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REGULARITY AND CONVERGENCE OF CRYSTALLINE MOTION*

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Abstract. We consider the motion of polygons by crystalline curvature. We show that "smooth" polygon evolves by crystalline curvature "smoothly" and that it shrinks to a point in finite time. We also establish the convergence of crystalline motion to the motion by mean curvature.

Key words. crystalline motion, motion by mean curvature, viscosity solutions

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1. Introduction.

Several models in phase transitions give rise to geometric equations relating the normal velocity of the interface to its curvature. The curvature term is related to surface tension and the surface energy is often an anisotropic function of the normal direction, indicating the preferred directions of the underlying crystal structure.

When the surface energy is isotropic, the resulting equation is the mean curvature flow and a variety of techniques have been used to analyze this flow. Huisken [25] showed that any convex set in higher than two space dimensions, shrinks to a point smoothly in finite time. We note that Huisken's method cannot be applied to the planar motion by mean curvature. Using different methods from those in [25], Gage and Hamilton [15] and Grayson [24] showed that a smooth planar embedded curve first becomes convex and then smoothly shrinks to a point in finite time. However, in general, in dimensions higher than two, embedded hypersurfaces may develop singularities and a weak formulation of the mean curvature flow is necessary to define the subsequent evolution after the onset of singularities. Brakke [8] was the first to study the mean curvature flow past the singularities. Using varifolds in geometric measure theory, he constructed global generalized solutions that are not necessarily unique. Almgren, Taylor, and Wang [2] used a time-step energy minimization approach together with geometric measure theory to analyze a very general class of equations.

An alternate approach, initially suggested in the physics literature by Ohta, Jasnaw, and Kawasaki [28], for numerical calculations by Osher and Sethian [26], represents the evolving surfaces as the level set of an auxiliary function solving an appropriate nonlinear differential equation. This level-set approach has been extensively developed by Chen, Giga, and Goto [9] and Evans and Spruck [12]. Evolution of hypersurfaces with codimension greater than one is studied by Ambrosio and Soner [3], and intrinsic definitions were developed by Soner [29] and Barles, Soner, and Souganidis [7]. Since the level-set equations are degenerate parabolic, the theory of viscosity solutions by Crandall and Lions [11] is used to define the level-set solutions. For more

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information on viscosity solutions see the survey by Crandall, Ishii, and Lions [10] and the book by Fleming and Soner [13].

When the surface energy is convex, the evolution law is still degenerate parabolic and much of the above theory generalizes to these equations as well.

Nonsmooth energies are also of interest, and an interesting class of surface energies—called *crystalline* energies—have polygonal Frank diagrams. For these energies, the corresponding solutions are also polygonal, and the evolution law is a system of ordinary differential equations for the length of each side of the solution (see (2.3) below). An excellent introduction to crystalline motion is given in the recent book of Gurtin [22] and in the surveys of Taylor [32] and Taylor, Cahn, and Handwerker [34]. Short time existence and the other properties of the planar solutions are proved by Angenent and Gurtin [4] and Taylor [33]. Almgren and Taylor [1] showed that the crystalline flow is consistent with the variational approach developed in [2]. In a recent preprint Giga, Gurtin, and Mathias [19] study the classical solutions in three space dimensions and a deep viscosity theory for graph-like solutions of very general geometric equations have been developed by Giga and Giga [16] and the references therein. We also refer to Gurtin, Soner, and Souganidis [23] and Ohnuma and Sato [27], which treat a relaxed formulation of evolving surfaces by nonconvex interfacial energies.

In this paper, we consider a two-dimensional problem with a crystalline energy whose level sets are regular n-polygons and show the convergence of these solutions to the unique smooth solution of the mean curvature flow. This convergence has already been proved by Girao [20] for convex solutions and by Girao and Kohn [21] for graph-like solutions. They also obtained the rate of convergence. Here we generalize the convergence results in [20, 21] to general curves that are not necessarily convex. Our proof is a set theoretic analogue of the weak viscosity approach of Barles and Perthame [5, 6]. To describe our approach, let $\{\Omega_n(t)\}_{t\in[0,T)}$ be a sequence of open polygons each solving a crystalline flow. We define two possible limits:

$$\widehat{\Omega}(t) := \limsup_{n\to\infty,\ s\to t} \Omega_n(s),$$
$$\underline{\Omega}(t) := \liminf_{n\to\infty,\ s\to t} \Omega_n(s).$$

(Precise definitions are given in (4.2) below.) Then, with *only L^∞ estimates*, the Barles–Perthame approach enables us to show that $\widehat{\Omega}$ is a viscosity subsolution of the mean curvature flow, and $\underline{\Omega}$ is a viscosity supersolution of the mean curvature flow. Since, in two space dimensions, there is a smooth solution to the mean curvature flow, we show that both of these sets are equal to the smooth solution. This yields the convergence of Ω_n in the Hausdorff topology.

The paper is organized as follows. In the next section, we give the definition of crystalline motion and prove the existence of a regular solution in section 3. We define the weak viscosity limits in section 4 and prove their viscosity properties. Convergence is proved in the final section. Some properties of the viscosity solutions are gathered in the appendix.

After this work was completed, we were informed of a recent work of Giga and Giga [17] related to ours. They proved the stability of the periodic graph-like solutions for the motion by nonlocal weighted curvature. They also proved the motion by crystalline energy is shown to approximate the motion by regular interfacial energy if the crystalline energy approximates the regular interfacial energy. We also refer to Fukui and Giga [14] for an approximation property of the motion by nonsmooth weighted energy.

2. Crystalline motion and n-smooth polygons. Here we recall several standard definitions and equations. Gurtin's book [22] provides an excellent introduction to this subject. Also, see [31, 33].

2.1. Surface energy. All geometric flows that we consider are, formally, the gradient flows of the surface energy functional

$$I(\Gamma) := \int_\Gamma f(\vec{n})\, ds, \tag{2.1}$$

where Γ is a Jordan curve in $\mathcal{R}^2$, $\vec{n}$ is its outward unit normal vector, and $f : S^1 \to [0,\infty)$ is the *surface energy* function. It is customary to extend f to the whole $\mathcal{R}^2$ as a homogeneous function of degree one,

$$f(x) = |x| f\left(\frac{x}{|x|}\right) \qquad \forall x \neq 0,$$

and define

$$\hat{f}(\theta) := f(\cos\theta, \sin\theta).$$

Then the twice differentiability of f on $\mathcal{R}^2\backslash\{0\}$ is equivalent to the twice differentiability of $\hat{f}$, and f is convex if and only if $\hat{f}(\theta) + \hat{f}_{\theta\,\theta}(\theta) \geq 0$ for all θ.

The *Frank diagram* of the surface energy f is simply the polar graph of $\hat{f}^{-1}$, or equivalently, it is the one-level set of f, i.e.,

$$\mathcal{F}(f) := \{x \in \mathcal{R}^2 \ : \ f(x) = 1\} = \{r\,(\cos\theta, \sin\theta) \ : \ r\hat{f}(\theta) = 1\}.$$

When the surface tension f is smooth and convex, the gradient flow for the functional I has the form

$$\beta(\theta) V = (\hat{f}(\theta) + \hat{f}_{\theta\,\theta}(\theta))\ \kappa, \tag{2.2}$$

where V, κ, $(\cos\theta, \sin\theta)$ are, respectively, the normal velocity, the curvature, and the normal vector of the solution $\Gamma(t)$, and the given nonnegative function β is the kinetic coefficient. The mean curvature flow corresponds to $\hat{f} \equiv \beta \equiv 1$, and the other cases with strictly convex surface energy are qualitatively very similar to the mean curvature flow.

If f is not convex, we need to modify *both* f and β to obtain the correct relaxed equation. This relaxation procedure and the analytical properties of the relaxed equation was studied by Gurtin, Soner, and Souganidis [23] and, independently, by Ohnuma and Sato [27]. The common critical hypothesis in these works is the continuous differentiability of the relaxed surface energy function.

2.2. Crystalline flow. Nonsmooth energy functions are of interest in models for crystal growth, as it is well known that solid crystals can exist in polygonal shapes. An interesting class of nonsmooth energies are the *crystalline* energies. The Frank diagram of crystalline energy is a polygon.

Although the crystalline energies are only Lipschitz continuous, an appropriate weak formulation of (2.2) is possible and is called the crystalline flow; see [22, section 12.5] for the precise definition. The crystalline flow was derived by Taylor [31] and, independently, from thermodynamical considerations by Angenent and Gurtin [4].

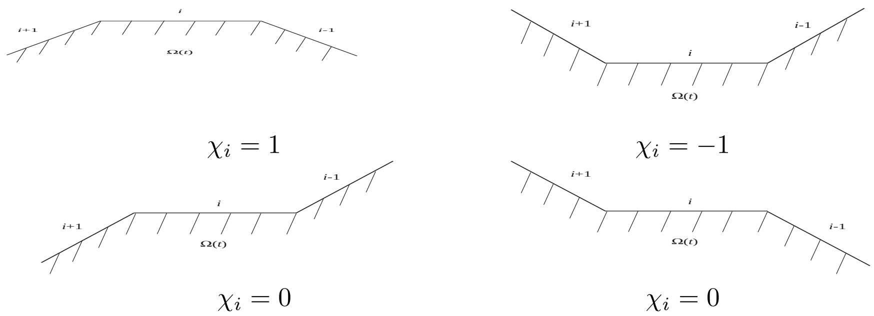

FIG. 1. *Definition of* χ_i.

Consider a crytalline energy function f, and let $\Theta := \{\theta_1, \dots, \theta_N\}$ be the angles corresponding to the corner points of the Frank digram of f. Suppose that the curve Γ is locally smooth around a point with a normal angle $\theta^* \notin \Theta$—say, $\theta^* \in (\theta_1, \theta_2)$. We can, then, decrease the energy $I(\Gamma)$ of Γ by infinitesimally alternating the normal angle between θ_1 and θ_2. Therefore, for crystalline energies, we consider only polygonal solutions with normal angles taking values in Θ.

In this paper, for simplicity, we consider only crystalline energies whose Frank diagrams are regular n-polygons, and kinetic coefficient $\beta \equiv 1$. Then

$$\Theta = \Theta_n := \left\{ \frac{2\pi k}{n} \ : \ k = 0, 1, \dots, (n-1) \right\}.$$

Here and hereafter $\theta \in \Theta$ means $\theta \equiv 2\pi k/n \mod 2\pi$ for some $k \in \{0, 1, \dots, n-1\}$. The evolution of side i, $L_i(t)$, is governed by

$$V_i(t) = -\frac{2\tan(\pi/n)}{l_i(t)}\, \chi_i, \tag{2.3}$$

where $V_i(t)$, $l_i(t)$, and χ_i, are, respectively, the normal velocity, the length, and the discrete curvature of $L_i(t)$. The discrete curvature $\chi_i \in \{-1, 0, +1\}$. It is equal to $+1$ if both edges of $L_i(t)$ have positive curvature, it is equal to -1 if both edges of $L_i(t)$ have negative curvature, and it is equal to zero otherwise; see Figure 1. ($\Omega(t)$ denotes the domain enclosed by $L_i(t)'s$.)

We close this subsection by stating the evolution rule for the length, $l_i(t)$, of the sides of a solution of the crystalline flow:

$$\frac{d}{dt} l_i(t) = \frac{1}{\cos^2(\pi/n)} \left(2\cos\left(\frac{2\pi}{n}\right) \cdot \frac{\chi_i^2}{l_i(t)} - \frac{\chi_{i+1}^2}{l_{i+1}(t)} - \frac{\chi_{i-1}^2}{l_{i-1}(t)} \right). \tag{2.4}$$

This equation follows from (2.3) and geometry; see [22, equation (12.39)].

2.3. *n*-smooth polygons. We continue by defining the notion of a "good" solution of (2.3). For a polygon Γ, let $N(\Gamma)$ be the total number of sides.

DEFINITION 2.1. *We say that a closed polygon Γ is an n-smooth polygon if $N(\Gamma)$ is finite and*

(1) Γ *encloses a simply-connected, bounded, open subset of* $\mathcal{R}^2$,
(2) *for every* $i = 1, \dots, N(\Gamma)$, *the normal angle* θ_i *of the side* i *belongs to* Θ_n,
(3) $|\theta_i - \theta_{i-1}| = 2\pi/n$ *for every* $i = 1, \dots, N(\Gamma)$, *where* $|\theta_i - \theta_{i-1}|$ *is understood as the infimum over its representatives.*

The third condition is formally equivalent to the "discrete continuity" of the normal angle, which explains the term "smooth."

By definition, any solution of (2.3) satisfies the second condition.

Let

$$N^+(\Gamma) := \{i \in \{1, \ldots, N(\Gamma)\} \ : \chi_i = 1\},$$

$$N^-(\Gamma) := \{i \in \{1, \ldots, N(\Gamma)\} \ : \chi_i = -1\},$$

$$N^0(\Gamma) := \{i \in \{1, \ldots, N(\Gamma)\} \ : \chi_i = 0\}.$$

Then for any n-smooth polygon Γ,

$$N^+(\Gamma) - N^-(\Gamma) = \sum_{i=1}^{N(\Gamma)} \chi_i = n \tag{2.5}$$

is an identity which is the discrete version of

$$\int_C \kappa \, ds = 2\pi$$

for a smooth Jordan curve C.

3. Regularity. In this section, we will show that there is a unique n-smooth solution of (2.3) which evolves smoothly in time (i.e., remains n-smooth) and shrinks to a point in finite time. This is the discrete analogue of a theorem of Grayson [24] and Gage and Hamilton [15]. A more general statement is proved by Taylor [33, Theorem 3.1]. For the reader's convenience, we provide all the details of this result.

THEOREM 3.1 (Taylor [33]). *Let Γ_0 be an n-smooth polygon enclosing an open set Ω_0. Then there exist n-smooth polygons $\{\Gamma(t)\}_{t\in[0,T)}$ solving* (2.3) *with the initial condition $\Gamma(0) = \Gamma_0$. Moreover $\Gamma(t)$ shrinks to a point as $t \uparrow T$, and*

$$T = \frac{|\Omega_0|}{2n\tan(\pi/n)}. \tag{3.1}$$

Remark 3.2. Uniqueness follows from Giga and Gurtin [18] and Taylor [33].

We start with several results toward the proof of Theorem 3.1.

Clearly, for a short time there is a solution $\Gamma(t)$ satisfying initial data. Let $t_1 > 0$ be the first time this solution is no longer n-smooth. Since, by definition, the normal angles of any solution take values in Θ_n (cf. section 2.2), there are two possibilities at t_1: either the length of one or more sides tend to zero or the solution self-intersects at t_1. We will first show that the latter does not happen. Our proof is very similar to [33, Theorem 3.2(1)].

LEMMA 3.3. *Let t_1 and $\{\Gamma(t) = \partial\Omega(t)\}_{t\in[0,t_1)}$ be as above. Then*

$$\liminf_{t\uparrow t_1} \inf\{l_i(s) \ : \ s \in [0,t], \ i = 1, \ldots, N(\Gamma(0))\} = 0.$$

Proof. Suppose the opposite. Then

$$\inf\{l_i(s) \ : \ s \in [0,t_1), \ i = 1, \ldots, N(\Gamma(0))\} > 0.$$

Then, by (2.4), each $l_i(\cdot)$ is smooth on $(0, t_1)$ and therefore

$$\Omega(t_1) = \lim_{t \uparrow t_1} \Omega(t)$$

exists in the Hausdorff topology. By the definition of t_1, $\Gamma(t_1)$ self-intersects. Moreover, for all $t \in [0, t_1]$,

$$|\theta_i - \theta_{i-1}| = \frac{2\pi}{n},\ i = 1, \ldots, N(\Gamma(t)) = N(\Gamma(0)), \tag{3.2}$$

so that at t_1 there are two possibilities: either two sides or two corner points touch each other. Note that, by (3.2), if a corner point touches a side, then necessarily two sides also touch each other. The following arguments are very similar to those in [18].

Case 1. Suppose that $L_i(t_1)$ intersects at $L_j(t_1)$.

Then a straightforward analysis argument shows that $(\chi_i, \chi_j) = (1, -1)$ or $(\chi_i, \chi_j) = (-1, 1)$. Since the analyses of these cases are symmetric, we may assume $(\chi_i, \chi_j) = (1, -1)$. Then $l_i(t_1) \leq l_j(t_1)$.

Subcase (1). $l_i(t_1) < l_j(t_1)$.

Then for some $\delta > 0$, $l_i(t) < l_j(t)$ in $(t_1 - \delta, t_1]$, and therefore,

$$\alpha(t) := \frac{2\tan(\pi/n)}{l_j(t)} - \frac{2\tan(\pi/n)}{l_i(t)} > 0, \qquad t \in (t_1 - \delta, t_1].$$

But $\alpha(t)$ is equal to the time derivative of the distance between $L_i(t)$ and $L_j(t)$ and this distance is equal to zero at t_1. Hence this case is not possible.

Subcase (2). $l_i(t_1) = l_j(t_1)$.

Then, the sides adjacent to $L_i(t)$ and $L_j(t)$ also touch each other at time t_1, and therefore, there have to be two sides satisfying the assumptions of the previous subcase, thus yielding a contradiction.

Case 2. Two corner points touch each other.

Let the intersection, $x_i(t)$ of $L_i(t)$ and $L_{i+1}(t)$ be the same as the intersection $x_j(t)$ of the sides $L_{j-1}(t)$ and $L_j(t)$. Then the angle between $L_i(t)$ and $L_j(t)$ and the one between $L_{i+1}(t)$ and $L_{j-1}(t)$ are equal to $2\pi/n$. By rotation, we may assume that $L_i(t)$ and $L_j(t)$ are parallel to the x-axis, and $L_{i+1}(t)$ is aligned with the $L_{j-1}(t)$ (cf. Figure 2). Moreover, $\chi_k \geq 0$ for $k = i, i+1, j, j-1$. Let $V_{x_i}(t)$ and $V_{x_j}(t)$ be the velocity vectors of the points $x_i(t)$ and $x_j(t)$, respectively. Then

$$(0, 1) \cdot (V_{x_j} - V_{x_i}) \geq 0,$$

and the inequality is strict unless $\chi_k = 0$ for all $k = i, i+1, j, j-1$. Since $x_i(t_1) = x_j(t_1)$, we conclude that $\chi_k = 0$ for all $k = i, i+1, j, j-1$. But then $V_{x_i}(t) = V_{x_j}(t) = 0$ for $t < t_1$ close to t_1 and this contradicts the definition of t_1. ∎

Our next result is the following lemma.

LEMMA 3.4. *Let t_1 and $\{\Gamma(t) = \partial\Omega(t)\}_{t \in [0,t)}$ be as above. Suppose t_1 is strictly less than the extinction time. Then as $t \to t_1$, $\Omega(t)$ converges to an n-smooth polygon $\Omega(t_1)$ in the Hausdorff topology.*

Proof. By the previous lemma, there is a side i^* such that

$$\liminf_{t \to t_1} l_{i^*}(t) = 0.$$

The main step in this proof is to show $\chi_{i^*} = 0$ if the side L_{i^*} disappears at t_1. So we suppose that it is equal to $+1$ or -1. Since the analyses of these cases are similar, we may assume that $\chi_{i^*} = 1$. Set $\theta = 2\pi/n$.

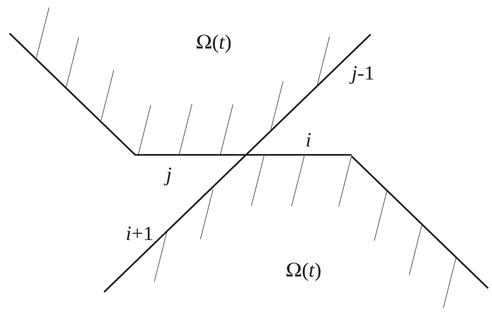

FIG. 2. *Case* 2.

1. In this step we will show that $l_{i^*}(\cdot)$ is continuous on $[0,t_1]$. For future reference, we will prove that, for any j, $l_j(\cdot)$ is continuous on $[0,t_1]$. By (2.4), all sides remain bounded, and we set

$$B := \limsup_{t\to t_1} l_j(t).$$

Suppose that

$$B > \liminf_{t\to t_1} l_j(t) := A.$$

Since $l_j(\cdot)$ is continuous in $[0,t_1)$, it crosses $(A+B)/2$ infinitely many times before t_1. In particular, by the mean value theorem, there is a sequence $t_k \uparrow t_1$ such that

$$l_j(t_k) \geq \frac{A+B}{2}, \quad \lim_{k\to+\infty} l_j'(t_k) = +\infty.$$

However, by (2.4),

$$l_j'(t_k) \leq \frac{2\cos\theta}{l_j(t_k)\cos^2(\theta/2)} \leq C$$

for some constant C independent of k. Hence $A = B$.

2. This step closely follows [33, Proposition 3.1].

Since t_1 is strictly less than the extinction time, there are at least two sides which have nonzero length at time t_1. Hence there are two sides L_{p_0} and L_{p_1} such that $p_0 < i^* < p_1$, $l_{p_0}(t)$ and $l_{p_1}(t)$ are uniformly positive in $[0,t_1]$, and

$$\lim_{t\uparrow t_1} l_j(t) = 0 \ \forall j = p_0+1,\ldots,p_1-1.$$

For any j, let $\mathcal{L}_j(t)$ be the line extending $L_j(t)$, $x_{j+1}(t)$ be the intersection between $\mathcal{L}_j(t)$ and $\mathcal{L}_{j+1}(t)$, and θ_j be the angle between the outward normal and the horizontal axis. Then, as $t \uparrow t_1$, all $x_{p_0+1}(t),\ldots,x_q(t)$ converge to the same point x^*.

We analyze several cases separately.

Case 1. $\chi_j \neq 0 \ \forall\ j = p_0+1,\ldots,p_1-1$.

Since we have assumed that $\chi_{i^*} = 1$, $\chi_j = 1 \ \forall\ j = p_0+1,\ldots,p_1-1$ and

$$x^* \in \bigcap_{0\leq t<t_1} \bigcap_{j=p_0}^{p_1} \{y \in \mathcal{R}^2 \ :\ (y - x_j(t))\cdot(\cos\theta_j, \sin\theta_j) \leq 0\}.$$

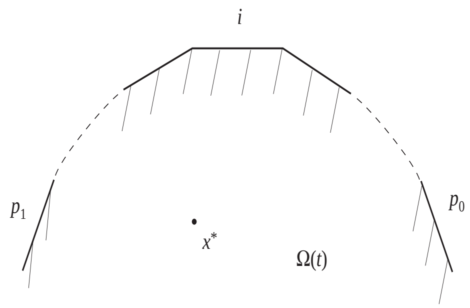

FIG. 3. *Position of* x^*.

See Figure 3.

By geometry, $|\theta_{p_0} - \theta_{p_1}| \le \pi$.

Subcase 1. $|\theta_{p_0} - \theta_{p_1}| < \pi$.

Let $y(t)$ be the intersection between $\mathcal{L}_{p_0}(t)$ and $\mathcal{L}_{p_1}(t)$. We define

$$d(t) = (y(t) - x^*) \cdot (\cos\theta_{p_0+1}, \sin\theta_{p_0+1}),$$
$$d_{p_0+1}(t) = \mathrm{dist}(x^*, \mathcal{L}_{p_0+1}(t)).$$

Then $d_{p_0+1}(t) \le d(t)\ \forall\ t \in [0, t_1)$ and $d_{p_0+1}(t_1) = d(t_1) = 0$. Moreover, $d(t)$ is Lipschitz continuous in t and

$$\frac{d}{dt} d_{p_0+1}(t) = V_{p_0+1}(t) = -\frac{2\tan(\theta/2)}{l_{p_0+1}(t)}.$$

Hence,

$$0 \ge -\int_t^{t_1} \frac{2\tan(\theta/2)}{l_{p_0+1}(\tau)} d\tau = d_{p_0+1}(t) \ge d(t) \ge -\|d'\|_{L^\infty(0,t_1)}(t_1 - t) \qquad \forall t < t_1.$$

This contradicts the fact $l_{p_0+1}(t) \to 0$ as $t \uparrow t_1$.

Subcase 2. $|\theta_{p_0} - \theta_{p_1}| = \pi$.

We repeat the argument used in the previous case with

$$\tilde{d}(t) = \mathrm{dist}(\mathcal{L}_{p_0}(t), \mathcal{L}_{p_1}(t)),$$
$$\tilde{d}_{p_0+1}(t) = \mathrm{dist}(L_{p_0+1}(t), \mathcal{L}_{p_1}(t)).$$

Case 2. $\chi_q = 0$ exactly for one $q \in \{p_0 + 1, \dots, p_1 - 1\}$.

Then, $\chi_j = 1$ for $j = p_0 + 1, \dots, q - 1$ and $\chi_j = -1$ for $j = q + 1, \dots, p_1 - 1$, or $\chi_j = -1$ for $j = p_0 + 1, \dots, q - 1$ and $\chi_j = 1$ for $j = q + 1, \dots, p_1 - 1$. Since the arguments in both cases are similar, without loss of generality, we consider only the first possibility.

If $|\theta_{p_0} - \theta_q| \le \pi$, we argue as in *Case* 1, using side $L_q(t)$ instead of $L_{p_1}(t)$. We also argue similarly, when $|\theta_q - \theta_{p_1}| \le \pi$. Therefore, we may assume that $|\theta_{p_0} - \theta_q| > \pi$ and that there is a side $L_j(t)$ with $q < j < p_1$, which is parallel to $L_{p_0}(t)$. Let $\mathcal{L}$ be the line going through x^* and parallel to both $L_{p_0}(t)$ and $L_j(t)$. Set

$$d(t) = \mathrm{dist}(L_{p_0}(t), \mathcal{L}) - \mathrm{dist}(L_j(t), \mathcal{L}).$$

Then $0 = d(t_1)$ and since $|\theta_{p_0} - \theta_q| > \pi$, $0 < d(t)\ \forall\ (0, t_1)$; see Figure 4.

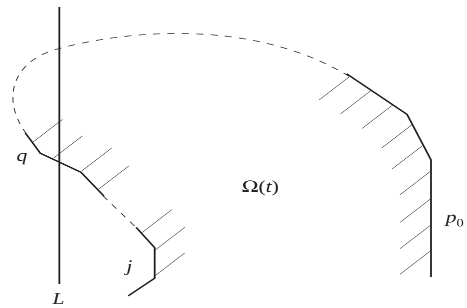

FIG. 4. *Case* 2.

However, this contradicts the fact that $d'(t) > 0 \; \forall \; t$ sufficiently close to t_1.

Case 3. $\chi_j = 0$ for more than one side.

Suppose that χ_q and χ_j are equal to zero. Then x^* belongs to both $L_q(t)$ and $L_j(t) \; \forall \; t$, and therefore, $j = q - 1$ or $q + 1$. Since $l_q(t)$ converges to zero, at least one side adjacent to $L_q(t)$ has nonzero discrete curvature. Hence there are two sides with zero discrete curvature and they are adjacent to each other. As in *Case* 1, all the other sides between $L_{p_0}(t)$ and $L_{p_1}(t)$ satisfy $\chi_k = 1$, and we argue as in *Case* 1.

Therefore, the case $\chi_{i^*} = 1$ is not possible. An entirely similar argument shows that the case $\chi_{i^*} = -1$ is not possible either. Hence $\chi_{i^*} = 0$ and L_{i^*-1} and L_{i^*+1} are parallel, and the normal angle of the "new" side is equal to that of these two ones. □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Since $\Gamma(0)$ is n-smooth for short time, there is an n-smooth solution $\Gamma(t)$. Moreover, by Lemma 3.4, this solution remains n-smooth until one side of $\Gamma(t)$ vanishes. Let t_1 be the first time a side vanishes. Then, $\Gamma(t)$ is n-smooth and $N(\Gamma(t)) = N(\Gamma(0)) \; \forall \; t \in [0, t_1)$. By Lemma 3.3, $\Gamma(t_1)$ is also n-smooth and $N(\Gamma(t_1)) \leq N(\Gamma(0)) - 2$. We repeat this procedure starting from $\Gamma(t_1)$. Since $N(\Gamma(0))$ is finite, we have only to repeat finitely many times.

Let $t_1 < t_2 < \cdots < t_N$ be the times at which a side vanishes. Let $t_N > 0$ be the time when $N^-(\Gamma(t_N)) = N^0(\Gamma(t_N)) = 0$. Then, by (2.5), $N^+(\Gamma(t_N)) = n$ and $\Gamma(t)$ is convex for all $t \geq t_N$.

We see that $\Gamma(t)$ shrinks to a point at finite time. Indeed, by (2.5), we can calculate the rate of change of $|\Omega(t)|$:

$$\begin{aligned} \frac{d}{dt}|\Omega(t)| &= \sum_i V_i l_i \\ &= -\sum_{i \in N^+(\Gamma(t))} 2\tan\frac{\pi}{n} + \sum_{i \in N^-(\Gamma(t))} 2\tan\frac{\pi}{n} \\ &= -2n\tan\frac{\pi}{n}. \end{aligned}$$

From the foregoing calculation, we conclude that the solution shrinks to a point at some time T. Moreover, at time T,

$$0 = |\Omega(T)| = |\Omega_0| - 2n\tan\frac{\pi}{n} \cdot T,$$

and (3.1) follows. □

4. Weak viscosity limits. In this section, we will study the properties of the set-theoretic analogue of the weak viscosity limits of Barles and Perthame [5, 6]. Let $\{\Gamma_n(t)\}_{t\in[0,T)}$ be a sequence of n-smooth solutions of (2.3), and let $\Omega_n(t)$ be the open set enclosed by $\Gamma_n(t)$. Assume that there is a constant $R > 0$, independent of n, satisfying

$$\Omega_n(t) \subset B(0, R), \tag{4.1}$$

where $B(x, r) = \{y \in \mathcal{R}^2 \ : \ |y - x| \le r\}$. Following [6, 29], for $t \in [0, T)$, we define

$$\widehat{\Omega}(t) := \bigcap_{\substack{r>0 \\ N\ge 1}} \mathrm{cl} \left(\bigcup_{\substack{|s-t|\le r,\ 0\le s<T \\ n\ge N}} \Omega_n(s) \right), \tag{4.2}$$

$$\underline{\Omega}(t) := \bigcup_{\substack{r>0 \\ N\ge 1}} \mathrm{int} \left(\bigcap_{\substack{|s-t|\le r,\ 0\le s<T \\ n\ge N}} \Omega_n(s) \right),$$

where $\mathrm{cl}\,A$ and $\mathrm{int}\,A$ are, respectively, the closure and the interior of the set A. In view of (4.1), $\widehat{\Omega}(t)$ is a bounded closed set and $\underline{\Omega}(t)$ is a bounded open set. We will show that, respectively, $\widehat{\Omega}(t)$ is a weak subsolution and $\underline{\Omega}(t)$ is a weak supersolution of the mean curvature flow.

This type of stability results are typical in the theory of viscosity solutions and, in general, they are a simple consequence of the maximum principle. However, the crystalline flow is not defined for smooth curves and this fact is the major difficulty in the following analysis.

The notion of viscosity solutions we use is first introduced by the second author in [29] and further developed in [7, 30]. Here we only recall the definition; other relevant definitions and results are gathered in the appendix.

We continue by recalling several definitions that will be used in the subsequent analysis. For subsets $\{\Omega(t)\}_{0\le t<T}$ in $\mathcal{R}^2$, the *upper semicontinuous* (*u.s.c.*) *envelope* and, respectively, the *lower semicontinuous* (*(l.s.c.*) *envelope* are defined by

$$\Omega^*(t) = \bigcap_{r>0} \mathrm{cl} \left(\bigcup_{\substack{|s-t|\le r \\ 0\le s<T}} \Omega(s) \right), \quad \Omega_*(t) = \bigcup_{r>0} \mathrm{int} \left(\bigcap_{\substack{|s-t|\le r \\ 0\le s<T}} \Omega(s) \right), \quad t \in [0, T).$$

Then, it is clear that $(\underline{\Omega})_* = \underline{\Omega}$ and $(\widehat{\Omega})^* = \widehat{\Omega}$. For other properties of these envelopes, see [29, Lemma 3.1].

For a collection of closed subsets $\{O(t)\}_{0\le t<T}$ with smooth boundary, $V_O(x, t)$ is the normal velocity of $\partial O(t)$ at x and $\kappa_O(x, t)$ is the curvature of $\partial O(t)$ at x. We use the convention that the curvature of a convex curve is nonnegative.

We are now in a position to give the weak (viscosity) definition of the mean curvature flow we will use. This definition is very similar to the one given in [29]; see the appendix for the connection between these two definitions.

Definition 4.1. *Let $\{\Omega(t)\}_{0\le t<T}$ be a collection of bounded subsets in $\mathcal{R}^2$ satisfying $\Omega_*(t) \ne \emptyset$ for every $t \in [0, T)$.*

We say $\{\Omega(t)\}_{0\le t<T}$ is a *weak subsolution* of the mean curvature flow, if for any closed, smooth subsets $\{O(t)\}_{0\le t<T}$,

$$V_O(x_0, t_0) \le -\kappa_O(x_0, t_0) \tag{4.3}$$

at each $t_0 \in (0,T)$ and $x_0 \in \partial O(t_0)$ satisfying

$$\Omega^*(t) \subset\subset O(t) \qquad \forall\, t \neq t_0, \tag{4.4}$$

$$\Omega^*(t_0) \subset O(t_0), \quad \partial\Omega^*(t_0) \cap \partial O(t_0) = \{x_0\}.$$

Similarly, we say $\{\Omega(t)\}_{0\le t<T}$ is a *weak supersolution* of the mean curvature flow if for any closed, smooth subsets $\{O(t)\}_{0\le t<T}$,

$$V_O(x_0,t_0) \ge -\kappa_O(x_0,t_0)$$

at each $t_0 \in (0,T)$ and $x_0 \in \partial O(t_0)$ satisfying

$$O(t) \subset\subset \Omega_*(t)\ \forall\, t \neq t_0, \quad O(t_0) \subset \Omega_*(t_0), \quad \partial\Omega_*(t_0) \cap \partial O(t_0) = \{x_0\}.$$

Condition (4.4) implies that $(x_0,t_0) \in \partial O(t_0) \times (0,T)$ is the *strict* maximizer of $-\mathrm{dist}(x, \partial\Omega^*(t))$ over all $(x,t) \in \partial O(t) \times (0,T)$. A similar conclusion also holds for supersolutions.

Following is the set theoretic analogue of the Barles and Perthame procedure [5, 6], [13, section 5], and it is the chief technical contribution of this paper.

Recall that $\Gamma_n(t) = \partial\Omega_n(t)$.

LEMMA 4.2. *$\widehat{\Omega}$ is a weak subsolution of the mean curvature flow, while $\underline{\Omega}$ is a weak supersolution.*

Before we give the proof of this lemma, we will first give a formal proof of the subsolution property.

In view of our definition of a weak solution, we start with smooth sets $\{O(t)\}_{0<t<T}$ and a point (x_0,t_0) satisfying (4.4). Our goal is to verify (4.3). By (4.4) there are a subsequence n_k and a sequence $(x_k,t_k) \to (x_0,t_0)$ satisfying $\Omega_{n_k}(t_k) \subset O(t_k)$ and that $x_k \in \Gamma_{n_k}(t_k)$. Although there are several other cases, assume that x_k is the intersection of $L_{i-1}(t_k)$ and $L_i(t_k)$ of $\Gamma_{n_k}(t_k)$, and $\chi_i = \chi_{i-1} = 1$. We choose a coordinate system so that x_k is the origin and the $L_i(t_k)$ side is included in the x_1-axis. Let $n_1 = (0,1)$, $n_2 = (\sin(2\pi/n_k), \cos(2\pi/n_k))$. Then, the unit normal vector of ∂O satisfies $n_O(x_k,t_k) = (\sin\alpha, \cos\alpha)$ for some $0 < \alpha < 2\pi/n_k$. By the crystalline equation (2.3),

$$V_{x_k} \cdot n_1 = V_i = -\frac{2\tan(\pi/n_k)}{l_i},$$
$$V_{x_k} \cdot n_2 = V_{i-1} = -\frac{2\tan(\pi/n_k)}{l_{i-1}},$$

and therefore,

$$V_{x_k} = 2\tan\frac{\pi}{n_k}\left(\frac{1}{\tan(2\pi/n_k)}\left(\frac{1}{l_i} - \frac{1}{l_{i-1}}\right), -\frac{1}{l_i}\right), \tag{4.5}$$

$$\begin{aligned} V_O(x_k,t_k) &= V_{x_k} \cdot n_O(x_k,t_k) \\ &= -\frac{1}{\cos^2(\pi/n_k)}\left(\frac{\sin(2\pi/n_k - \alpha)}{l_i} + \frac{\sin\alpha}{l_{i-1}}\right). \end{aligned} \tag{4.6}$$

Since $V_O(x_k,t_k) < 0$, we may assume $\inf_{k\in\mathcal{N}} \kappa_O(x_k,t_k) > 0$. This implies that, as $k \to \infty$, both l_i and l_{i-1} converge to zero. By elementary geometry, we obtain a sharper estimate: for every $\varepsilon > 0$,

$$l_i \le \frac{2\sin\alpha}{\kappa_O(x_k,t_k)-\varepsilon}, \qquad l_{i-1} \le \frac{2\sin(2\pi/n_k-\alpha)}{\kappa_O(x_k,t_k)-\varepsilon}$$

for sufficiently large k. Substitute these into (4.6):

$$\begin{aligned} V_O(x_k,t_k) &\le -\frac{\kappa_O(x_k,t_k)-\varepsilon}{2\cos^2(\pi/n_k)}\left(\frac{\sin(2\pi/n_k-\alpha)}{\sin\alpha}+\frac{\sin\alpha}{\sin(2\pi/n_k-\alpha)}\right) \\ &\le -\kappa_O(x_k,t_k)+\varepsilon. \end{aligned}$$

In the foregoing argument, we crucially used the assumption that x_k is a "convex" corner point of Γ_{n_k}. Although this is the most likely situation, other cases may also arise, and for that we will perturb the test sets O in the preceding proof.

Proof. We will prove only the subsolution property. Proof of the supersolution case is similar.

Let $\{O(t)\}_{0<t<T}$ and (t_0,x_0) be as in (4.4). Our goal is to verify (4.3), i.e.,

$$v := V_O(x_0,t_0) \le -\kappa := -\kappa_O(x_0,t_0).$$

If necessary, by perturbing $O(\cdot)$, we may assume that $\kappa \neq 0$. We analyze two cases separately.

Case 1. $\kappa > 0$.

For $\varepsilon > 0$, $x^* \in \mathcal{R}^2$, and a large constant K, let $D^\varepsilon(t:x^*)$ be the disk with center x^* and radius

$$R^\varepsilon(t) = \frac{1}{\kappa-\varepsilon} + v(t-t_0) + K(t-t_0)^2.$$

Set

$$x_0^\varepsilon := x_0 - R^\varepsilon(t_0)n_O(x_0,t_0).$$

By the smoothness of ∂O, for all sufficiently large K, there is a δ^ε such that

$$O(t)\cap B(x_0,2\delta^\varepsilon) \subset D^\varepsilon(t:x_0)\cap B(x_0,2\delta^\varepsilon) \tag{4.7}$$

for all $|t-t_0| \le 2\delta^\varepsilon$. We fix K large enough so that the above inequality holds.

Next we approximate $D^\varepsilon(t:x^*)$ by regions with polygonal boundaries. Let

$$C_n := \left\{ x\in\mathcal{R}^2 \;:\; x\cdot\left(\cos\left(\frac{2l\pi}{n}\right),\sin\left(\frac{2l\pi}{n}\right)\right) \le 1 \quad \forall\, l=0,1,\ldots,(n-1) \right\},$$

and, for any x^*, set

$$D_n^\varepsilon(t:x^*) := \{x^*\}\oplus R^\varepsilon(t)C_n.$$

Since $D_n^\varepsilon(\cdot:x_0^\varepsilon)$ approximates $D^\varepsilon(\cdot:x_0^\varepsilon)$, by (4.4) and (4.7), there are a subsequence n_k and sequences $(x_k,t_k)\to(x_0,t_0)$, $y_k\to x_0^\varepsilon$ satisfying

$$x_k \in \Gamma_{n_k}(t_k)\cap\partial D_{n_k}^\varepsilon(t_k:y_k),$$

$$\Omega_{n_k}(t) \cap B(x_0, \delta^\varepsilon) \subset D^\varepsilon_{n_k}(t : y_k) \cap B(x_0, \delta^\varepsilon) \qquad \forall\ |t - t_0| \le \delta^\varepsilon.$$

A proof of this fact is given in the appendix in Lemma 6.2. To simplify the notations, we assume that $n_k = k$ and write $D_k(t)$ for $D^\varepsilon_{n_k}(t : y_k)$.

Let x_k be on the ith side of $\Gamma_k(t_k)$. Then the normal velocity, V_i, of this side is equal to the normal velocity of D_k at t_k. Hence,

$$V_i = v + 2K(t_k - t_0).$$

Since $D_k(t_k)$ is a regular k-polygon, $\chi_i(t_k) = 1$ and, therefore, the length $l_i(t_k)$ of side i of $\Gamma_k(t_k)$ is less than or equal to the length of any side of $D_k(t_k)$:

$$l_i(t_k) \le 2R^\varepsilon(t_k) \sin \frac{\pi}{k}.$$

Then, by (2.3) and the foregoing discussion,

$$v + 2K(t_k - t_0) = V_i = -\frac{2\tan(\pi/k)}{l_i(t_k)} \le -\frac{1}{R^\varepsilon(t_k)\cos(\pi/k)}.$$

Since $R^\varepsilon(t_k)$ converges to $1/\kappa$ and $t_k \to t_0$, we obtain (4.3) by first letting $k \to \infty$ and then $\varepsilon \downarrow 0$.

Case 2. $\kappa < 0$.

For small $\varepsilon > 0$ and any x^*, let $x^\varepsilon_0 := x_0 + R^\varepsilon(t_0) n_O(x_0, t_0)$, and let $D^\varepsilon(t : x^*)$ be the complement of the disk with center x^*, radius

$$R^\varepsilon(t) = \frac{1}{-\kappa + \varepsilon} + v(t - t_0) - K(t - t_0)^2.$$

As in the previous case, there is a δ^ε such that

$$O(t) \cap B(x_0, 2\delta^\varepsilon) \subset D^\varepsilon(t : x^\varepsilon_0) \cap B(x_0, 2\delta^\varepsilon) \tag{4.8}$$

$\forall\ |t - t_0| \le 2\delta^\varepsilon$, and for any x^*, we set

$$D^\varepsilon_n(t : x^*) := \mathcal{R}^2 \setminus \{x^*\} \oplus R^\varepsilon(t) C_n.$$

Then, $D^\varepsilon_n(\cdot : x_0)$ approximates $D^\varepsilon(\cdot : x_0)$, and by (4.4) and (4.8), there are a subsequence n_k and sequences $(x_k, t_k) \to (x_0, t_0)$, $y_k \to x^\varepsilon_0$ satisfying

$$x_k \in \Gamma_{n_k}(t_k) \cap \partial D^\varepsilon_{n_k}(t_k : y_k),$$

$$\Omega_{n_k}(t) \cap B(x_0, \delta^\varepsilon) \subset D^\varepsilon_{n_k}(t : y_k) \cap B(x_0, \delta^\varepsilon) \qquad \forall\ |t - t_0| \le \delta^\varepsilon.$$

Again, we assume that $n_k = k$, write $D_k(t)$ for $D^\varepsilon_{n_k}(t : y_k)$, and let x_k belong to the ith side of $\Gamma_k(t_k)$. Since, in this case, the normal velocity of D_k at t_k is equal to $v - 2K(t_k - t_0)$,

$$V_i = v - 2K(t_k - t_0).$$

If $v \le 0$, (4.3) is immediately satisfied. Hence, we may assume that $v > 0$. So, for small $\varepsilon > 0$, $V_i > 0$, and by (2.3), $\chi_i = -1$. Consequently, $l_i(t_k)$ is greater than or equal to the length of any side of $D_k(t_k)$:

$$l_i(t_k) \ge 2R^\varepsilon(t_k) \sin \frac{\pi}{k},$$

and therefore,

$$v - C(t_k - t_0) = V_i = \frac{2\tan(\pi/k)}{l_i(t_k)} \le \frac{1}{R^\varepsilon(t_k)\cos(\pi/k)}.$$

We first let $k \to \infty$ and then $\varepsilon \downarrow 0$. Since $R^\varepsilon(t_k)$ converges to $1/|\kappa| = -1/\kappa$, the result is (4.3). $\square$

5. Convergence. Let $\Gamma_0 = \partial\Omega_0$ be a twice differentiable Jordan curve and $\Gamma_{n\,0} = \partial\Omega_{n\,0}$ be an n-smooth approximation of Γ_0 satisfying

$$\lim_{n\to\infty} d_H(\Omega_{n\,0}, \Omega_0) = 0, \tag{5.1}$$

where d_H is the Hausdorff distance. For each n, there is a unique n-smooth solution $\{\Gamma_n(t)\}_{t\in[0,T_n)}$ of (2.3) satisfying the initial condition $\Gamma_n(0) = \Gamma_{n\,0}$ by Theorem 3.1. Moreover,

$$T_n = \frac{|\Omega_{n\,0}|}{2n\tan(\pi/n)} \to T_0 := \frac{|\Omega_0|}{2\pi}, \qquad n \to +\infty. \tag{5.2}$$

Let $\widehat{\Omega}$ and $\underline{\Omega}$ be as in section 4 so that, by construction,

$$\mathrm{cl}\,\underline{\Omega}(t) \subset \widehat{\Omega}(t) \qquad \forall t \in [0, T_0). \tag{5.3}$$

Moreover, $\widehat{\Omega}$ is a weak subsolution of the mean curvature flow, and $\underline{\Omega}$ is a weak supersolution of the mean curvature flow. In general space dimension, there is no comparison between weak sub- and supersolutions; however, in dimension two, there is always a smooth solution of the mean curvature flow, $\Gamma(t) = \partial\Omega(t)$ and we will show that

$$\widehat{\Omega}(t) \subset \mathrm{cl}\,\Omega(t) \subset \mathrm{cl}\,\underline{\Omega}(t) \qquad \forall t \in [0, T_0). \tag{5.4}$$

Combining (5.3) and (5.4), we will then obtain the convergence of Ω_n to Ω in Hausdorff topology, thus generalizing the previous convergence results of Girao [20] and Girao and Kohn [21].

The foregoing outline of our convergence result is entirely analogous to the Barles and Perthame procedure of proving convergence with very weak L^∞ estimates [5, 6].

THEOREM 5.1. *Let $\Gamma_n(t) = \partial\Omega_n(t)$ be the solution of* (2.3) *with initial data $\Gamma_{n\,0}$, and let $\Gamma(t) = \partial\Omega(t)$ be the smooth solution of the mean curvature flow with initial data Ω_0. Assume* (5.1)*; then*

$$\lim_{n\to\infty} d_H(\Omega_n(t), \Omega(t)) = 0 \tag{5.5}$$

locally uniformly in $t \in [0, T_0)$.

We begin with the following lemma.

LEMMA 5.2. $\widehat{\Omega}(0) \subset \mathrm{cl}\,\Omega_0 \subset \mathrm{cl}\,\underline{\Omega}(0)$.

Proof. We will prove only the first inclusion. Proof of the second inclusion is similar.

Since $d_H(\Omega_n, \Omega_0) \to 0$, for any $x_0 \in \Omega_0$ there are $\delta_0 > 0$ and $n_0 \in \mathcal{N}$ satisfying

$$B(x_0, \delta_0) \subset\subset \Omega_n \qquad \forall n > n_0.$$

Let γ_n be the regular n-polygon enclosing $B(x_0, \delta_0)$. If necessary, by taking n_0 larger, we may assume that $\gamma_n \subset\subset \Omega_n \ \forall\ n > n_0$. Let $\gamma_n(t)$ be the solution of the crystalline flow (2.3) with initial data $\gamma_n(0) = \gamma_n$ and $\omega_n(t)$ be the open set enclosed by $\gamma_n(t)$. Then by the containment principle for crystalline motions (cf. Giga and Gurtin [18]),

$$B\left(x_0, \delta_0/2\right) \subset \omega_n(t) \subset \Omega_n(t) \qquad \forall n > n_0,\ 0 \le t \le \frac{1}{4}\delta_0^2.$$

Let $n \to +\infty$ and $t \downarrow 0$ to conclude that $B(x_0, \delta_0/2) \subset \underline{\Omega}(0)$; therefore $x_0 \in \underline{\Omega}(0)$. ☐

In our second step, we will show that the smooth mean curvature flow yields a viscosity sub- and supersolution of the following equation:

$$u_t + F(Du, D^2u) = 0, \qquad \mathcal{R}^2 \times (0, T),$$

where

$$F(p, X) = -\mathrm{tr}((I - \bar{p} \otimes \bar{p})X) \tag{5.6}$$

and $\bar{p} = p/|p|$. This step is very similar to Evans and Spruck [12, Section 6] and Ambrosio and Soner [3, section 3].

We refer to Crandall, Ishii, and Lions [10] and Fleming and Soner [13] for information on viscosity solutions and to Chen, Giga, and Goto [9], and Evans and Spruck [12] for the properties of the level set equations.

Let $\{\Gamma(t)\}_{0 \le t < T_0}$ be a unique smooth mean curvature flow satisfying $\Gamma(0) = \Omega_0$, and let $d(x,t)$ be the signed distance function to $\Gamma(t)$, i.e.,

$$d(x,t) = \begin{cases} \mathrm{dist}(x, \Gamma(t)) & \text{if } x \in \Omega(t), \\ -\mathrm{dist}(x, \Gamma(t)) & \text{otherwise,} \end{cases}$$

where $\Omega(t)$ is the open set enclosed by $\Gamma(t)$. For a scalar d, $d \wedge 0 = \min\{d, 0\}$ and $d \vee 0 = \max\{d, 0\}$.

Lemma 5.3. *For any $\delta > 0$, there are constants $\sigma = \sigma(\delta) > 0$ and $K = K(\delta) > 0$ so that the function $u(x,t) := e^{-Kt}[(d \vee 0)(x,t) \wedge \sigma]$ is a viscosity subsolution of*

$$u_t + F(Du, D^2u) = 0 \qquad \text{in} \quad \mathcal{R}^2 \times (0, T_0).$$

Proof. For $\delta > 0$, there exists a $\sigma = \sigma(\delta) > 0$ such that d is smooth in $\{x \in \mathcal{R}^2 : |d(x,t)| < 2\sigma\} \times [0, T_0 - \delta]$, and in this tubular set,

$$\Delta d(x,t) = \frac{\kappa(y,t)}{1 - \kappa(y,t)d(x,t)}, \tag{5.7}$$

where $y \in \Gamma(t)$ is a unique point satisfying $|d(x,t)| = |x - y|$ and $\kappa(y,t)$ is the curvature of $\Gamma(t)$ at y. Since $\{\Gamma(t)\}_{0 \le t < T_0}$ is a smooth mean curvature flow,

$$d_t - \Delta d = 0 \qquad \text{in} \quad \Gamma(t) \times (0, T_0). \tag{5.8}$$

Since

$$C(\delta) := \sup\{|\kappa(x,t)| \ : \ (x,t) \in \partial\Omega(t) \times [0, T_0 - \delta]\} < \infty,$$

by (5.7) and (5.8), d is a classical subsolution of

$$d_t - \Delta d - Kd \le 0 \qquad \text{on} \quad \{x \ : \ 0 \le d(x,t) \le 2\sigma\} \times (0, T_0 - \delta]$$

for some $K \geq C(\delta)$. Since $|Dd| = 1$, d is also a classical subsolution of

$$d_t + F(Dd, D^2 d) - Kd = 0 \qquad \text{on} \quad \{x \;:\; 0 \leq d(x,t) \leq 2\sigma\} \times (0, T_0 - \delta].$$

Let h^ϵ be a bounded smooth function satisfying $h^\epsilon(r) = 0$ for $r \leq 0$, $h^\epsilon(r) = \sigma$ for $r \geq \sigma$, and as $\epsilon \downarrow 0$, $h^\epsilon(r)$ converges to $(r \vee 0) \wedge \sigma$. Since F is geometric, i.e.,

$$F(\lambda p, \lambda A + \mu p \otimes p) = \lambda F(p, A), \qquad \lambda, \mu \geq 0,$$

by calculus, we conclude that $u^\epsilon = e^{-Kt} h^\epsilon(d)$ is a classical subsolution of

$$u_t^\epsilon + F(Du^\epsilon, D^2 u^\epsilon) \leq 0 \qquad \text{on} \quad \mathcal{R}^2 \times (0, T_0 - \delta].$$

We let $\epsilon \downarrow 0$, $\delta \downarrow 0$ and use the celebrated stability property of viscosity solutions. ◻

An entirely similar argument yields the following lemma.

LEMMA 5.4. *For any $\delta > 0$, there are constants $\sigma = \sigma(\delta) > 0$ and $K = K(\delta) > 0$ so that the function $u(x,t) := e^{Kt}[(d \wedge 0)(x,t) \vee (-\sigma)]$ is a viscosity supersolution of*

$$u_t + F(Du, D^2 u) = 0 \qquad \text{in} \quad \mathcal{R}^2 \times (0, T_0).$$

We are now in a position to complete the proof of Theorem 5.1.

Proof of Theorem 5.1. For notational convenience, we set $\Omega_n(t) = \emptyset \;\forall\; n > 1,\; t > T_n$. Let $\widehat{\Omega}$ and $\underline{\Omega}$ be as in section 4, and let $\widehat{T}$, $\underline{T}$ be, respectively, the extinction time of $\widehat{\Omega}(t)$ and $\underline{\Omega}(t)$. Set $\tilde{T} = \min\{\underline{T}, T_0, \widehat{T}\}$.

By Lemma 5.3, $u(x,t) = e^{-Kt}[(d \;\vee 0)(x,t) \wedge \sigma]$ is a viscosity subsolution of

$$u_t + F(Du, D^2 u) = 0 \qquad \text{in} \quad \mathcal{R}^2 \times (0, \tilde{T} - \delta), \tag{5.9}$$

and by Lemma 4.2 and Proposition 6.1, $v(x,t) = \text{dist}(x, \mathcal{R}^2 \backslash \underline{\Omega}(t))$ is a viscosity supersolution of (5.9). Moreover, by Lemma 5.2, $u(\cdot, 0) \leq v(\cdot, 0)$ in $\mathcal{R}^2$, and therefore the comparison principle for solutions of (5.9) (cf. Chen, Giga, and Goto [9], Evans and Spruck [12]) yields

$$u \leq v \qquad \text{in} \quad \mathcal{R}^2 \times [0, \tilde{T} - \delta).$$

We claim that this inequality implies that

$$\Omega(t) \subset \underline{\Omega}(t) \qquad \forall t \in [0, \tilde{T} - \delta).$$

Indeed, for $(x,t) \in \Omega(t) \times [0, \tilde{T} - \delta)$, $0 < u(x,t)$. Then, by the previous inequality, $0 < v(x,t)$ and, therefore, $x \in \underline{\Omega}(t)$.

Similarly, we show that $\widehat{\Omega}(t) \subset \text{cl}\, \Omega(t) \;\forall\; t \in [0, \tilde{T} - \delta)$, and then we let $\delta \to 0$ to obtain (5.4) on $[0, \tilde{T})$.

A lengthy elementary argument shows that (5.4) is equivalent to (5.5). Hence, (5.5) holds on $[0, \tilde{T})$.

By (5.2) and the construction, $\underline{T} \leq \widehat{T} \leq T_0$. The uniform convergence of Ω_n to Ω implies that $\tilde{T} = T_0$. ◻

6. Appendix. In this section we gather several properties of the weak solutions.

Let $\{\Omega_n(t)\}_{0 \leq t < T_n}$, $\{\widehat{\Omega}(t)\}_{0 \leq t < T}$, and $\{\underline{\Omega}(t)\}_{0 \leq t < T}$ be as in section 4, and let $d_n(x,t)$ (resp., $\widehat{d}(x,t)$ and $\underline{d}(x,t)$) be the signed distance function for $\{\Omega_n(t)\}_{0 \leq t < T_n}$

(resp., for $\{\widehat{\Omega}(t)\}_{0\le t<T}$ and $\{\underline{\Omega}(t)\}_{0\le t<T}$). Then the definitions of $\widehat{\Omega}(t)$ and $\underline{\Omega}(t)$ are equivalent to

$$(\widehat{d}\wedge 0)(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ n\to+\infty}} (d_n\wedge 0)(y,s),$$
$$(\underline{d}\vee 0)(x,t) = \liminf_{\substack{(y,s)\to(x,t)\\ n\to+\infty}} (d_n\vee 0)(y,s).$$

The following weak regularity result in t follows from an attendant modification of [29, Lemma 7.3]:

$$\limsup_{y\to x,\ s\uparrow t} (\widehat{d}\wedge 0)(y,s) = (\widehat{d}\wedge 0)(x,t), \qquad (x,t)\in\mathcal{R}^2\times(0,T), \tag{6.1}$$

$$\liminf_{y\to x,\ s\uparrow t} (\underline{d}\vee 0)(y,s) = (\underline{d}\vee 0)(x,t), \qquad (x,t)\in\mathcal{R}^2\times(0,T). \tag{6.2}$$

These identities and the techniques of [29, section 14] yield the equivalence between the weak solutions defined in section 4 and the distance solutions defined by Soner in [29]. Let F be as in (5.6).

PROPOSITION 6.1. *$\{\Omega(t)\}_{0\le t<T}$ is a weak subsolution of the mean curvature flow satisfying* (6.1) *if and only if $d_{\Omega^*}(x,t)\wedge 0$ is a viscosity subsolution of*

$$u_t + F(Du, D^2u) = 0 \qquad in \quad \mathcal{R}^2\times(0,T). \tag{6.3}$$

$\{\Omega(t)\}_{0\le t<T}$ is a weak supersolution of the mean curvature flow satisfying (6.2) *if and only if $d_{\Omega_*}(x,t)\vee 0$ is a viscosity supersolution of* (6.3)*.*

We close the appendix by proving an approximation result used in section 4.

LEMMA 6.2. *Let $\{O(t)\}_{0\le t<T}$ be a family of closed smooth sets, and let $t_0\in(0,T)$, $x_0\in\partial O(t_0)$ satisfy* (4.4)*. Let $D^\varepsilon(t)$ and $D_n^\varepsilon(t:x^*)$ be the same sets as in the proof of* Lemma 4.1*. Assume that $D^\varepsilon(t:x_0^\varepsilon)$ satisfies* (4.7)*. Then there are a subsequence n_k and sequences $(x_k,t_k)\to(x_0,t_0)$, $y_k\to x_0^\varepsilon$ as $k\to+\infty$ satisfying*

$$x_k\in\Gamma_{n_k}(t_k)\cap\partial D_{n_k}^\varepsilon(t_k:y_k),$$

$$\Omega_{n_k}(t)\cap B(x_0,\delta^\varepsilon)\subset D_{n_k}^\varepsilon(t:y_k)\cap B(x_0,\delta^\varepsilon) \qquad \forall\ |t-t_0|\le\delta^\varepsilon.$$

Proof. Fix $\varepsilon>0$ and recall $(\widehat{\Omega})^*=\widehat{\Omega}$. Let $d_n(x,t)$ be the signed distance to $D_n^\varepsilon(t:x_0^\varepsilon)$, $d(x,t)$ be the signed distance to $D^\varepsilon(t:x_0^\varepsilon)$, and let

$$\alpha_n := \inf_{|t-t_0|\le\delta^\varepsilon} \inf\{d_n(x,t)\ :\ x\in\Omega_n(t)\cap B(x_0,\delta^\varepsilon)\}.$$

Choose $t_n\in[t_0-\delta^\varepsilon, t_0+\delta^\varepsilon]$, $x_n\in\Omega_n(t_n)\cap B(x_0,\delta^\varepsilon)$ and $w_n\in\partial D_n^\varepsilon(t_n:x_0^\varepsilon)$ such that

$$|w_n-x_n| = |\alpha_n|.$$

Set

$$y_n = x_0^\varepsilon - (w_n - x_n),$$

so that

$$\Omega_n(t) \cap B(x_0, \delta^\varepsilon) \subset D_n^\varepsilon(t : y_n) \cap B(x_0, \delta^\varepsilon) \quad \forall\ |t - t_0| \le \delta^\varepsilon.$$

Since $x_0 \in \widehat{\Omega}(t_0)$, by the definition of $\widehat{\Omega}$, there are a subsequence n_k and sequences $(z_k, s_k) \to (x_0, t_0)$ such that

$$z_k \in \Omega_{n_k}(s_k).$$

Then

$$\limsup_{k\to\infty} \alpha_{n_k} \le \limsup_{k\to\infty} d_{n_k}(z_k, s_k) = d(x_0, t_0) = 0.$$

A similar argument, using (4.7), shows that $\liminf \alpha_{n_k} \ge 0$. Hence $\alpha_{n_k} \to 0$ and, therefore, $y_{n_k} \to x_0^\varepsilon$.

It remains to show that $(x_{n_k}, t_{n_k}) \to (x_0, t_0)$. Suppose that on a further subsequence, denoted by n_k again,

$$(x_{n_k}, t_{n_k}) \to (\bar{x}, \bar{t}) \in B(x_0, 2\delta^\varepsilon) \times [t_0 - \delta^\varepsilon, t_0 + \delta^\varepsilon].$$

Since d_n converges to d uniformly,

$$d(\bar{x}, \bar{t}) = \lim_{k\to\infty} \alpha_{n_k} = 0 \le \lim_{k\to\infty} d_{n_k}(z_k, s_k) = d(x_0, t_0).$$

Since (x_0, t_0) is the strict minimizer of d, this implies that $(\bar{x}, \bar{t}) = (x_0, t_0)$. □

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