Functions of Bounded Higher Variation

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ABSTRACT. We say that a function $u : \mathbb{R}^m \to \mathbb{R}^n$, with $m \geq n$, has bounded n-variation if $\mathrm{Det}(u_{x_{\alpha_1}}, \dots, u_{x_{\alpha_n}})$ is a measure for every $1 \leq \alpha_1 < \cdots < \alpha_n \leq m$. Here $\mathrm{Det}(v_1, \dots, v_n)$ denotes the distributional determinant of the matrix whose columns are the given vectors, arranged in the given order.

In this paper we establish a number of properties of BnV functions and related functions. We establish general (and rather weak) versions of the chain rule and the coarea formula; we show that stronger forms of the chain rule can fail, and we also demonstrate that BnV functions cannot, in general, be strongly approximated by smooth functions; and we prove that if $u \in BnV(\mathbb{R}^m, \mathbb{R}^n)$ and $|u| = 1$ a.e., then the Jacobian of u is an $m - n$-dimensional rectifiable current.

1. INTRODUCTION

Given a function $u : \mathbb{R}^m \supset U \to \mathbb{R}^n$ with $n \leq m$, we define the distributional Jacobian $[Ju]$ of u to be the pullback by u (in the sense of distributions) of the standard volume form on $\mathbb{R}^n$, so that

$$[Ju] = \sum_{\alpha \in I_{n,m}} \mathrm{Det}(u_{x_{\alpha_1}}, \dots, u_{x_{\alpha_n}})\, dx^\alpha. \tag{1.1}$$

Here $I_{n,m}$ is the set of all multiindices of the form $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $1 \leq \alpha_1 < \cdots < \alpha_n \leq m$. For such a multiindex, $dx^\alpha := dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_n}$. Det denotes the distributional determinant, the definition of which is given in Section 2. The definition of $[Ju]$ makes sense if for example $u \in W^{1,n-1}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(U; \mathbb{R}^n)$.

We say that a function has locally bounded n-variation in U if for every bounded open set $V \subset\subset U$ there exists some constant $C = C(V)$ such that

$$\sum_{\alpha \in I_{n,m}} \int \omega^\alpha \,\mathrm{Det}(u_{x_{\alpha_1}}, \dots, u_{x_{\alpha_n}}) := \langle \omega, [Ju] \rangle \leq C \|\omega\|_{C^0} \tag{1.2}$$

for all C^1 n-forms $\omega = \sum_{\alpha \in I_{n,n}} \omega^\alpha \, dx^\alpha$ with support in V. When this holds we write $u \in BnV_{\text{loc}}(U; \mathbb{R}^n)$.

If C can be chosen independently of V, then we say that u has bounded n-variation in U, and we write $u \in BnV(U; \mathbb{R}^n)$.

If $u \in BnV_{\text{loc}}(U; \mathbb{R}^n)$, the Riesz Representation Theorem asserts that there is a nonnegative Radon measure, which we denote $|Ju|$, and a $|Ju|$-measurable function $\nu : U \to \Lambda^n \mathbb{R}^m$ such that

$$|\nu(x)| = 1 \quad |Ju| \text{ almost every } x; \text{ and}$$

$$\langle \omega, [Ju] \rangle = \int \omega(x) \cdot \nu(x) \, d|Ju|.$$

Moreover, $|Ju|$ satisfies

$$|Ju|(V) = \sup\{\langle \omega, [Ju] \rangle \mid \omega \in C_c^1(V; \Lambda^n \mathbb{R}^m);\ |\omega| \le 1 \text{ in } V\} \tag{1.3}$$

for every open $V \subset U$.

When $u \in BnV_{\text{loc}}$ we also write $\int \phi \cdot [Ju]$ for $\langle \phi, [Ju] \rangle$.

We will chiefly be interested in the case where $[Ju]$ cannot be represented by an L^1 function.

Results. The definition of BnV more or less generalizes that of the classical space BV, and many results about BV have some sort of generalization in BnV. The first result we state is an analogue of the theorem of De Giorgi on the rectifiability of the reduced boundary of a set of finite perimeter.

We write $u \in BnV_{\text{loc}}(U; S^{n-1})$ to mean that $u \in BnV_{\text{loc}}(U; \mathbb{R}^n)$ with $|u| = 1$ almost everywhere. We write ω_n to denote the volume of the unit ball in $\mathbb{R}^n$.

Theorem 1.1. *If $u \in W^{1,n-1} \cap BnV_{\text{loc}}(U; S^{n-1})$, then there exists an $(m-n)$-dimensional rectifiable set Γ and a positive integer-valued $\mathcal{H}^{m-n}$-measurable function $\theta : \Gamma \to \mathbb{Z}$, such that*

$$\int \phi \cdot [Ju] = \omega_n \int_\Gamma \theta \phi \cdot \nu \, d\mathcal{H}^{m-n} \tag{1.4}$$

for all $\phi \in C_c^0(\mathbb{R}^m; \Lambda^n \mathbb{R}^m)$. In addition, $\nu(x)$ represents an oriented unit n-vector normal to Γ at $\mathcal{H}^{m-n}$ a.e. $x \in \Gamma$.

This theorem is an easy consequence of known results, but as far as we know it was not explicitly pointed out before our work. We will give two proofs, one of which is due to M. Giaquinta and G. Modica. After learning of our result, Lin and Hang [20] gave a quick proof of a result very similar to Theorem 1.1, for functions $u \in W^{1-1/n,n}(\mathbb{R}^m; S^{n-1})$ such that the distributional pullback of the volume form on $\mathbb{R}^n$ is a measure.

Rectifiable sets and related geometric background are discussed in many references; see for example [17], Section 2.1.4.

In the statement of our next result we use the notation

$$u_a(x) = \frac{u(x) - a}{|u(x) - a|} \quad \text{for } u : \mathbb{R}^m \to \mathbb{R}^n,\ a \in \mathbb{R}^n.$$

If u is a smooth function and a is a regular value of u, then simple examples lead one to expect that $[Ju_a]$ should be a unit multiplicity measure whose support is exactly the level set $\{x \mid u(x) = a\}$. More generally we think of $[Ju_a]$ as a weak, measure-theoretic representation of the level set $\{u(x) = a\}$; note in this context that in view of Theorem 1.1, if $u_a \in BnV$ then $[Ju_a]$ is carried by a $(m-n)$-dimensional rectifiable set.

These considerations lead us to interpret (1.5) as a weak form of the coarea formula.

Theorem 1.2. *If $u \in W^{1,n-1}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^m;\mathbb{R}^n)$, then $u_a \in W^{1,n-1}_{\mathrm{loc}} \cap L^\infty$ for a.e. $a \in \mathbb{R}^n$, and*

$$[Ju] = \frac{1}{\omega_n}\int_{a\in\mathbb{R}^n} [Ju_a]\, da \tag{1.5}$$

in the sense of distributions.

If $u \in W^{1,p}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^m;\mathbb{R}^n)$ for some $p > n-1$, and if $F \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)$ and $w = F(u)$, then

$$\langle \omega, [Jw]\rangle = \frac{1}{\omega_n}\int_{a\in\mathbb{R}^n} JF(a)\langle \omega, [Ju_a]\rangle\, da \tag{1.6}$$

for all $\omega \in C^1_c(\mathbb{R}^m;\Lambda^n\mathbb{R}^m)$. This remains true if $u \in W^{1,n-1}_{\mathrm{loc}} \cap C^0$, or if $u \in W^{1,n-1} \cap L^\infty_{\mathrm{loc}}$ and $F \in C^1$.

If, in addition, $u \in BnV(\mathbb{R}^m;\mathbb{R}^n)$ and either $u \in W^{1,p}$ for some $p > n$, or $|u| = 1$ a.e., then $u_a \in BnV(\mathbb{R}^m;S^{n-1})$ for a.e. $a \in \mathbb{R}^n$, and

$$|Ju|(A) = \frac{1}{\omega_n}\int_{a\in\mathbb{R}^n} |Ju_a|(A)\, da \tag{1.7}$$

for every Borel set $A \subset \mathbb{R}^m$.

We refer to (1.5) as the weak coarea formula, and to (1.7) as the strong coarea formula.

Remark 1.3. Observe that (1.5) and (1.6) do *not* require $u \in BnV_{\mathrm{loc}}$.

Our next result details some ways in which BnV fails to inherit certain nice properties of BV.

Theorem 1.4. *Suppose that $u \in BnV_{\mathrm{loc}} \cap L^\infty(\mathbb{R}^m;\mathbb{R}^n)$. If $V \subset \mathbb{R}^m$ is any open set, then*

$$
\begin{aligned}
(1.8)\qquad |Ju|(V) &\le \frac{1}{\omega_n}\int_{\mathbb{R}^n} |Ju_a|(V)\,da \\
(1.9)\qquad &\le \liminf_{\varepsilon\to 0}\{|Ju^\varepsilon|(V) : u^\varepsilon \text{ smooth, } \|u^\varepsilon\|_{L^\infty} \le C, \\
&\qquad\qquad u^\varepsilon \to u \text{ in } W^{1,n-1}\}.
\end{aligned}
$$

Either of the inequalities above can be strict if $n \ge 2$. In particular, there exist functions w, $\widetilde{w}$ such that, for any open set U containing the origin,

$$
(1.10)\qquad |Jw|(U) = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} |Jw_a|(U)\,da = +\infty,
$$

$$
(1.11)\qquad \int_{\mathbb{R}^n} |J\widetilde{w}_a|(U)\,da = 0 \quad \text{and}
$$

$$
\liminf_{\varepsilon\to 0}\{|Jv^\varepsilon|(U) : v^\varepsilon \text{ smooth, } \|v^\varepsilon\|_{L^\infty} \le C,\ v^\varepsilon \to \widetilde{w} \text{ in } W^{1,n-1}\} = +\infty.
$$

Finally, there exist functions $u \in BnV$ and $F \in C^\infty$ such that $F(u) \notin BnV$.

Several of the examples we give of possible pathological behavior are drawn from earlier work of Giaquinta, Modica, and Soucek [16].

Related work. A lot of attention has been devoted to distributional determinants in the past 10 years. In particular, Stefan Müller, motivated originally by the relevance of weak determinants in nonlinear elasticity (see for example Ball [5]) wrote a series of papers [24], [25], [26], [27] investigating questions of weak continuity, the relationship between pointwise and distributional determinants, integrability properties, and so on. Some of these questions were subsequently taken up by Coifman, Lions, Meyer, and Semmes [10], as part of a more general investigation of the relationship between compensated compactness and Hardy spaces.

Another source of interest in distributional determinants has been questions relating to harmonic maps with singularities, notably the work of Brezis, Coron, and Lieb [8], later recast in a more general setting by Almgren, Browder, and Lieb [1]. Later work of Bethuel [6], motivated by a conjecture of Brezis, showed that a map $u \in H^1(B^3, S^2)$ can be approximated by smooth S^2-valued functions if and only if the distributional Jacobian $[Ju]$ vanishes.

The work of Giaquinta, Modica, and Souček on Cartesian currents (see [17], [18]) provided an appropriate weak setting for non-scalar variational problems with global topological or geometric constraints, including problems in both nonlinear elasticity and harmonic maps. In many situations the distributional Jacobian $[Ju]$ of a map $u : \mathbb{R}^m \to \mathbb{R}^n$ is essentially the projection onto $\mathbb{R}^m$ of the vertical part of a Cartesian current that, roughly speaking, corresponds to the graph of u with holes "filled in". Thus our work is very closely related to Cartesian currents. Indeed, some of our ideas are implicit in the work of Giaquinta, Modica, and Souček, and some of our results become very transparent when viewed in the framework of Cartesian currents. These connections are discussed in more detail in Section 6.

Organization. Section 2 gives the definition and basic properties of distributional Jacobians, and also summarizes some notation. Section 3 contains a number of examples, some of which are part of the proof of Theorem 1.4. Section 4 is mainly devoted to the proof of Theorem 1.2, though it also cleans up some loose ends in Theorem 1.4. The proofs of these two theorems are summarized at the beginning of Section 4.

In Sections 5 and 6 we give two proofs of Theorem 1.1. The first uses slicing properties of $[Ju]$, and the second, due to M. Giaquinta and L. Modica, uses some ideas about Cartesian currents.

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2. Definitions and basic properties

Definitions. For sufficiently smooth functions $u : \mathbb{R}^m \to \mathbb{R}^n$, we define an $n-1$ form

$$j(u) := \sum_{\alpha \in I_{n-1,m}} \det(u, u_{x_{\alpha_1}}, \dots, u_{x_{\alpha_{n-1}}})\, dx^\alpha. \tag{2.1}$$

For our purposes, "sufficiently smooth" will mean that u belongs to a Sobolev space in which $j(u)$ is necessarily locally integrable, for example $u \in W^{1,n-1}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ or $u \in W^{1,p}_{\mathrm{loc}}$, $p \geq mn/(m+1)$. One could relax this requirement somewhat; we will not pursue this here.

We will write the components of $j(u)$ as $j^\alpha(u)$, so that

$$j(u) = \sum_{I_{n-1,m}} j^\alpha(u)\, dx^\alpha.$$

We next define

$$[Ju] := \frac{1}{n}\,dj(u) = \frac{1}{n}\sum_{i=1}^{m}\sum_{\alpha\in I_{n-1,m}} \partial_{x_i} j^{\alpha}(u)\,dx^i \wedge dx^{\alpha} \tag{2.2}$$

in the sense of distributions. Here d is the exterior derivative. Thus, for any $\omega \in C_c^1(\mathbb{R}^m;\Lambda^n\mathbb{R}^m)$

$$\langle \omega, [Ju]\rangle = \frac{1}{n}\int d^*\omega \cdot j(u), \tag{2.3}$$

where d^* is the formal adjoint of d. We will write the components of $[Ju]$ as $[J^\alpha u]$, so that $[Ju] = \sum_{\alpha\in I_{n,m}}[J^\alpha u]\,dx^\alpha$. The dot product "$\cdot$" appearing in (2.3) is defined as follows. We write $\Lambda^k\mathbb{R}^m$ to denote the linear space spanned by $\{dx^\alpha\}_{\alpha\in I_{k,m}}$. We endow $\Lambda^k\mathbb{R}^m$ with the Euclidean inner product, making $\{dx^\alpha\}_{\alpha\in I_{k,m}}$ an orthonormal basis. We will write the inner product as either $v\cdot w$ or (v,w), according to convenience. For $v\in\Lambda^k\mathbb{R}^m$, we write $|v|$ to mean the standard Euclidean norm $(v,v)^{1/2}$. We will normally identify $\Lambda^k\mathbb{R}^m$ with its dual, via the inner product.

If u is smooth enough, say C^2, then $[Ju] = (1/n)dj(u)$ can be computed by differentiating $j(u)$ pointwise. This is easy to carry out, using the multilinearity of the determinant. In this case all second derivatives cancel, and we obtain exactly

$$[Ju] = \sum_{\alpha\in I_{n,m}} \det(u_{x_{\alpha_1}},\dots,u_{x_{\alpha_n}})\,dx^\alpha.$$

(Strictly speaking, we should write the left-hand side as a Radon-Nikodym derivative $d[Ju]/d\mathcal{L}^m$). An approximation argument shows that this remains valid if $u\in W^{1,n}(\mathbb{R}^m;\mathbb{R}^n)$. More generally, Müller [25] has shown that this identity holds whenever $[Ju]$ can be represented as an L^1 function.

Basic properties. First note that BnV is not a linear space; indeed, it is not even convex. To see this, suppose that $u = (u^1,u^2)\in W^{1,1}\cap L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ is a function that does not belong to $B2V$. We can write $u = \frac{1}{2}v_1 + \frac{1}{2}v_2$, where $v_1 := (2u_1, 0)$ and $v_2 := (0, 2u_2)$. One easily sees that $[Jv_i] = 0$, $i = 1, 2$, so that $v_i\in B2V$ for $i = 1, 2$.

We next mention the following result.

Lemma 2.1 (Lower semicontinuity)**.** *Suppose $u_k \in BnV_{\mathrm{loc}}(\mathbb{R}^m;\mathbb{R}^n)$, and assume that $u_k \to u$ in L^1_{loc} and $j(u_k) \rightharpoonup j(u)$ weakly in L^1_{loc}.*

Then $u\in BnV_{\mathrm{loc}}$, and

$$|Ju|(V) \le \liminf |Ju_k|(V) \tag{2.4}$$

for every open set $V \subset \mathbb{R}^m$.

More generally, (2.4) *remains true if we merely know that* $[Ju_k] \rightharpoonup [Ju]$ *weak-** *in* $C_c^1(\mathbb{R}^m; \Lambda^n \mathbb{R}^m)^*$, (*where one or both sides of the inequality may now equal* $+\infty$).

Proof. We prove only the first assertion, as the proof of the second is essentially the same.

Suppose that $\omega \in C_c^1(V; \Lambda^n \mathbb{R}^m)$ and $|\omega| \leq 1$ in V. Then

$$\frac{1}{n}\int d^*\omega \cdot j(u) = \lim_k \frac{1}{n}\int d^*\omega \cdot j(u_k) = \lim_k \int \omega \cdot [Ju_k] \leq \liminf_k |Ju_k|(V).$$

In view of (1.3), this implies the conclusion of the proposition. ◻

It is well-known that Jacobians have certain weak continuity properties. Some of these are given in Lemma 4.6.

A simple but useful fact is the following result.

Lemma 2.2.

(i) *If* $u \in W^{1,m}(\mathbb{R}^m; S^{n-1})$, *then* $[Ju] = 0$.

(ii) *If* $u \in W^{1,n-1} \cap C^0(\mathbb{R}^m; S^{n-1})$, *then* $u \in BnV$ *and* $[Ju] = 0$.

Remark 2.3. For (ii) we need not assume that $u \in BnV$.

Proof. First assume that u is smooth. Then the condition $|u| \equiv 1$ implies that $Du(x)$ has rank at most $n-1$ at every x. It follows that all $n \times n$ minors of $Du(x)$ vanish and thus that $[Ju] = 0$.

It is well-known that smooth functions are dense in $W^{1,m}(\mathbb{R}^m; S^{n-1})$. For a proof, see for example [17], Section 5.5.1. It is also clear that smooth functions are dense in $W^{1,n-1} \cap C^0(\mathbb{R}^m; S^{n-1})$. So in either case the result follows by approximation. ◻

Stefan Müller [27] shows that given any integer $n > 1$ and any $\alpha \in (0, n)$, there exists a continuous function $u \in W^{1,p}(\mathbb{R}^n; \mathbb{R}^n)$ for all $p \in [1, n)$ such that, in our notation, $[Ju]$ is a nonnegative measure carried by a set S of Hausdorff dimension α. In our language, the functions constructed by Müller belong to $BnV(\mathbb{R}^n; \mathbb{R}^n)$.

Our results imply that, for general $u \in BnV(\mathbb{R}^m; \mathbb{R}^n)$, the s-dimensional density of $[Ju]$ is identically zero for every $s < m - n$. Müller's construction shows that for any $s \in [m-n, m]$, $[Ju]$ can have a nontrivial s-dimensional part.

Miscellaneous notation. We summarize some notation that we will use throughout this paper.

We will always use ω_n to denote the volume of the unit ball in $\mathbb{R}^n$. Thus $n\omega_n$ is the $(n-1)$-volume of the unit sphere S^{n-1}. We also sometimes use ω to denote a generic differential form; we believe that this will not cause any confusion.

We write $B_r^k(a)$ to denote the *open* ball $\{x \in \mathbb{R}^k : |x-a| < r\}$. We will normally omit the superscript k when the dimension of the ball is clear.

When considering functions $u : \mathbb{R}^n \to \mathbb{R}^n$ we use the terms distributional determinant and distributional Jacobian interchangeably. For $u : \mathbb{R}^m \to \mathbb{R}^n$ with $m > n$, we only use the term distributional Jacobian.

When A is a subset of some Euclidean space, we write χ_A to denote the characteristic function of A, defined by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if not.} \end{cases}$$

The Hodge star operator $\star : \Lambda^k\mathbb{R}^m \to \Lambda^{m-k}\mathbb{R}^m$ is defined by

$$\star\, dx^\alpha = \operatorname{sgn}(\alpha\beta)\, dx^\beta,$$

for the unique $\beta \in I_{m-k,m}$ such that $(\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{m-k})$ is a permutation of $(1, \dots, m)$. Here $\operatorname{sgn}(\alpha\beta)$ is the sign of the permutation. It is not hard to check that

$$dx^\alpha \wedge \star\, dx^\alpha = dx^1 \wedge \cdots \wedge dx^m, \quad \star\star\, dx^\alpha = (-1)^{k(m-k)}\, dx^\alpha.$$

If M is a smooth, oriented codimension k manifold of $\mathbb{R}^m$ and ν is an oriented normal k-vector at some point of M, then $\star\nu := \tau$ is the tangent $(m-k)$-vector at the same point, appropriately oriented.

There is a classical formula for action of the formal adjoint on k-forms, in terms of the Hodge-$\star$ operator:

$$d^* = (-1)^{m(m-k)-(m-k+1)} \star d \star. \tag{2.5}$$

3. EXAMPLES

In this section we collect some examples. Many of these are known to experts.

For $u : \mathbb{R}^2 \to \mathbb{R}^2$ it is convenient to use the identity

$$\int \phi[Ju] = \int \nabla \times \phi \cdot j(u)\, dx, \quad \nabla \times \phi := (\phi_{x_2}, -\phi_{x_1}).$$

One can check that this is equivalent to the definition we have given above.

Example 3.1. Suppose that $u : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ is a function which is homogeneous of degree zero and smooth away from the origin. Then u can be written using polar coordinates in the form $u(re^{i\theta}) = \gamma(\theta)$, for some smooth 2π-periodic function $\gamma : \mathbb{R} \to \mathbb{R}^2$. Using the chain rule,

$$u_{x_i} = \gamma'(\theta)\theta_{x_i} \quad \text{and so} \quad j(u) = D\theta \det(\gamma(\theta), \gamma'(\theta)).$$

From this, one computes that, for any $\phi \in C_c^1(\mathbb{R}^2)$,

$$\nabla \times \phi \cdot j(u) = -\frac{1}{r}\frac{\partial \phi}{\partial r}\det(\gamma(\theta), \gamma'(\theta)).$$

Integrating, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} \nabla \times \phi \cdot j(u) &= \int_0^{2\pi}\int_0^\infty \nabla \times \phi \cdot j(u) r\, dr\, d\theta \\ &= \int_0^{2\pi} \det(\gamma(\theta), \gamma'(\theta))\left(\int_0^\infty -\frac{\partial \phi}{\partial r}\, dr\right) d\theta \\ &= \phi(0)\int_0^{2\pi} \det(\gamma(\theta), \gamma'(\theta)).\end{aligned}$$

We conclude that

$$[Ju] = \frac{1}{2}\nabla \times j(u) = A\delta_0, \quad A := \frac{1}{2}\int_0^{2\pi} \det(\gamma(\theta), \gamma'(\theta)).$$

Note that A is just the area enclosed by the image of γ (counting sign and multiplicity).

The above example is a special case of the following example.

Example 3.2. Now consider a function $u : \mathbb{R}^n \to \mathbb{R}^n$ which is homogeneous of degree zero and smooth away from the origin. Given a smooth, compactly supported n-form $\phi(x)\, dx$, we get from the definitions that

$$\int \phi \cdot [Ju] = -\frac{1}{n}\int D\phi \cdot (\det(u, u_{x_2}, \dots, u_{x_n}), \dots, \det(u_{x_1}, \dots, u_{x_{n-1}}, u))\, dx.$$

We let $\tilde{j}_u$ denote the vector of determinants in the integrand. Since u is homogeneous of degree zero, $\tilde{j}_u$ is homogeneous of degree $1 - n$. We claim that

$$D\phi(x) \cdot \tilde{j}_u(x) = \left(D\phi \cdot \frac{x}{|x|}\right)\left(\tilde{j}_u(x) \cdot \frac{x}{|x|}\right). \tag{3.1}$$

This is obvious if x has the form $x = (a, 0, \dots, 0)$, $a \neq 0$, since then $u_{x_1}(x) = 0$ by homogeneity. The general case follows by a change of coordinates. Using (3.1) and again the homogeneity of u we get

$$\int \phi \cdot [Ju] = -\frac{1}{n}\int_0^\infty \int_{\partial B_1} (D\phi(ry) \cdot y)(\tilde{j}_u(y) \cdot y)\, d\mathcal{H}^{n-1}(y)\, dr.$$

Since $D\phi(ry) \cdot y = (d/dr)\phi(ry)$, integrating first with respect to r gives

$$\int \phi \cdot [Ju] = \phi(0)\frac{1}{n}\int_{\partial B_1} y \cdot \tilde{j}_u(y)\, d\mathcal{H}^{n-1}(y) =: V\phi(0),$$

where the constant V is defined in the obvious way. In other words, $[Ju] = V\delta_0$. We give a geometric interpretation to V as follows: Let $v(x) = |x|u(x)$, so that v is Lipschitz. Then

$$\int_{B_1} \det Dv\, dx = \frac{1}{n}\int_{B_1} \mathrm{div}\tilde{j}_v\, dx = \frac{1}{n}\int_{\partial B_1} \tilde{j}_v(y)\cdot y\, d\mathcal{H}^{n-1}(y).$$

However, one easily checks that for $y \in \partial B_1$, $\tilde{j}_v(y)\cdot y = \tilde{j}_u(y)\cdot y$. Thus $V = \int_{B_1}\det Dv$. If we write u in the form $u(x) = \gamma(x/|x|)$ for some smooth function $\gamma : S^{n-1} \to \mathbb{R}^n$, then one can think of V as the volume enclosed by the image of γ, counting sign and multiplicity.

Example 3.3. We record some consequences of the previous example.

First note that if u maps $\mathbb{R}^n$ into S^{n-1}, then the volume enclosed by the image of the unit sphere is an integer multiple of the volume ω_n of the unit ball, and so $[Ju] = k\omega_n\delta_0$ for some integer k.

Next consider a map $v : \mathbb{R}^n \to S^{n-1}$, smooth away from the origin. We let $v^\lambda(x) = v(\lambda x)$, and we assume that as $\lambda \to 0$, v^λ converges locally in $W^{1,n}$, say, to a limit u, which is necessarily homogeneous of degree zero. Lemma 2.2 implies that $[Jv^\lambda]$ vanishes away from the origin for every λ. Let ϕ be a smooth test function, and write $\tilde{\phi}^\lambda(x) = \phi(\lambda x)\chi(x)$ for some smooth, compactly supported function χ that equals one in the unit ball, From the definition, one checks that

$$\int \phi\cdot[Jv] = \int \tilde{\phi}^\lambda\cdot[Jv^\lambda] = \lim_{\lambda\to 0}\int \tilde{\phi}^\lambda\cdot[Jv^\lambda].$$

Clearly $\tilde{\phi}^\lambda \to \phi(0)\chi$ as $\lambda \to 0$, and we have assumed that v^λ converges to a homogeneous function u. Thus weak continuity properties of Jacobians, see Lemma 4.6, imply that $[Jv] = [Ju] = k\omega_n\delta_0$ for some integer k.

Finally, suppose that w is a map $\mathbb{R}^n \to S^{n-1}$, smooth away from a finite collection of singular points $\{a_1,\dots,a_m\}$, and that around each singularity w looks locally like a translate of a function of the form of v above. Again the Jacobian $[Jw]$ must vanish away from the singular points, and so we conclude that

$$[Jw] = \omega_n \sum d_i\delta_{a_i} \tag{3.2}$$

for certain integers $d_1,\dots,d_m$.

The rectifiability theorem, Theorem 1.1, shows that in fact (3.2) holds for every $w \in BnV(\mathbb{R}^n;S^{n-1})$, without *ad hoc* smoothness assumptions of the sort we introduced above.

Example 3.4. Suppose $u \in W^{1,1}(\mathbb{R}^3;S^1)$ is a map for which there is a smooth, connected, embedded, closed curve $\Gamma \subset \mathbb{R}^3$ with u smooth away from Γ, and such that u has winding number 1 on appropriately oriented curves around Γ. Then, for almost every $\theta \in [0, 2\pi)$, the set

$$M_\theta := \{x \in \mathbb{R}^3 \mid u(x) = e^{i\theta}\}$$

is a 2-dimensional submanifold with boundary Γ, and $j(u)/|Du|$ is a smooth oriented unit normal to M_θ, which we will denote ν_θ.

We will compute $\int \phi \cdot [Ju]$ using Federer's coarea formula (see for example [11, Section 3.4]). Note that the Jacobian of u as a map from $\mathbb{R}^3$ to S^1 is just $|Du| = |j(u)|$, and so the coarea formula yields

$$\begin{aligned} \int \phi \cdot [Ju] &= \frac{1}{2} \int \nabla \times \phi \cdot j(u) = \frac{1}{2} \int \nabla \times \phi \cdot \nu_\theta |Du| \\ &= \frac{1}{2} \int_0^{2\pi} \Big(\int_{M_\theta} \nabla \times \phi \cdot \nu_\theta \, d\mathcal{H}^2(x) \Big) \, d\theta, \end{aligned} \tag{3.3}$$

where $\mathcal{H}^2$ as usual is the 2-dimensional Hausdorff measure.

Since almost all the level sets M_θ share the same boundary Γ, Stokes' theorem implies that

$$\int_{M_\theta} \nabla \times \phi \cdot \nu_\theta \, d\mathcal{H}^2(x) = \int_\Gamma \phi \cdot \tau \, d\mathcal{H}^1(x)$$

for almost every θ, where τ is the appropriately oriented unit tangent vector along Γ. Thus (3.3) can be integrated to give

$$\int \phi \cdot [Ju] = \pi \int_\Gamma \phi \cdot \tau \, d\mathcal{H}^1(x).$$

Thus in this case $[Ju] = \pi \tau \mathcal{H}^1|_\Gamma$.

Theorem 1.1 shows that this situation is in some way typical for functions in $B2V(\mathbb{R}^3; S^1)$, and indeed for functions in $BnV(\mathbb{R}^m; S^{n-1})$ whenever $m > n$: the Jacobian measure $[Ju]$ is supported on a codimension n rectifiable set that may be thought of as as the topological singular set of u.

We now present several examples illustrating various pathologies, including the possible failure of the strong coarea formula. These constitute the bulk of the proof of Theorem 1.4. Example 3.8 is drawn from Giaquinta, Modica, and Souček [16], and many of the other examples are loosely inspired by the same paper.

Example 3.5. Define a homogeneous function $u(re^{i\theta}) = \gamma(\theta)$, where γ is a 2π-periodic function mapping onto $[0, 2\pi)$ onto a "figure 8", circling the ball $B_{\text{right}} = B_1((1,0))$ once in an orientation-reversing sense, and the ball $B_{\text{left}} = B_1((-1,0))$ once in the opposite sense, for example

$$\gamma(\theta) = \begin{cases} (-1,0) + (\cos 2\theta, \sin 2\theta) & \text{if } 0 \le \theta \le \pi, \\ (1,0) + (-\cos 2\theta, \sin 2\theta) & \text{if } \pi \le \theta \le 2\pi. \end{cases}$$

Then the signed area enclosed by γ is zero, and so $[Ju] = 0$, and hence $|Ju| = 0$.

Also, one easily sees that

$$[Ju_a] = \begin{cases} -\pi\delta_0 & \text{if } a \in B_{\text{right}}, \\ \pi\delta_0 & \text{if } a \in B_{\text{left}}, \\ 0 & \text{if } a \notin \bar{B}_{\text{right}} \cup \bar{B}_{\text{left}}, \end{cases} \tag{3.4}$$

so $|Ju_a| = \pi\delta_0$ if $a \in B_{\text{right}} \cup B_{\text{left}}$, and $|Ju_a| = 0$ otherwise. As a result,

$$0 = |Ju|(V) < \frac{1}{\omega_n}\int_{\mathbb{R}^2} |Ju_a|(V)\,da = 2\pi$$

for any V containing the origin.

Example 3.6. By a fairly standard construction, which we sketch below, one can build a "dipole" that has singularities like that of Example 3.5 above at two points, with opposite orientation, and which in addition is constant outside of a compact set. More precisely, given two points p and n in $\mathbb{R}^2$, for example $p = (\frac{1}{2}, 0) = -n$, we construct a function v which is constant outside the unit ball, and satisfying

$$[Jv] = 0$$

and

$$[Jv_a] = \begin{cases} \pi\delta_p - \pi\delta_n & \text{if } a \in B_{\text{right}}, \\ \pi\delta_n - \pi\delta_p & \text{if } a \in B_{\text{left}}, \\ 0 & \text{otherwise.} \end{cases}$$

We construct v as follows. Let u be a function of the form $u(re^{i\theta}) = \gamma(\theta)$, for some γ that covers the "figure 8" as in Example 3.5. Assume further that $\gamma(\theta) \equiv (0,0)$ for all $\theta \in [\pi/4, 7\pi/4]$; for example

$$\gamma(\theta) = \begin{cases} (-1,0) + (\cos 8\theta, \sin 8\theta) & \text{if } 0 \le \theta \le \pi/4, \\ (1,0) + (-\cos 8\theta, \sin 8\theta) & \text{if } -\pi/4 \le \theta \le 0, \\ (0,0) & \text{if } \pi/4 \le \theta \le 7\pi/4. \end{cases}$$

Now define

$$v(x_1, x_2) = \begin{cases} u\left(x_1 + \dfrac{1}{2}, x_2\right) & \text{in } \{x_1 \le 0\}, \\ u\left(\dfrac{1}{2} - x_1, x_2\right) & \text{in } \{x_1 \ge 0\}. \end{cases}$$

This function is Lipschitz away from $\{p, n\}$ and maps $\mathbb{R}^2 \setminus \{p, n\}$ into a set of measure zero, so it is clear that the support of $[Jv]$ is contained in $\{p, n\}$. It is then easy to see from Example 3.5 that in fact $[Jv] = 0$.

Similarly, if $a \notin \text{image}(\gamma)$, then v_a is Lipschitz away from $\{p, n\}$, so one easily verifies that the support of $[Jv_a]$ is contained in $\{p, n\}$. It is then easy to check, following Example 3.5, that $[Jv_a]$ is as claimed above.

Finally, it is not hard to see that $v = (0,0)$ outside the unit ball $B_1(0)$.

Example 3.7. Now we use scaling properties to construct, from the previous example, a function w such that $[Jw] = 0$ but $[Jw_a]$ is *not* a measure for any $a \in B_{\text{right}} \cup B_{\text{left}}$. Let v be the function constructed in Example 3.6 above, and for $\varepsilon > 0$ let $v^\varepsilon(x) := v(x/\varepsilon)$. Note that for each ε, $v^\varepsilon \equiv (0,0)$ outside the ball $B_\varepsilon(0)$, and

$$\int_{B_\varepsilon(0)} |v^\varepsilon|^p + |Dv^\varepsilon|^p \, dx \le \varepsilon^{2-p} \int_{B_1(0)} |v|^p + |Dv|^p \, dx = C\varepsilon^{2-p}.$$

Now let $\varepsilon_k := 2^{-k}$, define points $q_k \in \mathbb{R}^2$ by $q^k = (2^{2-k}, 0)$, and let

$$w(x) = \begin{cases} v^{\varepsilon_k}(x - q_k) & \text{if } |x - q_k| \le 2^{1-k} \text{ for some } k \ge 0, \\ (0,0) & \text{if } |x - q_k| > 2^{1-k} \text{ for all } k \ge 0. \end{cases}$$

Then $w \in W^{1,p}_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ for all $p < 2$ and $[Jw] = 0$, but

$$[Jw_a] = \begin{cases} \pi \sum(\delta_{p_k} - \delta_{n_k}) & \text{if } a \in B_{\text{right}}, \\ \pi \sum(\delta_{n_k} - \delta_{p_k}) & \text{if } a \in B_{\text{left}}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

for certain sequences of points n_k, $p_k \to 0$. In fact $p_k = q_k + \varepsilon_k p$ and $n_k = q_k + \varepsilon_k n$. In particular, if $a \in B_{\text{right}} \cup B_{\text{left}}$, $[Jw_a]$ is a distribution that belongs to $(C^1_c)^*$, but it is not a measure.

Thus w satisfies (1.10).

Example 3.8. We now indicate the construction of a function satisfying (1.11). The basic idea is due to Giaquinta, Modica, and Souček [16], to which we refer the reader for more details.

Their example is a function $u(re^{i\theta}) = \gamma(\theta)$ that is homogeneous of degree zero, such that the image of γ is a figure 8 curve, as in the previous examples. In this case, we assume that γ covers each half of the figure 8 twice, with opposite orientations. For example, we could take

$$\gamma(\theta) = \begin{cases} (-1,0) + (\cos 4\theta, \sin 4\theta) & \text{if } 0 \le \theta \le \pi/2, \\ (1,0) + (-\cos 4\theta, \sin 4\theta) & \text{if } \pi/2 \le \theta \le \pi, \\ (-1,0) + (\cos 4\theta, -\sin 4\theta) & \text{if } \pi \le \theta \le 3\pi/2, \\ (1,0) + (-\cos 4\theta, -\sin 4\theta) & \text{if } 3\pi/2 \le \theta \le 2\pi. \end{cases}$$

Then for each $a \in B_{\text{right}} \cup B_{\text{left}}$, γ circles a twice, with opposite orientations. It follows that $[Ju_a] = 0$ for almost every $a \in \mathbb{R}^2$, and hence that

$$\int_{a \in \mathbb{R}^2} |Ju_a|(V)\, da = 0$$

for every set V.

But Giaquinta, Modica, and Souček prove that, if V is any open set containing the origin and u^ε is any approximating sequence, then

$$\liminf |Ju^\varepsilon|(V) \geq 4\pi.$$

Following the arguments of Example 3.6, one can construct a function v which is constant outside a bounded set, say the unit ball, and having singularities like that of u at two points $\{p, n\}$. One can then use simple scaling arguments, as in Example 3.7, to create a new function $\widetilde{w} \in W^{1,p} \cap L^\infty \cap B2V(\mathbb{R}^2; \mathbb{R}^2)$ for all $p < 2$, which has an infinite sequence of such "dipoles" on smaller scales that accumulate, for example, at the origin. Such a function $\widetilde{w}$ satisfies (1.11).

Example 3.9. As in Example 3.5 let $B_{\text{right}} := B_1((1,0))$ and $B_{\text{left}} = B_1((-1,0))$. Let F be a smooth function $\mathbb{R}^2 \to \mathbb{R}^2$ and $c \in \mathbb{R}$ a *nonzero* number such that

$$\int_{B_{\text{left}}} JF(a) - \int_{B_{\text{right}}} JF(a) = c.$$

Now let $\hat{u} = F(u)$, where u is the function defined in Example 3.5. Using the distributional chain rule (1.6) and the explicit computation of $[Ju_a]$ given in (3.4), we find that

$$[J\hat{u}] = \frac{1}{\pi} \int_{\mathbb{R}^2} JF(a)[Ju_a]\, da = \delta_0 \Big(\int_{B_{\text{left}}} JF(a) - \int_{B_{\text{right}}} JF(a) \Big) = c\delta_0.$$

In particular, the property $[Ju] = 0$ is not preserved under composition with smooth functions.

In a similar spirit, define $\hat{w} = F(w)$, where w is the function constructed in Example 3.7. Using (3.5) and the distributional chain rule we compute that

$$[J\widetilde{w}] = c \sum (\delta_{n_k} - \delta_{p_k}).$$

Thus, $w \in BnV$ with $[Jw] = 0$, and F is smooth, but $\hat{w} = F(w) \notin BnV$.

4. COAREA FORMULA AND CHAIN RULE

In this section we give the proof of Theorem 1.2, and we complete the proof of Theorem 1.4. The main part of the proofs are spread around in a number of lemmas. We summarize here both proofs.

Proof of Theorem 1.2. The weak coarea formula (1.5) is proved in Lemma 4.3. The weak chain rule (1.6) is proved in Lemma 4.5. The strong coarea formula (1.7) is proved in Lemma 4.4 for $u \in W^{1,p}(\mathbb{R}^m;\mathbb{R}^n)$, $p > n$ and in Lemma 4.8 for $u \in BnV(\mathbb{R}^m;S^{n-1})$. ◻

Proof of Theorem 1.4. The inequality (1.8) is proved in the first step of the proof of Lemma 4.4.

To prove (1.9), fix bounded some open set V. The Approximation Lemma 4.9 guarantees that, after passing to a subsequence $\{u_k\}$, we may assume that $[Ju_k^a] \rightharpoonup [Ju_a]$ weak-* in $C_c^1(\mathbb{R}^m;\Lambda^n\mathbb{R}^m)^*$ for almost every $a \in \mathbb{R}^n$. Then from the lower semicontinuity of Jacobians, Lemma 2.1, we deduce that

$$|Ju_a|(V) \le \liminf_k |Ju_k^a|(V).$$

Hence Fatou's Lemma and the strong coarea formula (1.7) (which applies to the smooth functions u_k) allow us to conclude

$$\frac{1}{\omega_n}\int_{\mathbb{R}^n} |Ju_a|(V)\,da \le \liminf \frac{1}{\omega_n}\int_{\mathbb{R}^n} |Ju_k^a|(V)\,da = \liminf |Ju_k|(V).$$

The examples whose existence is asserted in Theorem 1.4 are all constructed in the previous section. In particular, see Example 3.7 for (1.10) and Example 3.8 for (1.11). Finally, Example 3.9 shows that membership in BnV need not be preserved under composition with smooth functions. ◻

We use the notation $u_a(x) := (u(x) - a)/(|u(x) - a|)$. Note the following result.

Lemma 4.1. *If* $u \in W^{1,n-1} \cap C^0(\mathbb{R}^m;\mathbb{R}^n)$ *and* $u_a \in W^{1,n-1}(\mathbb{R}^m;\mathbb{R}^n)$, *then the support of* $[Ju_a]$ *is contained in* $\{x \mid u(x) = a\}$.

Proof. Let O be the complement of $\{u = a\}$. The restriction of u_a to O belongs to $W^{1,n-1} \cap C^0(O;S^{n-1})$, so the conclusion follows immediately from Lemma 2.2. ◻

Remark 4.2. By comparing (1.7) with Federer's coarea formula and using Lemma 4.1, one can easily show that if $u \in W^{1,\infty}$, then $|Ju_a| = \omega_n \mathcal{H}^{m-n}|_{u^{-1}(a)}$ for a.e. $a \in \mathbb{R}^n$.

The key calculation in the proof of Theorem 1.2 is contained in the following lemma.

Lemma 4.3. *If* $u \in W^{1,n-1}_{\mathrm{loc}} \cap L^\infty(\mathbb{R}^m;\mathbb{R}^n)$, *then* $u_a \in W^{1,n-1}_{\mathrm{loc}} \cap L^\infty(\mathbb{R}^m;\mathbb{R}^n)$ *for a.e.* a, *and the distributional coarea formula* (1.5) *holds.*

Proof.

1. For every positive integer M, the set $\{x \in \mathbb{R}^m : |x| \le M,\ u(x) = a\}$ can have positive $\mathcal{L}^m$ measure for at most countably many $a \in \mathbb{R}^n$. Thus $\{x \in \mathbb{R}^m \mid$

$u(x) = a\}$ can have positive measure for at most countably many a. It follows that for a.e. $a \in \mathbb{R}^n$, $u_a(x)$ is well-defined for a.e $x \in \mathbb{R}^m$.

2. Let $R := \|u\|_{L^\infty} + 1$. For y, $a \in \mathbb{R}^n$, let $P_a(y) = (y-a)/|y-a|$. If $|a| \geq R$, then P_a is smooth on the essential range of u, and so in this case clearly $u_a(x) = P_a(u(x))$ belongs to $u_a \in W^{1,n-1}_{\text{loc}}$, and $Du_a = DP_a(u)\,Du$.

For $a \leq R$, let $v_a = DP_a(u)\,Du$. We want to show that v_a is the distributional gradient of u_a, and that $v_a \in L^{n-1}$ for a.e. a.

To verify the latter point, fix some bounded open set $U \subset \mathbb{R}^m$ and compute

$$\begin{aligned}\int_{\{a:|a|\leq R\}}\int_U |v_a|^{n-1}\,dx\,da &= C\int_{\{a:|a|\leq R\}}\int_U \frac{|Du|^{n-1}}{|u-a|^{n-1}}\,dx\,da\\ &= C\int_U\int_{\{a:|u(x)-a|\leq R\}} \frac{|Du|^{n-1}}{|a|^{n-1}}\,da\,dx\\ &\leq C\int_U\int_{\{a:|a|\leq 2R\}} \frac{|Du|^{n-1}}{|a|^{n-1}}\,da\,dx\\ &\leq C\int_U |Du|^{n-1}\,dx < \infty.\end{aligned}$$

Thus Fubini's theorem implies that $v_a \in L^{n-1}_{\text{loc}}$ for a.e. a.

Also, define

$$u_a^\varepsilon = P_a^\varepsilon \circ u, \quad P^\varepsilon(y) = \begin{cases}\dfrac{1}{\varepsilon}(y-a) & \text{if } |y-a| \leq \varepsilon,\\ P_a(y) & \text{if } |y-a| \geq \varepsilon.\end{cases}$$

Then the chain rule implies that $u_a^\varepsilon \in W^{1,n-1}$, and it is clear that $u_a^\varepsilon \to u_a$ almost everywhere as $\varepsilon \to 0$, as long as $\{x \in \mathbb{R}^m \mid u(x) = a\}$ has measure zero. If in addition $v_a \in L^{n-1}_{\text{loc}}$, then a short computation involving the dominated convergence theorem implies that $Du_a^\varepsilon \to v_a$ in L^{n-1}_{loc}. We conclude that for almost every a, v_a is the distributional gradient of u_a, and hence that $u_a \in W^{1,n-1}_{\text{loc}}$.

3. It is a fact that for any $y \in \mathbb{R}^n$ and $R > |y|$,

$$y = \frac{1}{\omega_n}\int_{\{z\in\mathbb{R}^n:|z|<R\}} \frac{y-z}{|y-z|^n}\,dz. \tag{4.1}$$

One way to see this is to differentiate both sides of the identity

$$\frac{1}{(2-n)\omega_n}\int_{B_R} |y-z|^{2-n}\,dz = \frac{1}{2}|y|^2 + \frac{n}{2(2-n)}R^2, \quad \text{if } R > |y|, n \geq 3,$$

which is due to Newton, at least for $n = 3$. A similar identity holds if $n = 2$.

Our choice of R implies that $|u| < R$ a.e., so we can use (4.1) to write, for a.e. $x \in \mathbb{R}^m$,

$$\begin{aligned} j(u) &= \sum_{\alpha \in I_{n-1,N}} \det(u, u_{x_{\alpha_1}}, \dots, u_{x_{\alpha_{n-1}}})\, dx^\alpha \\ &= \sum_{\alpha \in I_{n-1,N}} \det\left(\left(\frac{1}{\omega_n} \int_{|a| \le R} \frac{u-a}{|u-a|^n}\, da\right), u_{x_{\alpha_1}}, \dots, u_{x_{\alpha_{n-1}}}\right) dx^\alpha. \end{aligned}$$

Moreover, using the fact that the determinant is linear in each column,

$$\begin{aligned} &\det\left(\left(\frac{1}{\omega_n} \int_{|a| \le R} \frac{u-a}{|u-a|^n}\, da\right), u_{x_{\alpha_1}}, \dots, u_{x_{\alpha_{n-1}}}\right) \\ &\quad = \frac{1}{\omega_n} \int_{|a| \le R} \det\left(\frac{u-a}{|u-a|^n}, u_{x_{\alpha_1}}, \dots, u_{x_{\alpha_{n-1}}}\right) da \\ &\quad = \frac{1}{\omega_n} \int_{|a| \le R} \det\left(\frac{u-a}{|u-a|}, \frac{u_{x_{\alpha_1}}}{|u-a|}, \dots, \frac{u_{x_{\alpha_{n-1}}}}{|u-a|}\right) da \\ &\quad = \frac{1}{\omega_n} \int_{|a| \le R} \det\left(u_a, u_{a,x_{\alpha_1}}, \dots, u_{a,x_{\alpha_n}}\right) da. \end{aligned}$$

The last equality follows from that fact that

$$u_{a,x_i} = \frac{u_{x_i}}{|u-a|} + u_a\left(u_a \cdot \frac{u_{x_i}}{|u-a|}\right) = \frac{u_{x_i}}{|u-a|} + \text{vector parallel to } u_a.$$

The terms involving vectors parallel to u_a contribute nothing, because of course $\det(v_1, \dots, v_n) = 0$ if two columns v_i, v_j are parallel.

4. Assembling the above calculations, we find that

$$j(u) = \frac{1}{\omega_n} \int_{|a| \le R} j(u_a)\, da,$$

which implies that

$$[Ju] = \frac{1}{\omega_n} \int_{|a| \le R} [Ju_a]\, da$$

as distributions.

To complete the proof of (1.5), we only need to check that $[Ju_a] = 0$ for all a such that $|a| \ge R = \|u\|_{L^\infty} + 1$.

For such a, define $u^\varepsilon := \eta^\varepsilon * u$, where η^ε is a standard mollifier. Then $|u^\varepsilon - a|$ is bounded away from zero for all sufficiently small ε, so one easily sees that $u_a^\varepsilon := (u^\varepsilon - a)/|u^\varepsilon - a|$ is globally C^∞ and converges to u_a strongly in $W^{1,n-1}_{\text{loc}}$. Since $|u_a^\varepsilon| \equiv 1$, Lemma 2.2 implies that $[Ju_a^\varepsilon] = 0$. Moreover, the convergence of u_a^ε to u_a implies that $[Ju_a^\varepsilon] \to [Ju_a]$ in the sense of distributions, so we are finished. ◻

We next demonstrate that the strong coarea formula (1.7) holds for sufficiently differentiable functions.

Lemma 4.4. *If* $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^m;\mathbb{R}^n)$ *for some* $p > n$*, then*

$$u_a \in BnV_{\mathrm{loc}}(\mathbb{R}^m; S^{n-1})$$

for a.e. $a \in \mathbb{R}^n$*, and the strong coarea formula* (1.7) *holds.*

Proof.

1. For any Radon measure μ and any Borel set A,

$$\mu(A) = \inf\{\mu(O) \mid O \text{ open},\ A \subset O\}.$$

So it suffices to prove (1.7) under the assumption that A is open. For such sets,

$$\begin{aligned}|Ju|(A) &= \sup\left\{\int \omega\cdot[Ju] \mid \omega \in C^1_c(A;\Lambda^n\mathbb{R}^m),\ |\omega|\le 1\right\}\\ &= \sup\left\{\frac{1}{\omega_n}\int_{a\in\mathbb{R}^n}\int \omega\cdot[Ju_a]\,da \mid \omega\in C^1_c(A;\Lambda^n\mathbb{R}^m),\ |\omega|\le 1\right\}\\ &\le \frac{1}{\omega_n}\int_{a\in\mathbb{R}^n}|Ju_a|(A)\,da.\end{aligned}$$

This argument in fact is valid for any $u \in W^{1,n-1}\cap L^\infty$. Thus in particular we have established (1.8).

2. We now prove the other inequality. For the time being assume that u is C^1, and fix an open set $A \subset \mathbb{R}^m$ such that $|Ju|(A) < \infty$. We define a measure μ on $\mathbb{R}^n$ by setting $\mu(B) = |Ju|(u^{-1}(B)\cap A)$.

Let μ_{ac} denote the part of μ that is absolutely continuous with respect to the Lebesgue measure. General theorems on differentiation of Radon measures (see for example Evans and Gariepy [11, Chapter 1]) imply that

$$\mu_{\mathrm{ac}} = m(a)\mathcal{L}^n(da) = m(a)\,da, \quad \text{for } m(a) := \lim_{\varepsilon\to 0}\frac{\mu(B_\varepsilon(a))}{\omega_n\varepsilon^n}.$$

In particular, the limit that defines $m(a)$ exists and is finite for $\mathcal{L}^n$ almost every a.

Now define

$$u^\varepsilon_a(x) := \begin{cases}\dfrac{1}{\varepsilon}(u-a) & \text{if } |u-a|\le\varepsilon,\\[2mm] \dfrac{u-a}{|u-a|} & \text{if } |u-a|\ge\varepsilon.\end{cases} \tag{4.2}$$

Since u is C^1, it is clear that

$$Ju_a^\varepsilon(x) := \begin{cases} \dfrac{1}{\varepsilon^n} Ju & \text{if } |u-a| < \varepsilon, \\ 0 & \text{if } |u-a| > \varepsilon. \end{cases}$$

It follows that for every $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{\varepsilon^n}\mu(B_\varepsilon(a)) &= \frac{1}{\varepsilon^n}\int_{\{x\in A: |u(x)-a|<\varepsilon\}} |Ju|\,dx \\ &= \int_{x\in A} |Ju_a^\varepsilon|\,dx = |Ju_a^\varepsilon|(A). \end{aligned} \tag{4.3}$$

Recall also that $u_a \in W^{1,n-1}_{\text{loc}}$ for a.e. $a \in \mathbb{R}^n$, by Lemma 4.3. For such a, one readily checks that $u_a^\varepsilon \to u_a$ in $W^{1,n-1}_{\text{loc}}$ and $u_a^\varepsilon \rightharpoonup u_a$ weak-* L^∞ as $\varepsilon \to 0$, and these imply that $j(u_a^\varepsilon) \rightharpoonup j(u_a)$ weakly in L^1. By lower semicontinuity it follows that

$$\liminf_{\varepsilon\to 0} |Ju_a^\varepsilon|(A) \ge |Ju_a|(A), \quad \text{a.e. } a \in \mathbb{R}^n. \tag{4.4}$$

At this stage it is still possible that both sides of the above inequality are infinite.

3. Thus for a.e. $a \in \mathbb{R}^n$,

$$m(a) = \lim_{\varepsilon\to 0} \frac{\mu(B_\varepsilon(a))}{\omega_n \varepsilon^n} = \lim_{\varepsilon\to 0} \frac{1}{\omega_n} |Ju_a^\varepsilon|(A) \ge \frac{1}{\omega_n} |Ju_a|(A), \tag{4.5}$$

using (4.3) and (4.4). In particular this shows that $|Ju_a|$ is a measure for a.e. $a \in \mathbb{R}^n$. So

$$|Ju|(A) \ge \mu_{\text{ac}}(\mathbb{R}^n) = \int_{a\in\mathbb{R}^n} m(a)\,da \ge \frac{1}{\omega_n}\int_{a\in\mathbb{R}^n} |Ju_a|(A)\,da. \tag{4.6}$$

4. It remains to prove (4.6) for u merely belonging to $W^{1,p}_{\text{loc}}(\mathbb{R}^m;\mathbb{R}^n)$, for $p > n$.

Fix such a function u, let A be any open set, and let $\{u^k\}$ be a sequence of smooth approximators converging to u locally in $W^{1,p}$. Since u, $u^k \in W^{1,p}$, it is clear that $|Ju|$ and $|Ju^k|$ are absolutely continuous with respect to $\mathcal{L}^m$. We write $|Ju(x)|$ to indicate the Radon-Nikodym derivative $\partial|Ju|/\partial\mathcal{L}^m$, and similarly $|Ju^k(x)|$. Then the $W^{1,p}$ convergence implies that

$$|Ju|(A) = \int_A |Ju(x)|\,dx = \lim_k \int_A |Ju^k(x)|\,dx = \lim_k |Ju^k|(A). \tag{4.7}$$

Moreover, Lemma 4.9, to be proven at the end of this section, together with Lemma 2.1, shows that for a.e $a \in \mathbb{R}^n$,

$$|Ju_a|(A) \le \liminf_k |Ju_k^a|(A).$$

Thus by Fatou's lemma and the results of Step 3,

$$\frac{1}{\omega_n}\int_{\mathbb{R}^n} |Ju_a|(A)\,da \le \liminf_k \frac{1}{\omega_n}\int_{\mathbb{R}^n} |Ju_k^a|(A)\,da \le \liminf_k |Ju^k|(A). \tag{4.8}$$

The desired result follows immediately from (4.7) and (4.8). □

Next we establish the distributional chain rule. It will follow from the distributional coarea formula and an approximation argument.

As above, we write $u_a := (u-a)/|u-a|$.

Lemma 4.5. *The distributional chain rule* (1.6) *holds under any of the following assumptions:*

(i) $F \in C^1(\mathbb{R}^m;\mathbb{R}^n)$ *and* $u \in W^{1,n-1}_{\mathrm{loc}} \cap L^\infty(\mathbb{R}^m;\mathbb{R}^n)$.
(ii) $F \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)$ *and* $u \in W^{1,p}_{\mathrm{loc}} \cap L^\infty(\mathbb{R}^m;\mathbb{R}^n)$, $p > n-1$.
(iii) $F \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)$ *and* $u \in W^{1,n-1}_{\mathrm{loc}} \cap C^0(\mathbb{R}^m;\mathbb{R}^n)$.

For the proof we will need the following well-known lemma.

Lemma 4.6 (Weak continuity of Jacobians)**.**

(i) *Suppose that* $u_k \rightharpoonup u$ *weakly in* $W^{1,p_1}_{\mathrm{loc}}(\mathbb{R}^m;\mathbb{R}^n)$ *and* $u_k \to u$ *strongly in* $L^{p_2}_{\mathrm{loc}}$, *for* $p_1, p_2 \ge 1$ *such that* $(n-1)/p_1 + 1/p_2 := 1/q < 1$. *Then* $j(u_k) \rightharpoonup j(u)$ *in* L^q_{loc}, *and*

$$[Ju_k] \rightharpoonup [Ju] \tag{4.9}$$

weak- in* $(C^1)^*$. *The convergence of* $[Ju_k]$ *takes place in the weak topology on* $L^{p_1/n}$ *if* $p_1 > n$, *and weak-* in* $(C^0_c)^*$ *if* $p_1 = n$.

(ii) *If* $u_k \rightharpoonup u$ *weakly in* $W^{1,n-1}_{\mathrm{loc}}(\mathbb{R}^m;\mathbb{R}^n)$, $u_k \to u$ *strongly in* L^∞_{loc}, *and* u_k, u *are continuous, then* $j(u_k) \rightharpoonup j(u)$ *weak-* in* $(C^0_c)^*$, *and* (4.9) *is again satisfied.*

Remark 4.7. Note that the Sobolev embedding theorem implies that hypothesis (i) above holds if $u_k \rightharpoonup u$ weakly in $W^{1,p}$ for any $p > mn/(m+1)$.

Proof. Under hypothesis (i), the lemma is standard. A proof under slightly different hypotheses can be found for example in Giaquinta, Modica, and Souček [17, Proposition 1 in Section 3.3.1]. That the hypotheses of this proposition are satisfied in our situation follows from Theorem 1 in Section 3.3.2 in the same reference.

If (ii) holds, then the $(n-1)\times(n-1)$ minors of Du_k converge weakly as measures to the corresponding minors of Du; the proof, which again is standard, essentially appears in [17, Section 3.3.1]. The remaining claims follow from standard facts about products of strongly and weakly converging sequences. □

Proof of Lemma 4.5.

1. We first prove the result under the assumption that both F and u are C^∞. Let $w = F(u)$.

For $\omega \in C_c^1(\mathbb{R}^m;\Lambda^n\mathbb{R}^m)$ fixed we compute

$$\langle\omega,[Jw]\rangle = \int_{\mathbb{R}^m}\omega(x)\cdot Jw(x)\,dx = \int_{\mathbb{R}^m}\omega(x)\cdot Ju(x)JF(u(x))\,dx.$$

Let $\widetilde{\omega}(x) := JF(u(x))\omega(x) \in C_c^\infty(\mathbb{R}^m;\Lambda^n\mathbb{R}^m)$. Then we can rewrite the above as

$$\int_{\mathbb{R}^m}\omega(x)\cdot Ju(x)JF(u(x))\,dx = \langle\widetilde{\omega},[Ju]\rangle = \frac{1}{\omega_n}\int_{a\in\mathbb{R}^n}\langle\widetilde{\omega},[Ju_a]\rangle\,da,$$

by the coarea formula (1.5). But $u(x) = a$ on the support of $[Ju_a]$, so $\widetilde{\omega}(x) = JF(a)\omega(x)$ on the support of $[Ju_a]$. It follows that $\langle\widetilde{\omega},[Ju_a]\rangle = JF(a)\langle\omega,[Ju_a]\rangle$ a.e. $a\in\mathbb{R}^n$, using the fact that $[Ju_a]$ is a measure a.e. a, which is proved for smooth functions in Lemma 4.4.

Thus (1.6) holds if F and u are smooth.

2. Now suppose that F is C^1, but u merely belongs to $W^{1,n-1}_{\mathrm{loc}}\cap L^\infty(\mathbb{R}^m;\mathbb{R}^n)$. By mollifying u we can produce a uniformly bounded sequence of smooth functions u^k such that $u^k \to u$ in $W^{1,n-1}_{\mathrm{loc}}$. It follows that $w^k := F(u^k)$ converges to w in the same norm. Thus, using the Approximation Lemma 4.9, which is proven at the end of this section, we deduce from Step 1 that the chain rule (1.6) continues to hold under these hypotheses. Thus the lemma is proven under assumption (i).

3. Now assume that F is merely Lipschitz. By smoothing F we obtain a sequence of smooth functions F^k such that $F^k \to F$ in $W^{1,p}$ for all $p<\infty$, $\|DF^k\|_\infty \le C$, and moreover, $DF^k \to DF$ almost everywhere. Define $v^k = F^k(u)$. By Step 2,

$$\langle\omega,[Jv^k]\rangle = \frac{1}{\omega_n}\int_{a\in\mathbb{R}^n}JF^k(a)\langle\omega,[Ju_a]\rangle\,da$$

for every k. The function $a\mapsto\langle\omega,[Ju_a]\rangle$ is integrable, so the dominated convergence theorem allows us to pass to limits on the right-hand side.

It remains only to show that

$$\langle\omega,[Jv^k]\rangle \to \langle\omega,[Jw]\rangle \tag{4.10}$$

as $k\to\infty$.

If $u \in W^{1,p}$, $p > n-1$, then $v^k \to w$ in L^∞, and $\{Dv^k\}$ is uniformly bounded in L^p, and hence weakly precompact in L^p. So the conclusion follows from (i) of Lemma 4.6.

If $u \in W^{1,n-1} \cap C^0$, then similarly (4.10) follows from (ii) of Lemma 4.6. $\square$

In the following lemma, we verify that the strong coarea formula holds for functions in BnV that are S^{n-1}-valued.

Lemma 4.8. *Suppose that* $u \in W^{1,n-1}_{\mathrm{loc}} \cap BnV_{\mathrm{loc}}(\mathbb{R}^m; S^{n-1})$*, and write* $u_a = (u-a)/|u-a|$ *as usual. Then*

$$[Ju_a] = \begin{cases} [Ju] & \text{if } |a| < 1, \\ 0 & \text{if } |a| > 1. \end{cases}$$

In particular, $u_a \in BnV_{\mathrm{loc}}(\mathbb{R}^m; S^{n-1})$ *for a.e.* $a \in \mathbb{R}^n$*, and the strong coarea formula* (1.7) *holds.*

Proof. We start by defining

$$F_a^\varepsilon(\alpha) = F\left(\frac{\alpha - a}{\varepsilon}\right), \quad F(\alpha) = f(|\alpha|)\frac{\alpha}{|\alpha|},$$

for $f \in C^\infty(\mathbb{R}^+; \mathbb{R}^+)$ satisfying

$$f(r) \equiv 0 \text{ for } r < \frac{1}{3}, \quad f(r) \equiv 1 \text{ for } r > \frac{2}{3}, \quad f' \geq 0 \ \forall r.$$

Then one can verify that $\eta := (1/\omega_n)JF$ is smooth, nonnegative, supported in the unit ball, and satisfies $\int_{\mathbb{R}^n} \eta(\alpha)\, d\alpha = 1$. Moreover, one easily checks that $JF_a^\varepsilon(\alpha) = \eta^\varepsilon(\alpha - a)$, for $\eta^\varepsilon(\alpha) := (1/\varepsilon^n)\eta(\alpha/\varepsilon)$.

Fix any a such that $|a| < 1$. For $\varepsilon < 1 - |a|$ we can define a smooth function $G_a^\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$G_a^\varepsilon(\alpha) = \begin{cases} F^\varepsilon(\alpha) & \text{if } \alpha \in B_\varepsilon(a), \\ G_a^\varepsilon(\alpha) = \alpha & \text{if } |\alpha| = 1, \end{cases}$$

and

$$G_a^\varepsilon \text{ maps } \mathbb{R}^n \setminus B_\varepsilon(a) \text{ into } S^{n-1}.$$

Finally, define $u_a^\varepsilon := G_a^\varepsilon(u)$. Note that $JG_a^\varepsilon \equiv JF_a^\varepsilon$, since $G_a^\varepsilon = F_a^\varepsilon$ in $B_\varepsilon(A)$ and $JG_a^\varepsilon = JF_a^\varepsilon = 0$ elsewhere.

Since $|u| = 1$ a.e., the definition of G_a^ε implies that $u_a^\varepsilon = u$ a.e., and hence that $[Ju_a^\varepsilon] = [Ju]$. Combining this with the distributional coarea formula, we obtain

$$[Ju] = [Ju_a^\varepsilon] = \frac{1}{\omega_n}\int JF_a^\varepsilon(\alpha)[Ju_\alpha]\, d\alpha = \int \eta^\varepsilon(\alpha - a)[Ju_\alpha]\, d\alpha.$$

Since this holds for all sufficiently small ε, we conclude that $[Ju] = [Ju_a]$ if $|a| < 1$. It is clear from Lemma 4.1 that $[Ju_a] = 0$ if $|a| > 1$. $\square$

Finally we prove the approximation result used several times above. Informally, the point is that almost every level set of the approximating functions converges to the corresponding level set of the limiting function, in a weak sense.

As usual, given a function u_k and a point $a \in \mathbb{R}^n$, we write $u_k^a := (u_k - a)/|u_k - a|$.

Lemma 4.9. *Suppose that $u \in W^{1,n-1}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^m;\mathbb{R}^n)$, and that u_k is a sequence of smooth, locally uniformly bounded functions such that $u_k \to u$ in $W^{1,n-1}_{\mathrm{loc}}$. Then*

$$j(u_k) \rightharpoonup j(u) \text{ weakly in } L^1_{\mathrm{loc}}, \quad [Ju_k] \rightharpoonup [Ju] \text{ weakly in } (C^1_c)^*;$$

and for every $\omega \in C^1_c(\mathbb{R}^m, \Lambda^n\mathbb{R}^m)$,

$$\lim_k \int |\langle \omega, [Ju_k^a]\rangle - \langle \omega, [Ju_a]\rangle| \, da = 0. \tag{4.11}$$

Also, after passing to a subsequence we can arrange that

$$\langle \omega, [Ju_k^a]\rangle \to \langle \omega, [Ju_a]\rangle \quad \text{a.e. } a \in \mathbb{R}^n. \tag{4.12}$$

Remark 4.10. Given u as above, such a sequence u^k can be produced by mollification.

Proof.

1. It is clear that $u_k \rightharpoonup u$ weak-* L^∞. With the assumption $u_k \to u$ in $W^{1,n-1}_{\mathrm{loc}}$, this implies that $j(u_k) \to j(u)$ weakly in L^1_{loc}. It immediately follows that $[Ju_k] \rightharpoonup [Ju]$ weakly in $(C^1_c)^*$.

Also, note that (4.11) easily implies (4.12). Indeed, (4.11) states that, for any $\omega \in C^1_c(\mathbb{R}^m;\Lambda^n\mathbb{R}^m)$, the functions $a \mapsto \langle \omega, [Ju_k^a]\rangle$ converge in $L^1(da)$ as $k \to \infty$ to the limit $a \mapsto \langle \omega, [Ju_a]\rangle$. We may thus pass to a subsequence such that, for every ω_ν in a countable dense subset $\{\omega_\nu\}_{\nu=1}^\infty$ of $C^1_c(\mathbb{R}^m;\Lambda^n\mathbb{R}^m)$, $\langle \omega_\nu, [Ju_k^a]\rangle \to \langle \omega_\nu, [Ju_a]\rangle$ for almost every a. This implies (4.12).

So we only need to prove (4.11).

Since (4.11) involves integrating against compactly supported test functions ω, we may assume that $u \in L^\infty$ and that the approximating sequence u^k is uniformly bounded.

Arguing as in Step 1 of the proof of Lemma 4.3, we see that for a.e. $a \in \mathbb{R}^n$,

$$u_k^a \in W^{1,n-1}_{\mathrm{loc}} \quad \text{for all } k, \text{ and } u^a \in W^{1,n-1}_{\mathrm{loc}}.$$

It suffices to show that every subsequence of $\{u_k\}$ has a further subsequence for which (4.11) holds. We may therefore pass to subsequences and assume that

$$u_k \to u \text{ a.e.}, \quad Du_k \to Du \text{ a.e.},$$

which implies that $j(u_k) \to j(u)$ a.e. $x \in \mathbb{R}^m$.

2. The desired conclusion (4.11) will easily follow once we demonstrate that

$$\int_U \int_V |j(u_k^a) - j(u_a)| \, da \, dx \to 0 \tag{4.13}$$

as $k \to \infty$, for every bounded open $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$.

From Step 1 we see that $j(u_k^a)(x) \to j(u_a)(x)$ for a.e. $(x, a) \in \mathbb{R}^m \times \mathbb{R}^n$. Also it is clear from the definition of $j(u)$ that there is some constant C such that

$$|j(u_k^a)| \le h_k := \frac{C|Du_k|^{n-1}}{|u_k - a|^{n-1}}, \quad |j(u_a)| \le h := \frac{C|Du|^{n-1}}{|u - a|^{n-1}}.$$

Here h_k and h are functions of $(x, a) \in \mathbb{R}^m \times \mathbb{R}^n$. Step 1 implies that $h_k \to h$ for a.e.(x, a). By a well-known variant of the dominated convergence theorem, it suffices to show that

$$\int_U \int_V h_k \, da \, dx \to \int_U \int_V h \, da \, dx.$$

Toward this end, note that

$$\int_U \int_V (h_k - h) \; dx = I + II,$$

for

$$\begin{aligned} I &:= C \int_U \int_V \frac{|Du_k|^{n-1} - |Du|^{n-1}}{|u_k - a|^{n-1}} \, da \, dx, \\ II &:= C \int_U \int_V \left(\frac{1}{|u_k - a|^{n-1}} - \frac{1}{|u - a|^{n-1}} \right) |Du|^{n-1} \, da \, dx. \end{aligned}$$

3. Integrating first with respect to a, we easily estimate

$$|I| \le C(V) \int_U \big| |Du_k|^{n-1} - |Du|^{n-1} \big| \, dx \to 0, \quad \text{as } m \to \infty.$$

To show that $II \to 0$ as $m \to \infty$, define a function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(z) := \int_V \frac{1}{|z - a|^{n-1}} \, da = \int_{\mathbb{R}^n} \frac{1}{|a|^{n-1}} \chi_V(z - a) \, da.$$

The dominated convergence theorem implies that f is continuous. It follows that $f(u_k(x)) \to f(u(x))$ for a.e. $x \in U$. Since V is bounded, one easily sees that f is bounded, and so the dominated convergence theorem implies that

$$\int_U \int_V \frac{|Du|^{n-1}}{|u_k - a|^{n-1}}\, da\, dx. = \int_U |Du|^{n-1} f(u_k)\, dx$$
$$\to \int_U \int_V \frac{|Du|^{n-1}}{|u - a|^{n-1}}\, da\, dx.$$

Thus $II \to 0$ and so we have established (4.13).

4. To prove (4.11), note that

$$\int |\langle \omega, [Ju_k^a]\rangle - \langle \omega, [Ju_a]\rangle|\, da = \int_V |\langle \omega, [Ju_k^a]\rangle - \langle \omega, [Ju_a]\rangle|\, da$$
$$\leq \|d^* \omega\|_\infty \int_U \int_V |j(u_k^a) - j(u_a)|\, dx\, da,$$

where $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are bounded open sets such that $\operatorname{supp}(\omega) \subset U$ and $B_R(0) \subset V$ for $R := \sup_k \|u^k\|_{L^\infty(U)}$. ◻

5. Rectifiability via Slicing

In this section we give a proof of Theorem 1.1 that relies on the Rectifiable Slices Theorem. We start by stating this theorem.

Suppose that $\mathbf{J}$ is a k-dimensional current in $\mathbb{R}^m$, and $f : \mathbb{R}^m \to \mathbb{R}^\ell$ is a Lipschitz function, where $\ell \leq k$. A family of $k - \ell$-dimensional currents $\{\langle \mathbf{J}, f, y\rangle\}_{y \in \mathbb{R}^\ell}$ in $\mathbb{R}^m$ are slices of $\mathbf{J}$ by f if

$$\mathbf{J}(\phi \wedge f^\sharp(dy)) = \int_{\mathbb{R}^\ell} \langle \mathbf{J}, f, y\rangle(\phi)\, dy$$

for all smooth $k - \ell$-forms ϕ. Here $f^\sharp(dy)$ denotes the pullback by f of the standard volume form on $\mathbb{R}^\ell$.

Theorem 5.1. *Suppose* $\mathbf{J}$ *is a k dimensional normal current in* $\mathbb{R}^m$ *(i.e., both* $\mathbf{J}$ *and* $\partial\mathbf{J}$ *have finite mass).*

Then $\mathbf{J}$ *is integer multiplicity rectifiable if and only if for every projection P onto a k-dimensional subspace of* $\mathbb{R}^m$*, the slices* $\langle \mathbf{J}, P, y\rangle$ *are* 0*-dimensional integer multiplicity rectifiable for a.e. y.*

This is a special case of a theorem in White [30], which in fact applies in the more general setting of flat k-chains with coefficients in an arbitrary group.

Currents $\mathbf{J}$ as in the statement of the theorem can always be sliced by Lipschitz functions, and $\langle \mathbf{J}, P, y\rangle$ is a normal current for a.e. y (see for example Federer [12, 4.3.1-2]).

When we wrote the first version of this paper, we were unaware of White's work, and we developed our own proof of Theorem 5.1 in more or less the form stated here. A sketch of our proof appears in [23], and we present the full version of our proof of Theorem 5.1 in [21]. Elements of our argument were later used by Ambrosio and Kirchheim [3] in developing a general theory of rectifiable currents in metric spaces.

To apply this theorem in our situation, we let $\mathbf{j_u}$ represent the $m-n+1$-dimensional current on U defined by

$$\mathbf{j_u}(\phi) = \int j(u) \wedge \phi,$$

and let $\mathbf{J_u} = (1/n)\partial \mathbf{j_u}$. Then Theorem 1.1 states that $(1/\omega_n)\mathbf{J_u}$ is an integer multiplicity $m-n$-dimensional rectifiable current.

Proof of Theorem 1.1. We verify that $(1/\omega_n)\mathbf{J_u}$ satisfies the hypotheses of Theorem 5.1 with $k = m-n$. It is clear that $(1/\omega_n)\mathbf{J_u}$ is a normal current, since it has no boundary and has finite mass by hypothesis. Let P be any projection $\mathbb{R}^m \to \mathbb{R}^{m-n}$. By making a suitable change of basis, we may assume that $P(x_1, \dots, x_m) = (x_1, \dots, x_{m-n})$.

1. We first claim that for a.e. $y \in \mathbb{R}^{m-n}$, one can identify $\langle \mathbf{J}, P, y \rangle$ with the Jacobian of the restriction of u to $P^{-1}(y)$, that is,

$$\langle \mathbf{J}, P, y \rangle(\phi) = \int_{z \in \mathbb{R}^n} \phi(y, z) \operatorname{Det} D_z u(y, z)\, dz. \tag{5.1}$$

Here we are writing $x \in \mathbb{R}^m$ as $(y, z) \in \mathbb{R}^{m-n} \times \mathbb{R}^n$, and D_z indicates differentiation only with respect to the z variables.

Using the definition of slices, one finds that (5.1) is equivalent to the assertion that

$$\begin{aligned}
&\int_{\mathbb{R}^m} \phi \operatorname{Det}(u_{x_{m-n+1}}, \dots, u_{x_m})\, dx \\
&= \int_{\mathbb{R}^{m-n}} \left(\int_{\mathbb{R}^n} \phi \operatorname{Det}(u_{x_{m-n+1}}, \dots, u_{x_m})\, dx_{m-n+1} \dots dx_m \right) dx_1 \dots dx_{m-n},
\end{aligned}$$

where the determinants are understood in the sense of distributions. This is an immediate consequence of Fubini's Theorem and the definition of the distributional determinant.

2. Thus to apply Theorem 5.1 we need to verify that

$$\left| \begin{array}{l} \dfrac{1}{\omega_n} \operatorname{Det} D_z u(y, z) \text{ is a finite sum of integer} \\ \text{multiplicity point masses for a.e. } y. \end{array} \right. \tag{5.2}$$

As remarked above, almost every slice of a normal current is again a normal current, and this implies that $\operatorname{Det} D_z u(y,z)$ has finite mass for almost every y, and thus that

$$z \mapsto u(y,z) \text{ belongs to } BnV(\mathbb{R}^n; S^{n-1}) \text{ for almost every } y. \tag{5.3}$$

In Proposition 5.2 below we prove that Theorem 1.1 holds when $m = n$. This fact, together with (5.3), implies that (5.2) holds. ◻

We now establish the propositions used above. First we prove Theorem 1.1 in the special case $m = n$.

Proposition 5.2. *Assume $u \in BnV(\mathbb{R}^n; S^{n-1})$. Then there are a finite collection of points $(a_1, \ldots, a_m) \in \mathbb{R}^n$ and integers $(d_1, \ldots, d_m) \in \mathbb{Z}$ such that*

$$[Ju] = \omega_n \sum d_i \delta_{a_i}\, dx.$$

Proof.

1. For $r > 0$ and $x \in \mathbb{R}^n$, let $u_{r,x}$ be the restriction of u to $\partial B_r(x)$. For every x, $u_{r,x}$ belongs to $W^{1,n-1}(\partial B_r; S^{n-1})$ for almost every r. For these values of r, x we define

$$d(r,x) := \frac{1}{n\omega_n} \int_{\partial B_{r(x)}} j(u) \cdot \tau_{\partial B_r}.$$

Results of Brezis and Nirenberg [9] show that $d(r,x)$ is exactly the degree of $u_{r,x}$ as a map from $\partial B_r(x)$ to S^{n-1}. If u is smooth, this is essentially a classical integral formula for the degree; the point here is that $j(u)$ is the pullback by u of the standard volume form on S^{n-1}. The paper of Brezis and Nirenberg mentioned above shows that integral representations of this sort remain valid for example in $W^{1,n-1}(S^{n-1}; S^{n-1})$, which is our situation.

In particular, for every $x \in \mathbb{R}^n$, $d(r,x)$ is an integer for a.e. $r > 0$.

For convenience we now consider balls around $x = 0$, and we write B_r for $B_r(0)$, the *open* ball of radius r about the origin. Similarly we write $d(r)$ as a shorthand for $d(r,0)$.

2. Let $f \in C^1_c([0,\infty); \mathbb{R})$ satisfy $f'(0) = 0$ and $\|f\|_\infty \le 1$. Then $\varphi := f(|x|)\, dx$ is a smooth n-form, where $dx := dx^1 \wedge \cdots \wedge dx^n$ is the standard volume form. Using (2.5) we then compute

$$d^*\varphi = (-1)^n \star df(|x|) = (-1)^n \star \left(f'(|x|) \frac{x_i}{|x|}\, dx^i \right).$$

For every $r > 0$, the vector field $(x_i/|x|)\, dx^i$ is an oriented unit normal to ∂B_r, so $\star((x_i/|x|)\, dx^i) := \tau$ is the appropriately oriented unit tangent $(n-1)$-vector field to ∂B_r. (Recall that we are not distinguishing between vectors and covectors.)

Thus

$$\int \varphi \cdot [Ju] = \frac{1}{n}\int j(u)\cdot d^*\varphi \tag{5.4}$$
$$= \frac{(-1)^n}{n}\int_0^\infty f'(r)\Big(\int_{\partial B_r} j(u)\cdot \tau_{\partial B_r}\Big)\,dr$$
$$= (-1)^n\omega_n \int f'(r)d(r)\,dr.$$

Also, since $u \in BnV$ and $|\varphi| \le 1$,

$$\infty > |Ju|(\mathbb{R}^n) \ge \int \varphi \cdot [Ju] = (-1)^n\omega_n \int f'(r)d(r)\,dr.$$

Since this holds for all f as above, we deduce that $d(\cdot) \in BV([0,\infty))$.

3. We next define, for $r > 0$,

$$\tilde{d}(r) := \lim_{h\to 0}\frac{1}{h}\int_{r-h}^{r} d(s)\,ds.$$

Since $d(\cdot)$ is a BV function, this limit exists for *all* $r > 0$. Then $\tilde{d}(\cdot)$ is a left continuous function that agrees with $d(\cdot)$ almost everywhere. In particular $\tilde{d}(\cdot)$ is integer-valued. It is also a function of bounded variation in the classical, pointwise sense.

Fix some $r > 0$. Since $[Ju]$ can be identified with a signed measure, $[Ju](B_r)$ is certainly well-defined, and can be computed by

$$[Ju](B_r) = \lim_{k\to\infty}\int \varphi_k \cdot [Ju],$$

where φ_k is a sequence of smooth n-forms defined by $\varphi_k(x) = f_k(|x|)\,dx$, for smooth, nonincreasing functions $f_k : [0,\infty) \to [0,1]$ that satisfy

$$f_k(s) = \begin{cases} 1 & \text{if } s \le r - 1/k, \\ 0 & \text{if } s \ge r. \end{cases}$$

Using the fact that $\tilde{d}(\cdot)$ is left continuous and (5.4)

$$[Ju](B_r) = (-1)^n\omega_n \lim_{k\to\infty}\int f_k'(s)d(s)\,ds = (-1)^{n+1}\omega_n\tilde{d}(r).$$

4. Because $\widetilde{d}(\cdot)$ has bounded variation and is integer valued, there exists some $r_0 > 0$ such that

$$[Ju](B_r) = (-1)^{n+1}\omega_n \widetilde{d}(r) \quad \text{is constant } \forall r < r_0.$$

We have established this fact for balls $B_r(0)$ around the origin, but it clearly remains valid for balls about any center $x \in \mathbb{R}^n$. For each $x \in \mathbb{R}^n$, we may thus define

$$(-1)^{n+1}\omega_n\, d_0(x) := \lim_{r\to 0}[Ju](B_r(x)) = [Ju](B(r)) \quad \text{for all } r < r_0(x).$$

By the same argument used above, $d_0(x)$ is an integer for *every* x. Since $[Ju]$ is finite, it is clear that $d_0(x)$ can be nonzero at only finitely many points x. Once we know this, it is easy to see that for any open set U,

$$[Ju](U) = (-1)^{n+1}\omega_n \sum_{x\in U: d_0(x)\neq 0} d_0(x).$$

The same identity then holds for all sets U, which proves the theorem. ◻

6. Rectifiability via Cartesian currents

In this section we give a quick proof of Theorem 1.1 that was suggested by M. Giaquinta and G. Modica.

Given $U \subset \mathbb{R}^m$ and $u \in W^{1,n-1}(U, S^{n-1})$, let G_u represent the m-dimensional current associated with integration over the graph of u in the product space $U \times \mathbb{R}^n$.

We write x to denote a typical point in U and ξ a point in $\mathbb{R}^n$. We will write $dx^\alpha \wedge d\xi^\beta$ to denote multivectors in the product space, where α and β are multiindices. A differential form of type (j,k) is one of the form

$$\phi = \sum_{|\alpha|=j,\,|\beta|=k} \phi^{\alpha\beta}\, dx^\alpha \wedge d\xi^\beta.$$

Let $\mathbf{j_u}$ represent the $m-n+1$ dimensional current on U defined by

$$\mathbf{j_u}(\phi) = \int j(u) \wedge \phi,$$

and let $\mathbf{J_u} = (1/n)\partial \mathbf{j_u}$.

Finally, let $[S^{n-1}]$ denote the $n-1$-dimensional current associated with integration over S^{n-1} in $\mathbb{R}^n$.

Giaquinta and Modica note that Theorem 1.1 follows almost immediately from the following lemma, which is essentially proven in [18].

Lemma 6.1. *If* $u \in W^{1,n-1}(U, S^{n-1})$, *then* $\partial G_u = (1/\omega_n)\mathbf{J_u} \times [S^{n-1}]$.

Using the lemma, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. First we claim that G_u is an integer multiplicity rectifiable current. To verify this, it suffices to check that all $k \times k$ minors of Du are locally integrable. (See for example [17, Section 3.2.1.].) This is clear if $k \le n-1$, since $u \in W^{1,n-1}$. And since $Du(x)$ is for a.e. x a linear map from $\mathbb{R}^m$ into $T_{u(x)}S^{n-1}$, it must have rank at most $n-1$ almost everywhere. This implies that all $n \times n$ minors of Du vanish a.e.

Next, if $u \in BnV$ then Lemma 6.1 implies that ∂G_u has finite mass. The boundary rectifiability theorem of Federer and Fleming [13] then states that ∂G_u is integer multiplicity rectifiable. It immediately follows from the lemma that $(1/\omega_n)\mathbf{J_u}$ is also integer multiplicity rectifiable. This is Theorem 1.1. □

Lemma 6.1 is not presented in exactly the form stated here in Giaquinta, Modica, and Souček [18], so for the reader's convenience we give the proof. Note that the lemma does not assume $u \in BnV$.

Proof of Lemma 6.1.

1. First, for any $v \in W^{1,n-1} \cap L^\infty(U;\mathbb{R}^n)$, one can explicitly write out the action of G_v on an m form ϕ to find that

$$G_v(\phi) = \int_U \phi(x, v(x)) \cdot (dx_1 + v^{i_1}_{x_1}\, d\xi_{i_1}) \wedge \cdots \wedge (dx_m + v^{i_m}_{x_m}\, d\xi_{i_m})\, dx. \tag{6.1}$$

Thus in particular, $G_v(\phi)$ has the form

$$\sum_j \int_U \sum_{\substack{|\alpha| = m-j,\\ |\beta| = j}} \phi^{\alpha\beta}(x, v(x)) M_{\alpha\beta}(Dv(x))\, dx, \tag{6.2}$$

where $M_{\alpha\beta}(Dv)$ is a $|\beta| \times |\beta|$ minor of Dv.

Since the target S^{n-1} is $(n-1)$-dimensional, $G_u(\phi) = 0$ for forms ϕ of type $(m-j, j)$ with $j \ge n$. Thus $\partial G_u(\phi) = 0$ for forms of type $(m-j-1, j)$ with $j \ge n$.

We now claim that $\partial G_u(\phi) = 0$ for forms of type $(m-j-1, j)$ with $j \le n-2$. To see this, let u^ε be a sequence of smooth functions, in general not S^{n-1}-valued, converging to u in $W^{1,n-1}$, and let G_{u^ε} denote the current associated with integration over the graph of u^ε. For $j \le n-1$, the $j \times j$ minors of u^ε converge strongly in L^1_{loc} to the $j \times j$ minors of Du, so one easily deduces from (6.2) that $\lim_\varepsilon G_{u^\varepsilon}(\phi) = G_u(\phi)$ for any form ϕ of type $(m-j, j)$, with $j \le n-1$. Since the functions u^ε are smooth, the associated currents G_{u^ε} have vanishing boundary. Thus if ϕ is a form of type $(m-j-1, j)$ with $j \le n-2$, then

$$\partial G_u(\phi) = G_u(d\phi) = \lim_\varepsilon G_{u^\varepsilon}(d\phi) = \lim_\varepsilon \partial G_{u^\varepsilon}(\phi) = 0.$$

So ∂G_u is nonzero only on forms of type $(m-n, n-1)$.

2. Because G_u is supported in $U \times S^{n-1} \subset U \times \mathbb{R}^n$, we may think of it as a current in the manifold $U \times S^{n-1}$, and similarly ∂G_u. Let ϕ be a differential form of type $(m-n, n-1)$ with compact support in $U \times S^{n-1}$. Since $\Lambda^{n-1} TS^{n-1}$ is one-dimensional, we can write ϕ in the form $\sum_{\alpha \in I_{m-n,m}} \phi^\alpha(x,\xi)\, dx^\alpha \wedge \tau$, where for $(x,\xi) \in U \times S^{n-1}$, τ is the unit $n-1$ vector in $\Lambda^{n-1} T_\xi S^{n-1}$ representing the oriented tangent to S^{n-1} at ξ. Since $\Lambda^n TS^{n-1}$ is trivial, $d_\xi \phi = 0$, so Hodge theory implies that ϕ can be written

$$\phi(x,\xi) = \sum_{\alpha \in I_{m-n,m}} \bar{\phi}^\alpha(x)\, dx^\alpha \wedge \tau + d_\xi \psi^\alpha(x,\xi) \wedge dx^\alpha$$

where, for each $\alpha \in I_{m-n,m}$, ψ^α is a form of type $(0, n-2)$ with compact support in $U \times S^{n-1}$, and

$$\bar{\phi}^\alpha(x) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \phi^\alpha(x,\xi)\, dH^{n-1}(\xi). \tag{6.3}$$

Defining $\bar{\phi} = \sum_\alpha \bar{\phi}^\alpha\, dx^\alpha$, we write the above more concisely as $\phi = \bar{\phi} \wedge \tau + d_\xi \psi$. This can be rewritten $\phi = \bar{\phi} \wedge \tau + d\psi - d_x \psi$. Since $d_x \psi$ is a form of type $(m-n+1, n-2)$, we deduce from Step 1 that

$$\partial G_u(\phi) = \partial G_u(\bar{\phi} \wedge \tau).$$

One can then compute from (6.1) that

$$\partial G_u(\phi) = G_u(d\bar{\phi} \wedge \tau) = \int_U d\bar{\phi}(x) \wedge j(u) = \mathbf{j_u}(d\bar{\phi}). \tag{6.4}$$

3. On the other hand, the definition of a product of currents implies that $\mathbf{j_u} \times [S^{n-1}]$ is nonzero only on forms of type $(m-n+1, n-1)$. Let ψ be such a form, where $\psi = \sum_{\alpha \in I_{m-n+1,m}} \psi^\alpha(x,\xi)\, dx \wedge \tau$. Then by definition,

$$\mathbf{j_u} \times \frac{1}{|S^{n-1}|}[S^{n-1}](\psi) = \mathbf{j_u}(\bar{\psi}),$$

where $\bar{\psi}$ is the $(m-n+1)$-form with compact support in U defined as in (6.3) by averaging over the vertical variables. In particular this holds for $\psi = d\phi$. Since $|S^{n-1}| = n\omega_n$ and $\partial[S^{n-1}] = 0$, we deduce from (6.4) that

$$\begin{aligned}\partial G_u(\phi) = \mathbf{j_u}(d\bar{\phi}) &= \frac{1}{n\omega_n} \partial(\mathbf{j_u} \times [S^{n-1}])(\phi)\\ &= \frac{1}{n\omega_n} \partial \mathbf{j_u} \times [S^{n-1}](\phi) = \frac{1}{\omega_n} \mathbf{J_u} \times [S^{n-1}](\phi). \qquad \square\end{aligned}$$

REFERENCES

[1] F. ALMGREN, F. BROWDER & E.H. LIEB, *Co-area, liquid crystals, and minimal surfaces*, Partial Differential Equations (Tianjin, 1986), Lectures Notes in Math Volume 1306, Springer, Berlin, 1988, pp. 1-22.

[2] L. AMBROSIO, *Metric space valued functions of bounded variation*, Ann. Scuola. Norm. Sup. Pisa Cl. Sci (4) **17** (1990), 439-478.

[3] L. AMBROSIO & B. KIRCHHEIM, *Currents in metric spaces*, Acta Math. **185** (2000), 1-80.

[4] L. AMBROSIO & G. DAL MASO, *A general chain rule for distributional derivatives*, Proc. Amer. Math. Soc **108** (1990), 691-702.

[5] J. BALL, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rat. Mech. Anal **63** (1977), 337-403.

[6] F. BETHUEL, *A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps*, Ann. Inst. H. Poincaré Anal. Non Lineaire **7** (1990), 269-286.

[7] H. BREZIS, personal communication, 1998.

[8] H. BREZIS, J.-M. CORON & E. LIEB, *Harmonic maps with defects*, Commun. Math. Phys **107** (1986), 649-705.

[9] H. BREZIS & L. NIRENBERG, *Degree theory and BMO: Part i: compact manifolds without boundaries*, Selecta Math. (N.S.) **1** (1995), 197-263.

[10] R. COIFMAN, P.-L. LIONS, Y. MEYER & S. SEMMES, *Compensated compactness and hardy spaces*, J. Math. Pures Appl.(9) **72** (1993), 247-286.

[11] L. C. EVANS & R.F. GARIEPY, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, Florida, 1992.

[12] H. FEDERER, *Geometric Measure Theory*, Springer-Verlag, Berlin-Heidelberg-New York, 1969.

[13] H. FEDERER & W. FLEMING, *Normal and integral currents*, Ann. of Math. (2) **72** (1960), 458-520.

[14] W. FLEMING & R. RISHEL, *An integral formula for the total variation*, Arch. Math. **11** (1960), 218-222.

[15] M. GIAQUINTA, G. MODICA & J. SOUČEK, *Cartesian currents, weak diffeomorphisms and existence theorems in nonlinear elasticity*, Arch. Rat. Mech. Anal. **106** (1989), 97-159.

[16] ______, *Graphs of finite mass which cannot be approximated in area by smooth graphs*, Manuscripta Math. **78** (1993), 259-271.

[17] ______, *Cartesian Currents in the Calculus of Variations. I. Cartesian Currents*, Springer-Verlag, 1998.

[18] ______, *Cartesian Currents in the Calculus of Variations. II. Variational Integrals*, Springer-Verlag, 1998.

[19] E. GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, 1984.

[20] F. HANG & F.-H. LIN, *A remark on the Jacobians*, Commun. Contemp. Math **2** (2000), 35-46.

[21] R.L. JERRARD, *A new proof of the Rectifiable Slices Theorem*, preprint, 2001.

[22] R.L. JERRARD & H.M. SONER, *The Jacobian and the Ginzburg-Landau energy*, Calc. Var. PDE, to appear.

[23] R.L. JERRARD & H.M. SONER, *Rectifiability of the distributional Jacobian for a class of functions*, C.R. Acad. Sci. Paris, Série I, **329** (1999), 683-688.

[24] S. MÜLLER, *Weak continuity of determinants and nonlinear elasticity*, C. R. Acad. Sci Paris Sér. I Math **307** (1988), 501-506.

[25] ———, Det = det. *A remark on the distributional determinant*, C.R.A.S. **311** (1990), 13-17.

[26] ———, *Higher integrability of determinants and weak convergence in L^1*, J. Reine Angew. Math. **412** (1990), 20-34.

[27] ———, *On the singular support of the distributional determinant*, Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), 657-696.

[28] R. SCHOEN *&* K. UHLENBECK, *A regularity theory for harmonic maps*, J. Differential Geometry **17** (1982), 307-335.

[29] L. SIMON, *Lectures on geometric measure theory*, Australian National University, 1984.

[30] B. WHITE, *Rectifiability of flat chains*, Ann. of Math. (2) **150** (1999), 165-184.

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