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Dynamics of Ginzburg-Landau Vortices

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Contents

1. Introduction

It is formally expected that, for a large class of Ginzburg-Landau-type reaction diffusion equations, the dynamics of the nodal set asymptotically depends only on the local geometry of the nodal set. More interestingly, the asymptotic behavior of the solution is determined by the nodal set, thus dominating the other properties of the solutions. In the case of scalar solutions, this phenomenon is well understood for several canonical equations. Typically, the solutions form sharp interfaces, called domain walls, and the time evolution of these sets is governed by geometric equations. See for instance, [16, 1, 17] and the references therein.

Neu [13] demonstrated this scenario for complex-valued solutions of a nonlinear Schrödinger equation and a Ginzburg-Landau-type equation. By formal asymptotics, he showed that the zeroes of these complex solutions, which he calls vortices, persist in time, keeping their original winding number, and the asymptotic vortex dynamics reduce to a set of ordinary differential equations for the vortex positions. In particular, vortices with opposite signs attract each other, while the ones with the same sign repel. His results were extended to the full Ginzburg-Landau model by Peres & Rubinstein [14] and later by E [6].

The main goal here is to rigorously study the asymptotics of the sequence of solutions u^ε considered by Neu [13] and E [6], in the limit $\varepsilon \downarrow 0$. Functions u^ε solve a Ginzburg-Landau-type reaction diffusion system

(1.1) $$u_t^\varepsilon - \Delta u^\varepsilon = \frac{u^\varepsilon}{\varepsilon^2}\,(1-|u^\varepsilon|^2) \quad \text{in } \Omega\times(0,\infty)$$

and the boundary condition

(1.2) $$u^\varepsilon(x,t) = g(x), \quad \forall\, x\in\partial\Omega,$$

where Ω is an open, bounded set in $\mathbb{R}^2$ and g is a given function with $|g|=1$. Equation (1.1) is the gradient flow of the Ginzburg-Landau functional

(1.3) $$I^\varepsilon(w) = I^\varepsilon_\Omega(w) := \int_\Omega e_\varepsilon(w)dx,$$

where, for $\mathbb{R}^2$-valued functions of Ω, the energy density $e_\varepsilon(w)$ is given by

(1.4) $$e_\varepsilon(w) := \frac{1}{2}|\nabla w|^2 + \frac{1}{\varepsilon^2}W(w), \quad W(w) = \frac{1}{4}(1-|w|^2)^2.$$

Recently, BETHUEL, BREZIS & HELEIN [3, 4] obtained a very detailed characterization of the Ginzburg-Landau functional in the limit $\varepsilon\downarrow 0$. In particular, they showed that, for any open subset O of $\mathbb{R}^2$ and a $w: O\to\mathbb{R}^2$ with $|w|=1$ on ∂O,

(1.5) $$I^\varepsilon_O(w) \geqq \pi|\deg(w;O)||\ln\varepsilon| - C,$$

where C is a constant depending on O and the boundary values of w. Suppose that $d=\deg(g;\Omega)>0$. BETHUEL, BREZIS & HELEIN also showed that

$$\begin{aligned}&\inf\left\{I^\varepsilon_\Omega(w) \;:\; w=g \text{ on } \partial O\right\}\\ &\qquad = d\pi|\ln\varepsilon| + \min\left\{W_g(\vec{y}) \;:\; \vec{y}=\{y^1,\dots,y^d\}\subset\Omega\right\} + o(\varepsilon),\end{aligned}$$

where, as $\varepsilon\downarrow 0$, $|o(\varepsilon)|/\varepsilon\to 0$ and W_g is the renormalized energy defined in [4]: see §2, below. Moreover, the zeroes of the minimizers converge, along a subsequence, to a minimizer of W_g. In [4], it is assumed that Ω is star-shaped and this restriction is later removed by STRUWE [21], who also gave alternative proofs for several results of [4]. Further results were obtained by LIN [12] and JERRARD [7]. In particular, JERRARD [7] proved the lower energy bound (1.5) for a smaller class of functions w, but with a constant C *independent* of the boundary values of w. In our proof, we will use this version of (1.5) as stated in Lemma 4.1, below.

Formal analyses indicate that, if initially u^ε has isolated vortices, then these vortices move with velocities of the order of $|\ln\varepsilon|^{-1}$. Therefore to obtain nontrivial vortex dynamics, we rescale the time variable by a factor of $|\ln\varepsilon|$ and set

$$v^\varepsilon(x,t) := u^\varepsilon(x,|\ln(\varepsilon)|t).$$

Then v^ε satisfies the boundary condition (1.2) and

(1.6) $$k_\varepsilon v_t^\varepsilon - \Delta v^\varepsilon = \frac{v^\varepsilon}{\varepsilon^2}\,(1-|v^\varepsilon|^2) \quad \text{in } \Omega\times(0,\infty),$$

where

$$k_\varepsilon = (|\ln(\varepsilon)|)^{-1}.$$

Our chief result, Theorem 2.1, is this: Assume that initially there are M isolated vortices with degree ± 1. Then, in the limit, these vortices persist and satisfy a set of ordinary differential equations (2.10) as long as they remain separated. The vortex equation (2.10) is, in fact, the gradient flow of the renormalized energy W_g. Our key assumption is an energy upper bound:

$$\int_{\Omega} e_\varepsilon(v^\varepsilon(\cdot, 0))(x)dx \leqq M\pi |\ln \varepsilon| + C. \tag{1.7}$$

Then, by the standard energy estimate (3.4), this upper bound holds for all later time.

BAUMAN, CHEN, PHILLIPS & STERNBERG [2] obtained the first result in this direction. They studied the large-time asymptotics of (1.1) on $\mathbb{R}^2$, with $\varepsilon = 1$ and showed that, as $t \to \infty$, the solution converges to a point on the unit circle. RUBINSTEIN & STERNBERG [15], studied the dynamics of one vortex in the limit $\varepsilon \downarrow 0$ under several a priori assumptions on the behavior of the solution around the vortex. They proved that the speed of the vortex, in the original time scaling, is of order $|\ln \varepsilon|^{-1}$. In particular, they assumed that, for all time t, there is exactly one zero of $u^\varepsilon(\cdot, t)$ and the degree around this point is equal to 1. Later, LIN [11] studied the dynamics of $|d|$ vortices, where d is the degree of the boundary data g. In this case, all vortices have the same sign and LIN proved that, in the original time scaling, they move with a speed of order $|\ln \varepsilon|^{-1}$. Our result differs from these in two key points. We do not assume that $M = |d|$, and we rigorously derive the vortex equation.

One key step in the proof is the lower energy bound

$$\int_{\Omega} e_\varepsilon(u^\varepsilon(\cdot, t))(x)dx \geqq \pi M |\ln \varepsilon| - C(t) \quad \forall t \geqq 0,$$

so that the unbounded part of the upper and lower bounds agree. When $M = |d|$, this lower bound follows from the stationary results. However, in the general case, one needs to localize the estimates around each vortex. We prove this lower bound by using the local stationary result of JERRARD and a local regularity result which states that a local integral bound uniform in ε of the energy density implies a uniform pointwise estimate of the energy in a slightly smaller region. This result, proved by us in [8], is stated in Lemma 4.2. These two results imply the desired lower bound, as long as the vortices stay isolated. Then, we show that the vortices remain separated by the following energy estimate:

$$\int \eta d\mu_t^\varepsilon \leqq \int \eta d\mu_0^\varepsilon + |\ln \varepsilon| \int_0^t \mu_s^\varepsilon(O^\varepsilon)ds,$$

where O^ε is an open set not containing the vortices,

$$d\mu_t^\varepsilon(x) := E^\varepsilon(x,t)dx,$$

$$E^\varepsilon(x,t) := e_\varepsilon(v^\varepsilon(\cdot, t))(x) = \frac{1}{2}|\nabla v^\varepsilon(x,t)|^2 + \frac{1}{\varepsilon^2}W(v^\varepsilon(x,t)),$$

and η is a smooth, positive function which is equal to a quadratic function around each vortex; see (3.5). This estimate with $\eta(x) = |x|^2$ was first used in [2] and later in [15]. Our argument is similar to that of [15].

In Lemmas 5.1 and 5.2, we combine all these to conclude that there are vortices $y^i(t)$, depending continuously on t, such that, along a subsequence ε_n,

$$\nu_t^{\varepsilon_n} \overset{*}{\rightharpoonup} \pi \sum_{i=1}^{M} \delta_{\{y^i(t)\}},$$

where

$$\nu_t^{\varepsilon_n} := k_\varepsilon \mu_t^{\varepsilon_n}.$$

In Lemma 5.3, we show that away from the vortices v^{ε_n} converges uniformly to a function $v(x,t)$, which is explicitly defined in §2. Moreover, E^{ε_n} also converges to $\frac{1}{2}|\nabla v|^2$, away from the vortices. Finally, this convergence result and the energy identity (3.3), with an appropriately chosen test function, yield the ordinary differential equation (2.10) satisfied by the vortices.

After the completion of this work, we have learned that, independently, LIN [10] also derived the vortex equation in the case when all the vortices have the same sign, or equivalently, when $M = d$.

2. Main Result

We assume that initial data $v_0^\varepsilon := v^\varepsilon(0,\cdot)$ satisfy the following property: There are M distinct points $\{a_1^\varepsilon, ..., a_M^\varepsilon\} \subset \Omega$ and a constant c^* satisfying: $v_0^\varepsilon = g$ on $\partial\Omega$,

(2.1) $$R_0 := \frac{1}{3} \min_{0<\varepsilon\leqq 1} \left\{ \min_{i\neq j} \left\{ |a_i^\varepsilon - a_j^\varepsilon| \right\},\ \min_i \left\{ \operatorname{dist}(a_i^\varepsilon, \partial\Omega) \right\} \right\} > 0,$$

(2.2) $$d_i := \deg(v_0^\varepsilon; \partial B_{R_0}(a_i^\varepsilon)) \in \{-1, +1\}, \qquad i = 1, ..., M,$$

(2.3) $$\sup \left\{ E^\varepsilon(x,0) : |x - a_i^\varepsilon| \geqq \tfrac{1}{2} R_0,\ i = 1, \ldots, M,\ \varepsilon \in (0,1] \right\} \leqq c^*,$$

(2.4) $$\inf \left\{ |v_0^\varepsilon(x,0)| : |x - a_i^\varepsilon| \geqq \tfrac{1}{2} R_0,\ \forall i = 1, \ldots, M,\ \varepsilon \in (0,1] \right\} \geqq \tfrac{3}{4},$$

(2.5) $$\int_\Omega E^\varepsilon(x,0)dx \leqq M\pi |\ln\varepsilon| + c^*,$$

(2.6) $$|v_0^\varepsilon| \leqq 1, \quad \varepsilon \left\| \nabla v_0^\varepsilon \right\|_\infty + \varepsilon^2 \left\| D^2 v_0^\varepsilon \right\|_\infty \leqq c^*.$$

We further assume that

$$\lim_{\varepsilon\to 0} a_i^\varepsilon = a_i$$

exists for all $i = 1, \ldots, M$ and

(2.7) $$\nu_0^\varepsilon(dx) := k_\varepsilon E^\varepsilon(x,0)dx \overset{*}{\rightharpoonup} \pi \sum_{i=1}^{M} \delta_{\{a_i\}}(dx).$$

In view of (2.2) and (2.3),

$$d := \deg(v_0^\varepsilon; \partial\Omega) = \deg(g; \partial\Omega) = \sum_{i=1}^{M} d_i . \tag{2.8}$$

The assumption (2.7) is not restrictive. Indeed it follows from the stationary results stated in Section 4 and a slightly stronger version of (2.3).

There are initial data satisfying these hypotheses; see Remark 2.1, below.

We continue by introducing several functions. For $\theta \in \mathbb{R}^1$ and $\xi = (b, c) \in \mathbb{R}^2$, let

$$\xi^\perp := (-c, b), \quad \vec{n}(\theta) := (\cos(\theta), \sin(\theta)), \quad \vec{t}(\theta) := (\vec{n}(\theta))^\perp,$$

and for a non-zero vector, x, let $\theta(x)$ be the multi-valued function satisfying

$$\vec{n}(\theta(x)) = \frac{x}{|x|} \quad \forall x \neq 0.$$

Note that, locally on $\mathbb{R}^2 \setminus \{0\}$, there are smooth, single-valued representatives of $\theta(\cdot)$ and, moreover, each representative satisfies

$$\nabla\theta(x) = \frac{\vec{t}(\theta(x))}{|x|} = \frac{x^\perp}{|x|^2} \quad \forall x \neq 0.$$

For M distinct points $\vec{y} := \{y^1, \ldots, y^M\} \subset \Omega$, set

$$\Theta(x; \vec{y}) := \sum_{i=1}^{M} d_i \theta(x - y^i), \quad x \neq y^i.$$

Since $|v_0^\varepsilon| = |g| = 1$ on $\partial\Omega$, for every $\vec{y} \subset \Omega$ there is, by (2.8), a single-valued smooth function φ_0 defined on $\partial\Omega$ satisfying

$$\vec{n}\,(\varphi_0 + \Theta(x; \vec{y})) = v_0^\varepsilon = g(x), \quad x \in \partial\Omega. \tag{2.9}$$

Let $\varphi(x) = \varphi(x; \vec{y})$ be the solution of

$$\Delta\varphi = 0 \quad \text{in} \;\; \Omega$$

and $\varphi = \varphi_0$ on $\partial\Omega$.

Finally, set

$$R(\vec{y}) := \tfrac{1}{3} \min\left\{ \min_{i \neq j}\{|y^j(t) - y^i(t)|\},\; \min_i\{\operatorname{dist}(y^i(t), \partial\Omega)\}\right\},$$

and let $\vec{y}(t) := \{y^1(t), \ldots, y^M(t)\}$ be the solution of

$$\frac{d}{dt} y^i(t) = -2d_i \left(\big(\nabla\varphi\big(y^i(t); \vec{y}(t)\big)\big)^\perp + \sum_{m \neq i} d_m \frac{y^m(t) - y^i(t)}{|y^m(t) - y^i(t)|^2} \right), \tag{2.10}$$

on $(0, T_0)$ with initial data $y^i(0) = a_i$, where

$$T_0 := \inf\{t > 0 \;:\; R(\vec{y}(t)) = 0\}.$$

Our chief result is

Theorem 2.1. *As* $\varepsilon \downarrow 0$,

(2.11) $$v_t^\varepsilon(dx) := k_\varepsilon E^\varepsilon(x,t)dx \overset{*}{\rightharpoonup} \pi \sum_{i=1}^M \delta_{\{y^i(t)\}}(dx) \quad \forall t \in [0, T_0),$$

and v^ε *converges to*

$$\vec{n}(\varphi(x;\vec{y}(t)) + \Theta(x;\vec{y}(t))),$$

uniformly on any compact subset of $\left\{(x,t) \in \Omega \times [0,T_0) \ : \ x \neq y^i(t)\right\}$. *Moreover, there are zeroes,* $y^{i,\varepsilon}(t)$, *of* $v^\varepsilon(\cdot, t)$ *such that*

$$y^i(t) = \lim_{\varepsilon \downarrow 0} y^{i,\varepsilon}(t) \quad \forall t \in [0, T_0).$$

A lengthy computation shows that the differential equation (2.10) can be rewritten as

$$\frac{d}{dt} y^i(t) = -2\nabla_{y^i(t)} W(\vec{y}(t))$$

where $W(\vec{y}) = W_g(\vec{y}d_1, \ldots, d_M)$ is the renormalized energy defined by BETHUEL, BREZIS & HELEIN [4]: Given $\vec{y}$, let $F(x)$ be the harmonic function satisfying

$$\nabla F \cdot n = g \wedge g_\tau - \sum_{i=1}^M d_i \frac{(x - y^i) \cdot n}{|x - y^i|^2}, \qquad \partial\Omega,$$

where n is the unit, outward normal vector and g_τ is the tangential derivative. Note that $\nabla F(x) = (\nabla\varphi(x;\vec{y}))^\perp$. On $\partial\Omega$, set

$$\Phi(x) = F(x) + \sum_{i=1}^M d_i \ln|x - y^i|;$$

then the renormalized energy is given by ([4, (47) page 21])

$$W(\vec{y}) = -\sum_{i \neq j} d_i d_j \ln|y^i - y^j| + \frac{1}{2\pi}\int_{\partial\Omega} \Phi(g \wedge g_\tau) d\mathscr{H}^1 - \sum_{i=1}^M d_i F(y^i).$$

Remark 2.1. Given any sequence $\vec{a}^\varepsilon := \{a_1^\varepsilon, \ldots, a_M^\varepsilon\}$, there are functions v_0^ε satisfying (2.3)–(2.7) and the boundary condition (1.2). Indeed, let

$$\Theta^\varepsilon(x) := \Theta(x;\vec{a}^\varepsilon) = \sum_{i=1}^M d_i \theta(x - a_i^\varepsilon),$$

and let φ^ε be a smooth, single-valued function satisfying

$$\vec{n}(\varphi^\varepsilon + \Theta^\varepsilon) = g(x), \quad x \in \partial\Omega.$$

Define

$$v_0^\varepsilon(x) = \prod_{i=1}^M H\left(\frac{|x - a_i^\varepsilon|}{\varepsilon}\right) \vec{n}(\varphi^\varepsilon + \Theta^\varepsilon),$$

where $H : \mathbb{R}^1 \to [0, 1]$ is any smooth, non-decreasing function with $H(0) = 0$ and $H(1) = 1$.

Remark 2.2. At T_0, two vortices, say y^{M-1} and y^M, of opposite sign collide (i.e.,

$$y^{M-1}(T_0) = y^M(T_0), \quad d_M d_{M-1} = -1.$$

Suppose that, at T_0, all other vortices are away from $y^{M-1}(T_0) = y^M(T_0)$. Then it is expected that these two vortices cancel each other and the remaining vortices satisfy the differential equation obtained by deleting these two vortices. Analysis of this cancellation is an interesting open question. The difficulty is this: at the ε level, the total energy is expected to decrease by $2\pi|\ln\varepsilon|$ at T_0. Since in our analysis, it is crucial that the $|\ln\varepsilon|$ part of the upper and lower energy estimates agree, our proof fails after T_0.

A related question is to understand the breakup of initial vortices with degree greater than one. It is expected that such vortices break up into several degree-one vortices and then satisfy an augmented differential equation. Our energy-type estimates of §3, in particular (3.5), show that, in the original time scaling, this breakup does not happen in finite time.

3. Energy Estimates

Let E^ε, μ_t^ε and k_ε be as in the Introduction. Then (1.6) gives

$$E_t^\varepsilon = \operatorname{div} p^\varepsilon - k_\varepsilon |v_t^\varepsilon|^2, \tag{3.1}$$

$$\nabla E^\varepsilon = -k_\varepsilon p^\varepsilon + \operatorname{div}(\sigma^\varepsilon), \tag{3.2}$$

where for $i, j = 1, 2$,

$$p_j^\varepsilon = \sum_{\alpha=1}^{2} v_t^{\varepsilon,\alpha}\, v_{x_j}^{\varepsilon,\alpha}, \quad \sigma_{ij}^\varepsilon = \sum_{\alpha=1}^{2} v_{x_i}^{\varepsilon,\alpha}\, v_{x_j}^{\varepsilon,\alpha}.$$

Let $\eta(x)$ be a smooth, positive function with $\nabla\eta(x) = 0$ for $x \in \partial\Omega$. As in [8, §2], multiply (3.1) by η, (3.2) by $\nabla\eta$ and subtract the two identities. After integrating by parts, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t} \textstyle\int \eta\, d\mu_t^\varepsilon \\ &\quad = -k_\varepsilon \int \eta \left|v_t^\varepsilon\right|^2 dx + |\ln\varepsilon| \int \left(D^2\eta\nabla v^\varepsilon \cdot \nabla v^\varepsilon - \Delta\eta E^\varepsilon\right) dx. \end{aligned} \tag{3.3}$$

For $\eta \equiv 1$, the foregoing computation and (2.5) yield the standard energy estimate

$$\begin{aligned} &\int_\Omega E^\varepsilon(x,t)dx + k_\varepsilon \int_0^t \int_\Omega |v_t^\varepsilon|^2 dx \\ &\qquad = \int_\Omega E^\varepsilon(x,0)dx \leqq M\pi|\ln\varepsilon| + c^*. \end{aligned} \tag{3.4}$$

The energy estimate (3.3) with $\eta(x) = |x|^2$ was first used by BAUMAN, CHEN, PHILLIPS & STERNBERG [2] and later by RUBINSTEIN & STERNBERG [15]. We modify the quadratic function in the following way. Let R_0 be as in (2.1) and choose η so that

$$\begin{aligned} \eta(x) &= \tfrac{1}{2}\left|x - a_i^\varepsilon\right|^2, && x \in B_{R_0}(a_i^\varepsilon), \\ \eta(x) &\geqq \eta_0 = \tfrac{1}{4}R_0^2, && x \in O^\varepsilon := \Omega \setminus \bigcup_i B_{R_0}(a_i^\varepsilon), \\ \nabla\eta(x) &= 0, && x \in \partial\Omega, \\ \left\|D^2\eta\right\|_\infty &\leqq C. \end{aligned}$$

Then $D^2\eta = I$ in $\bigcup_i B_{R_0}(a_i^\varepsilon)$ and therefore

$$D^2\eta\nabla v^\varepsilon \cdot \nabla v^\varepsilon - \Delta\eta E^\varepsilon = -\frac{2}{\varepsilon^2}W(v^\varepsilon) \quad \text{in } \cup_i B_{R_0}(a_i^\varepsilon).$$

Moreover, for $x \in O^\varepsilon$,

$$D^2\eta\nabla v^\varepsilon \cdot \nabla v^\varepsilon - \Delta\eta E^\varepsilon \leqq CE^\varepsilon.$$

Hence

$$\frac{\partial}{\partial t}\int \eta d\mu_t^\varepsilon \leqq C|\ln\varepsilon|\mu_t^\varepsilon(O^\varepsilon),$$

with an appropriate constant C, independent of ε. We integrate this inequality to obtain

$$\int \eta d\mu_t^\varepsilon \leqq \int \eta d\mu_0^\varepsilon + |\ln\varepsilon| \int_0^t \mu_s^\varepsilon(O^\varepsilon)ds. \tag{3.5}$$

We close this section by stating pointwise estimates that follow from (2.6) and the heat kernel representation of the solution v^ε (for details see [18, §3]):

$$|v^\varepsilon| \leqq 1, \quad \varepsilon\left\|\nabla v^\varepsilon\right\|_\infty + \varepsilon^2\left\|D^2v^\varepsilon\right\|_\infty \leqq C. \tag{3.6}$$

4. Stationary Results and Regularity

In this section, we recall and summarize several technical results that will be used in the next section. The first result is a local lower bound for the energy functional I^ε. BETHUEL, BREZIS & HELEIN [4] studied the minimizers of I^ε with given boundary data. They obtained lower bounds and the exact asymptotic behavior of the minimizers in star-shaped domains. Later, STRUWE [21] removed this restriction. Further results were obtained by LIN [12, 11] and JERRARD [7]. The following lemma is a special case of the local lower bound proved by JERRARD [7].

Lemma 4.1. *Let* $0 < \varepsilon \leqq 1$, $\varepsilon < R$ *and* $w : \bar{B}_{2R} \to B_1$ *be a continuously differentiable function satisfying*

$$|\nabla w| < \frac{k_1}{\varepsilon}, \quad \deg(w; \partial B_R) \neq 0, \quad |w(x)| \geqq \frac{1}{2} \quad \forall |x| \in [R, 2R].$$

Then there is a constant $C(k_1)$*, depending only on* k_1*, such that*

$$\int_{B_{2R}} e_\varepsilon(w)\, dx \geqq \pi \ln\left(\frac{R}{\varepsilon}\right) - C(k_1).$$

Moreover, there exists $x^* \in B_R$ *such that* $w(x^*) = 0$ *and for every* $\lambda \in [\varepsilon, R]$

$$\int_{B_\lambda(x^*)} e_\varepsilon(w)\, dx \geqq \pi \ln\left(\frac{\lambda}{\varepsilon}\right) - C(k_1).$$

The following pointwise gradient estimate is proved by us in [8].

Lemma 4.2 (Regularity). *Let* $0 < \varepsilon \leqq 1$, $\varepsilon < R$ *and* u^ε *be a solution of* (1.1) *in* $B_R \times (0, 4R^2)$. *Suppose that*

$$\sup\left\{\int_{B_{2R}} e_\varepsilon(u^\varepsilon(\cdot, t))\, dx \ : \ t \in [0, 4R^2]\right\} \leqq k_1. \tag{4.1}$$

Then there is a constant $C(k_1)$*, depending only on* k_1*, such that*

$$e_\varepsilon(u^\varepsilon(\cdot, t))(x) \leqq \frac{C(k_1)}{R^2} \quad \forall |x| \leqq R,\ t \in [R^2, 4R^2].$$

Further assume that

$$e_\varepsilon(u^\varepsilon(\cdot, 0))(x) \leqq k_1 \quad \forall |x| \leqq 2R.$$

Then

$$e_\varepsilon(u^\varepsilon(\cdot, t))(x) \leqq \frac{C(k_1)}{R^2} \quad \forall |x| \leqq R,\ t \in [0, 4R^2].$$

The proof of this lemma consists of two main steps: first, by a monotonicity result of STRUWE [19], we establish this result for small k_1 and then we use a blow-up argument, similar to the one used by STRUWE [20]. For related results in bounded domains, we refer to CHEN & LIN [5].

The following result uses the fact that the range of the limit function is the circle. It is the key step in proving the convergence of v^ε away from the vortices. Our proof closely follows LIN [12, 11].

For $\vec{y} = \{y^1, \ldots, y^M\} \subset \Omega$, recall that

$$R(\vec{y}) := \tfrac{1}{3} \min\left\{\min_{i \neq j}\{|y^i - y^j|\},\ \min_i\{\operatorname{dist}(y^i, \partial\Omega)\}\right\},$$

and, for $\{r_1, \ldots, r_M\} \subset (0, R(\vec{y}) \wedge 1]$ and $r_0 \in [0, R(\vec{y}) \wedge 1]$, set

$$\Omega_{r_0} := \{x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) > r_0\},$$

$$r = \min\{r_i \ : \ i = 1, \ldots, M\},$$

$$O := \{x \in \Omega_{r_0} \ : \ |x - y^i| > r_i \quad \forall\, i = 1, \ldots, M\}.$$

In the following lemma, we consider a smooth function

$$w : \Omega \to B_1$$

satisfying the boundary data (1.2). We define d_i, $\Theta(x) = \Theta(x; \vec{y})$, $\theta_i(x) = \theta(x - y^i)$ as in §2 and assume that $\varepsilon \in (0, 1]$.

Lemma 4.3. *Suppose that $|w| \geqq \frac{1}{2}$ on O, and that there is a constant k, independent of r, satisfying*

$$\int_O e_\varepsilon(w) dx \leqq \pi \sum_{i=1}^{M} |\ln r_i| + k, \tag{4.2}$$

$$\int_{\partial B_{r_i}(y^i)} e_\varepsilon(w) d\mathscr{H}^1(x) \leqq \frac{k}{r_i^3} \quad \forall\, i = 1, \ldots, M, \tag{4.3}$$

$$\int_{\partial \Omega_{r_0}} e_\varepsilon(w) d\mathscr{H}^1(x) \leqq k. \tag{4.4}$$

Then, there is a single-valued, smooth function φ defined on O such that

$$w(x) = |w(x)| \vec{n}(\varphi(x) + \Theta(x)), \quad x \in O, \tag{4.5}$$

$$\int_O |\nabla \varphi|^2 \leqq C + C\varepsilon \frac{\sqrt{|\ln r|}}{r^2},$$

with a constant C depending only on k, $R(\vec{y})$ and the boundary data g.

Proof. 1. Since $|w| \geqq \frac{1}{2}$ on O, the definition of $\Theta(x; \vec{y})$ implies that there is a single-valued, smooth function φ defined on O, satisfying (4.5).

Set $\rho := |w|$ so that, by (4.2) and (4.5),

$$\int_O \rho^2 \big[\tfrac{1}{2} |\nabla \Theta|^2 + \tfrac{1}{2} |\nabla \varphi|^2 + \nabla \varphi \cdot \nabla \Theta\big] dx \leqq \pi \textstyle\sum_{i=1}^{M} |\ln r_i| + k. \tag{4.6}$$

Since Θ is harmonic in O, by integration by parts,

$$\int_O \nabla \varphi \cdot \nabla \Theta = \int_{\partial \Omega_{r_0}} \varphi \nabla \Theta \cdot n + \sum_{i=1}^{M} \int_{\partial B_{r_i}(y^i)} \varphi \nabla \Theta \cdot n^i,$$

where n and n^i are, respectively, the outward unit normal vectors of $\partial \Omega_{r_0}$ and $\partial B_{r_i}(y^i)$. The definition of Θ yields

$$\int_{\partial B_{r_i}(y^i)} \nabla \Theta \cdot n^i = 0,$$

and therefore, for any λ,

$$\left| \int_{\partial B_{r_i}(y^i)} \varphi \nabla\Theta \cdot n^i \right| = \left| \int_{\partial B_{r_i}(y^i)} [\varphi - \lambda] \nabla\Theta \cdot n^i \right|$$
$$\leqq C r_i \sup_{\partial B_{r_i}(y^i)} |\nabla\Theta \cdot n^i| \sup_{\partial B_{r_i}(y^i)} |\varphi - \lambda|.$$

Fix i and choose
$$\lambda = \frac{1}{2\pi r_i} \int_{\partial B_{r_i}(y^i)} \varphi\,.$$

Then, on $\partial B_{r_i}(y^i)$,
$$|\varphi - \lambda| \leqq C \int_{\partial B_{r_i}(y^i)} |\nabla\varphi|.$$

Since $|w| \geqq \frac{1}{2}$ on O, by (4.3) and (4.5),
$$|\varphi - \lambda| \leqq C\big(|\partial B_{r_i}(y^i)|\big)^{1/2} \bigg(\int_{\partial B_{r_i}(y^i)} |\nabla\varphi|^2 \bigg)^{1/2} \leqq \frac{C}{r_i},$$

with an appropriate constant C. Since $n^i = -\vec{n}(\theta_i)$,
$$\nabla\Theta(x) \cdot n^i(x) = \sum_{k=1}^{M} d_k \nabla\theta_k(x) \cdot n^i(x) = -\sum_{k=1}^{M} d_k \frac{(\vec{n}(\theta_k(x)))^{\perp} \cdot \vec{n}(\theta_i(x))}{|x - y^k|}.$$

Therefore, on $\partial B_{r_i}(y^i)$,
$$\big|\nabla\Theta(x) \cdot n^i(x)\big| \leqq \sum_{k \neq i}^{M} d_k \frac{\big|(\vec{n}(\theta_k(x)))^{\perp} \cdot \vec{n}(\theta_i(x))\big|}{|x - y^k|} \leqq \frac{C}{R(\vec{y})},$$

which yields
$$\left| \sum_{i=1}^{M} \int_{\partial B_{r_i}(y^i)} \varphi \nabla\Theta \cdot n^i \right| \leqq C$$

with a constant C depending only on k, $R(\vec{y})$ and g.
2. Since Θ is harmonic in O,
$$\int_{\partial\Omega_{r_0}} \varphi \nabla\Theta \cdot n = \int_{\partial\Omega_{r_0}} [\varphi - \lambda] \nabla\Theta \cdot n$$

for any λ. Choose
$$\lambda = \frac{1}{|\partial\Omega_{r_0}|} \int_{\partial\Omega_{r_0}} \varphi$$

so that, by (4.4),
$$\left| \int_{\partial\Omega_{r_0}} [\varphi - \lambda] \nabla\Theta \cdot n \right| \leqq C \sup_{\partial\Omega_{r_0}} |\nabla\Theta| \int_{\partial\Omega_{r_0}} |\nabla\varphi| \leqq C.$$

Combine the previous two steps to obtain

$$\left| \int_O \nabla\varphi \cdot \nabla\Theta \right| \leqq C, \tag{4.7}$$

with a constant C depending only on k, $R(\vec{y})$ and g.

3. Set $R^* = R(\vec{y}) \wedge 1$. The definition of Θ yields

$$\begin{aligned} \int_O \tfrac{1}{2}|\nabla\Theta|^2 dx &\geqq \sum_{i=1}^{M} \int_{r_i}^{R^*} \int_{\partial B_\tau(y^i)} \tfrac{1}{2}|\nabla\theta_i|^2 d\mathscr{H}^1(x) d\tau \\ &= \sum_{i=1}^{M} \int_{r_i}^{R^*} \frac{\pi}{\tau} d\tau = \pi \sum_{i=1}^{M} |\ln r_i| - C, \end{aligned}$$

where $C = \pi M |\ln R^*|$. Substitute this and (4.7) into (4.6) and use the fact that $|w| \geqq \frac{1}{2}$ on O to obtain

$$\begin{aligned} \int_O \tfrac{1}{8}|\nabla\varphi|^2 &\leqq \int_O \rho^2 \big[\tfrac{1}{2}|\nabla\Theta|^2 + \tfrac{1}{2}|\nabla\varphi|^2 + \nabla\varphi\cdot\nabla\Theta\big] dx + C - \textstyle\int_O \tfrac{1}{2}\rho^2|\nabla\Theta|^2 \\ &\leqq C + \pi \sum_{i=1}^{M} |\ln r_i| - \int_O \tfrac{1}{2}\rho^2 |\nabla\Theta|^2 \\ &\leqq C + \int_O \tfrac{1}{2}\left(1-\rho^2\right)|\nabla\Theta|^2. \end{aligned}$$

Since

$$|\nabla\Theta(x,t)| \leqq \frac{C}{r}, \quad x \in O,$$

we conclude by using (4.2) that

$$\int_O \left(1-\rho^2\right)|\nabla\Theta|^2 \leqq \frac{C}{r^2}\left(\int_O W(w)dx\right)^{1/2} \leqq C\varepsilon \frac{\sqrt{|\ln r|}}{r^2}. \qquad \square$$

5. Proof of the Main Theorem

We start by showing the localization of the energy.

Lemma 5.1. *There are constants $t_0 > 0$, C and functions*

$$y^{i,\varepsilon} : [0, t_0] \to B_{R_0/2}(a_i^\varepsilon), \quad i = 1, \ldots, M,$$

such that $v^\varepsilon(y^{i,\varepsilon}(t), t) = 0$ and for any $\varepsilon \in (0, 1]$, $t \in [0, t_0]$, $\lambda \in [\varepsilon, R_0]$

$$\mu_t^\varepsilon\big(B_\lambda(y^{i,\varepsilon}(t))\big) \geqq \pi \ln\left(\frac{\lambda}{\varepsilon}\right) - C \quad \forall\, i = 1, \ldots, M. \tag{5.1}$$

Proof. Set
$$\Omega_1^\varepsilon := \bigcup_{i=1}^M \{x \in \Omega \ : \ |x - a_i^\varepsilon| \in (R_0, 2R_0)\}.$$

1. For $\varepsilon \in (0, 1]$, set
$$t_\varepsilon := \sup\{T \geqq 0 \ : \ |v^\varepsilon(x, t)| \geqq \tfrac{1}{2}, \quad \forall\, (x, t) \in \Omega_1^\varepsilon \times [0, T]\}.$$

By assumption (2.4), $t_\varepsilon > 0$. The continuity of v^ε, (2.2), and the properties of the topological degree imply that
$$|\deg(v^\varepsilon(\cdot, t); \partial B_{R_0}(a_i^\varepsilon))| = 1 \quad \forall\, t \leqq t_\varepsilon, \ \ \varepsilon \in (0, 1], \ i = 1, \ldots, M.$$

We apply Lemma 4.1 to $w = v^\varepsilon(\cdot, t)$ with $R = R_0$. The gradient estimate (3.6) and Lemma 4.1 imply that for every $t \in [0, t_\varepsilon]$, $\varepsilon \in (0, 1]$, and $i = 1, \ldots, M$ there exists
$$y^{i,\varepsilon}(t) \in B_{R_0}(a_i^\varepsilon)$$
satisfying $v^\varepsilon(y^{i,\varepsilon}(t), t) = 0$ and (5.1) for all $\lambda \in [\varepsilon, R_0]$ with a constant C independent of ε. Then the global energy estimate (3.4) yields

$$\mu_t^\varepsilon(\{x : |x - y^{i,\varepsilon}(t)| \geqq \lambda, \ \ \forall\, i = 1, \ldots, M\}) \leqq C + \pi M \ln(R_0/\lambda) \tag{5.2}$$

for all $t \in [0, t_\varepsilon]$, $\varepsilon \in (0, 1]$, $\lambda \in [\varepsilon, R_0]$ and $i = 1, \ldots, M$. Set
$$T_\varepsilon := \sup\{T \in [0, t_\varepsilon] \ : \ y^{i,\varepsilon}(t) \in B_{R_0/2}(a_i^\varepsilon) \quad \forall\, t \in [0, T], \ i = 1, \ldots, M\}.$$

Since $v^\varepsilon(y^{i,\varepsilon}(t), t) = 0$, by (2.4), $T_\varepsilon > 0$ for all $\varepsilon \in (0, 1]$.

2. Let η be as in §3 and let O^ε be as in (3.5). By taking $\lambda = \frac{1}{2} R_0$ in (5.2), we get
$$\begin{aligned}\mu_t^\varepsilon(O^\varepsilon) &\leqq \mu_t^\varepsilon(\{x \ : \ |x - y^{i,\varepsilon}(t)| \geqq \tfrac{1}{2} R_0, \quad \forall\, i = 1, \ldots, M\}) \\ &\leqq C \quad \forall\, t < T_\varepsilon.\end{aligned}$$

Then, by (3.5),
$$\int \eta\, d\mu_t^\varepsilon \leqq \int \eta\, d\mu_0^\varepsilon + C|\ln\varepsilon| \int_0^t \mu_s^\varepsilon(O^\varepsilon) ds \leqq \int \eta d\mu_0^\varepsilon + C|\ln\varepsilon|\, t$$
for all $t \leqq T_\varepsilon$. Since, by (2.7),
$$\lim_{\varepsilon\downarrow 0} k_\varepsilon \int \eta\, d\mu_0^\varepsilon = 0,$$
there is a sequence $c(\varepsilon)$, such that, as $\varepsilon \downarrow 0$, $c(\varepsilon) \to 0$ and
$$\int \eta\, d\mu_t^\varepsilon \leqq [c(\varepsilon) + Ct]|\ln\varepsilon| \quad \forall\, t \leqq T_\varepsilon.$$

3. Suppose that $T_\varepsilon < \infty$. We claim that there exists a constant $\varepsilon_1 > 0$, explicitly constructed below, such that, if $\varepsilon \in (0, \varepsilon_1]$, then $|y^{i,\varepsilon}(T_\varepsilon) - a_i^\varepsilon| \geqq \frac{1}{2} R_0$ for some $i \in \{1, \ldots, M\}$. Indeed for all $t < T_\varepsilon$ and $i \in \{1, \ldots, M\}$, $y^{i,\varepsilon}(t) \in B_{R_0/2}(a_i^\varepsilon)$ and for $\lambda = \frac{1}{4} R_0$ in (5.2), we get

$$\mu_t^\varepsilon(\{x : |x - a_i^\varepsilon| \geqq \tfrac{3}{4}R_0 \quad \forall\, i = 1,\ldots,M\})$$
$$\leqq \mu_t^\varepsilon(\{x : |x - y^{i,\varepsilon}(t)| \geqq \tfrac{1}{4}R_0 \quad \forall\, i = 1,\ldots,M\})$$
$$\leqq C \quad \forall\, t < T_\varepsilon,\ \varepsilon \in (0,1].$$

By the regularity result Lemma 4.2 and (2.3), there is a constant C satisfying

$$E^\varepsilon(x,t) \leqq C^2 \quad \forall\,(x,t) \in \Omega_1^\varepsilon \times [0,T_\varepsilon), \quad \varepsilon \in (0,1].$$

In particular, in $\Omega_1^\varepsilon \times [0,T_\varepsilon)$, $W(v^\varepsilon(x,t)) \leqq C^2\varepsilon^2$ and therefore

$$|v^\varepsilon(x,t)|^2 \geqq 1 - 2C\varepsilon \quad \forall\,(x,t) \in \Omega_1^\varepsilon \times [0,T_\varepsilon).$$

Set $\varepsilon_1 = \min\{1, 1/(8C)\}$ so that $|v^\varepsilon(x,t)|^2 \geqq \frac{3}{4}$ for all $\varepsilon \in (0,\varepsilon_1]$, $(x,t) \in \Omega_1^\varepsilon \times [0,T_\varepsilon)$. By the continuity of v^ε, we conclude that $t_\varepsilon > T_\varepsilon$, and therefore $|y^{i,\varepsilon}(T_\varepsilon) - a_i^\varepsilon| \geqq \frac{1}{2}R_0$ for some $i \in \{1,\ldots,M\}$.

4. By the previous step,

$$\eta(x) \geqq c_1 := \frac{(R_0)^2}{32} \quad \forall\, x \in B_{R_0/4}(y^{i,\varepsilon}(T_\varepsilon)),$$

and, by (5.1),

$$\int \eta\, d\mu_{T_\varepsilon}^\varepsilon \geqq c_1\mu_{T_\varepsilon}^\varepsilon\big(B_{R_0/4}(y^{i,\varepsilon}(T_\varepsilon))\big) \geqq c_2|\ln\varepsilon| - c_3,$$

with appropriate constants c_2 and c_3. In view of Step 2,

$$[c(\varepsilon) + CT_\varepsilon]|\ln\varepsilon| \geqq c_2|\ln\varepsilon| - c_3.$$

Choose $\varepsilon_2 \in (0,\varepsilon_1]$ and $\hat{t}_0 > 0$ so that

$$[C\hat{t}_0 + c(\varepsilon)]\,|\ln\varepsilon| \ \leqq\ c_2|\ln\varepsilon| - c_3$$

for all $\varepsilon \in (0,\varepsilon_2]$. Therefore $\hat{t}_0 \leqq T_\varepsilon$ and

$$y^{i,\varepsilon}(t) \in B_{R_0/2}(a_i^\varepsilon) \quad \forall\, t \in [0,\hat{t}_0],\ \varepsilon \in (0,\varepsilon_2],\ i = 1,\ldots,M.$$

In the foregoing argument we assumed that $T_\varepsilon < \infty$; however, if $T_\varepsilon = \infty$, the above conclusion is immediate.

5. Hence, (5.1) holds with $\hat{t}_0$ for all $\varepsilon \in (0,\varepsilon_2]$. However, by (2.4),

$$t_0 := \hat{t}_0 \wedge \min\{T_\varepsilon : \varepsilon \in [\varepsilon_2, 1]\} > 0. \ \square$$

Let t_0 be as in Lemma 5.1 and Q be a dense, countable subset of $[0,t_0]$. By a diagonalization argument, we choose a subsequence, $\varepsilon_n \downarrow 0$, so that

$$y^i(t) := \lim_{n\to\infty} y^{i,\varepsilon_n}(t) \tag{5.3}$$

exists for all $t \in Q$ and $i \in \{1,\ldots,M\}$. Set

$$\nu_t^n(dx) := \nu_t^{\varepsilon_n}(dx) = k_{\varepsilon_n} E^{\varepsilon_n}(x,t)\,dx,$$

so that as $n \to \infty$, by (5.1) and (3.4),

$$\nu_t^n \overset{*}{\rightharpoonup} \pi \sum_{i=1}^{M} \delta_{\{y^i(t)\}} \quad \forall\, t \in Q. \tag{5.4}$$

Lemma 5.2. *For every* $i \in \{1, \dots, M\}$, $y^i(\cdot)$ *extends to a Hölder continuous function, with exponent* $\frac{1}{2}$, *on* $[0, t_0]$ *and* (5.4) *holds for every* $t \in [0, t_0]$. *Moreover,* y^{i,ε_n} *converges to* y^i *uniformly on* $[0, t_0]$.

Proof. 1. Fix i and let $\phi(x)$ be a smooth, positive function with compact support in $B_{R_0}(a_i)$. Then for any $t \in Q$,

$$\phi(y^i(t)) = \lim_{n\to\infty} \frac{1}{\pi} \int \phi\, d\nu_t^n.$$

2. Since $d\nu_t^\varepsilon = k_\varepsilon E^\varepsilon(x,t)\,dx$, by (3.1),

$$\frac{d}{dt}\int \phi\, d\nu_t^\varepsilon = -k_\varepsilon \left[\int k_\varepsilon \phi |v_t^\varepsilon|^2 + \nabla\phi \cdot p^\varepsilon\, dx\right] \leqq k_\varepsilon\, \|\nabla\phi\|_\infty \int |p^\varepsilon|\, dx$$

and therefore, for $0 \leqq s \leqq t$,

$$\int \phi\, d\nu_t^\varepsilon - \int \phi\, d\nu_s^\varepsilon \leqq \|\nabla\phi\|_\infty\, k_\varepsilon \left(\int_s^t \int_\Omega |\nabla v^\varepsilon|^2 dx\, dt\right)^{1/2} \left(\int_s^t \int_\Omega |v_t^\varepsilon|^2 dx\, dt\right)^{1/2}.$$

3. The energy estimate (3.4) yields

$$\int_\Omega |\nabla v^\varepsilon|^2 dx \leqq C[|\ln\varepsilon| + 1] \quad \forall\, t \geqq 0,$$

$$\int_s^t \int_\Omega |v_t^\varepsilon|^2 dx\, dt \leqq |\ln\varepsilon| \left[\int_\Omega E^\varepsilon(x,s)dx - \int_\Omega E^\varepsilon(x,t)dx\right].$$

Using (5.1), with $\lambda = R_0$, and the energy estimate, we conclude that

$$\int_s^t \int_\Omega |v_t^\varepsilon|^2 dx\, dt \leqq C[|\ln\varepsilon| + 1] \quad \forall\, 0 \leqq s \leqq t \leqq t_0.$$

4. Combine the previous two steps to obtain

$$\int \phi\, d\nu_t^\varepsilon - \int \phi\, d\nu_s^\varepsilon \leqq C\, \|\nabla\phi\|_\infty \sqrt{t-s}, \quad \forall\, 0 \leqq s \leqq t \leqq t_0,$$

and, by Step 1,

$$\phi(y^i(t)) - \phi(y^i(s)) \leqq C\, \|\nabla\phi\|_\infty \sqrt{t-s}, \quad \forall\, s \leqq t,\ s, t \in Q.$$

For any $i \in \{1, \dots, M\}$, $s < t$, $s, t \in Q$ and s sufficiently close to t, there is a smooth function ϕ, with compact support in $B_{R_0}(a_i)$, satisfying

$$\phi(y^i(t)) = 2, \quad \phi(y^i(s)) = 1, \quad \|\nabla\phi\|_\infty = |y^i(t) - y^i(s)|^{-1}.$$

Hence for all $s < t$ sufficiently close to t and for $s, t \in Q$

$$|y^i(t) - y^i(s)| \leqq C\sqrt{t-s},$$

and therefore, y^i is a Hölder continuous function on Q. We extend y^i as a Hölder continuous function on $[0, t_0]$.

5. To prove the uniform convergence, let t_n be a sequence in $[0, t_0]$. Choose a further subsequence n_k so that t_{n_k} and $y^{i,\varepsilon_{n_k}}(t_{n_k})$ converge, respectively, to t and $y^{i,*}$ for all $i \in \{1, \ldots, M\}$. Lemma 5.1 implies that, as $k \to \infty$,

$$\nu_{t_{n_k}}^{n_k} \overset{*}{\rightharpoonup} \pi \sum_{i=1}^{M} \delta_{\{y^{i,*}\}}.$$

Then, for any $s < t$, $s \in Q$, $i \in \{1, \ldots, M\}$ and ϕ as before,

$$\phi(y^{i,*}) - \phi(y^i(s)) \leqq C\, \|\nabla\phi\|_\infty \sqrt{t-s},$$

and therefore $y^{i,*} = y^i(t)$. □

Our next result is about the behavior of v^ε away from the vortices. Let $\vec{n}$, $\varphi(x; \vec{y})$ and $\Theta(x; \vec{y})$ be as in §3. For $r \in (0, R_0]$, $\lambda \in (0, 1]$, set

$$\Omega_r := \{x \in \Omega : \text{dist}(x, \partial\Omega) > r\},$$

$$Q_{r,\lambda} := \left\{(x,t) \in \Omega_r \times [0, t_0] : |x - y^i(t)| > \lambda R_0 \quad \forall\, i = 1, ..., M\right\},$$

$$Q_{r,\lambda}(t) := \left\{x \in \Omega_r : (x,t) \in Q_{r,\lambda}\right\},$$

$$Q_{r,\lambda}^n := \left\{(x,t) \in \Omega_r \times [0, t_0] : |x - y^{i,\varepsilon_n}(t)| > 2\lambda R_0 \quad \forall\, i = 1, ..., M\right\}.$$

The uniform convergence of y^{i,ε_n} imply that, for sufficiently large n, $Q_{r,\lambda}^n \subset Q_{r,\lambda}$. Moreover, the energy estimate (5.2) and the regularity result Lemma 4.2 imply that

$$\sup_{Q_{r,\lambda}} E^{\varepsilon_n} \leqq \frac{C(\lambda)}{r^2}. \tag{5.5}$$

In particular, there is $\varepsilon(r, \lambda) > 0$ such that

$$Q_{r,\lambda} \subset \Gamma^{\varepsilon_n}, \quad \varepsilon_n \in (0, \varepsilon(r,\lambda)],$$

where

$$\Gamma^\varepsilon := \left\{(x,t) \in \bar{\Omega} \times [0, t_0] \;:\; |v^\varepsilon(x,t)| \geqq \tfrac{1}{2}\right\}.$$

Then, for $\varepsilon_n \in (0, \varepsilon(r, \lambda)]$ there exists a single-valued, smooth function, $\varphi^{\varepsilon_n} : Q_{r,\lambda} \to \mathbb{R}^1$, satisfying

$$v^\varepsilon(x,t) = |v^\varepsilon(x,t)|\, \vec{n}(\varphi^\varepsilon(x,t) + \Theta(x; \vec{y}^\varepsilon(t))), \quad (x,t) \in Q_{r,\lambda},$$

where $\vec{y}^\varepsilon(t) = \{y^{1,\varepsilon}(t), \ldots, y^{M,\varepsilon}(t)\}$. Moreover, we may choose φ^{ε_n} to be independent of λ, r.

Lemma 5.3. *For* $\lambda \in (0, \frac{1}{2}]$*, there are constants* $C > 0$ *and* $C(\lambda) > 0$ *satisfying*

$$\int_{Q_{r,\lambda}(t)} |\nabla \varphi^{\varepsilon_n}|^2 \, dx \leqq C + C(\lambda)\varepsilon_n,$$

for every $t \in [0, t_0]$*,* $r \in (0, \frac{1}{2}R_0]$*, and* $\varepsilon_n \in (0, \varepsilon(r, \lambda)]$*.*

Proof. We suppress the subscript n in our notation and write ε for ε_n.
1. Fix $\lambda \in (0, \frac{1}{2}]$ and for $i \in \{1, \dots, M\}$, let

$$k(i) = k(t, i, \varepsilon, \lambda) := \inf \left\{ r \int_{\partial B_r(y^{i,\varepsilon}(t))} E^\varepsilon(x, t) d\mathscr{H}^1(x) \ : \ r \in [\lambda^2 R_0, \lambda R_0] \right\}.$$

By (5.2) with $\lambda = \lambda^2 R_0$,

$$\begin{aligned} I(i) &:= \mu_t^\varepsilon(\{x \in \Omega : |x - y^{i,\varepsilon}(t)| \in [\lambda^2 R_0, \lambda R_0]\}) \\ &\leqq \mu_t^\varepsilon(\{x \in \Omega : |x - y^{j,\varepsilon}(t)| \geqq \lambda^2 R_0 \quad \forall\, j = 1, \dots, M\}) \\ &\leqq C + 2\pi M |\ln \lambda|, \end{aligned}$$

where C is a constant independent of λ, ε, i and t. The definition of $k(i)$ yields

$$I(i) \geqq \int_{\lambda^2 R_0}^{\lambda R_0} \frac{k(i)}{r} \, dr = k(i) |\ln \lambda|.$$

Hence, $k(i) \leqq C^* := 2\pi M + C/\ln 2$ and therefore there exists $r_i = r_i(\lambda, t, \varepsilon) \in [\lambda^2 R_0, \lambda R_0]$ satisfying

$$\int_{\partial B_{r_i}(y^{i,\varepsilon}(t))} E^\varepsilon(x, t) \, d\mathscr{H}^1(x) \leqq \frac{C^*}{r_i}.$$

The above argument was first used in this context by STRUWE [21].
2. Set $r_0 = \frac{1}{2}R_0$ and fix λ, t and $\varepsilon \in (0, \varepsilon(r_0, \lambda)]$. Set

$$\Theta^\varepsilon(x) = \Theta(x; \vec{y}^\varepsilon(t)), \quad \theta_i^\varepsilon(x) = \theta(x - y^{i,\varepsilon}(t)),$$

$$O := \{x \in \Omega_{r_0} : x \notin \bigcup_i B_{r_i}(y^{i,\varepsilon}(t))\}.$$

The local lower bound (5.1) with $\lambda = r_i$, and the energy estimate (3.4) yield

$$\begin{aligned} \mu_t^\varepsilon(O) &\leqq \mu_t^\varepsilon(\Omega) - \sum_{i=1}^M \mu_t^\varepsilon(B_{r_i}(y^{i,\varepsilon}(t))) \\ &\leqq C + \pi M |\ln \varepsilon| - \pi \sum_{i=1}^M \ln(r_i/\varepsilon) \\ &\leqq C + \pi \sum_{i=1}^M |\ln r_i|. \end{aligned}$$

Hence the hypotheses of Lemma 4.3 are satisfied and consequently,

$$\sup_{t\in[0,t_0]}\int_O |\nabla\varphi^\varepsilon|^2\,dx \leqq C + C(\lambda)\varepsilon \quad \varepsilon\in(0,\varepsilon(r_0,\lambda)],$$

with constants $C(\lambda)$, independent of ε, and with C independent of ε and λ. Since $\varepsilon(r,\lambda)\leqq\varepsilon(r_0,\lambda)$ for all $r<r_0$, and $Q_{r_0,\lambda}(t)\subset O$ for all sufficiently small ε,

$$\sup_{t\in[0,t_0]}\int_{Q_{r_0,\lambda}(t)} |\nabla\varphi^\varepsilon|^2\,dx \leqq C + C(\lambda)\varepsilon, \quad \varepsilon\in(0,\varepsilon(r,\lambda)].$$

3. Since $\Omega_r\setminus\Omega_{r_0}\subset Q_{r,\lambda}(t)\subset\Gamma^\varepsilon(t)$ for $\varepsilon\in(0,\varepsilon(r,\lambda)]$, on $\Omega_r\setminus\Omega_{r_0}$,

$$|\nabla\varphi^{\varepsilon_n}|\leqq C[1+|\nabla v^{\varepsilon_n}|].$$

Hence

$$\int_{\Omega_r\setminus\Omega_{r_0}}|\nabla\varphi^{\varepsilon_n}|^2\leqq C\left[1+\int_{\Omega_r\setminus\Omega_{r_0}}|\nabla v^{\varepsilon_n}|^2\right]\leqq C. \quad \square$$

By redefining $\varepsilon(r,\lambda)$, if necessary, we may assume that $C(\lambda)\varepsilon(r,\lambda)\leqq C$ and therefore

$$\int_{Q_{r,\lambda}(t)}|\nabla\varphi^{\varepsilon_n}|^2\,dx\leqq C, \tag{5.6}$$

for every $t\in[0,t_0]$, $r\in(0,R_0]$, $\lambda\in(0,1]$, and $\varepsilon_n\in(0,\varepsilon(r,\lambda)]$.

We estimate the L^2 norm of φ^{ε_n} next. Given the gradient bound (5.6), it is enough to control φ^{ε_n} near the boundary, as in

Lemma 5.4. *There are constants $C>0$ and $r_0>0$ satisfying*

$$\int_{\partial\Omega_r}|\varphi^{\varepsilon_n}(x,t)-\varphi(x;\vec{y}^{\varepsilon_n}(t))|^2\,d\mathscr{H}^1(x)\leqq C\left(r+\frac{\varepsilon_n}{r^2}\right)$$

for every $t\in[0,t_0]$, $r\in(0,r_0]$, and $\varepsilon_n\in(0,\varepsilon(r,1)]$.

Proof. We suppress the subscript n in our notation and write ε for ε_n. Fix $t\in[0,t_0]$.

1. Let $s^*:=|\partial\Omega|$ and $p\ :\ [0,s^*]\to\partial\Omega$ be the arc-length parametrization of $\partial\Omega$, i.e., $|p'(s)|=1$ and

$$\partial\Omega=\{p(s)\ :\ s\in[0,s^*]\}.$$

Since $\partial\Omega$ is smooth, there is $r_0>0$ such that, for every $r\in[0,r_0]$,

$$\partial\Omega_r=\{p(s)-rn(s)\ :\ s\in[0,s^*]\},$$

where $n(s)$ is the unit outward normal to $\partial\Omega$.

2. Since

$$\sup\{\mu_t^\varepsilon(\Omega\setminus\Omega_{r_0})\ :\ t\in[0,t_0],\ \varepsilon\in(0,1]\}<\infty, \tag{5.7}$$

by a covering argument (see [4, §IV.1]), there are $\{s^{1,\varepsilon},\ldots,s^{N_\varepsilon,\varepsilon}\}\subset[0,s^*]$ and constants C, N^* satisfying

$$N_\varepsilon\leqq N^*,$$

$$\{x \in \Omega \setminus \Omega_{r_0} \;:\; |v^\varepsilon(x,t)| < \tfrac{1}{2}\} \subset \{p(s) - rn(s) \;:\; r \in [0, r_0],\; s \in I^\varepsilon\},$$

where

$$I^\varepsilon := \bigcup_i [s^{i,\varepsilon} - C\varepsilon, s^{i,\varepsilon} + C\varepsilon] \cap [0, s^*].$$

3. Fix $r \in (0, r_0]$. For $\varepsilon \in (0, \varepsilon(r,1)]$, we extend φ^ε to a smooth, single-valued function on

$$\Omega_{r,1} \cup \{(x,t) \in \bar{\Omega} \times [0, t_0] \;:\; x = p(s) - \rho n(s) \quad \text{for some } s \notin I^\varepsilon \; \rho \in [0, r]\}.$$

Moreover, we may choose φ^ε so that $\varphi^\varepsilon(x,t) = \varphi(x; \vec{y}^\varepsilon(t))$ for $x \in \partial\Omega$ and, as $\varepsilon \downarrow 0$,

$$\varphi(x; \vec{y}^\varepsilon(t)) \to \varphi(x; \vec{y}(t)),$$

uniformly in $x \in \partial\Omega$.

Since $\varphi(x; \vec{y}^\varepsilon(t))$ is smooth,

$$\int_0^{s^*} |\varphi(p(s); \vec{y}^\varepsilon(t)) - \varphi(p(s) - rn(s); \vec{y}^\varepsilon(t))|^2 \, ds \leqq Cr^2$$

and therefore

$$\alpha \leqq \hat{\alpha} + Cr^2,$$

where

$$\alpha := \int_{[0,s^*]\setminus I^\varepsilon} |\varphi^\varepsilon(p(s) - rn(s), t) - \varphi(p(s) - rn(s); \vec{y}^\varepsilon(t))|^2 ds,$$

$$\hat{\alpha} := 2 \int_{[0,s^*]\setminus I^\varepsilon} |\varphi^\varepsilon(p(s), t) - \varphi^\varepsilon(p(s) - rn(s), t)|^2 \, ds.$$

4. For $s \notin I^\varepsilon$, $|v^\varepsilon(p(s) - rn(s), t)| \geqq \frac{1}{2}$ and $\varphi^\varepsilon(p(s) - rn(s), t)$ is defined. Moreover, at $(p(s) - rn(s), t)$

$$|\nabla \varphi^\varepsilon| \leqq C[1 + |\nabla v^\varepsilon|].$$

By (5.7),

$$\begin{aligned}
\alpha &\leqq \hat{\alpha} + Cr^2 \\
&\leqq 2r \int_{[0,s^*]\setminus I^\varepsilon} \int_0^r |\nabla \varphi^\varepsilon(p(s) - \xi n(s), t)|^2 \, d\xi \, ds + Cr^2 \\
&\leqq Cr \int_{[0,s^*]\setminus I^\varepsilon} \int_0^r [1 + |\nabla v^\varepsilon(p(s) - \xi n(s), t)|^2] \, d\xi \, ds + Cr^2 \qquad (5.8) \\
&\leqq Cr \int_{\Omega\setminus\Omega_{r_0}} [1 + |\nabla \varphi^\varepsilon|^2] \, dx + Cr^2 \\
&\leqq Cr.
\end{aligned}$$

5. Since $|I^\varepsilon| \leqq N^* C\varepsilon$, by (5.8), there is $\hat{s} \in [0, s^*] \setminus I^\varepsilon$ such that

$$|\varphi^\varepsilon(p(\hat{s}) - rn(\hat{s}), t) - \varphi(p(\hat{s}) - rn(\hat{s}); \vec{y}^\varepsilon(t))|^2 \leqq Cr.$$

By (5.5), for any $s \in [0, s^*]$

$$\begin{aligned}|\varphi^\varepsilon(p(s) - rn(s), t) - \varphi(p(s) - rn(s); \vec{y}^\varepsilon(t))|^2 \\ &\leqq \frac{C}{r^2} + 2|\varphi^\varepsilon(p(\hat{s}) - rn(\hat{s}), t) - \varphi(p(\hat{s}) - rn(\hat{s}); \vec{y}^\varepsilon(t))|^2 \\ &\leqq \frac{C}{r^2} + Cr.\end{aligned}$$

Hence

$$\begin{aligned}\int_{I^\varepsilon} |\varphi^\varepsilon(p(s) - rn(s), t) - \varphi(p(s) - rn(s); \vec{y}^\varepsilon(t))|^2\, ds \\ \leqq |I^\varepsilon| \left(\frac{C}{r^2} + Cr\right) \leqq C\left(\frac{1}{r^2} + r\right)\varepsilon. \quad \square\end{aligned}$$

Again we may assume that $\varepsilon(r, 1)r^{-2} \leqq r$ and therefore

$$\int_{\partial\Omega_r} |\varphi^{\varepsilon_n}(x, t) - \varphi(x; \vec{y}^{\varepsilon_n}(t))|^2\, d\mathscr{H}^1(x) \leqq Cr \tag{5.9}$$

for every $t \in [0, t_0]$, $r \in (0, R_0]$, and $\varepsilon_n \in (0, \varepsilon(r, 1)]$. By the Sobolev embedding theorem, (5.6) and (5.9) yield

$$\int_{Q_{r,\lambda}(t)} |\varphi^{\varepsilon_n}(x, t)|^2\, dx \leqq C \tag{5.10}$$

for every $t \in [0, t_0]$, $r \in (0, R_0]$, $\lambda \in (0, 1]$ and $\varepsilon_n \in (0, \varepsilon(r, \lambda)]$.

Set

$$U := \{(x, t) \in \Omega \times [0, t_0] \,:\, x \neq y^i(t) \qquad \forall\, i = 1, \ldots, M\}.$$

Proposition 5.5. *As $n \to \infty$, v^{ε_n} converges to*

$$v(x, t) = \vec{n}(\varphi(x; \vec{y}(t)) + \Theta(x; \vec{y}(t))),$$

uniformly on compact subsets of U. Moreover, $|\nabla v^{\varepsilon_n}|^2$ and $2E^{\varepsilon_n}$ converge to $|\nabla v|^2$ strongly in $L^1_{\text{loc}}(U)$.

Proof. We suppress the subscript n in our notation and write ε for ε_n.

1. Let r_m be any sequence tending to zero and set $Q_m := Q_{r_m, r_m}$, $\varepsilon(m) := \varepsilon(r_m, r_m)$ and so forth. By (5.6) and (5.10),

$$\begin{aligned}\sup\Big\{\int_{Q_m(t)} |\varphi^\varepsilon(x, t)|^2 + |\nabla\varphi^\varepsilon(x, t)|^2 dx \;:\; \\ t \in [0, t_0],\ \varepsilon \in (0, \varepsilon(m)],\ m = 1, 2, \ldots\Big\} < \infty,\end{aligned} \tag{5.11}$$

where $Q_m(t)$ is the t cross section of Q_m. We use this estimate and (5.5) in a diagonal argument to construct a subsequence, $\varepsilon_k \downarrow 0$, and φ such that, for every m,

$$\varphi^{\varepsilon_k} \to \varphi \qquad \text{strongly in } L^2(Q_m),$$
$$\nabla\varphi^{\varepsilon_k} \rightharpoonup \nabla\varphi \qquad \text{in weak }^*L^\infty(Q_m).$$

Since $U = \lim Q_m$, it follows that φ is defined on U. In view of (5.11) φ extends to $\Omega \times [0, t_0]$ and it satisfies

$$\sup_{t\in[0,t_0]} \int_\Omega |\varphi|^2 + |\nabla\varphi|^2\, dx < \infty.$$

Moreover, for every m,

$$v^{\varepsilon_k}(x,t) \to v(x,t) := \vec{n}(\varphi(x,t) + \Theta(x;\vec{y}(t))) \quad \text{in} \;\; L^2(Q_m),$$

and ∇v^{ε_k} converges to ∇v in the weak* topology of $L^\infty(Q_m)$.

2. Fix m and recall that $k_\varepsilon = |\ln\varepsilon|^{-1}$. We claim that $\varphi(x,t) = \varphi(x;\vec{y}(t))$ and that φ^{ε_k} converges to φ uniformly on Q_m. Indeed, let $t_k \to t^* \in (0, t_0]$. For all k satisfying $\varepsilon_k \leqq \varepsilon(m)$, set

$$w^k(x,t) := \varphi^{\varepsilon_k}(x, t\,k_{\varepsilon_k} + t_k), \quad (x,t) \in G^k_m,$$
$$\Theta^k(x,t) := \Theta^{\varepsilon_k}(x, t\,k_{\varepsilon_k} + t_k), \quad (x,t) \in G^k_m,$$
$$\rho^k(x,t) := |v^{\varepsilon_k}(x, t\,k_{\varepsilon_k} + t_k)|, \quad (x,t) \in G^k_m,$$

where

$$G^k_m = \{(x,t) \;:\; (x, t\,k_{\varepsilon_k} + t^*) \in Q_m\},$$

and, for sufficiently large k,

$$Q^*_m \times [-t^*_k, 0] \subset G^k_m, \quad t^*_k = t^*|\ln\varepsilon_k|,$$

$$Q^*_m = \{x \in \Omega_{r_m} \;:\; |x - y^i(t^*)| \geqq r_{m+1}R_0 \quad \forall\, i = 1, \ldots, M\}.$$

Moreover, w^k satisfies

$$u^{\varepsilon_k}(x, t + t_k|\ln\varepsilon_k|) = \rho^k(x,t)\vec{n}(w^k(x,t) + \Theta^k(x,t)),$$

where u^ε is the solution of the Ginzburg-Landau equation (1.1) in the original unscaled variables. From (1.1) we obtain

$$(\rho^k)^2\, w^k_t - \nabla\cdot\Big((\rho^k)^2\nabla(w^k + \Theta^k)\Big) = 0 \quad \text{in} \;\; G^k_m \tag{5.12}$$

and, by Step 3 of Lemma 5.2,

$$\int_0^{t^*_k}\int_\Omega |u^{\varepsilon_k}_t|^2\, dx\, dt < C,$$

with a constant C independent of k. Since $\rho^k \geqq \frac{1}{2}$ on G^k_m,

$$\int_{-t^*_k}^0 \int_{Q^*_m} |w^{\varepsilon_k}_t + \Theta^k_t|^2\, dx\, dt < C.$$

From (5.11) and (5.5), we also know that

$$\sup_{m,k,t}\left\{\int_{Q_m^*}|w^k|^2+|\nabla w^k|^2\,dx\right\}<\infty,$$

and, by (5.5),

$$\sup_k\|\nabla w^k\|_{L^\infty(G_m^k)}\leqq C(m).$$

Since, on $Q_m^*\times(-\infty,0]$, Θ^k is uniformly smooth in the x variable, the family $\{w^k+\Theta^k\}_{k=1}^\infty$ is precompact in $C^{1/4}_{\mathrm{loc}}(Q_m^*\times(-\infty,0])$. Moreover, as $k\to\infty$, Θ^k uniformly converges to

$$\Theta(x):=\Theta(x;\vec{y}(t^*)).$$

Hence there are a subsequence, denoted by k again, and a bounded function w defined on $\Omega\times(-\infty,0]$ such that for every m, w^k converges uniformly to w on every compact subset of $Q_m^*\times(-\infty,0]$ and

$$\sup_{t\leqq 0}\int_\Omega|w|^2+|\nabla w|^2\,dx<\infty,\qquad\int_{-\infty}^0\int_\Omega|w_t|^2\,dx\,dt<\infty.$$

Note that Θ is harmonic in U and, by (5.5), ρ^k converges to 1 in $H^1_{\mathrm{loc}}(Q_m^*\times(-\infty,0))$. We let $k\to\infty$ in (5.12) and conclude that w satisfies the heat equation on $Q_m^*\times(-\infty,0]$. In view of our estimates, w is a solution in $\Omega\times(-\infty,0]$. Moreover, by (5.9),

$$w(x,t)=\varphi(x;\vec{y}(t^*)),\qquad x\in\partial\Omega.$$

Since, by definition, $\varphi(x;\vec{y}(t^*))$ is harmonic in Ω, standard uniqueness results for the heat equation imply that

$$w(x,t)=\varphi(x;\vec{y}(t^*)),\qquad(x,t)\in\Omega\times(-\infty,0].$$

This proves our claim that $\varphi(x,t)=\varphi(x;\vec{y}(t))$ and that φ^{ε_k} converges uniformly to φ. Moreover, since the limit is independent of the subsequence, φ^ε is convergent along the original sequence.

3. Let $t_\varepsilon\to t^*$ be given. We claim that, for any m and $T>0$,

$$\lim_{\varepsilon\downarrow 0}\int_{-T}^0\int_{Q_m(t_\varepsilon)}E^\varepsilon(x,t\,k_\varepsilon+t_\varepsilon)\,dx\,dt=T\int_{Q_m(t^*)}\tfrac12|\nabla v(x,t^*)|^2\,dx.$$

This convergence result is very similar to the convergence results proved by us [8, Lemma 6.1], so we only give the outline of its proof. For ε sufficiently small, set $O=Q_m(t_\varepsilon)$,

$$\tilde{E}^\varepsilon(x,t):=E^\varepsilon(x,t\,k_\varepsilon+t_\varepsilon)=e_\varepsilon(u^\varepsilon(\cdot,t+|\ln\varepsilon|t_\varepsilon))(x),\qquad(x,t)\in\Omega\times[-T,0].$$

We compute

$$\tilde{E}^\varepsilon_t-\Delta\tilde{E}^\varepsilon+|D^2u^\varepsilon|^2+\frac{4}{\varepsilon^2}|u^\varepsilon\nabla u^\varepsilon|^2+\frac{4|u^\varepsilon|^2}{\varepsilon^4}W(u^\varepsilon)=\frac{2}{\varepsilon^2}(1-|u^\varepsilon|^2)|\nabla u^\varepsilon|^2$$

(see [8] for details). Moreover, by the regularity result, there is an open set $\hat{Q}$, containing $\bar{O} \times [-T, 0]$, so that $\tilde{E}^\varepsilon$ is bounded on $\hat{Q}$, uniformly in ε. Hence,

$$\tilde{E}^\varepsilon_t - \Delta \tilde{E}^\varepsilon + |D^2 u^\varepsilon|^2 + \frac{W(u^\varepsilon)}{\varepsilon^4} \leqq C, \tag{5.13}$$

and therefore

$$\sup_\varepsilon \int_{-T}^0 \int_O |D^2 u^\varepsilon|^2 dx\, dt < \infty,$$

$$\lim_{\varepsilon \downarrow 0} \int_{-T}^0 \int_O \frac{W(u^\varepsilon)}{\varepsilon^2}\, dx\, dt = 0.$$

These estimates, together with the uniform gradient and time derivative estimates of u^ε, imply the claimed convergence of the energy; see [8, Lemma 6.1].
4. To complete the proof of this lemma, it suffices to show that

$$\lim_{\varepsilon \downarrow 0} \int_{Q_m} E^\varepsilon(x,t)\, dx\, dt = \int_{Q_m} \tfrac{1}{2} |\nabla v|^2\, dx\, dt. \tag{5.14}$$

For sufficiently small ε, let M_ε be the smallest integer greater than $t_0 |\ln \varepsilon|$,

$$t^l_\varepsilon = \frac{l}{|\ln \varepsilon|}, \quad l = 0, 1, \ldots, M_\varepsilon,$$

and, for $t \in [t^{l-1}_\varepsilon, t^l_\varepsilon]$,

$$\begin{aligned} g_\varepsilon(t) &:= |\ln \varepsilon| \int_{t^{l-1}_\varepsilon}^{t^l_\varepsilon} \int_{Q_m(t^l_\varepsilon)} E^\varepsilon(x,s)\, dx\, ds \\ &= \int_{-1}^0 \int_{Q_m(t^l_\varepsilon)} E^\varepsilon(x, \tau\, k_\varepsilon + t^l_\varepsilon)\, dx\, d\tau. \end{aligned}$$

Since E^ε is bounded in Q_m,

$$\lim_{\varepsilon \downarrow 0} \int_{Q_m} E^\varepsilon\, dx\, dt = \lim_{\varepsilon \downarrow 0} \int_0^{t_0} g_\varepsilon(t)\, dt.$$

For $t \in [0, t_0]$, let $l(t, \varepsilon)$ be the smallest integer greater than t. Then, as $\varepsilon \downarrow 0$, $t^{l(t,\varepsilon)}_\varepsilon \to t$ and, by Step 6,

$$g_\varepsilon(t) \to \int_{Q_m(t)} \tfrac{1}{2} |\nabla v(x,t)|^2 dx\, dt.$$

Moreover, g_ε is bounded by a constant $K(m)$ independent of ε. Therefore, (5.14) follows from the dominated convergence theorem. □

Proof of Theorem 2.1. Let t_0 be as in Lemma 5.1 and ε_n, $y^i(t)$ be as in (5.3). By (2.7), $y^i(0) = a_i$ for each i. We first show that $\vec{y}(\cdot)$ is a solution of (2.10) on $[0, t_0]$.

Fix i and $t \in [0, t_0]$. Without loss of generality, assume that $i = 1$ and $y^1(t) = 0$.

1. Let ϕ be a smooth function with $\nabla\phi$ compactly supported in Ω. Since $d\nu_t^\varepsilon = k_\varepsilon E^\varepsilon(x,t)dx$, by (3.3),

$$\frac{d}{dt}\int_O \phi d\nu_t^\varepsilon = -\int_O (k_\varepsilon)^2\phi|v_t^\varepsilon|^2 + \int_O D^2\phi\nabla v^\varepsilon\cdot\nabla v^\varepsilon - \Delta\phi\, E^\varepsilon\, dx.$$

Steps 1 and 3 of Lemma 5.2 yield

$$\phi(y^1(s)) - \phi(0) = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_t^s\int_O D^2\phi\nabla v^\varepsilon\cdot\nabla v^\varepsilon - \Delta\phi E^\varepsilon dx d\tau \quad \forall s\in[0,t_0].$$

If the support of $D^2\phi$ does not include $\{y^1(\tau),\dots,y^M(\tau)\}$ for all $\tau\in[t,s]$, by Lemma 5.3,

$$\phi(y^1(s)) - \phi(0) = \frac{1}{\pi}\int_t^s\int_O D^2\phi\nabla v\cdot\nabla v - \frac{1}{2}\Delta\phi|\nabla v|^2 dx\, d\tau. \tag{5.15}$$

2. For $A\in\mathbb{R}^2$ and $\delta\in(0,\frac{1}{4}R_0]$, let $\phi_\delta = (A\cdot x)\,H(|x|)$, where, for $r \geqq 0$,

$$H(r) := \begin{cases} 1, & r\in[0,\delta],\\ 2 - r/\delta, & r\in[\delta,2\delta]\\ 0, & r\geqq 2\delta.\end{cases}$$

We calculate

$$D^2\phi_\delta = (A\cdot n)n\otimes n\, rH''(r) + [n\otimes A + A\otimes n + (A\cdot n)(I - n\otimes n)]\,H'(r),$$

where $r = |x|$, $n = \vec{n}(\theta) = x/r$ and $\otimes$ is the tensor product. Although ϕ_δ is not smooth enough, by an approximation argument, we use (5.15) with $\phi = \phi_\delta$. For all s sufficiently close to t, we find that

$$\begin{aligned}\phi(y^1(s)) - \phi(0) &= [y^1(s) - y^1(t)]\cdot A\\ &= \frac{1}{\pi}\int_t^s [I_1(\tau,\delta) + I_2(\tau,\delta) + I_3(\tau,\delta)]\,d\tau\cdot A,\end{aligned} \tag{5.16}$$

where

$$I_1(s,\delta) = \frac{1}{\delta}\int_{B_{2\delta}\setminus B_\delta}(\tfrac{1}{2}|\nabla v|^2 + |\nabla v\cdot n|^2)\,n \;-\; 2(\nabla v\cdot n)\cdot\nabla v\,dx,$$

$$I_2(s,\delta) = 2\int_{\partial B_{2\delta}}(|\nabla v\cdot n|^2 - \tfrac{1}{2}|\nabla v|^2)n\,d\mathscr{H}^1(x),$$

$$I_3(s,\delta) = -\int_{\partial B_\delta}(|\nabla v\cdot n|^2 - \tfrac{1}{2}|\nabla v|^2)n\,d\mathscr{H}^1(x).$$

Since the left-hand side of (5.16) is independent of δ and since (5.16) holds for all $A\in\mathbb{R}^2$,

$$y^1(s) - y^1(t) = \lim_{\delta\downarrow 0}\frac{1}{\pi}\int_t^s [I_1(\tau,\delta) + I_2(\tau,\delta) + I_3(\tau,\delta)]\,d\tau \tag{5.17}$$

for all s sufficiently close to t.

3. Recall that $\vec{t}(\theta) := (\vec{n}(\theta))^{\perp}$ and

$$\Phi := \varphi + \sum_{i=1}^{M} d_i \theta_i, \quad \theta_i(x,s) := \theta(x - y^i(s)).$$

Then, $v = \vec{n}(\Phi)$ and, for $\alpha = 1, 2$,

$$\begin{aligned}\nabla v^{\alpha} &= \Big(\nabla\varphi + \sum_{i=1}^{M} d_i \nabla\theta_i\Big)(\vec{t}(\Phi))_{\alpha} \\ &= \Big(\nabla\varphi + \sum_{i=1}^{M} d_i \vec{t}(\theta_i)|x - y^i(s)|^{-1}\Big)\Big(\vec{t}(\Phi)\Big)_{\alpha}.\end{aligned}$$

We evaluate the following functions at $x = r\vec{n}(\theta)$:

$$\begin{aligned}|\nabla v|^2 &= \frac{1}{r^2} + \frac{2d_1}{r}\Big(- (\nabla\varphi)^{\perp} + \sum_{i=2}^{M} d_i \frac{\vec{n}(\theta_i)}{|x - y^i(s)|}\Big) \cdot \vec{n}(\theta) + E_1(x), \\ \nabla v^{\alpha} \cdot \vec{n}(\theta) &= \Big(\nabla\varphi + \sum_{i=2}^{M} d_i \frac{\vec{t}(\theta_i)}{|x - y^i(s)|}\Big) \cdot \vec{n}(\theta)\Big(\vec{t}(\Phi)\Big)_{\alpha}, \\ |\nabla v \cdot \vec{n}(\theta)|^2 &= E_2(x), \\ (\nabla v \cdot \vec{n}(\theta)) \cdot \nabla v &= \sum_{\alpha=1}^{2} (\nabla v^{\alpha} \cdot \vec{n}(\theta))\nabla v^{\alpha} \\ &= \frac{d_1}{r}\Big(\nabla\varphi + \sum_{i=2}^{M} d_i \frac{\vec{t}(\theta_i)}{|x - y^i(s)|}\Big) \cdot \vec{n}(\theta)\ \vec{t}(\theta) + E_3(x),\end{aligned}$$

where E_1, E_2 and E_3 are bounded functions. We use these identities in the definition of $I_1(s,\delta)$:

$$\begin{aligned}I_1(s,\delta) = k_1(s,\delta)\,\delta + \frac{2d_1}{\delta}\int_{\delta}^{2\delta}\int_0^{2\pi} \frac{1}{2}(B(s,x) \cdot \vec{n}(\theta))\vec{n}(\theta) \\ - (A(s,x) \cdot \vec{n}(\theta))\vec{t}(\theta)\, d\theta\, dr,\end{aligned}$$

where $k_1(s,\delta)$ is bounded and

$$B(s,x) = -(\nabla\varphi)^{\perp} + \sum_{i=2}^{M} d_i \frac{\vec{n}(\theta_i)}{|x - y^i(s)|},$$

$$A(s,x) = \nabla\varphi + \sum_{i=2}^{M} d_i \frac{\vec{t}(\theta_i)}{|x - y^i(s)|};$$

observe that $(A(s,x))^{\perp} = -B(s,x)$. Since for a fixed $\gamma \in \mathbb{R}^2$,

$$\int_0^{2\pi} (\gamma \cdot \vec{n}(\theta))\vec{n}(\theta)\, d\theta = \pi\gamma, \quad \int_0^{2\pi} (\gamma \cdot \vec{n}(\theta))\vec{t}(\theta)\, d\theta = \pi\gamma^{\perp},$$

as $\delta \downarrow 0$, it follows that

$$I_1(s,\delta) \to 3\pi d_1 B(s,0),$$

uniformly in $s \in [0, t_0]$. A similar calculation shows that

$$I_2(s,\delta) = k_2(s,\delta)\,\delta - 2d_1 \int_0^{2\pi} (B(s,x) \cdot \vec{n}(\theta))\vec{n}(\theta)\, d\theta$$

and, therefore, as $\delta \downarrow 0$, $I_2(s,\delta) \to -2\pi d_1 B(s,0)$, uniformly in $s \in [0, t_0]$. Similarly, as $\delta \downarrow 0$, $I_3(s,\delta) \to \pi d_1 B(s,0)$, uniformly in $s \in [0, t_0]$ and, by (5.17),

$$y^1(s) - y^1(t) = 2d_1 \int_t^s B(\tau, 0)d\tau,$$

for all s sufficiently close to t. Since B is continuous,

$$\frac{d}{dt} y^1(t) = 2d_1 B(t, y^1(t)) \quad \forall\, t \in [0, t_0].$$

4. In the previous steps, we have proved Theorem 2.1 on $[0, t_0]$. Since the family of functions $\{v^\varepsilon(\cdot, t_0)\}$ satisfies the assumptions (2.2)–(2.7), we complete the proof of the theorem by an iterative argument. □

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