq -w_t - \frac{\sigma^2}{2} p_n^2 w_{pp}$$

$$- \frac{\sigma^2}{2(1+\delta)^2 a^2} \left[A^2 C_{rr}(\cdots) + \left(r^{\epsilon_n,\delta} + AC_r(\cdots) + F_2 \right)^2 \right] + F_1 .$$

5. Let $o(1)$ be any sequence converging to 0 as $n \to \infty$. By (B.5), $r^{\epsilon_n,\delta} = o(1)/\sqrt{\epsilon_n}$, and consequently, $E_1 = \sqrt{\epsilon_n} o(1)$, $E_2 = o(1)$, $E_3 = o(1)$, and

$$\lim_{n \to \infty} |F_1| + |F_2| = 0.$$

A similar argument shows that, $A - A_0 = o(1)$.

Moreover, by Proposition 2.1,

$$\mu p_n = a\sqrt{\epsilon_n}\ p_n \geq |z_y^{\epsilon_n}| = |a\sqrt{\epsilon_n}\, p_n (1+\delta) C_r(r^{\epsilon_n,\delta}; A_0) + 4(w_p - y_n)^3|$$

In Appendix A, it is shown that $rC_r(r,A) \geq 0$ for any $r \in \mathscr{R}^1$ and $A \neq 0$. Therefore, $C_r(r^{\epsilon_n,\delta}; A_0)$ and $4(w_p - y_n)^3$ have the same sign and the above inequality implies that

$$|C_r(r^{\epsilon_n,\delta}; A_0)| < 1.$$

Hence, by (A.4),

$$|r^{\epsilon_n,\delta}| \leq g(A_0).$$

This inequality implies that $r^{\epsilon_n,\delta}$ remains bounded, independently of n and δ.

6. Since $|C_r(r^{\epsilon_n,\delta}; A_0)| < 1$, by (A.1),

$$A_0^2 C_{rr}(r^{\epsilon_n,\delta}; A_0) + \left(r^{\epsilon_n,\delta} + A_0 C_r(r^{\epsilon_n,\delta}; A_0) \right)^2 = A_0 S(A_0).$$

Therefore, by the estimates of Step 5,

$$A^2 C_{rr}(\cdots) + \left(r^{\epsilon_n,\delta} + AC_r(\cdots) + F_2 \right)^2 \leq A_0 S(A_0) + o(1).$$

Here we have strongly used the fact that $r^{\epsilon_n,\delta}$ is bounded independently of n and δ.

We use this in (B.6). The result is:

$$-w_t - \frac{\sigma^2}{2} p_n^2 w_{pp} - \frac{\sigma^2}{2} p_0^2 w_{pp}(p_0, t_0) S\left(a^2 (1+\delta) p_0^2 w_{pp}(p_0, t_0) \right)$$

$$\leq \frac{\sigma^2}{2(1+\delta)^2 a^2}\ o(1) + F_1 .$$

We complete the proof of the lemma, after letting $n \to \infty$ and then $\delta \downarrow 0$. □

We continue by proving the supersolution property to z_*. As remarked earlier, parts of the following proof is substantially different than the proof of Lemma 4.1.

Lemma B.2. z_* *is a viscosity supersolution of* (1.2)-(1.3) *with* $r = 0$.

Proof. Since z^ϵ is larger than the Black-Scholes price φ minus $\epsilon\alpha^2/2\sigma^2$,

$$z_*(p,T) \geq \varphi(p,T) = (p-q)^+$$

and, therefore, z_* is a supersolution of (1.3).

To prove the viscosity property, we have to show the following: let $w(p,t)$ be a smooth test function and $(p_0,t_0) \in (0,\infty) \times [0,T)$ be a *strict global* minimizer of the difference $z^* - w$ on $[0,\infty) \times [0,T]$, then we need to show that

$$\text{(B.7)} \qquad -w_t - \frac{1}{2}\sigma^2 p^2 w_{pp} \left[1 + S\left(a^2 p^2 w_{pp}\right)\right] \geq 0,$$

at (p_0,t_0).

1. For ϵ and $0 < \delta \ll 1$, set

$$A := A^\delta(p,t) = a^2\ p^2\ (1-\delta)^2\ w_{pp}(p,t),$$

$$A_0 = A^\delta(p_0,t_0),$$

and

$$r^{\epsilon,\delta}(y,p,t) := (1-\delta)ap\ \frac{(w_p(p,t)-y)}{\sqrt{\epsilon}}.$$

As in the proof of Lemma 4.1, we may assume that $w_{pp}(p_0,t_0) \neq 0$ and therefore $A_0 \neq 0$.

Chief difference between this proof and the proof of Lemma 4.1 is this: in Step 3 of that proof, we used the perturbation $|y-w(p,t)|^4$ to construct a sequence of approximate maximizers. For technical reasons that will become clear in Step 3 below, we can not employ such a perturbation technique in this proof. We overcome this difficulty by using Lemma 2.2 and by appropriately truncating C. For this purpose, let $C(r;A)$ be as in Appendix A and let $\chi : \mathscr{R}^1 \to \mathscr{R}^1$ be a smooth concave increasing function satisfying:

$$\chi(t) = t \quad \text{if } t \leq R, \qquad\qquad \frac{d}{dt}\chi(t) = 0 \quad \text{if } t \geq 2R,$$

where $R > C(g(A_0);A_0)$ is chosen so that $\tilde{C} := \chi(C)$ satisfies

$$-A_0^2\tilde{C}_{rr} - (r + A_0\tilde{C}_r(r,A_0))^2 \leq -A_0 S(A_0)\ ,$$

for all r. The cut-off function χ truncates C only in the region where $|C_r| = 1$ while keeping the main properties of C. The existence of χ and R follows from the explicit construction of C given in Appendix A.

Define

$$w^{\epsilon,\delta}(y,p,t) := w(p,t) + \epsilon\tilde{C}(r^{\epsilon,\delta}(y,p,t);A_0\ ).$$

2. Set $2\eta = \min(T-t_0,p_0)$ so that $0 < \eta \leq T$. Consider the function $z^\epsilon - w^{\epsilon,\delta}$ in

$$Q_\eta := \mathscr{R}^1 \times [p_0-\eta, p_0+\eta] \times [t_0-\eta, t_0+\eta]\ .$$

Since $\tilde{C}$ is bounded, Lemma 2.2 implies that the difference $z^\epsilon - w^{\epsilon,\delta}$ has a local minimizer in Q_η. Then, by the arguments of Barles and Perthame [2], there exists a sequence $\epsilon_n \downarrow 0$ and local minimizers $(y_n, p_n, t_n) \in Q_\eta$ of $z^{\epsilon_n} - w^{\epsilon_n,\delta}$ satisfying $(p_n, t_n) \to (p_0, t_0)$, $z^{\epsilon_n}(y_n, p_n, t_n) \to z_*(p_0, t_0)$. Recall that, in the definition of z_*, there are no restrictions on the y variable and, therefore, we only have weak information on $|y_n|$, as n tends to infinity. Indeed, a careful analysis reveals that $\sqrt{\epsilon_n}|y_n|$ or equivalently $\epsilon_n r^{\epsilon_n,\delta}$ remain bounded uniformly in n. However, this estimate will not be used in the subsequent analysis.

3. By calculus, at (y_n, p_n, t_n),

$$|z_y^{\epsilon_n}| = a\sqrt{\epsilon_n}\, p_n(1-\delta)|\tilde{C}_r(r^{\epsilon_n,\delta}; A_0)|.$$

Since $|\tilde{C}_r(r; A_0)| \le 1$ for any $r \in \mathscr{R}^1$,

$$|z_y^{\epsilon_n}| \le a\sqrt{\epsilon_n}\, p_n(1-\delta) < \sqrt{\epsilon_n}\, a p_n = \mu p_n.$$

Hence, by Proposition 2.1,

$$0 \le -z_t^{\epsilon_n} - \frac{1}{2}\sigma^2 p_n^2 z_{pp}^{\epsilon_n} - \frac{1}{2\epsilon_n}\sigma^2 p_n^2 (z_p^{\epsilon_n} - y_n)^2 - \alpha(z_p^{\epsilon_n} - y_n)$$

at (y_n, p_n, t_n).

4. We proceed as in Step 4 of Lemma 4.1. The result is this:

$$\text{(B.8)} \quad -w_t - \frac{\sigma^2}{2} p_n^2 w_{pp}$$

$$\ge \frac{\sigma^2}{2(1-\delta)^2 a^2}\left[A^2 \tilde{C}_{rr}(\cdots) + \left(r^{\epsilon_n,\delta} + A\tilde{C}_r(\cdots) + F_2\right)^2\right] + F_1 \cdot ,$$

where F_1 and F_2 are as in Step 4 of Lemma 4.1 with $(1+\delta)$ replaced by $(1-\delta)$ and without the terms related to $(y - w_p)^4$.

Since $\chi(t)$ is constant for $t > 0$ large, there exists $\tilde{R} > 0$ such that, if $|r| \ge \tilde{R}$, then

$$\tilde{C}_r(r, A_0) = \tilde{C}_{rr}(r, A_0) = 0.$$

By increasing $\tilde{R}$, if necessary, we may assume that

$$\text{(B.9)} \qquad \tilde{R}^2 \ge A_0 S(A_0)\ .$$

Suppose that $|r^{\epsilon_n,\delta}| \le \tilde{R} + 1$ on a subsequence. We estimate the error terms exactly as in the previous lemma and obtain (B.7), by using the properties of $\tilde{C}$.

Now suppose that $|r^{\epsilon_n,\delta}| > \tilde{R} + 1$. Then, by definition of $\tilde{R}$, $\tilde{C}_r = \tilde{C}_{rr} = 0$, and the error terms F_1 and F_2 converge to zero. Hence, the right-hand side of (B.8) is equal to

$$\frac{\sigma^2}{2(1-\delta)^2 a^2}\left(r^{\epsilon_n,\delta} + o(1)\right)^2 + o(1)\ .$$

Since $|r^{\epsilon_n,\delta}| > \tilde{R} + 1$, $|r^{\epsilon_n,\delta} + o(1)| \ge \tilde{R}$ for all sufficiently large n and therefore, by (B.9),

$$\left(r^{\epsilon_n,\delta} + o(1)\right)^2 \geq \tilde{R}^2 \geq A_0 S(A_0)\ .$$

We complete the proof of Lemma 4.1 as in Step 6 of the previous proof. □

The final step in the proof of Theorem 3.1 is to show that $z^* \leq z_*$ on $[0,\infty) \times [0,T]$. We need a comparison result to achieve this.

Proof of Theorem 3.1. We first rewrite the nonlinear Black-Scholes equation (1.2) in the following way

$$\Psi_t + F(p^2\Psi_{pp}) = 0\ ,$$

in $(0,\infty) \times (0,T)$ where

$$F(M) = \frac{1}{2}\sigma^2 M\ \left[1 + S(a^2 M)\right]\ ,$$

for $M \in \mathscr{R}^1$. Since S has a linear growth at infinity (see the properties of S we provide after the statement of Theorem 3.1), F has a quadratic growth at infinity. In particular, F is not a uniformly continuous function in $\mathscr{R}^1$ and therefore we can not immediately use a standard comparison theorem from the theory of viscosity solutions, even if the equation were set in a bounded domain. But, the equation (1.2) is also set in an unbounded domain with unbounded solutions and this is a second difficulty.

We first overcome this second difficulty by using the condition (3.1). For $\eta > 0$, we set

$$z_\eta(p,t) = z^*(p,t) - \eta(p+1)\ ,$$

in $[0,\infty) \times [0,T]$. Then, z_η is still a subsolution of (1.2)-(1.3) and, in view of (3.1),

$$\lim_{p\to\infty} (z_\eta(p,t) - z_*(p,t)) = -\infty\ ,$$

uniformly for $t \in [0,T]$. Moreover, by (B.1), $z_\eta(0,t) = z_*(0,t) - \eta$ for any $t \in [0,T]$. Now, suppose that

$$\max\{\ z^*(p,t) - z_*(p,t)\ :\ (p,t) \in [0,\infty) \times [0,T]\ \} \geq 0.$$

Then, there is a maximizer $(p_0,t_0) \in (0,\infty) \times [0,T]$ of the difference $z_\eta - z_*$. Note that $p_0 > 0$.

To overcome the difficulty coming from the nonlinearity F, we introduce the change of variables $p = e^x$, i.e., for $x \in \mathscr{R}^1$ and $t \in [0,T]$, let

$$u_\eta(x,t) := z_\eta(e^x,t), \qquad u_*(x,t) := z_*(e^x,t),$$

so that, u_η and u_* are, respectively, a supersolution and a subsolution of

$$-u_t - F(u_{xx} - u_x) = 0,$$

in $\mathscr{R}^1 \times (0,T)$. Moreover, $x_0 := \ln p_0$ and t_0 is a maximizer of the difference $u_\eta - u_*$. Now a standard a comparison theorem applies to the preceding equation because the nonlinearity is now independent of x (see Crandall et al. [10]) and implies that

$$u_\eta(x_0,t_0) - u_*(x_0,t_0) \le 0.$$

Therefore $z_\eta \le z_*$ and, by letting η to zero, we arrive at $z^* \le z_*$ on $[0,\infty)\times[0,T]$. Since, by construction, $z_* \le z^*$, we conclude that $z^* = z_*$ and since z^* is an upper-semicontinuous subsolution of (1.2)-(1.3) and z_* is lower-semicontinuous semicontinuous supersolution of (1.2)-(1.3), the function Ψ defined by $\Psi := z_* = z^*$ on $[0,\infty)\times[0,T]$, is a continuous solution of (1.2)-(1.3). Moreover, by classical arguments, the equality $z_* = z^*$ in $[0,\infty)\times[0,T]$ implies the local uniform convergence of z^ϵ to Ψ. □

We close this section by proving Lemma 2.2.

Proof of Lemma 2.2. Fix $\eta > 0$ and let $g : \mathscr{R}^1 \to \mathscr{R}^1$ be a smooth, nondecreasing function satisfying: $g(t) \equiv 0$ if $t \le 0$ and $g(t) \equiv 1$ if $t \ge \eta$. Set

$$\psi(y,p,t) = g(T-t)\mu p|y| - K\epsilon(T-t)\ ,$$

where K is a positive constant to be chosen later.

We compute:

$$
\begin{aligned}
&-\psi_t - \frac{1}{2}\sigma^2 p^2 \psi_{pp} - \frac{1}{2\epsilon}\sigma^2 p^2 (\psi_p - y)^2 \ - \alpha p(\psi_p - y)\\
&= \ -\epsilon K + g'(T-t)\mu p|y| - \frac{1}{2\epsilon}\sigma^2 p^2 (g(T-t)\mu|y| - y)^2\\
&\quad -\alpha p(g(T-t)\mu|y| - y)\\
&\le \ -\epsilon K + g'(T-t)\mu p|y| - \frac{1}{2\epsilon}\sigma^2 p^2 (1-\mu)^2|y|^2 \ + \alpha p|y|.
\end{aligned}
$$

Since

$$g'(T-t)\mu p|y| \le \epsilon \left[\frac{g'(T-t)\mu}{\sigma(1-\mu)}\right]^2 + \frac{1}{4\epsilon}\sigma^2 p^2 (1-\mu)^2 |y|^2\ ,$$

and

$$\alpha p|y| \le \epsilon \frac{\alpha^2}{\sigma^2(1-\mu)^2} + \frac{1}{4\epsilon}\sigma^2 p^2(1-\mu)^2|y|^2,$$

there exists a constant K, depending only on η, so that

$$-\psi_t - \frac{1}{2}\sigma^2 p^2 \psi_{pp} - \frac{1}{2\epsilon}\sigma^2 p^2 (\psi_p - y)^2 \ - \alpha p(\psi_p - y) \le 0\ .$$

Moreover,

$$|\psi_y(y,p,t)| \le \mu p \quad \text{for } y \ne 0\ .$$

If necessary, by increasing K, we may assume that $K \ge \alpha^2/2\sigma^2$ and, by Proposition 2.1,

$$\psi(y,p,t) \le z^\epsilon(y,p,t) \quad \text{if } y = 0 \text{ or } t = T.$$

We now follow the proof of Proposition 2.1 to prove that $\psi \le z^\epsilon$ in $\mathscr{R}^1 \times [0,\infty)\times[0,T]$. Then, the stated lower bound follows from the properties of g. □

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