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SUPERREPLICATION UNDER GAMMA CONSTRAINTS*

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Abstract. In a financial market consisting of a nonrisky asset and a risky one, we study the minimal initial capital needed in order to superreplicate a given contingent claim under a gamma constraint. This is a constraint on the unbounded variation part of the hedging portfolio. We first consider the case in which the prices are given as general Markov diffusion processes and prove a verification theorem which characterizes the superreplication cost as the unique solution of a quasi-variational inequality. In the context of the Black–Scholes model (i.e., when volatility is constant), this theorem allows us to derive an explicit solution of the problem. These results are based on a new dynamic programming principle for general "stochastic target" problems.

Key words. stochastic control, viscosity solutions, stochastic analysis, superreplication, gamma constraint

AMS subject classifications. 35K55, 49J20, 60H30, 90A09

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1. Introduction.

We study the problem of superreplicating a contingent claim under a *gamma* constraint. This is a constraint on the unbounded part of the hedging portfolio.

To explain this constraint and the idea of superreplication, let us first consider the classical Black–Scholes framework with one riskless asset which is normalized to $S^0 = 1$ and one risky asset whose price process evolves according to the stochastic differential equation $dS(t)/S(t) = \mu dt + \sigma dW(t)$. Then given a European contingent claim of the type $g(S(T))$, the unconstrained superreplication cost $v^{BS}(0, S(0))$ is defined as the minimal initial capital which allows us to hedge $g(S(T))$ through some portfolio strategy on the assets S^0 and S. It is known that the solution of this problem coincides with the Black–Scholes arbitrage price of $g(S(T))$ and therefore it is given by $v^{BS}(t, s) = E^Q[g(S(T))|S(t) = s]$. Here $E^Q(.)$ is the expectation operator under the equivalent martingale measure, i.e., Q is the probability measure equivalent to P under which the process S is a martingale. Then the optimal hedging strategy consists of holding $\Delta(t, S(t)) := v_s^{BS}(t, S(t))$ units of the risky asset at each time $t \in [0, T]$.

In practice, traders are faced with shortselling, borrowing, or another type of constraint. These restrictions render this optimal strategy impossible to use in practice, and the notion of superreplication is introduced to replace the no-arbitrage price of Black and Scholes, in the presence of such constraints. We refer to Jouini and Kallal (1995) and Cvitanić and Karatzas (1993) for the superreplication problem with general portfolio constraints. They provide a characterization of the minimal superreplication cost as the value of a stochastic optimal control problem. Broadie, Cvitanić, and Soner (1998) observe that, for a contingent claim of the type $g(S(T))$, this con-

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trol problem can be explicitly solved by proving that the minimal superreplication cost is the unconstrained Black–Scholes price of a modified claim. For the stochastic volatility model, a similar explicit solution is provided in Cvitanič, Pham, and Touzi (1999).

Another problem which in practice faces traders is the variation of the optimal hedging strategy. The gamma associated to the optimal hedging strategy is defined by $\gamma(t, S(t)) := v_{ss}^{BS}(t, S(t))$ and describes the variation of the holdings in S, in the optimal hedging strategy, with respect to an infinitesimal change of the process S. Since traders act only in discrete-time, a large γ induces an important risk exposure between two transaction dates. This problem was raised by Broadie, Cvitanić, and Soner (1998) who provided an upper bound for the superreplication cost under gamma constraint, as well as the associated hedging strategy. However, they did not formulate a precise statement of the problem.

The chief goals of this paper are first to define the superreplication problem under a gamma constraint and then to obtain an explicit solution.

Formulation of the problem is obtained by observing that the gamma constraint is equivalent to a bound on the variation of the hedging portfolio. We then provide a simple solution to this problem. To describe this solution, let $\hat{g}$ be the smallest function greater than g which satisfies the gamma constraint. Then the minimal superreplicating cost with a gamma constraint solves a variational inequality with terminal condition $\hat{g}$. When the volatility is a given constant, the solution of the problem is given by $E^Q[\hat{g}(S(t))]$, i.e., the Black and Scholes no-arbitrage price of the contingent claim $\hat{g}(S(T))$. We explicitly calculate the $\hat{g}$ function for several standard options such as European calls, puts, and digital options.

Previously, the convex duality argument was used to characterize the minimal superreplicating cost. In this approach, the dual formulation of the problem is obtained by suitable changes of measure. However, in the case of gamma constraints, it seems that the diffusion coefficients need to be modified in order to follow a similar technique. Since this cannot be accomplished by equivalent changes of measure, we were not able to use the convex duality arguments. Instead, we introduce a dynamic programming argument to identify the superreplication cost as the viscosity solution of a differential inequality. To our knowledge, this is the first use of dynamic programming in this context. We believe that this is a powerful tool in analyzing "stochastic target" problems and establishing the connection between the backward-forward stochastic differential equations and viscosity solutions as developed in an accompanying paper by the authors Soner and Touzi (2000).

A technical contribution of this paper is a result on the behavior of double stochastic integrals with respect to Brownian motion. This is needed because our formulation of the problem involves a nonclassical constraint on the unbounded variation part of the portfolio process, which is itself the integrand of the martingale part of the state process.

This paper is organized as follows. Section 2 describes the general problem. We introduce the modified terminal data in section 3 and state the assumptions in section 4. After stating the dynamic programming in section 5, we state and prove the main result in section 6. Section 7 focuses on the constant coefficient (i.e., the Black–Scholes) case, and several examples are discussed in section 8. The remainder of the paper is devoted to technical results: section 9 proves the viscosity property, a comparison result is proved in section 10, and finally a property of stochastic double integrals is proved in section 11.

2. Problem. We consider a financial market which consists of one bank account, with constant price process $S^0(t) = 1$ for all $t \in [0, T]$, and one risky asset with price process evolving according to the following stochastic differential equation:

$$S_{t,s}(t) = s \quad \text{and} \quad \frac{dS_{t,s}(u)}{S_{t,s}(u)} = \mu(u, S_{t,s}(u))dt + \sigma(u, S_{t,s}(u))dW^0(u), \qquad t \le u \le T.$$

Here W^0 is a standard Brownian motion in $\mathbb{R}$ defined on a complete probability space $(\Omega, \mathcal{F}, P^0)$. We shall denote by $\mathbb{F} = \{\mathcal{F}(t),\ 0 \le t \le T\}$ the P^0-augmentation of the filtration generated by W^0. The drift and the volatility functions $s\mu(t,s)$ and $s\sigma(t,s)$ satisfy the usual Lipschitz and linear growth conditions in order for the process $S_{t,s}$ to be well defined; we also assume that $\sigma(t,s) > 0$ for all $(t,s) \in [0,T] \times (0, \infty)$ and

$$E^{P^0}\left[\mathcal{E}\left(-\int_0^T \frac{\mu(t, S_{0,s}(t))}{\sigma(t, S_{0,s}(t))} dW^0(t)\right)\right] = 1,$$

where $E^{P^0}(.)$ is the expectation operator under the probability measure P^0 and $\mathcal{E}(.)$ is the Doléans–Dade exponential martingale, i.e.,

$$\mathcal{E}\left(\int_0^T b(t)dW^0(t)\right) = \exp\left(\int_0^T b(t)dW^0(t) - \frac{1}{2}\int_0^T b^2(t)dt\right).$$

As usual, the assumption that the interest rate of the bank account is zero can be easily dispensed with by appropriate discounting.

Consider now an economic agent, endowed with an initial capital x at time t, who invests at each time $u \in [t, T]$ an amount $Y(u)S(u)$ of his wealth in the risky asset and the remaining wealth in the bank account. The process $Y = \{Y(u),\ t \le u \le T\}$ represents the number of shares of risky asset S held by the agent during the time interval $[t, T]$. Then, by the self-financing condition, the wealth process evolves according to the stochastic differential equation

$$X(t) = x \qquad \text{and} \qquad dX(u) = Y(u)dS(u), \qquad t \le u \le T.$$

The purpose of this paper is to introduce constraints on the variations of the hedging portfolio Y. We consider portfolios which are continuous semimartingales with respect to the filtration $\mathbb{F}$. Since $\mathbb{F}$ is the Brownian filtration, we define the controlled portfolio strategy $Y^{\alpha,\gamma}_{t,s,y}$ by

$$\begin{aligned} Y^{\alpha,\gamma}_{t,s,y}(t) &= y, \\ dY^{\alpha,\gamma}_{t,s,y}(u) &= \alpha(u)du + \gamma(u)\frac{dS_{t,s}(u)}{S_{t,s}(u)}, \quad t \le u \le T, \end{aligned} \tag{2.1}$$

where $y \in \mathbb{R}$ is the initial portfolio and the control pair (α, γ) takes values in

$$\mathcal{D}_t := (L^\infty([t,T] \times \Omega;\ \text{Lebesgue} \otimes P^0))^2.$$

Hence a *trading strategy* is defined by the triple (y, α, γ) with $y \in \mathbb{R}$ and $(\alpha, \gamma) \in \mathcal{D}_t$. Then the associated wealth process, denoted by $X^{\alpha,\gamma}_{t,x,s,y}$, satisfies

$$X^{\alpha,\gamma}_{t,x,s,y}(u) = x + \int_t^u Y^{\alpha,\gamma}_{t,s,y}(r)dS_{t,s}(r), \qquad t \le u \le T. \tag{2.2}$$

We shall formulate the gamma constraint by requiring that the process γ be bounded from above. Before making this definition precise, we give a formal discussion. Formally, we expect the hedging portfolio to satisfy

$$Y(u) = v_s(u, S_{t,s}(u)),$$

where v is the minimal superreplication cost. Indeed, this is true in the classical Black–Scholes theory as well as in the case of portfolio constraints; see Broadie, Cvitanič, and Soner (1998). Assuming enough regularity, we apply the Itô formula. The result is

$$dY(u) = A(u)du + \sigma(u, S_{t,s}(u))S_{t,s}(u)v_{ss}(u, S_{t,s}(u))dW^0(u),$$

where $A(u)$ is given in terms of derivatives of v. Compare this equation with (2.1) to conclude that

$$\gamma(u) = S_{t,s}(u)\ v_{ss}(u, S_{t,s}(u)).$$

Therefore a bound on the process γ translates to a bound on sv_{ss}. Notice that, by changing the definition of the process γ in (2.1), we may bound v_{ss} instead of sv_{ss}. However, we choose to study sv_{ss} because it is a dimensionless quantity, i.e., if all the parameters in the problem are increased by the same factor, sv_{ss} remains unchanged.

We now formulate the gamma constraint in the following way. Let Γ be a constant fixed throughout the paper. Given some initial capital $x > 0$, a trading strategy (y, α, γ) is said to be x-admissible if it satisfies the gamma constraint $\gamma(u) \leq \Gamma$ for all $t \leq u \leq T$ almost surely (a.s.) and the associated wealth process $X^{\alpha,\gamma}_{t,x,s,y}$ is nonnegative. We shall denote by

$$\mathcal{A}_{t,s}(x) := \left\{(y, \alpha, \gamma) \in \mathbb{R} \times \mathcal{D}_t : \gamma(.) \leq \Gamma \ \text{ and } \ X^{\alpha,\gamma}_{t,x,s,y}(.) \geq 0\right\}$$

the set of all admissible trading strategies.

We consider a European-type contingent claim $g(S_{t,s}(T))$ defined by the terminal payoff function g. Given such a contingent claim, we then consider the infimum $v(t,s)$ of initial capitals x which induce a wealth process $X^{\alpha,\gamma}_{t,x,s,y}$ through some admissible trading strategy (y, α, γ) such that $X^{\alpha,\gamma}_{t,x,s,y}$ hedges $g(S_{t,s}(T))$, i.e.,

$$v(t,s) = \inf\left\{x : \exists\ (y, \alpha, \gamma) \in \mathcal{A}_{t,s}(x),\ \ X^{\alpha,\gamma}_{t,x,s,y}(T) \geq g(S_{t,s}(T)) \ \text{ a.s.}\right\}. \tag{2.3}$$

Note that if g is convex so is v in the s-variable; hence, in this case, gamma is bounded from below as well.

Our goal is to prove that function $v(t,s)$ solves a variational inequality and that its terminal value is given by some function $\hat{g}$ dominating g. When we focus on the constant volatility case, these observations allow us to derive an explicit solution of the hedging problem (2.3): $v(t,s)$ is the (unconstrained) Black and Scholes price of the modified contingent claim $\hat{g}(S_{t,s}(T))$. This function $\hat{g}$ can be easily computed and several examples are provided in section 7.

Throughout this paper, we shall introduce a probability measure $P \sim P^0$ defined by

$$P(A) = E^{P^0}\left[1_A \mathcal{E}\left(-\int_0^T \frac{\mu(t, S_{0,s}(t))}{\sigma(t, S_{0,s}(t))} dW^0(t)\right)\right] \qquad \text{for all } A \in \mathcal{F}.$$

We shall denote by $E(.)$ the expectation operator under the probability measure P. By Girsanov's theorem, the process W defined by

$$W(u) := W^0(u) + \int_t^u \frac{\mu(r, S_{t,s}(r))}{\sigma(r, S_{t,s}(r))} dr, \qquad t \le u \le T,$$

is a Brownian motion under P. In terms of the Brownian motion W, the risky asset price process is defined by

$$S_{t,s}(t) = s \qquad \text{and} \qquad \frac{dS_{t,s}(u)}{S_{t,s}(u)} = \sigma(u, S_{t,s}(u))dW(u), \qquad t \le u \le T. \tag{2.4}$$

3. Modified terminal data. Due to the constraint, the limit of the value function $v(t,s)$ of (2.3), as t tends to the terminal time T, may not be equal to the contingent claim g. Indeed the determination of this limit is an important step toward the solution of the problem.

We will show in the following sections that the following function $\hat{g}$ is equal to the limit

$$\hat{g}(s) := h^{\text{conc}}(s) + \Gamma s \ln(s), \quad s > 0,$$

where $h(s) = g(s) - \Gamma s \ln(s)$ and h^{conc} is the concave envelope of h, i.e., the smallest concave function greater than h. In other words, function $\hat{g}(s) - \Gamma s \ln(s)$ is the concave envelope of function $g(s) - \Gamma s \ln(s)$. The chief property of $\hat{g}$ that we will use is the following.

LEMMA 3.1. *$\hat{g}$ is the smallest function satisfying the following two conditions:* (i) $\hat{g} \ge g$ *and* (ii) $\hat{g}(s) - \Gamma s \ln(s)$ *is concave.*

Proof. Clearly, $\hat{g}$ satisfies these conditions. Let u be another function satisfying both of them. Set $w(s) := u(s) - \Gamma s \ln(s)$. Clearly $w \ge h$. Since w is concave and $w \ge h$, by the definition of the concave envelope of h, $w \ge h^{\text{conc}}$. Therefore,

$$u(s) = w(s) + \Gamma s \ln(s) \ge h^{\text{conc}}(s) + \Gamma s \ln(s) = \hat{g}(s). \qquad \square$$

In section 6 below, we will show that the terminal data of the minimal superreplicating cost are equal to $\hat{g}$. The formal reason for this is that if v is sufficiently smooth, we formally expect $v(t,s)$ to satisfy the gamma constraint $s v_{ss}(t,s) \le \Gamma$. This is equivalent to the statement that $v(t,s) + \Gamma s \ln(s)$ is concave. Therefore we formally expect the terminal data $\lim_{t \uparrow T} v(T,s)$ to be the smallest function satisfying the two conditions of the previous lemma.

4. Assumptions. We always assume that

$$\text{function } g \text{ is nonnegative and lower semicontinuous.} \tag{4.1}$$

We start with several assumptions on the payoff function g and $\hat{g}$. In sections 7 and 8 below, we will verify that all these assumptions are satisfied by standard claims in the Black–Scholes model, i.e., in the case of a constant volatility function $\sigma(t,s)$; see section 7 below.

Assumption 4.1. We assume that $\hat{g}$ is finite and there exists a nonnegative, strictly concave function $\phi \in C^2$ with $\lim_{s \to \infty} \phi(s) = \infty$ such that

$$\limsup_{s \to \infty} \frac{\hat{g}^{\text{conc}}(s) - E\left[\hat{g}^{\text{conc}}(S_{t,s}(T))\right]}{\phi(s)} < \infty. \tag{4.2}$$

Remark 4.2. Any function g which is growing at most linearly at infinity satisfies $\hat{g}^{\text{conc}}(.) < \infty$. Indeed in this case, $h(s) \leq H(s) := K - \Gamma s \ln(s)/2$ for some constant K. Since H is concave, $h^{\text{conc}} \leq H$, and therefore $\hat{g}$ is finite.

The main use of Assumption 4.1 is to prove a comparison result. The statement and the proof of this result are given in section 10.

Our final assumption is the existence of a smooth solution to the variational inequality

$$\min\{-\mathcal{L}u;\ \Gamma - su_{ss}\}(t,s) = 0 \quad \text{on } [0,T) \times (0,\infty) \tag{4.3}$$

together with the terminal condition

$$u(T,s) = \hat{g}(s) \qquad \text{for all } s > 0, \tag{4.4}$$

where $\mathcal{L}$ is the parabolic operator related to the infinitesimal generator of the stock price process,

$$\mathcal{L} := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2(t,s)s^2\frac{\partial^2}{\partial s^2}.$$

We will prove in section 6 that this solution is equal to the minimal superreplication cost.

Assumption 4.3. The variational inequality (4.3)–(4.4) has a $C^{1,2}([0,T),(0,\infty))$ solution $\hat{v}$ satisfying

(i) $\hat{v}(t,0) = \hat{g}(0)$ for all $t \in [0,T]$,
(ii) $\hat{v}$ is polynomially growing in its s variable at infinity,
(iii) $s\hat{v}_{ss}$, $\mathcal{L}\hat{v}$ are bounded,
(iv) v_s is a $W^{1,2}$ function with generalized derivatives satisfying $\mathcal{L}\hat{v}_s$ bounded.

In section 7, for the constant volatility model, we verify this assumption by providing an explicit solution.

Remark 4.4. By a classical comparison theorem for the equation $\mathcal{L}v = 0$ (see, for instance, Friedman (1964)), we see that $\hat{v}(t,s) \geq E[\hat{g}(S_{t,s}(T))]$. Since g is nonnegative, so is $\hat{g}$; therefore we have

$$\hat{v}(t,s) \geq E\left[\hat{g}(S_{t,s}(T))\right] \geq 0 \qquad \text{for all } (t,s) \in [0,T] \times (0,\infty).$$

5. Dynamic programming. The following is the analogue of the principle of dynamic programming which is standard in the theory of stochastic optimal control theory first proved by R. Bellman.

LEMMA 5.1. *Let $(t,s) \in [0,T) \times [0,\infty)$ and consider an arbitrary stopping time θ valued in $[t,T]$. Suppose that $X^{\alpha,\gamma}_{t,x,s,y}(T) \geq g(S_{t,s}(T))$ P-a.s. for some $(\alpha,\gamma) \in \mathcal{A}_{t,s}(x)$, $y \in \mathbb{R}$, and initial wealth $x \in \mathbb{R}$. Then, for the value function v of* (2.3)*, we have*

$$X^{\alpha,\gamma}_{t,x,s,y}(\theta) \geq v\left(\theta, S_{t,s}(\theta)\right), \quad P\text{-a.s.}$$

Proof. Let x, y, θ, and (α,γ) be as in the above statement. Set $\hat{x} = X^{\alpha,\gamma}_{t,x,s,y}(\theta)$, $\hat{s} = S_{t,s}(\theta)$, $\hat{y} = Y^{\alpha,\gamma}_{t,s,y}(\theta)$. Clearly $Y_{t,s,y} = Y_{\theta,\hat{s},\hat{y}}$. By definition of the wealth process (2.2), this provides

$$X^{\alpha,\gamma}_{t,x,s,y}(T) = X^{\alpha,\gamma}_{\theta,\hat{x},\hat{s},\hat{y}}(T).$$

Also, by uniqueness of the solution for the stochastic differential equation defining the stock price S, we have $S_{t,s} = S_{\theta,\hat{s}}$. Since $X^{\alpha,\gamma}_{\theta,\hat{x},\hat{s},\hat{y}}(T) = X^{\alpha,\gamma}_{t,x,s,y}(T) \geq g\left(S_{t,s}(T)\right)$

$= g(S_{\theta,\hat{s}}(T))$, it follows that $\hat{x} \geq v(\theta, \hat{s})$ by definition of the control problem $v(\theta, \hat{s})$. $\square$

Remark 5.2. As in optimal control theory, the second part of the dynamic programming is also available. A systematic study of dynamic programming is given in an accompanying paper by the authors. Since we do not need the second part of the dynamic programming in this paper, we refer the reader to Soner and Touzi (2000) for a discussion of the full dynamic programming.

6. Main result. Let $\hat{v}$ be the solution of the variational inequality (4.3)–(4.4) introduced in Assumption 4.3.

THEOREM 6.1. *Let Assumptions* 4.1 *and* 4.3 *hold. Then, the value function v of the hedging problem* (2.3) *is equal to the unique smooth solution of the variational inequality* (4.3)–(4.4)*, i.e.,*

$$v = \hat{v}.$$

Notice that the variational inequality (4.3)–(4.4) was not assumed to have a unique solution satisfying the requirement of Assumption 4.3. Uniqueness is obtained as a consequence of the above theorem.

Let v_* be the lower semicontinuous envelope of v:

$$v_*(t,s) := \liminf_{(t',s') \to (t,s)} v(t,s). \tag{6.1}$$

We prove the theorem after assuming two properties of the value function v.

P1. Function $s \longmapsto v_*(t,s) - \Gamma s \ln(s)$ is concave for all $t \in [0,T)$.

P2. v_* is a viscosity supersolution of the equation $-\mathcal{L}u = 0$ on $[0,T) \times (0,\infty)$.

These properties will be verified in section 9 below.

Proof. We start with the inequality $v \leq \hat{v}$. For $t \leq u \leq T$, set

$$y = \hat{v}_s(t,s), \qquad \alpha(u) = \mathcal{L}\hat{v}_s(u, S(u)), \qquad \gamma(u) = S_{t,s}(u)\hat{v}_{ss}(u, S(u)).$$

Since $\mathcal{L}\hat{v} \leq 0$,

$$\begin{aligned} g(S_{t,s}(T)) &\leq \hat{g}(S_{t,s}(T)) = \hat{v}(T, S_{t,s}(T)) \\ &= \hat{v}(t,s) + \int_t^T \mathcal{L}\hat{v}(u, S_{t,s}(u))du + \hat{v}_s(u, S_{t,s}(u))dS_{t,s}(u) \\ &\leq \hat{v}(t,s) + \int_t^T Y_{t,y}^{\alpha,\gamma}(u)dS_{t,s}(u); \end{aligned}$$

in the last step we applied the generalized Itô's formula to $v_s \in W^{1,2}$. (See Krylov (1980, Theorem 1, p. 122) for Itô's formula with generalized derivatives.) By Assumption 4.3, $(\alpha, \gamma) \in \mathcal{D}_t$. Furthermore, since $\hat{v}$ solves the variational inequality (4.3), $\gamma(u) \leq \Gamma$ for all $u \in [t,T]$. By Remark 4.4, $\hat{v}(u, S_{t,s}(u)) = X_{t,x,s,y}^{\alpha,\gamma}(u) \geq 0$ with $x = \hat{v}(t,s)$. Hence $(y, \alpha, \gamma) \in \mathcal{A}_{t,s}(\hat{v}(t,s))$, and by the definition of the minimal replicating price, we conclude that $v \leq \hat{v}$.

We now prove the reverse inequality. Fix $(t,s) \in (0,T) \times (0,\infty)$, and $\delta > 0$. By the definition of v, there exist an initial wealth $x \in [v(t,s), v(t,s)+\delta)$ and a trading strategy $(y^t, \alpha^t, \gamma^t) \in \mathcal{A}_{t,s}(x)$ satisfying

$$X_{t,x,s,y^t}^{\alpha^t,\gamma^t}(T) \geq g(S_{t,s}(T)) \quad P\text{-a.s.}$$

Therefore,

$$\delta + v(t,s) + \int_t^T Y_{t,s,y^t}^{\alpha^t,\gamma^t}(u)dS_{t,s}(u) \geq g\left(S_{t,s}(T)\right) \quad P\text{-a.s.}$$

By the definition of $\mathcal{A}_{t,s}(x)$, the local martingale $\{\int_t^u Y_{t,s,y^t}^{\alpha^t,\gamma^t}(r)dS_{t,s}(r),\, u \geq t\}$ is bounded from below and is therefore a supermartingale. We take the expected value in the last inequality and then use this fact. The result is

$$\delta + v(t,s) \geq E\left[g(S_{t,s}(T))\right].$$

Since $\delta > 0$ is arbitrary and g is lower semicontinuous, Fatou's lemma yields

$$v_*(T,s) = \liminf_{(t,s')\to(T,s)} v(t,s') \geq g(s) \qquad \text{for all } s > 0.$$

In view of the property P1, $v_*(T,\cdot)$ satisfies both conditions stated in Lemma 3.1, and therefore $v_*(T,s) \geq \hat{g}(s)$.

By dynamic programming, for any $(y,\alpha,\gamma) \in \mathcal{A}_{t,s}(x)$ satisfying $X_{t,x,s,y}^{\alpha,\gamma}(T) \geq g(S_{t,s}(T))$,

$$X_{t,x,s,y}^{\alpha,\gamma}(u) \geq v(u, S_{t,s}(u)) \qquad \text{for all } u \in [t,T].$$

Since we have shown that $v_*(T,s) \geq \hat{g}(s)$, by taking the limit as u tends to T, we conclude that

$$X_{t,x,s,y}^{\alpha,\gamma}(T) \geq \hat{g}(S_{t,s}(T)).$$

Therefore, any strategy that dominates g also dominates $\hat{g}$. Since $\hat{g} \geq g$, this provides

$$v(t,s) = \inf\left\{x : \exists\, (y,\alpha,\gamma) \in \mathcal{A}_{t,s}(x),\;\; X_{t,x,s,y}^{\alpha,\gamma}(T) \geq \hat{g}(S_{t,s}(T)) \text{ a.s.}\right\}, \tag{6.2}$$

i.e., v is the minimal superreplication cost for the claim $\hat{g}$. By definition, the Black–Scholes price (i.e., unconstrained superreplication cost) is always smaller than the superreplication cost with gamma constraint,

$$v(t,s) \geq E\left[\hat{g}(S_{t,s}(T))\right] \qquad \text{for all } (t,s) \in [0,T) \times (0,\infty). \tag{6.3}$$

Moreover, by (6.2), $v(t,0+) = \hat{g}(0)$ for all $t \in [0,T)$. Therefore, $v_*(t,0) \leq v(t,0) = \hat{g}(0)$. Also (6.3) together with Fatou's lemma yield $v_*(t,0) \geq \hat{g}(0)$. Hence $v_*(t,0) = \hat{g}(0)$.

In view of Lemma 9.2 below, v_* is a lower semicontinuous viscosity supersolution of (4.3)–(4.4). By Theorem 10.1, $v_* \geq \hat{v}$. This completes the proof of the theorem since $v \geq v_*$. □

Remark 6.2. In the first part of the above proof, the optimal hedging strategy (y,α,γ) is expressed explicitly in terms of the derivatives of the minimal superreplication cost function $\hat{v}$.

Remark 6.3. In the proof above, it is shown (without appealing to Theorem 10.1) that the (unconstrained) Black and Scholes price of $\hat{g}(S_{t,s}(T))$ is a trivial lower bound for v

$$v(t,s) \geq E\left[\hat{g}(S_{t,s}(T))\right] \qquad \text{for all } (t,s) \in [0,T) \times (0,\infty).$$

We shall use this lower bound in the proof of the comparison Theorem 10.1.

7. The Black and Scholes model. In this section, we focus on a discussion of the Black and Scholes model in which the volatility function $\sigma(t,s)$ is constant, i.e., $\sigma(t,s)=\sigma$ for all $(t,s)\in[0,T]\times(0,\infty)$.

We shall provide an explicit solution to the hedging problem (2.3) under the following condition.

Assumption 7.1. Function $s\longmapsto h^{\mathrm{conc}}(s)-Cs\ln(s)$ is convex for some constant C.

Remark 7.2. Suppose that function g is such that $s\longmapsto g(s)+As\ln(s)$ is convex for some constant A. Then, since $h(s)=g(s)+As\ln(s)-(\Gamma+A)s\ln(s)$, it follows from the construction of the concave envelope that Assumption 7.1 is satisfied by $C=\Gamma+A$.

THEOREM 7.3. *Let Assumptions* 4.1 *and* 7.1 *hold. Then, Assumption* 4.3 *holds and the value function* v *of the hedging problem* (2.3) *is simply the unconstrained Black and Scholes price* $\hat{v}$ *of the contingent claim* $\hat{g}(S_{t,s}(T))$, *i.e.,*

$$v(t,s)=\hat{v}(t,s)=E\left[\hat{g}(S_{t,s}(T))\right]\qquad \textit{for all } (t,s)\in[0,T]\times(0,\infty).$$

Proof. Denote $\tilde{v}(t,s):=E\left[\hat{g}(S_{t,s}(T))\right]$. Then $\tilde{v}$ is a classical solution to the equation

$$-\mathcal{L}u=0\quad\text{on } [0,T)\times(0,\infty)\qquad\text{and}\qquad u(T,s)=\hat{g}(s),\quad s>0.$$

Furthermore, by the definition of $\hat{g}$,

$$\tilde{v}(t,s)-\Gamma s\ln(s)=E\left[h^{\mathrm{conc}}(S_{t,s}(T))\right]+\frac{1}{2}\sigma^2(T-t)\Gamma s. \tag{7.1}$$

Since h^{conc} is concave and $S_{t,s}(T)$ is linear in s, this proves that for all $t\in[0,T]$, function $s\to\tilde{v}(t,s)-\Gamma s\ln(s)$ is concave, and therefore $s\tilde{v}_{ss}(t,s)\le\Gamma$ for all $(t,s)\in[0,T)\times(0,\infty)$. A similar argument using Assumption 7.1 shows that $s\tilde{v}_{ss}(t,s)\ge C$.

Consequently $\tilde{v}=\hat{v}$ is a classical solution of the variational inequality (4.3)–(4.4). By Friedman (1964, Theorem 10, p. 72), function $\hat{v}_s$ is $C^{1,2}$, which provides all the regularity required in Assumption 4.3, except the property (iii). To verify Assumption 4.3 (iii), we differentiate the equation $\mathcal{L}\hat{v}=\hat{v}_t(t,s)+\sigma^2s^2\hat{v}_{ss}(t,s)=0$ to obtain $\mathcal{L}\hat{v}_s(t,s)=\sigma^2s\hat{v}_{ss}(t,s)$. Since we have already proved that $s\hat{v}_{ss}$ is bounded, so is $\mathcal{L}\hat{v}$. ∎

Remark 7.4. Observe that Assumption 4.1 is only used in the proof of the comparison Theorem 10.1 which is needed to show that $\hat{v}\le v$. Since in the Black and Scholes case $\hat{v}(t,s)=E[\hat{g}(S_{t,s}(T))]$, the variational inequality (4.3) reduces to the linear equation $-\mathcal{L}v=0$. Then we can appeal to the standard comparison theorem for this equation, and Assumption 4.1 can be relaxed by requiring only that $\hat{g}(.)<\infty$.

8. Examples.

European call option. Let $g(s)=(s-K)^+$, $s>0$. Since g is convex, Assumption 7.1 is satisfied; see Remark 7.2. Next, it is easily checked that the concave envelope of function $h(s)=(s-K)^+-\Gamma s\ln(s)$ is given by

$$h^{\mathrm{conc}}(s)=\begin{cases} h(s), & s\in(0,\infty)\setminus[s_1,s_2],\\ h(s_1)+h'(s_1)(s-s_1), & s\in[s_1,s_2],\end{cases}$$

i.e., h^{conc} coincides with h outside the interval $[s_1,s_2]$ and is defined by a straight line in $[s_1,s_2]$. The values s_1 and s_2 are characterized by

$$s_1<K<s_2\, h'(s_1)=h'(s_2)\qquad\text{and}\qquad h(s_2)=h(s_1)+h'(s_1)(s_2-s_1).$$

A direct calculation yields

$$s_1 = \frac{K}{\Gamma(e^{1/\Gamma} - 1)} \qquad \text{and} \qquad s_2 = \frac{Ke^{1/\Gamma}}{\Gamma(e^{1/\Gamma} - 1)}.$$

Therefore,

$$\hat{g}(s) = \begin{cases} (s-K)^+, & s \in (0,\infty) \setminus [s_1, s_2], \\ \Gamma\left(s \ln \frac{s}{s_1} + s_1 - s\right), & s \in [s_1, s_2]. \end{cases}$$

Since $\hat{g}^{\mathrm{conc}}(s) = s$ for all $s > 0$, Assumption 4.1 is clearly satisfied and Theorem 7.3 applies.

European put option. We now consider the case $g(s) = (K-s)^+$, $s > 0$. As in the previous example, g is convex, and therefore Assumption 7.1 is satisfied. The concave envelope of function $h(s) = (K-s)^+ - \Gamma s \ln(s)$ is given by

$$h^{\mathrm{conc}}(s) = \begin{cases} h(s), & s \in (0,\infty) \setminus [s_1, s_2], \\ h(s_1) + h'(s_1)(s - s_1), & s \in [s_1, s_2], \end{cases}$$

i.e., h^{conc} coincides with h outside the interval $[s_1, s_2]$ and is defined by a straight line in $[s_1, s_2]$. The values s_1 and s_2 are characterized by

$$s_1 < K < s_2 \, h'(s_1) = h'(s_2) \qquad \text{and} \qquad h(s_2) = h(s_1) + h'(s_1)(s_2 - s_1).$$

We directly calculate that

$$s_1 = \frac{K}{\Gamma(e^{1/\Gamma} - 1)} \qquad \text{and} \qquad s_2 = \frac{Ke^{1/\Gamma}}{\Gamma(e^{1/\Gamma} - 1)}$$

(the same values as in the first example) and

$$\hat{g}(s) = \begin{cases} (K-s)^+, & s \in (0,\infty) \setminus [s_1, s_2], \\ K - s + \Gamma\left(s \ln \frac{s}{s_1} + s_1 - s\right), & s \in [s_1, s_2]. \end{cases}$$

Since $\hat{g}$ is bounded, Assumption 4.1 holds and therefore Theorem 7.3 applies.

Straddle option. We now study the contingent claim defined by $g(s) = (s-K)^+ + (K-s)^+$, $s > 0$. The same argument as in the previous examples yields

$$\hat{g}(s) = \begin{cases} (s-K)^+ + (K-s)^+, & s \in (0,\infty) \setminus [s_1, s_2], \\ K - s + \Gamma\left(s \ln \frac{s}{s_1} + s_1 - s\right), & s \in [s_1, s_2], \end{cases}$$

where $s_1 = \frac{2K}{\Gamma(e^{2/\Gamma}-1)}$ and $s_2 = s_1 e^{2/\Gamma}$.

Digital option. Our last example is the contingent claim defined by $g(s) = 1_{\{s>K\}}$, $s > 0$. Then, it is easily seen that the concave envelope of function $h(s) = 1_{s>K} - \Gamma s \ln(s)$ is given by

$$h^{\mathrm{conc}}(s) = \begin{cases} h(s), & s \in (0,\infty) \setminus [s^*, K], \\ h(s^*) + h'(s^*)(s - s^*), & s^* \le s \le K, \end{cases}$$

where s^* is the unique solution of

$$0 < s^* < \Gamma \qquad \text{and} \qquad s^* - K \ln(s^*) = K - K \ln(K) + \frac{1}{\Gamma}.$$

Clearly, the above function satisfies Assumption 7.1. This provides the candidate for the hedging problem under the gamma constraint:

$$\hat{g}(s) = \begin{cases} 0, & s \ \le s^*, \\ \Gamma s \ln(s) + h(s^*) + h'(s^*)(s - s^*), & s^* \le s \le K, \\ 1, & s \ \ge K. \end{cases}$$

Since $\hat{g} \le 1$, we have $\hat{g}^{\text{conc}} \le 1$ and Assumption 4.1 holds. Then, Theorem 7.3 again applies.

9. Viscosity property. In this section, we prove properties P1 and P2 of section 6.

THEOREM 9.1. *v_* is a viscosity supersolution of the variational inequality*

$$\min\{-\mathcal{L}u(t,s),\ \Gamma - s u_{ss}(t,s)\} = 0 \tag{9.1}$$

on $(0,T) \times (0,\infty)$.

Proof. For $\varepsilon \in (0,1]$, set

$$\mathcal{A}^{\varepsilon}_{t,s}(x) := \left\{ (y,\alpha,\gamma) \in \mathcal{A}_{t,s}(x) : |\alpha(.)| + |\gamma(.)| \le \varepsilon^{-1} \right\},$$

and

$$v^{\varepsilon}(t,s) = \inf\left\{ x : \exists\, (y,\alpha,\gamma) \in \mathcal{A}^{\varepsilon}_{t,s}(x),\ \ X^{\alpha,\gamma}_{t,x,s,y}(T) \ge g(S_{t,s}(T)) \ \text{ a.s.} \right\}.$$

Let v^{ε}_* be the lower semicontinuous envelope of v^{ε}; cf. (6.1). It is clear that v^{ε} also satisfies the dynamic programming equation of Lemma 5.1.

First we will show that v^{ε}_* is a viscosity supersolution of (9.1). Let $\varphi \in C^{\infty}(\mathbb{R}^2)$ and $(t_0,s_0) \in (0,T) \times (0,\infty)$ satisfy

$$(v^{\varepsilon}_* - \varphi)(t_0,s_0) = \min_{(t,s)\in(0,T)\times(0,\infty)} (v^{\varepsilon}_* - \varphi)(t,s).$$

We need to show that

$$-\mathcal{L}\varphi(t_0,s_0) \ge 0 \qquad \text{and} \qquad s_0\varphi_{ss}(t_0,s_0) \le \Gamma. \tag{9.2}$$

We may assume that $(v^{\varepsilon}_* - \varphi)(t_0,s_0) = 0$ so that $v^{\varepsilon}_* \ge \varphi$.

Choose $(t_n,s_n) \to (t_0,s_0)$ so that $v^{\varepsilon}(t_n,s_n)$ converges to $v^{\varepsilon}_*(t_0,s_0)$. For each n, by the definition of v^{ε} and the dynamic programming, there are $x_n \in [v^{\varepsilon}(t_n,s_n), v^{\varepsilon}(t_n,s_n)+1/n]$ hedging strategies $(y_n,\alpha_n,\gamma_n) \in \mathcal{A}^{\varepsilon}_{t_n,s_n}(x_n)$ satisfying

$$X^{\alpha_n,\gamma_n}_{t_n,x_n,s_n,y_n}(t_n+t) - v^{\varepsilon}(t_n+t, S_{t_n,s_n}(t_n+t)) \ge 0$$

for every $t > 0$. Since $v^{\varepsilon} \ge v^{\varepsilon}_* \ge \varphi$,

$$x_n + \int_{t_n}^{t_n+t} Y^{\alpha_n,\gamma_n}_{t_n,s_n,y_n}(u)\,dS_{t_n,s_n}(u) - \varphi(t_n+t, S_{t_n,s_n}(t_n+t)) \ge 0.$$

Set

$$\beta_n := x_n - \varphi(t_n,s_n)$$

and observe that $\beta_n \to 0$ as $n \to \infty$, since $\varphi(t_n,s_n) \to \varphi(t_0,s_0) = v^{\varepsilon}_*(t_0,s_0)$, $|x_n - v^{\varepsilon}(t_n,s_n)| \le 1/n$, and $v^{\varepsilon}(t_n,s_n) \longrightarrow v^{\varepsilon}_*(t_0,s_0)$.

By Itô's lemma,

$$M_n(t) \leq D_n(t) + \beta_n \tag{9.3}$$

for every $t \geq 0$, where

$$M_n(t) = \int_0^t \left[\varphi_s(t_n+u, S_{t_n,s_n}(t_n+u)) - Y^{\alpha_n,\gamma_n}_{t_n,s_n,y_n}(t_n+u)\right] dS_{t_n,s_n}(t_n+u),$$
$$D_n(t) = -\int_0^t \mathcal{L}\varphi(t_n+u, S_{t_n,s_n}(t_n+u))du.$$

For some sufficiently large positive constant λ, define the stopping time $t_n+\theta_n$ by

$$\theta_n := \inf\left\{u > 0 : \left|\ln\left(S_{t_n,s_n}(t_n+u)/s_n\right)\right| \geq \lambda\right\}$$

and observe that the sequence of stopping times (θ_n) satisfies

$$\liminf_{n\to\infty} t\wedge\theta_n \geq \frac{1}{2}t\wedge\theta_0 \qquad P\text{-a.s.}$$

for all $t > 0$; see Remark 11.2. By the smoothness of $\mathcal{L}\varphi$, the integrand in the definition of M_n is bounded up to the stopping time θ_n and therefore, taking the expectation in (9.3) provides

$$-E\left[\int_0^{t\wedge\theta_n} \mathcal{L}\varphi(t_n+u, S_{t_n,s_n}(t_n+u))du\right] \geq -\beta_n.$$

By sending n to infinity, we obtain

$$-E\left[\int_0^{t\wedge\theta_0} \mathcal{L}\varphi(t_0+u, S_{t_0,s_0}(t_0+u))du\right] \geq 0$$

by dominated convergence and continuity of $\mathcal{L}\varphi$. Then, dividing by t and taking the limit as $t \searrow 0$, we get by dominated convergence

$$-\mathcal{L}\varphi(t_0, s_0) \geq 0,$$

which is the first part of (9.2). It remains to prove the second inequality.

By another application of Itô's lemma, it follows that

$$M_n(t) = \int_0^t \left(z_n + \int_0^u a_n(r)dr + \int_0^u b_n(r)dS_{t_n,s_n}(t_n+r)\right) dS_{t_n,s_n}(t_n+u),$$

where

$$\begin{aligned} z_n &= \varphi_s(t_n, s_n) - y_n, \\ a_n(r) &= \mathcal{L}\varphi_s(t_n+r, S_{t_n,s_n}(t_n+r)) - \alpha_n(t_n+r), \\ b_n(r) &= \varphi_{ss}(t_n+r, S_{t_n,s_n}(t_n+r)) - \frac{\gamma_n(t_n+r)}{S_{t_n,s_n}(t_n+r)}. \end{aligned}$$

Observe that the processes $a_n(.\wedge\theta_n)$ and $b_n(.\wedge\theta_n)$ are bounded uniformly in n since $\mathcal{L}\varphi_s$ and φ_{ss} are smooth functions. By (9.3),

$$M_n(t\wedge\theta_n) \leq D_n(t\wedge\theta_n) + \beta_n \leq Ct\wedge\theta_n + \beta_n$$

for some positive constant C. We now apply the results of Propositions 11.5 and 11.6 to the martingales M_n. The result is

$$\lim_{n\to\infty} y_n = \varphi_s(t_0, y_0) \qquad \text{and} \qquad \liminf_{n\to\infty,\ t\searrow 0} b(t) \le 0,$$

where b is the L^2 weak limit of the sequence (b_n). The remaining inequality in (9.2) is obtained after recalling that $\gamma_n(t) \le \Gamma$.

Hence v_*^ε is a viscosity supersolution of (9.1). Since

$$v_*(t,s) = \liminf_* v^\varepsilon(t,s) = \liminf_{\varepsilon\to 0,(t',s')\to(t,s)} v_*^\varepsilon(t',s'),$$

the Barles–Perthame technique implies that v_* is a viscosity supersolution of (9.1) as well. ∎

The following result completes the proof of the properties P1 and P2 of section 6.

LEMMA 9.2. *Let f be a lower semicontinuous function defined on $(0,\infty)$. Then, f is a viscosity supersolution of $\Gamma - s f_{ss}(s) \ge 0$ if and only if $f(s) - \Gamma s \ln(s)$ is concave.*

Proof. Suppose that $h(s) := f(s) - \Gamma s \ln(s)$ is a concave function and a smooth test function φ and $s_0 > 0$ satisfy

$$0 = (f-\varphi)(s_0) = \min\ \{\ (f-\varphi)(s)\ :\ s \ge 0\ \}.$$

Set $\psi(s) := \varphi(s) - \Gamma s \ln(s)$, so that for any $\delta > 0$,

$$\psi(s_0+\delta) + \psi(s_0-\delta) - 2\psi(s_0) \le h(s_0+\delta) + h(s_0-\delta) - 2h(s_0) \le 0.$$

We divide by δ^2 and let δ go to zero. The result is $\varphi_{ss}(s_0) \le \Gamma/s_0$. Hence, f is a viscosity supersolution of $-s f_{ss}(s) + \Gamma \ge 0$.

Now suppose that f is a viscosity supersolution of $-s f_{ss}(s) + \Gamma \ge 0$. We need to show that

$$h(s+\delta) + h(s-\delta) - 2h(s) \le 0$$

for any $\delta > 0$. Suppose that there exist s_0 and $\delta > 0$ such that

$$\alpha := h(s_0+\delta) + h(s_0-\delta) - 2h(s_0) > 0.$$

Set

$$\psi(s) := h(s_0) + \frac{h(s_0+\delta) - h(s_0-\delta)}{2\delta}\ (s-s_0) + \frac{\alpha}{4\delta^2}(s-s_0)^2.$$

Then, $(h-\psi)(s_0) = 0$ and

$$(h-\psi)(s_0 \pm \delta) = \frac{1}{2}\,[h(s_0+\delta) + h(s_0-\delta) - 2h(s_0)] - \frac{\alpha}{4} = \frac{\alpha}{4}.$$

Hence, $(h-\psi)$ attains a local minimum in $(s_0-\delta, s_0+\delta)$. Set $\varphi(s) := \psi(s) + \Gamma s \ln(s)$ so that $(f-\varphi)$ attains a local minimum in the same interval, say at s^*. We calculate that

$$\Gamma - s^* \varphi_{ss}(s^*) = -s^* \frac{\alpha}{2\delta^2} < 0.$$

This contradicts the supersolution property of f. ∎

10. The comparison result. This section is devoted to the proof of a comparison theorem which was used in the proof of our main result. We refer to Crandall, Ishii, and Lions (1992) and Fleming and Soner (1993) for the definition and the properties of viscosity solutions.

THEOREM 10.1. *Let Assumption* 4.1 *hold. Suppose that the variational inequality* (4.3)–(4.4) *has a solution* $\hat{v} \in C^{1,2}([0,T] \times (0,\infty))$ *which is polynomially growing and has bounded* $\mathcal{L}\hat{v}$. *Let* u *be a lower semicontinuous viscosity supersolution of* (4.3) *satisfying* $u(T,\cdot) \geq \hat{g}$, $u(\cdot,0) \geq \hat{v}(\cdot,0)$, *and* $u(t,s) \geq E\left[\hat{g}(S_{t,s}(T)\right]$. *Then,*

$$u \geq \hat{v} \qquad \text{on } [0,T] \times (0,\infty).$$

We start with deriving an upper bound for the solution $\hat{v}$ of (4.3)–(4.4).

LEMMA 10.2. *For all* $(t,s) \in [0,T] \times (0,\infty)$, $\hat{v}(t,s) \leq \hat{g}^{\mathrm{conc}}(s)$.

Proof. To prove this result, we first show that $\hat{v}$ is related to some stochastic control problem. Let $\mathcal{N}$ be the set of all bounded nonnegative progressively measurable processes. For all $\nu \in \mathcal{N}$, consider the controlled process $S^\nu_{t,s}$ defined by

$$\frac{dS^\nu_{t,s}(u)}{S^\nu_{t,s}(u)} = \left[\frac{\nu(u)}{1+S^\nu_{t,s}(u)} + \sigma^2\left(t, S^\nu_{t,s}(u)\right)\right]^{1/2} dW(u).$$

Notice that the random function $s \longmapsto s\left[\nu(1+s)^{-1} + \sigma^2(t,s)\right]^{1/2}$ is Lipschitz uniformly in t and therefore the process S^ν is well defined. Next, for some small parameter $\eta > 0$, define the stochastic control problem

$$u(t,s) := \sup_{\nu \in \mathcal{N}} E\left[\hat{g}(S^\nu_{t,s}(T)) - \frac{1}{2}(\Gamma - \eta)\int_t^T \nu(u)\frac{S^\nu_{t,s}(u)}{1+S^\nu_{t,s}(u)}du\right]$$

and consider the approximating problems

$$u^n(t,s) := \sup_{\nu \in \mathcal{N}^n} E\left[\hat{g}(S^\nu_{t,s}(T)) - \frac{1}{2}(\Gamma - \eta)\int_t^T \nu(u)\frac{S^\nu_{t,s}(u)}{1+S^\nu_{t,s}(u)}du\right]$$

with $\mathcal{N}^n$ consisting of elements in $\mathcal{N}$ which are bounded by n. Clearly, for every n we have $u^n(t,s) \leq u(t,s)$ for all $(t,s) \in [0,T] \times (0,\infty)$. By classical arguments, it is easily checked that u^n is a viscosity solution of the Hamilton–Jacobi–Bellman (HJB) equation

$$-\sup_{0 \leq \nu \leq n}\left\{w_t + \frac{1}{2}s^2\left(\sigma^2(t,s) + \frac{\nu}{1+s}\right)w_{ss} - \frac{1}{2}(\Gamma-\eta)\nu\frac{s}{1+s}\right\} = 0$$

which can be written as

$$-\mathcal{L}w - \frac{1}{2}n\frac{s}{1+s}[sw_{ss} - (\Gamma-\eta)]^+ = 0 \qquad \text{on } [0,T) \times (0,\infty). \tag{10.1}$$

Now recall that $\hat{v}$ is a classical solution to (4.3).

Case 1. $s\hat{v}_{ss} < \Gamma$, then $\mathcal{L}\hat{v} = 0$ and therefore $-\mathcal{L}\hat{v} - \frac{1}{2}n\frac{s}{1+s}[s\hat{v}_{ss} - (\Gamma-\eta)]^+ \leq 0$.

Case 2. $s\hat{v}_{ss} = \Gamma$, then $\mathcal{L}\hat{v} \geq 0$ and $-\mathcal{L}\hat{v} - \frac{1}{2}n\frac{s}{1+s}[s\hat{v}_{ss} - (\Gamma-\eta)]^+ \leq -\mathcal{L}\hat{v} - \frac{1}{2}n\eta\frac{s}{1+s}$ ≤ 0 for sufficiently large n; recall that $\mathcal{L}\hat{v}$ is assumed to be bounded uniformly in (t,s).

We have then proved that $\hat{v}$ is a subsolution of the HJB equation (10.1) for sufficiently large n. Since $\hat{v}(T,s) = u^n(T,s) = \hat{g}(s)$, it follows from the comparison theorem (which will be verified at the end of this proof) that $\hat{v} \leq u^n$ and therefore

$$\hat{v}(t,s) \leq u(t,s) \qquad \text{for all } (t,s) \in [0,T] \times (0,\infty).$$

We then have

$$\hat{v}(t,s) \le \sup_{\nu\in\mathcal{N}} E\left[\hat{g}(S^{\nu}_{t,s}(T))\right] \le \sup_{\nu\in\mathcal{N}} E\left[\hat{g}^{\text{conc}}(S^{\nu}_{t,s}(T))\right].$$

By the Jensen inequality and the martingale property of the process $S^{\nu}_{t,s}$, this provides

$$\hat{v}(t,s) \le \sup_{\nu\in\mathcal{N}} \hat{g}^{\text{conc}}\left(E\left[S^{\nu}_{t,s}(T)\right]\right) = \hat{g}^{\text{conc}}(s).$$

It remains to prove the comparison theorem for (10.1). Let m be the growth rate of $\hat{v}$, i.e., $\hat{v}(t,s) \le C(1+s^m)$ for some constant C. Take some $\lambda \ge m(m+1)\sigma^2(t,s)/2$ (recall that $s \longmapsto s\sigma(t,s)$ is Lipschitz uniformly in t and therefore σ is bounded). Choose a minimizer at (t_0,s_0) of

$$\psi(t,s) = e^{\lambda t}u^n(t,s) - e^{\lambda t}\hat{v}(t,s) + \varepsilon s^{m+1},$$

where ε is a small positive parameter. Since $u^n \ge 0$ and $\hat{v}$ is growing at the rate m, ϕ attains its minimum. If $s_0 = 0$ or $t_0 = T$, then $\psi(t_0,s_0) \ge 0$ by the boundary conditions. Now, suppose that $s_0 > 0$ and $t_0 < T$. Since u^n is a viscosity solution of (10.1) and $\hat{v}$ is a classical subsolution of (10.1), it follows that

$$\begin{aligned}
&\lambda e^{\lambda t_0}[u^n(t_0,s_0) - \hat{v}(t_0,s_0)] + \varepsilon\frac{1}{2}\sigma^2(t_0,s_0)m(m+1)s_0^{m(m+1)} \\
&\quad \ge e^{\lambda t_0}\frac{n}{2}\frac{s_0}{1+s_0}\left\{[s_0\hat{v}_{ss}(t_0,s_0) - \Gamma]^+ - [s_0\hat{v}_{ss}(t_0,s_0) - e^{-\lambda t_0}\Gamma - \varepsilon m(m+1)s_0^m]^+\right\} \\
&\quad \ge 0.
\end{aligned}$$

Then $\psi(t_0,s_0) \ge 0$ from the choice of the parameter λ. By sending ε to zero, we obtain the comparison result for (10.1). $\square$

Proof of Theorem 10.1. Fix some positive scalar λ and set $\hat{w}(t,s) = \hat{v}(t,s)e^{-\lambda t}$ and $w(t,s) = u(t,s)e^{-\lambda t}$ for all $(t,s) \in [0,T]\times(0,\infty)$. Then $\hat{w}$ is a $C^{1,2}([0,T)\times(0,\infty))$ solution of the variational inequality

$$\begin{aligned}
\min\left\{\lambda\hat{w} - \mathcal{L}\hat{w};\ \Gamma e^{-\lambda t} - s\hat{w}_{ss}\right\} = 0 \quad \text{on } [0,T)\times(0,\infty), \\
\hat{w}(T,s) = \hat{g}(s)e^{-\lambda T}, \quad s > 0,
\end{aligned} \tag{10.2}$$

and w is a lower semicontinuous viscosity supersolution of the above equation. Given $\varepsilon > 0$, define the test function

$$\varphi(t,s) = \hat{w}(t,s) - \varepsilon\phi(s), \quad (t,s) \in [0,T)\times(0,\infty),$$

where ϕ is the function introduced in Assumption 4.1; recall that ϕ is positive, C^2 is strictly concave, and $\lim_{s\to\infty}\phi(s) = +\infty$. By Remark 6.3, we have $v(t,s) \ge E\left[\hat{g}(S_{t,s}(T))\right]$. Moreover, since g is nonnegative, we have $\hat{g} \ge 0$ and by the definition of the concave envelope, it follows that $\hat{g} \ge \hat{g}^{\text{conc}} - C$ for some positive constant C. Then, from Lemma 10.2 together with condition (4.2), we can conclude that

$$\liminf_{s\to\infty}(w-\varphi)(t,s) \ge \liminf_{s\to\infty}\left\{E\left[\hat{g}(S_{t,s}(T))\right] - \hat{g}^{\text{conc}}(s) + \varepsilon\phi(s)\right\} = +\infty$$

for all $t \in [0,T]$. Then there exists $(t_0,s_0) \in [0,T]\times[0,\infty)$ such that

$$(w-\varphi)(t_0,s_0) = \min_{[0,T]\times[0,\infty)}(w-\varphi).$$

In order to prove the required result, we have to show that

$$(w - \varphi)(t_0, s_0) \geq 0, \tag{10.3}$$

which implies that $w(t,s) - \hat{w}(t,s) + \varepsilon\phi(s) \geq 0$ for all $(t,s) \in [0,T] \times (0,\infty)$ and the result of the theorem follows by sending ε to zero.

Inequality (10.3) is trivially satisfied if $s_0 = 0$ or $t_0 = T$. We then concentrate on the case $t_0 < T$ and $s_0 > 0$. Since (t_0, s_0) is an interior minimum, it follows from the viscosity supersolution property of w that

$$\lambda w(t_0, s_0) - \mathcal{L}\varphi(t_0, s_0) \geq 0 \qquad \text{and} \qquad \Gamma e^{-\lambda t_0} - s_0 \varphi_{ss}(t_0, s_0) \geq 0. \tag{10.4}$$

Recalling the definition of φ, the second inequality provides

$$\Gamma e^{-\lambda t_0} - s_0 \hat{w}_{ss}(t_0, s_0) \geq -\varepsilon \phi_{ss}(s_0) > 0.$$

By (10.2), we then see that $\mathcal{L}\hat{w}(t_0, s_0) = \lambda \hat{w}(t_0, s_0)$. Plugging this into the first inequality of (10.4) provides

$$\lambda(w - \varphi)(t_0, s_0) \geq \varepsilon \left[\lambda \phi(s_0) - \frac{1}{2}\sigma^2(t_0, s_0)\phi_{ss}(t_0, s_0) \right] \geq 0,$$

which is the required inequality (10.3). □

11. Appendix: Properties of stochastic integrals. In this section we prove several properties of double stochastic integrals with respect to Brownian motion. The key idea in our analysis was provided by Professor F. Delbaen. Our main result is Proposition 11.6 below.

It is known that if

$$\int_0^{t\wedge\theta} h(u)dW(u) \leq C t \wedge \theta \qquad \text{for all } t \geq 0, \tag{11.1}$$

for some continuous adapted process $h(\cdot)$, standard Brownian motion $W(\cdot)$, positive stopping time θ, and a constant C, then $h(0) = 0$. This result is contained in Soner, Shreve, and Cvitanič (1995).[1]

In the analysis of gamma constraints, in particular in proving the viscosity property of the value function in section 9, we are led to study a similar situation for double stochastic integrals such as

$$\int_0^t \int_0^u b(r)dW(r)\ dW(u) \leq Ct. \tag{11.2}$$

In this section, we analyze several inequalities of the type (11.2) ordered by increasing difficulty.

First, suppose that the process $b(\cdot)$ in (11.2) is equal to a constant b_0. Then,

$$\frac{b_0}{2}[W^2(t) - t] = \int_0^t \int_0^u b(r)dW(r)\ dW(u) \leq Ct.$$

[1] Here is an alternative simple proof of this result. Given an arbitrary $\nu \in \mathbb{R}$, introduce the exponential martingale $Z^\nu = \mathcal{E}(\nu W)$. Then, multiplying both sides of (11.1) by $Z^\nu(t\wedge\theta)$, and taking expectations, it follows from the optional sampling theorem that $\nu E[Z^\nu(t\wedge\theta)\int_0^{t\wedge\theta} h(u)du] \leq CE[Z^\nu(t\wedge\theta)(t\wedge\theta)]$. Dividing by t, sending t to zero, and recalling that the process $h(.)$ is continuous, and the stopping time θ is positive P-a.s., we see that $\nu h(0) \leq C$. By arbitrariness of ν, this proves that $h(0) = 0$.

Hence, $b_0 \leq 0$ by the law of iterated logarithm.

Next, suppose that b is a bounded, progressively measurable process and (11.2) holds for all $t \in [0, \eta]$ where η is a positive constant. Delbaen proved the following:

$$P\left[\inf_{0 \leq u \leq t} b(u) \geq c\right] < 1 \qquad \text{for all } c > 0,\ t \leq \eta. \tag{11.3}$$

Suppose to the contrary, i.e., suppose that there are $c > 0, t \leq \eta$ such that $b(u) \geq c$ for all $u \in [0, t]$. Let $Z^\nu(t) := \exp\big(\nu W(t) - (\nu^2 t/2)\big)$. A direct calculation shows that

$$E\left[Z^\nu(t) \int_0^t \int_0^u b(v) dW(v) dW(u)\right] = \nu^2 E\left[\int_0^t \int_0^u b(v) Z^\nu(v) dv du\right] \geq c\nu^2 t^2/2.$$

By (11.2),

$$E\left[Z^\nu(t) \int_0^t \int_0^u b(v) dW(v) dW(u)\right] \leq Ct.$$

Hence, $c\nu^2 t^2/2 \leq Ct$ for all ν, which cannot happen. This proves (11.3).

We continue the analysis when (11.2) holds only up to a stopping time.

LEMMA 11.1. *Let θ be some bounded positive stopping time and $\{b(t),\ t \geq 0\}$ be a bounded progressively measurable process satisfying* (11.2) *for all $t \leq \theta$. Then,*

$$\liminf_{t \searrow 0} b(t) \leq 0. \tag{11.4}$$

Proof. Suppose to the contrary. Then, there exist a positive stopping time τ and a constant $c > 0$ such that $b(t \wedge \tau) \geq c$ for all t. Rename the stopping time $\tau \wedge \theta$ to be θ.

Step 1. We employ a time change and then use standard properties of Brownian motion to obtain a contradiction. Set

$$h(t) := \int_0^t [b(u)^2 + 1_{\{u > \theta\}}] du, \quad t \geq 0,$$

so that h is a continuous strictly increasing function on $[0, \theta]$. Let

$$\hat{W}(t) := \int_0^{h^{-1}(t)} b(u) dW(u), \quad t \geq 0,$$

and $\mathbb{G} = \{\mathcal{G}_t,\ t \geq 0\}$ be given by $\mathcal{G}_t := \mathcal{F}_{h^{-1}(t)}$. Then the time-changed process $(\hat{W}, \mathbb{G})$ is a standard Brownian motion. By the time-change formula (see, e.g., Karatzas and Shreve (1991, Proposition 4.8, p. 176)), we rewrite (11.2) as

$$\begin{aligned} Ct \wedge \theta \geq \int_0^{t \wedge \theta} \int_0^u b(r)\ dW(r)\ dW(u) &= \int_0^{t \wedge \theta} \hat{W}(h(u))\ dW(u) \\ &= \int_0^{h(t \wedge \theta)} \phi(u) \hat{W}(u)\ d\hat{W}(u) \\ &= \frac{1}{2} \int_0^{h(t \wedge \theta)} \phi(u) d[\hat{W}(u)^2] - \frac{1}{2} \int_0^{h(t \wedge \theta)} \phi(u) du, \end{aligned}$$

where $\phi(u) := 1/b(h^{-1}(u))$. Since b is bounded away from zero, ϕ is bounded and

$$\int_0^{h(t\wedge\theta)} \phi(u) d[\hat{W}(u)^2] \leq C' t\wedge\theta, \qquad t \geq 0, \tag{11.5}$$

for some constant C'.

Step 2. By the law of iterated logarithm, there exists a sequence of bounded positive $\mathbb{F}$-stopping times $(\tau_n)_n$ converging to zero such that

$$\frac{\hat{W}(\tau_n)^2}{\tau_n} \longrightarrow +\infty \qquad P\text{-a.s.}$$

Set

$$\theta_n := \theta \wedge h^{-1}(\tau_n)\ .$$

Since θ is positive, for sufficiently large n, $h(\theta_n) = h(h^{-1}(\tau_n)) = \tau_n$. Hence,

$$\frac{\hat{W}\left(h(\theta_n)\right)^2}{h(\theta_n)} \longrightarrow +\infty \qquad P\text{-a.s.} \tag{11.6}$$

Step 3. Choose M so that $|b| < M$. Let ϕ be as in Step 1. Since $b > 0$ on $[0, \theta]$, we have $\phi > 1/M$ on this interval.

Set $\ell := \liminf_{t\downarrow 0} \frac{2}{t}\int_0^t [\phi(u) - \frac{1}{M}] d[W^2(u)]$, and let $(\zeta_n)_n$ be a sequence of positive stopping times converging to zero P-a.s. such that

$$\int_0^{\zeta_n} \left[\phi(u) - \frac{1}{M}\right] d[W^2(u)] \leq \ell\zeta_n.$$

Direct calculation provides

$$\ell\ E[\zeta_n] \geq E\left[\int_0^{\zeta_n}\left[\phi(u) - \frac{1}{M}\right] d[W^2(u)]\right] = E\left[\int_0^{\zeta_n}\left[\phi(u) - \frac{1}{M}\right] du\right] \geq 0.$$

This proves that $\ell \geq 0$, and consequently

$$\liminf_{t\downarrow 0} \frac{\int_0^t \phi(u) d[\hat{W}(u)^2]}{\hat{W}(t)^2} \geq \frac{1}{M}.$$

Let θ_n be the sequence constructed in Step 2. Since θ_n tends to zero as n approaches to zero,

$$\liminf_{n\to\infty} \frac{\int_0^{h(\theta\wedge\tau_n)} \phi(u) d[\hat{W}(u)^2]}{\hat{W}\left(h(\theta\wedge\tau_n)\right)^2} \geq \frac{1}{M}. \tag{11.7}$$

Step 4. Since $b(\theta \wedge t) \geq c$, the definition of h implies that

$$\lim_{n\to\infty} \frac{h(\theta_n)}{\theta_n} \geq c^2.$$

Combining this inequality with (11.6) and (11.7), we arrive at

$$\limsup_{n\to\infty} \frac{h(\theta_n)}{\theta_n} \frac{\hat{W}\left(h(\theta_n)\right)^2}{h(\theta_n)} \frac{\int_0^{h(\theta_n)} \phi(u) d[\hat{W}(u)^2]}{\hat{W}\left(h(\theta_n)\right)^2} = +\infty.$$

Step 5. By (11.5), we have

$$\frac{h(\theta_n)}{\theta_n}\frac{\hat{W}\left(h(\theta_n)\right)^2}{h(\theta_n)}\frac{\int_0^{h(\theta_n)}\phi(u)d[\hat{W}(u)^2]}{\hat{W}\left(h(\theta_n)\right)^2} \le C'\frac{\theta_n}{\theta_n}.$$

Clearly this is in contradiction with the previous step. ∎

Our next generalization is to replace W in (11.2) by the stock price process.

We introduce some notation that will be used throughout this section. Let (t_n, s_n) be a sequence converging to some $(t_0, s_0) \in [0, T) \times (0, \infty)$. To simplify the notation, we set

$$S_n(t) := S_{t_n,s_n}(t) \qquad \text{and} \qquad \bar{\sigma}_n(t) := S_{t_n,s_n}(t)\sigma\left(t, S_{t_n,s_n}(t)\right).$$

Since the processes S_n may take very large values, we need to introduce a sequence of stopping times defined as follows. For a large constant $\lambda > 0$ let

$$\bar{\tau}_n := \inf\left\{t > t_n : \left|\ln\left(S_n(t)/s_n\right)\right| \ge \lambda\right\}. \tag{11.8}$$

In our notation, we do not show the dependence of $\bar{\tau}_n$ on λ.

Remark 11.2. The sequence of stopping times $(\bar{\tau}_n)_n$ satisfies

$$\liminf_{n\to\infty} t \wedge \bar{\tau}_n \ge \frac{1}{2} t \wedge \bar{\tau}_0 \qquad P\text{-a.s.}$$

Indeed, since $(t_n, s_n) \longrightarrow (t_0, s_0)$, it follows from Protter (1990, Theorem 37, p. 246) that for almost everywhere (a.e.) $\omega \in \Omega$, we have

$$S_{t_n,s_n} \longrightarrow S_{t_0,s_0} \qquad \text{uniformly on } [t_0, t_0 + t],$$

which implies the announced claim.

LEMMA 11.3. *Let b, θ, C be as in Lemma* 11.1. *Suppose that*

$$\int_0^{t\wedge\theta}\int_0^r b(r)\ dS_0(r)\ dS_0(u) \le Ct \wedge \theta \qquad \textit{for all } t \ge 0.$$

Then, b satisfies 11.4.

Proof. We follow the proof of Lemma 11.1. We replace θ by the stopping time $\bar{\theta} := \theta \wedge \bar{\tau}_0$ and the time-change function h by

$$\bar{h}(t) := \int_0^t [b(u)^2\bar{\sigma}(u)^2 + 1_{\{u > \bar{\theta}\}}]du.$$

We define the time-changed Brownian motion $\hat{W}$ in the obvious way. Then, the time-change formula implies that

$$\int_0^{t\wedge\bar{\theta}}\int_0^u b(r)dS_0(r)\ dS_0(u) = \int_0^{t\wedge\bar{\theta}}\hat{W}(u)\ dS_0(u) = \int_0^{h(t\wedge\bar{\theta})}\bar{\phi}(u)\hat{W}(u)d\hat{W}(u),$$

where $\bar{\phi} = 1/[\bar{b}(\bar{h}^{-1})]$. We then proceed as in Lemma 11.1. ∎

Remark 11.4. The conclusion of Lemma 11.1 is still valid if t is substituted for $t\wedge\theta$ in the right-hand side of inequality (11.2). This is easily checked by going through the proof. The same observation prevails for Lemma 11.3.

Finally, we provide two results which deal with a slightly general double integral:

$$M_n(t\wedge\theta_n) := \int_{t_n}^{t_n+t\wedge\theta_n}\left(z_n+\int_{t_n}^{u}a_n(r)dr+\int_{t_n}^{u}b_n(r)dS_n(r)\right)dS_n(u)\le \beta_n+Ct. \tag{11.9}$$

We will first show that if β_n tends to zero, then z_n also converges to zero. This is a slight generalization of the result on single stochastic integrals stated in the beginning of this section. The second result provides information on the limit behavior of the sequence $(b_n)_n$.

Proposition 11.5. *Let $(\{a_n(u),\ u\ge 0\})_n$ and $(\{b_n(u),\ u\ge 0\})_n$ be two sequences of real-valued, progressively measurable processes that are uniformly bounded in n. Suppose that (11.9) holds with real numbers $(z_n)_n$, $(\beta_n)_n$, and stopping times $(\theta_n)_n$. Assume further that, as n tends to zero,*

$$\beta_n \longrightarrow 0 \qquad \text{and} \qquad t\wedge\theta_n \longrightarrow t\wedge\theta_0 \quad P\text{-a.s.},$$

where θ_0 is a strictly positive stopping time. Then

$$\lim_{n\to\infty} z_n = 0.$$

Proof. For each $n\ge 0$, define the stopping time

$$\tau_n := 1\wedge\bar\tau_n\wedge\theta_n.$$

By Remark 11.2, $\liminf_n t\wedge\tau_n \ge t\wedge\tau_0/2$ with probability one. Let ν be an arbitrary real parameter and define the local martingales Z_n^ν by

$$Z_n^\nu(t) = \mathcal{E}\left(\int_0^t \frac{\nu dW(u)}{\bar\sigma_n(u)}\right), \quad t\ge 0.$$

By the definition of τ_n in (11.8), the process $\{Z_n^\nu(t\wedge\tau_n),\ t\ge 0\}$ is a P-martingale. We then define the probability measure P_n^ν equivalent to P by its density process $\{Z_n^\nu(t\wedge\tau_n),\ t\ge 0\}$ with respect to P. We shall denote by E_n^ν the expectation operator under P_n^ν. By Girsanov's theorem, the process $W_n^\nu(.\wedge\tau_n)$ defined by

$$W_n^\nu(t) = W(t) - \int_0^t \frac{\nu\, du}{\bar\sigma_n(u)}, \quad t\ge 0,$$

is a Brownian motion under P_n^ν. We also define the local martingale Z^ν by

$$Z^\nu(t) = \mathcal{E}\left(\int_0^t \frac{\nu dW(u)}{\bar\sigma_0(u)}\right), \quad t\ge 0.$$

By the same argument as above, the process $\{Z^\nu(t\wedge\tau_0),\ t\ge 0\}$ is a P-martingale and is therefore the density process of some probability measure P^ν equivalent to P. We shall denote by E^ν the expectation operator under P^ν. It is easily checked that $Z_n^\nu(.) \longrightarrow Z^\nu(.)$ P-a.s. Then, since $t\wedge\tau_0/2 \le \liminf_n t\wedge\tau_n \le \limsup_n t\wedge\tau_n \le t$, it follows from the continuity of Z_n^ν and Z^ν that

$$Z_\infty^\nu := \liminf_{n\to\infty} Z_n^\nu(t\wedge\tau_n) > 0. \tag{11.10}$$

Rewrite $M_n(t\wedge\tau_n)$ in terms of W_n^ν,

$$\begin{aligned} M_n(t\wedge\tau_n) &= \mathrm{mart}(P_n^\nu) + \nu z_n t\wedge\tau_n + \nu\int_{t_n}^{t_n+t\wedge\tau_n}\int_{t_n}^{u} a_n(r)\,dr\,du \\ &\quad + \nu t\wedge\tau_n \int_{t_n}^{t_n+t\wedge\tau_n} b_n(r)\bar{\sigma}_n(r)\,dW_n^\nu(r) + \nu^2\int_{t_n}^{t_n+t\wedge\tau_n}\int_{t_n}^{u} b_n(r)\,dr\,du, \end{aligned}$$

where $\mathrm{mart}(P_n^\nu)$ is a martingale under P_n^ν starting from zero. Take the expectation under P_n^ν, apply the Cauchy–Schwartz inequality for the third term on the right-hand side, and also utilize the bounds on $(a_n)_n$ and $(b_n)_n$ to obtain

$$\begin{aligned} \nu z_n E_n^\nu[t\wedge\tau_n] &\le \beta_n + C'\Big(E_n^\nu[t\wedge\tau_n] + (|\nu|+\nu^2)E_n^\nu[(t\wedge\tau_n)^2]^{3/4}\Big) \\ &\le \beta_n + C'\Big(t + (|\nu|+\nu^2)t^{3/4}\Big). \end{aligned}$$

Let ℓ denote either $\liminf_n z_n$ or $\limsup_n z_n$, and restrict ν to have the same sign as ℓ, so that $\nu\ell \ge 0$. Now, let n go to infinity. Then, it follows from Fatou's lemma together with (11.2) and (11.10) that

$$\frac{1}{2}\nu\ell E[t\wedge\tau_0 Z_\infty^\nu] \le C'\Big(t + (|\nu|+\nu^2)t^{3/4}\Big).$$

We now divide by t and take the limit as $t\searrow 0$. Since τ_0 and Z_∞^ν are positive P (and then P^ν)-a.s., we get by dominated convergence

$$\nu\ell \le C' \qquad \text{for all } \nu\in\mathbb{R}.$$

Since ν is arbitrary, we conclude that $\liminf_n z_n = \limsup_n z_n = 0$. ∎

The following result is a stronger version of Lemma 11.1 which was used in section 9. We shall denote by $\mathbb{H}^2$ the Hilbert space of all progressively measurable Lebesgue(0,T)$\otimes P$-square integrable processes.

Let $(b_n)_n$ be as in Lemma 11.5. By assumption, $(b_n)_n$ is bounded in L^∞(Lebesgue$(0,T)\otimes P$). Then it is bounded in $\mathbb{H}^2$ and, therefore, converges weakly to some b, possibly along a subsequence.

PROPOSITION 11.6. *Assume the hypothesis of Lemma* 11.5. *Let b be as above. Then*

$$\liminf_{u\searrow 0} b(u) \le 0.$$

Proof. Define the stopping times τ_n as in the proof of Lemma 11.5. To simplify the notation, we rename process $b_n(t)1_{t_n\le t\le t_n+t\wedge\tau_n}$ by $b_n(t)$. By Mazur's lemma, there exists a sequence of coefficients $(\lambda_k^n,\ k\ge n)_n$ with $\lambda_k^n\ge 0$ and $\sum_{k\ge n}\lambda_k^n = 1$ such that

$$\hat{b}_n := \sum_{k\ge n}\lambda_k^n b_k \longrightarrow b \qquad \text{strongly in } \mathbb{H}^2. \tag{11.11}$$

Integrating by parts and using the bound on a_n and $S_n(.\wedge\tau_n)$ provide

$$\begin{aligned}M_n(t\wedge\tau_n) &= z_n[S_n(t_n+t\wedge\tau_n)-s_n]\\ &\quad + S_n(t_n+t\wedge\tau_n)\int_{t_n}^{t_n+t\wedge\tau_n} a_n(r)dr-\int_{t_n}^{t_n+t\wedge\tau_n} S_n(u)a_n(u)du\\ &\quad + \int_{t_n}^{t_n+t\wedge\tau_n}\int_{t_n}^{u} b_n(r)dS_n(r)dS_n(u)\\ &\geq -C't\wedge\tau_n-|z_n|s_n(e^{\lambda}-1)+\int_{t_n}^{t_n+t\wedge\tau_n}\int_{t_n}^{u} b_n(r)dS_n(r)dS_n(u).\end{aligned}$$

Set $\hat\beta_n:=\beta_n+|z_n|s_n(e^{\lambda}-1)$. Then, from Lemma 11.5, $\hat\beta_n\longrightarrow 0$ as $n\to\infty$ and we get from the inequality satisfied by M_n

$$\int_{t_n}^{t_n+t\wedge\tau_n}\int_{t_n}^{u} b_n(r)dS_n(r)dS_n(u)\leq\hat\beta_n+Kt\wedge\tau_n \tag{11.12}$$

for some positive constant K. Set

$$\varepsilon_n(t):=\int_{t_n}^{t_n+t\wedge\tau_n}\int_{t_n}^{u} b_n(r)dS_n(r)dS_n(u)-\int_{t_0}^{t_0+t\wedge\tau_0}\int_{t_0}^{u} b_n(r)dS_0(r)dS_0(u).$$

We shall later prove that

$$\varepsilon_n(t)\longrightarrow 0 \qquad P\text{-a.s.} \tag{11.13}$$

possibly along a subsequence. Take convex combinations in (11.12) to conclude that

$$\sum_{k\geq n}\lambda_k^n\varepsilon_k(t)+\int_{t_0}^{t_0+t\wedge\tau_0}\int_{t_0}^{u}\hat b_n(r)dS_0(r)dS_0(u)\leq\sum_{k\geq n}\lambda_k^n\left(\hat\beta_k+Kt\wedge\tau_k\right). \tag{11.14}$$

We directly calculate that

$$\begin{aligned}&E\left[\left(\int_{t_0}^{t_0+t\wedge\tau_0}\int_{t_0}^{u}\left(\hat b_n(r)-b(r)\right)dS_0(r)dS_0(u)\right)^2\right]\\ &\quad = E\left[\int_{t_0}^{t_0+t\wedge\tau_0}\left(\int_{t_0}^{u}\left(\hat b_n(r)-b(r)\right)dS_0(r)\right)^2\bar\sigma_0(u)^2du\right]\\ &\quad\leq C_1\,E\left[\int_{t_0}^{t_0+t}\left(\int_{t_0}^{u}\left(\hat b_n(r)-b(r)\right)dS_0(r)\right)^2du\right]\\ &\quad\leq C_2\,E\left[\int_{t_0}^{t_0+t}\int_{t_0}^{u}\left(\hat b_n(r)-b(r)\right)^2drdu\right]\\ &\quad\leq C_3t\|\hat b_n-b\|^2_{\mathbb{H}^2},\end{aligned}$$

where C_i's are constants independent of n. This proves that

$$\int_{t_0}^{t_0+t\wedge\tau_0}\int_{t_0}^{u}\hat b_n(r)dS_0(r)dS_0(u)\longrightarrow\int_{t_0}^{t_0+t\wedge\tau_0}\int_{t_0}^{u} b(r)dS_0(r)dS_0(u)\quad\text{ as } n\to\infty$$

in $L^2(P)$, and therefore P-a.s. along some subsequence. Then, taking a.s. limits in (11.14) and using (11.13), we get

$$\int_{t_0}^{t_0+t\wedge\tau_0}\int_{t_0}^{u} b(r)dS_0(r)dS_0(u) \le Kt.$$

Since the limit process b inherits the bound on b_n, we apply the result of Lemma 11.3 to complete the proof; see also Remark 11.4.

It remains to prove the convergence result stated in (11.13). Set $\zeta_n = t_n + t\wedge\tau_n$ for $n \ge 0$. By Itô's lemma,

$$\varepsilon_n(t) = A_n + B_n + C_n,$$

where

$$\begin{aligned}
A_n &= [S_n(\zeta_n) - s_n]\int_{t_n}^{\zeta_n} b_n(u)dS_n(u) - [S_0(\zeta_0) - s_0]\int_{t_0}^{\zeta_0} b_n(u)dS_0(u),\\
B_n &= -\int_{t_n}^{\zeta_n} b_n(u)S_n(u)dS_n(u) + \int_{t_0}^{\zeta_0} b_n(u)S_0(u)dS_0(u),\\
C_n &= -\int_{t_n}^{\zeta_n} b_n(u)\bar{\sigma}_n(u)^2du + \int_{t_0}^{\zeta_0} b_n(u)\bar{\sigma}_0(u)^2du.
\end{aligned}$$

It suffices to prove that A_n, B_n, and C_n converge to zero P-a.s. along some subsequence. We prove only the convergence of A_n; the remaining claims are proved similarly.

(i) To simplify the presentation, set $\bar{\sigma}(.) = 0$ outside the stochastic interval $[t_n, \zeta_n]$ and observe that

$$S_n(\zeta_n) - S_0(\zeta_0) = s_n + \int_{t_n}^{\zeta_n} \bar{\sigma}_n(u)dW(u).$$

Since $\bar{\sigma}_n$ is bounded inside the stochastic interval $[t_n, \zeta_n]$, by dominated convergence,

$$\begin{aligned}
&E\left[\left(\int_{t_n}^{\zeta_n}\bar{\sigma}_n(u)dW(u) - \int_{t_0}^{\zeta_0}\bar{\sigma}_0(u)dW(u)\right)^2\right]\\
&\qquad = E\left[\int_{t_0\wedge t_n}^{\zeta_0\vee\zeta_n}\left(\bar{\sigma}_n(u)-\bar{\sigma}_0(u)\right)^2du\right] \longrightarrow 0.
\end{aligned}$$

This proves that

$$S_n(\zeta_n) \longrightarrow S_0(\zeta_0) \qquad P\text{-a.s.}$$

along some subsequence.

(ii) Recall that we have set $b_n(.) = 0$ outside the interval $[t_n, \zeta_n]$. Thus,

$$\begin{aligned}
\int_{t_n}^{\zeta_n} b_n(u)dS_n(u) - \int_{t_0}^{\zeta_0} b_n(u)dS_0(u) &= \int_{t_n}^{t_0\vee t_n} b_n(u)dS_0(u) + \int_{\zeta_0\wedge\zeta_n}^{\zeta_n} b_n(u)dS_0(u)\\
&\quad + \int_{t_0\vee t_n}^{\zeta_0\wedge\zeta_n} b_n(u)\left(\bar{\sigma}_n(u)-\bar{\sigma}_0(u)\right)dW(u).
\end{aligned}$$

From the bound on b_n, the first two terms on the right-hand side converge to zero in $L^2(P)$ and therefore P-a.s. along some subsequence. As for the third term,

$$E\left[\left(\int_{t_0\vee t_n}^{\zeta_0\wedge\zeta_n} b_n(u)\left(\bar{\sigma}_n(u)-\bar{\sigma}_0(u)\right)dW(u)\right)^2\right]$$

$$= E\left[\int_{t_0\vee t_n}^{\zeta_0\wedge\zeta_n} b_n(u)^2\left(\bar{\sigma}_n(u)-\bar{\sigma}_0(u)\right)^2 du\right]$$

$$\leq C_1 E\left[\int_{t_0\vee t_n}^{\zeta_0\wedge\zeta_n} \left(\bar{\sigma}_n(u)-\bar{\sigma}_0(u)\right)^2 du\right]$$

$$\leq C_2 E\left[\int_{t_0}^{\zeta_0} \left(\bar{\sigma}_n(u)-\bar{\sigma}_0(u)\right)^2 du\right],$$

where C_i's are constants and we have set $\sigma_n(.) = 0$ outside the stochastic interval $[t_n, \zeta_n]$. Since $\bar{\sigma}_n$ is bounded, we see by dominated convergence that the third term of interest converges to zero in $L^2(P)$ and therefore P-a.s. along some subsequence. This proves that

$$\int_{t_n}^{t_n+t\wedge\tau_n} b_n(u)dS_n(u) - \int_{t_0}^{t_0+t\wedge\tau_0} b_n(u)dS_0(u) \longrightarrow 0 \qquad P\text{-a.s.}$$

along some subsequence.

By (i) and (ii), $A_n \to 0$ P-a.s. along some subsequence. □

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