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Équations aux dérivées partielles/*Partial Differential Equations*

Rectifiability of the distributional Jacobian for a class of functions

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Abstract. We define the class B2V of functions of bounded 2-dimensional variation, and we sketch the proof of a theorem showing that, for $u \in \mathrm{B2V}(\mathbb{R}^3\,;\,\mathrm{S}^1)$, the distributional Jacobian of u is supported on a 1-dimensional rectifiable set. © 1999 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Rectifiabilité de la mesure jacobienne pour une classe de fonctions

Résumé. *Nous définissons la classe* B2V *des fonctions à variation bornée à deux dimensions, et nous esquissons la démonstration d'un théorème montrant que, pour* $u \in \mathrm{B2V}(\mathbb{R}^3\,;\,\mathrm{S}^1)$, *la mesure jacobienne de* u *est supportée par un ensemble rectifiable de dimension* 1. © 1999 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Dans cette Note, nous introduisons la classe des fonctions B2V, ou fonctions à variation bornée à deux dimensions. Ce sont les fonctions pour lesquelles tous les déterminants 2×2 sont des mesures. En tant que telles, ce sont les exemples les plus simples d'une famille de généralisations des fonctions BV. Nous esquissons également la preuve d'un théorème montrant que, pour une fonction $u \in \mathrm{B2V}(\mathbb{R}^3\,;\,\mathbb{R}^2)$ telle que $|u| = 1$ p.p., la mesure jacobienne – c'est-à-dire la collection de tous les déterminants 2×2 – est un ensemble rectifiable à une dimension.

Une théorie plus générale des fonctions à variation bornée à n dimensions, où n est un entier quelconque positif, est développée plus en détail dans un article écrit par les auteurs [8].

Notations et définitions. – Soit $\mathcal{H}^1$ la mesure de Hausdorff à une dimension.

Pour des vecteurs $v, w \in \mathbb{R}^2$ nous définissons $v \times w := v^1 w^2 - v^2 w^1$. Nous définissons $\nabla \times \phi := (\partial_{x_2}\phi, -\partial_{x_1}\phi)$ pour une fonction scalaire $\phi : \mathbb{R}^2 \to \mathbb{R}$ et $\nabla \times \eta := \partial_{x_1}\eta^2 - \partial_{x_2}\eta^1$ pour une fonction à valeurs vectorielles $\eta = (\eta^1, \eta^2) : \mathbb{R}^2 \to \mathbb{R}^2$.

Note présentée par Haïm BRÉZIS.

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Nous définissons $j(u) = \big(\det(u, \partial_{x_1} u), \ldots, \det(u, \partial_{x_m} u)\big)$, pour $u \in \mathrm{W}^{1,1}_{\mathrm{loc}} \cap \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^m, \mathbb{R}^2)$, $m = 2$ ou 3. Nous définissons également le Jacobien $[\mathrm{J}u] := \frac{1}{2} \nabla \times j(u)$ au sens des distributions. On peut facilement vérifier que si $u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^2 ; \mathbb{R}^2)$, alors $[\mathrm{J}u] = \det \mathrm{D}u$, et si $u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3 ; \mathbb{R}^2)$, alors $[\mathrm{J}u] = \big(\det(\partial_{x_2} u, \partial_{x_3} u), \det(\partial_{x_3} u, \partial_{x_1} u), \det(\partial_{x_1} u, \partial_{x_2} u)\big)$.

On dit qu'une fonction $u \in \mathrm{W}^{1,1}_{\mathrm{loc}} \cap \mathrm{L}^\infty(\mathbb{R}^m ; \mathbb{R}^2)$ est à 2-variation bornée s'il existe une constante C telle que $\langle \phi, [\mathrm{J}u] \rangle \leq C \|\phi\|_{\mathrm{L}^\infty}$ pour tout $\phi \in \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^m ; \mathbb{R}^{m(m-1)/2})$. Pour une telle fonction, on écrira $u \in \mathrm{B2V}(\mathbb{R}^m ; \mathbb{R}^2)$. On notera $u \in \mathrm{B2V}(\mathbb{R}^m ; \mathrm{S}^1)$ pour $u \in \mathrm{B2V}(\mathbb{R}^m ; \mathbb{R}^2)$ avec $|u| = 1$ presque partout.

Si $u \in \mathrm{B2V}(\mathbb{R}^m ; \mathbb{R}^2)$, le théorème de représentation de Riesz affirme qu'il existe une mesure de Radon non négative sur $\mathbb{R}^m$, que nous appelons la mesure de 2-variation totale de u et que nous notons $|\mathrm{J}u|$, et une fonction mesurable τ à valeurs dans $\mathbb{R}^{m(m-1)/2}$ telle que $|\tau(x)| = 1$ pour $|\mathrm{J}u|$ presque tout x, et $\langle \phi, [\mathrm{J}u] \rangle = \int \phi(x) \cdot \tau(x) \, |\mathrm{J}u|(\mathrm{d}x)$. Lorsque $u \in \mathrm{B2V}$ nous écrirons parfois $\int \phi \cdot [\mathrm{J}u]$ au lieu de $\langle \phi, [\mathrm{J}u] \rangle$.

Dans [8], nous démontrons :

THÉORÈME 1. – *Si* $u \in \mathrm{B2V}(\mathbb{R}^2 ; \mathrm{S}^1)$, *alors il existe un nombre* $m > 0$, *des entiers* $\{d_1, \ldots, d_m\}$ *et des points* $\{a_1, \ldots, a_m\}$ *tels que* $[\mathrm{J}u] = \pi \sum_i d_i \, \delta_{a_i}$.

La preuve utilise le fait que pour tout $a \in \mathbb{R}^2$, $[\mathrm{J}u](\mathrm{B}_r(a)) = \pi \deg(u ; \partial \mathrm{B}_r(a))$ pour presque tout r, pour montrer que $r \mapsto [\mathrm{J}u](\mathrm{B}_r(a))$ est une fonction à variation bornée. □

Le principal résultat de [8] se réduit, dans ce contexte, au théorème suivant :

THÉORÈME 2. –*Si* $u \in \mathrm{B2V}(\mathbb{R}^3 ; \mathrm{S}^1)$, *alors il existe une fonction* $|\mathrm{J}u|$*-mesurable, strictement positive* $m : \operatorname{supp} |\mathrm{J}u| \to \mathbb{Z}$ *telle que, pour* $|\mathrm{J}u|$ *presque tout* $x_0 \in \operatorname{supp} |\mathrm{J}u|$,

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \int \phi\left(\frac{x - x_0}{\lambda}\right) \cdot [\mathrm{J}u](\mathrm{d}x) = \pi \, m(x_0) \int_{\{x = s\tau(x_0),\, s \in \mathbb{R}\}} \phi(x) \cdot \tau(x_0) \mathcal{H}^1(\mathrm{d}x) \qquad \forall \phi \in \mathrm{C}^0_{\mathrm{c}}(\mathbb{R}^3 ; \mathbb{R}^3). \tag{1}$$

De plus, il existe également un ensemble rectifiable Γ *à une dimension tel que*

$$\int \phi \cdot [\mathrm{J}u] = \pi \int_\Gamma m(x) \phi(x) \cdot \tau(x) \mathcal{H}^1(\mathrm{d}x) \qquad \forall \phi \in \mathrm{C}^0_{\mathrm{c}}(\mathbb{R}^3 ; \mathbb{R}^3). \tag{2}$$

Rappelons qu'un ensemble Γ est un ensemble rectifiable à une dimension s'il peut être écrit comme la réunion dénombrable de sous-ensembles $\mathcal{H}^1$-mesurables de courbes Lipschitz et d'un ensemble de mesure $\mathcal{H}^1$ nulle. Des résultats bien connus de la théorie de la mesure géométrique (*cf.* [13], Theorem 11.8) impliquent que (1) implique en fait l'existence d'un ensemble rectifiable Γ tel que (2) soit satisfait. Ainsi, il suffit de prouver (1).

Une observation essentielle dans la démonstration est la suivante : si l'on écrit $x \in \mathbb{R}^3$, comme $(y, z) \in \mathbb{R} \times \mathbb{R}^2$ et $u_y(z) = u(y, z)$, alors on peut démontrer que la fonction $y \mapsto [\mathrm{J}u_y]$ est à variation bornée. Plus de détails sont donnés dans la version anglaise.

1. Introduction

In this Note we define the class of B2V functions, or functions of bounded 2-dimensional variation. Informally, these are functions for which all 2 by 2 distributional determinants are measures. As such they are the simplest example of a family of higher-order generalizations of BV functions. We also sketch the proof of a theorem showing that, for a function $u \in \mathrm{B2V}(\mathbb{R}^3 ; \mathbb{R}^2)$ such that $|u| = 1$ a.e., the Jacobian measure—that is, the collection of all 2 by 2 distributional determinants—is in fact supported on a 1-dimensional rectifiable set. A general theory of functions of bounded n-dimensional variation, where n can be any positive integer, is developed in detail in a longer paper by the authors [8]. This paper includes complete proofs of the results sketched here, as well as some additional results, including general versions of the chain rule and the coarea formula.

Distributional determinants arise naturally in nonlinear elasticity (*see* [2]), and also in problems involving singularities of harmonic maps, as was pointed out by Brézis, Coron, and Lieb, [4]. They have been widely studied in recent years (*see* for example [3], [5], [9], [10], [11], [12]).

Our results are closely related to the work of Giaquinta, Modica, and Souček on Cartesian currents (*see* for example [6], [7]). Indeed, some of our results become very transparent when viewed in the framework of Cartesian currents (*see* Remark 1 below).

2. Notation and definitions

We write $\mathcal{H}^1$ to denote 1-dimensional Hausdorff measure.

For vectors v, $w \in \mathbb{R}^2$ we define $v \times w := v^1 w^2 - v^2 w^1$. We define $\nabla \times \phi := (\partial_{x_2}\phi, -\partial_{x_1}\phi)$ for scalar $\phi : \mathbb{R}^2 \to \mathbb{R}$, and $\nabla \times \eta := \partial_{x_1}\eta^2 - \partial_{x_2}\eta^1$ for vector-valued $\eta = (\eta^1, \eta^2) : \mathbb{R}^2 \to \mathbb{R}^2$.

For $u \in \mathrm{W}^{1,1}_{\mathrm{loc}} \cap \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^m, \mathbb{R}^2)$, $m = 2$ or 3, we define $j(u) = \big(\det(u, \partial_{x_1} u), \ldots, \det(u, \partial_{x_m} u)\big)$. We also define the distributional Jacobian $[\mathrm{J}u] := \frac{1}{2}\nabla \times j(u)$ in the sense of distributions so that

$$\langle \phi, [\mathrm{J}u]\rangle = \frac{1}{2}\int \nabla \times \phi \cdot j(u) \ \text{ for } \begin{cases} \phi \in \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^2 ; \mathbb{R}) & \text{if } m = 2, \\ \phi \in \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^3 ; \mathbb{R}^3) & \text{if } m = 3. \end{cases} \tag{1}$$

One easily checks that $[\mathrm{J}u] = \begin{cases} \det \mathrm{D}u & \text{if } m = 2, \\ \big(\det(\partial_{x_2}u, \partial_{x_3}u),\ \det(\partial_{x_3}u, \partial_{x_1}u),\ \det(\partial_{x_1}u, \partial_{x_2}u)\big) & \text{if } m = 3, \end{cases}$ whenever $u \in \mathrm{H}^1_{\mathrm{loc}}$.

We say that a function $u \in \mathrm{W}^{1,1}_{\mathrm{loc}} \cap \mathrm{L}^\infty(\mathbb{R}^m ; \mathbb{R}^2)$ has bounded 2-variation if there exists a constant C such that $\langle \phi, [\mathrm{J}u]\rangle \le C\|\phi\|_{\mathrm{L}^\infty}$ for all $\phi \in \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^m ; \mathbb{R}^{m(m-1)/2})$. When this holds we write $u \in \mathrm{B2V}(\mathbb{R}^m ; \mathbb{R}^2)$. More generally, given any subset $U \subset \mathbb{R}^m$ one can define in the obvious way the spaces $\mathrm{B2V}(U ; \mathbb{R}^2)$ and $\mathrm{B2V}_{\mathrm{loc}}(U ; \mathbb{R}^2)$. We will write $u \in \mathrm{B2V}(\mathbb{R}^m ; \mathrm{S}^1)$ to mean that $u \in \mathrm{B2V}(\mathbb{R}^m ; \mathbb{R}^2)$ with $|u| = 1$ almost everywhere.

If $u \in \mathrm{B2V}(\mathbb{R}^m ; \mathbb{R}^2)$, the Riesz representation theorem asserts that there is a finite nonnegative Radon measure on $\mathbb{R}^m$, which we call the total 2-variation measure of u and denote $|\mathrm{J}u|$, and a $|\mathrm{J}u|$-measurable function τ taking values in $\mathbb{R}^{m(m-1)/2}$ such that $|\tau(x)| = 1$ for $|\mathrm{J}u|$ almost every x, and $\langle \phi, [\mathrm{J}u]\rangle = \int \phi(x)\cdot\tau(x)\,|\mathrm{J}u|(\mathrm{d}x)$. When $u \in \mathrm{B2V}$ we will sometimes write $\int \phi\cdot[\mathrm{J}u]$ for $\langle \phi, [\mathrm{J}u]\rangle$.

3. Results

In this paper we sketch the proofs of some theorems that describe the structure of the Jacobian measure $[\mathrm{J}u] = \tau|\mathrm{J}u|$ for functions $u \in \mathrm{B2V}(\mathbb{R}^m ; \mathrm{S}^1)$. We first treat the case $m = 2$.

THEOREM 1. – *If* $u \in \mathrm{B2V}(\mathbb{R}^2 ; \mathrm{S}^1)$ *then there exist a number* $m > 0$, *integers* $\{d_1, \ldots, d_m\}$ *and points* $\{a_1, \ldots, a_m\}$ *such that* $[\mathrm{J}u] = \pi \sum_i d_i\, \delta_{a_i}$.

For every $a \in \mathbb{R}^2$, $[Ju](B_r(a)) = \pi\deg(u; \partial B_r(a))$ for a.e. $r > 0$. The proof uses this fact and an appropriate choice of test function to show that $r \to [Ju](B_r(a))$ is a BV function for every a. Once this is know the theorem is not hard to prove. □

The main result of [8] reduces, in the present context, to the following

THEOREM 2. – *If* $u \in \mathrm{B2V}(\mathbb{R}^3\,;\,\mathrm{S}^1)$, *then there exists a positive* $|\mathrm{J}u|$*-measurable function* $m : \operatorname{supp}|\mathrm{J}u| \to \mathbb{Z}$ *such that, for* $|\mathrm{J}u|$ *almost every* $x_0 \in \operatorname{supp}|\mathrm{J}u|$,

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\int \phi\Big(\frac{x-x_0}{\lambda}\Big)\cdot[\mathrm{J}u](\mathrm{d}x) = \pi m(x_0)\int_{\{x=s\tau(x_0),\,s\in\mathbb{R}\}}\phi(x)\cdot\tau(x_0)\mathcal{H}^1(\mathrm{d}x) \qquad \forall\phi\in \mathrm{C}^0_\mathrm{c}(\mathbb{R}^3\,;\,\mathbb{R}^3). \tag{2}$$

Moreover, there also exists a 1*-dimensional rectifiable set* Γ *such that*

$$\int \phi\cdot[\mathrm{J}u] = \pi\int_\Gamma m(x)\phi(x)\cdot\tau(x)\mathcal{H}^1(\mathrm{d}x) \qquad \forall\phi\in \mathrm{C}^0_\mathrm{c}(\mathbb{R}^3\,;\,\mathbb{R}^3). \tag{3}$$

Recall that a set Γ is said to be a 1-dimensional rectifiable set if it can be written as a countable union of $\mathcal{H}^1$ measurable subsets of Lipschitz curves and a set of $\mathcal{H}^1$ measure zero. It is well-known (*see* [13], Theorem 11.8) that if (2) holds, then in fact there exists a rectifiable set Γ such that (3) is satisfied. Thus it suffices to prove (2). The remainder of this Note sketches the proof.

Remark 1. – After this paper was submitted, M. Giaquinta and G. Modica independently pointed out to us that Theorem 2 follows as an immediate consequence of the boundary rectifiability theorem of Federer and Fleming, together with a formula relating $[\mathrm{J}u]$ to the boundary of the current associated with integration over the graph of G_u, essentially proven in [7]. We present the details of their argument in [8].

Step 1. – *Measure theoretic preliminaries* : we will write points in $x \in \mathbb{R}^3$ in the form $(y,z) \in \mathbb{R}^1\times\mathbb{R}^2$, and we use ∂_{x_2} and ∂_{z_1} (for example) interchangeably. For each $y \in \mathbb{R}$ we define a function $u_y : \mathbb{R}^2 \to \mathrm{S}^1$ by $u_y(z) = u(y,z)$. For a.e. y we can then define $[\mathrm{J}u_y]$ as an element of $\mathrm{C}^1_\mathrm{c}(\mathbb{R}^2)^*$ in the standard way, via (1). We also write $[\mathrm{J}u] = ([\mathrm{J}^1u],[\mathrm{J}^2u],[\mathrm{J}^3u])$, where $[\mathrm{J}^1u] = \frac{1}{2}\left(\partial_{x_2}\det(u,\partial_{x_3}u) - \partial_{x_3}\det(u,\partial_{x_2}u)\right) = \frac{1}{2}\left(\partial_{z_1}\det(u,\partial_{z_2}u) - \partial_{z_2}\det(u,\partial_{z_1}u)\right)$, and so on.

It is clear that, for any scalar test function ϕ,

$$\int_{\mathbb{R}^3}\phi[\mathrm{J}^1u] = \int\bigg(\frac{1}{2}\int_{\mathbb{R}^2}\partial_{z_1}\phi\det(u,\partial_{z_2}u) - \partial_{z_2}\phi\det(u,\partial_{z_1}u)\,\mathrm{d}z\bigg)\mathrm{d}y = \int_{\mathbb{R}}\langle\phi,[\mathrm{J}u_y]\rangle\,\mathrm{d}y. \tag{4}$$

Starting from this identity one can show that $u_y \in \mathrm{B2V}(\mathbb{R}^2\,;\,\mathrm{S}^1)$ for a.e. $y\in\mathbb{R}$, and that

$$\int\phi|\mathrm{J}^1u| = \int_{\mathbb{R}}\int_{\mathbb{R}^2}\phi|\mathrm{J}u_y|(\mathrm{d}z)\,\mathrm{d}y. \tag{5}$$

This implies that $y\mapsto|\mathrm{J}u_y|(\mathbb{R}^2)$ is an integrable function on $\mathbb{R}$.

For any $\nu, x_0 \in \mathbb{R}^3$ and $\lambda > 0$ we define the cylinder $\mathrm{C}^\nu_\lambda(x_0) = \{x\in\mathbb{R}^3 : |\nu\cdot(x_0-x)| \le \lambda, |\nu\times(x_0-x)| \le \lambda\}$. We now define the *reduced boundary* $\partial^* j$ to be the set of points x_0 in the support of $[\mathrm{J}u]$ satisfying $|\tau(x_0)| = 1$,

$$\lim_{\lambda\to0}\frac{1}{|\mathrm{J}u|(\mathrm{C}^\nu_\lambda(x_0))}\int_{\mathrm{C}^\nu_\lambda(x_0)}|\tau(x_0)-\tau(x)|\,|\mathrm{J}u|(\mathrm{d}x) = 0 \qquad \text{for all } \nu\in\mathbb{R}^3\text{; and} \tag{6}$$

$$\limsup_{r\to0}\frac{|\mathrm{J}u|(\mathrm{B}_r(x_0))}{r} < \infty. \tag{7}$$

One can verify that $|\mathrm{J}u|$ almost every $x\in\mathbb{R}^3$ belongs to the reduced boundary $\partial^* j$.

Step 2. – *Measure-valued functions of bounded variation* : given a signed Radon measure μ on $\mathbb{R}^2$ and $V \subset \mathbb{R}^2$ we define $\|\mu\|_{\widehat{\mathrm{C}}^1_c(V)^*} := \sup\left\{ \int_V \phi(z)\mu(\mathrm{d}z) \ : \ \phi \in \mathrm{C}^\infty_c(V),\ \|\mathrm{D}\phi\|_{\mathrm{L}^\infty} \leq 1 \right\}$. Next, let $\{\mu_y\}_{y\in\mathbb{R}}$ be a family of signed Radon measures on $\mathbb{R}^2$ such that the mapping $y \mapsto \mu_y$ is weak-* measurable. For connected $U \subset \mathbb{R}$ and $V \subset \mathbb{R}^2$, we define

$$\mathrm{Var}\big(\mu_{(\cdot)}\,;\, U, \widehat{\mathrm{C}}^1_c(V)^*\big) = \sup_{\phi\in\mathrm{C}^\infty_c(U\times V),\, \|\mathrm{D}_z\phi\|_{\mathrm{L}^\infty}\leq 1} \int_U \int_V \frac{\partial}{\partial y}\phi(y,z)\mu_y(\mathrm{d}z)\,\mathrm{d}y. \tag{8}$$

Here $\mathrm{D}_z\phi$ denotes the gradient with respect to the z variables only. This measures the total variation in the $\widehat{\mathrm{C}}^1_c(V)^*$ norm of the measure-valued function $y \mapsto \mu_y$. Indeed, one can show that

$$\begin{aligned}&\mathrm{Var}\big(\mu_{(\cdot)}\,;\, U, \widehat{\mathrm{C}}^1_c(V)^*\big)\\ &\quad := \inf_{\{\tilde\mu_{(\cdot)}:\tilde\mu_y=\mu_y \ \text{a.e. } y\}} \sup\Big\{ \sum \|\tilde\mu_{y_i} - \tilde\mu_{y_{i+1}}\|_{\widehat{\mathrm{C}}^1_c(V)^*} \ : \ \cdots < y_{-1} < y_0 < y_1 < \cdots \Big\}.\end{aligned} \tag{9}$$

This is not hard to prove if $y \to \mu_y$ is sufficiently smooth, and for general weak-* measurable $\mu_{(\cdot)}$ it can be proven by an approximation argument.

More general results of the same character are proven in Ambrosio [1].

Remark 2. – we typically work with measures of essentially the form $\mu = \sum_{i=1}^n (\delta_{\xi_i} - \delta_{\eta_i})$ for $\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n \in \mathbb{R}^2$, not necessarily distinct. Brézis, Coron and Lieb [4] show that for such measures, $\|\mu\|_{\widehat{\mathrm{C}}^1_c(\mathbb{R}^2)^*} = \min_{\pi\in\mathfrak{S}_n} \sum_{i=1}^n |\xi_i - \eta_{\pi(i)}|$, where $\mathfrak{S}_n$ is the group of permutations of n objects.

Step 3. – *Blow-up argument* : we now fix some point $x_0 \in \partial^* j$. After a translation and a rotation we may assume that $x_0 = 0$ and that $\tau(x_0) = (1,0,0) := e_1$, the unit vector pointing in the y direction. After these normalizations, (2) can be rewritten

$$\lim_{\lambda\to 0} \frac{1}{\lambda} \int \psi\Big(\frac{x}{\lambda}\Big)[\mathrm{J}^i u](\mathrm{d}x) = \begin{cases} \pi m(0) \int_{\{x=se_1,\, s\in\mathbb{R}\}} \psi(x)\mathcal{H}^1(\mathrm{d}x) & \text{if } i = 1,\\ 0 & \text{if } i = 2,\ 3,\end{cases} \tag{10}$$

for $\psi \in \mathrm{C}^0_c(\mathbb{R}^3\,;\,\mathbb{R})$. In fact, (7) implies that it suffices to show that (10) holds for $\psi \in \mathrm{C}^\infty_c(\mathbb{R}^3)$. We discuss the proof of (10) for $i = 1$; the other cases follow quite easily from (6) and (7).

For each $\lambda > 0$ we define a family of measures on $\mathbb{R}^2$, $\{[\mathrm{J}^\lambda_y]\}_{y\in\mathbb{R}}$ by $[\mathrm{J}^\lambda_y](A) = [\mathrm{J}u_{\lambda y}](\lambda A)$. Then, writing $x = (y,z) \in \mathbb{R}\times\mathbb{R}^2$ as above, one can check using (4) that

$$\frac{1}{\lambda} \int_{\mathbb{R}^3} \psi\Big(\frac{x}{\lambda}\Big)[\mathrm{J}^1 u](\mathrm{d}x) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \psi(y,z)[\mathrm{J}^\lambda_y](\mathrm{d}z)\,\mathrm{d}y. \tag{11}$$

Thus, to establish (10) and prove Theorem 2 it suffices to show that there exists some positive integer m such that for every bounded open $U \subset \mathbb{R}$ and every $V \subset\subset \mathbb{R}^2$,

$$\operatorname*{ess\,sup}_{y\in U} \big\| [\mathrm{J}^\lambda_y] - m\pi\delta_0 \big\|_{\widehat{\mathrm{C}}^1_c(V)^*} \longrightarrow 0 \quad \text{as } \lambda \to 0. \tag{12}$$

The analytic control needed to establish (12) comes from the following estimate: fix $R > 0$ and let $U = (-R,R) \subset \mathbb{R}$ and $V = \mathrm{B}_R(0) \subset \mathbb{R}^2$. Then

$$\mathrm{Var}\big([\mathrm{J}^\lambda_{(\cdot)}]\,;\, U, \widehat{\mathrm{C}}^1_c(V)^*\big) \longrightarrow 0 \quad \text{as } \lambda \to 0. \tag{13}$$

Step 4. – *The BV estimate* : we now prove (13), which will follow from the fact that, for any scalar function ψ,

$$\int \mathrm{D}\psi \cdot [\mathrm{J}u] = \int \mathrm{D}\psi \cdot \tau |\mathrm{J}u| = 0. \tag{14}$$

This in turn is an immediate consequence of (1).

To prove (13), fix any $\psi \in \mathrm{C}_c^\infty(U \times V)$ such that $\|\mathrm{D}_z \psi\|_{\mathrm{L}^\infty} \leq 1$, and use (11) to compute

$$\int_U \int_V \frac{\partial}{\partial y} \psi(y,z)[\mathrm{J}_y^\lambda](\mathrm{d}z)\,\mathrm{d}y = \int_{\lambda U \times \lambda V} \frac{\partial}{\partial y} \psi^\lambda(x)[\mathrm{J}^1 u](\mathrm{d}x) \tag{15}$$

for $\psi^\lambda(x) := \psi(\frac{x}{\lambda})$. We now use (14) to rewrite the right-hand side of (15) in the form:

$$\int_{\lambda U \times \lambda V} \mathrm{D}\psi^\lambda \cdot (\tau_1, 0, 0)|\mathrm{J}u|(\mathrm{d}x) = -\int_{\lambda U \times \lambda V} \mathrm{D}\psi^\lambda \cdot (0, \tau^2, \tau^3)|\mathrm{J}u|(\mathrm{d}x).$$

Note that $\lambda U \times \lambda V$ is just the cylinder $\mathrm{C}_{\lambda R}^{e_1}(0)$, which is contained in the ball $\mathrm{B}_{2\lambda R}(0)$. Moreover, $|(0, \tau_2, \tau_3)| \leq |(1 - \tau_1, \tau_2, \tau_3)| = |\tau(0) - \tau|$. Observing that $\|\mathrm{D}_z \psi^\lambda\|_{\mathrm{L}^\infty} \leq \frac{1}{\lambda}$, we deduce that

$$\begin{aligned}\left| \int \mathrm{D}\psi^\lambda \cdot (0, \tau^2, \tau^3)|\mathrm{J}u|(\mathrm{d}x) \right| &\leq \frac{1}{\lambda} \int_{\mathrm{C}_{\lambda R}^{e_1}} |\tau(0) - \tau(x)|\, |\mathrm{J}u|(\mathrm{d}x) \\ &\leq \frac{|\mathrm{J}u|(\mathrm{B}_{2R\lambda})}{\lambda} \frac{1}{|\mathrm{J}u|(\mathrm{C}_{\lambda R}^{e_1})} \int_{\mathrm{C}_{\lambda R}^{e_1}} |\tau(0) - \tau(x)|\, |\mathrm{J}u|(\mathrm{d}x).\end{aligned}$$

The right-hand side vanishes as $\lambda \to 0$, due to (6) and (7). Since this holds for all ψ as above, we have proven (13). □

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