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A STOCHASTIC REPRESENTATION FOR THE LEVEL SET EQUATIONS

H. Mete Soner[1] and Nizar Touzi[2]

[1]Koç University, Department of Mathematics,
Rumelifener Yolu, Sariyer, Istanbul 80910, Turkey
E-mail: msoner@ku.edu.tr
[2]CREST, 15 Bd Gabriel Peri,
92245 Malakoff, Paris, France
E-mail: touzi@ensae.fr

ABSTRACT

A Feynman-Kac representation is proved for geometric partial differential equations. This representation is in terms of a stochastic target problem. In this problem the controller tries to steer a controlled process into a given target by judicial choices of controls. The sublevel sets of the unique level set solution of the geometric equation is shown to coincide with the reachability sets of the target problem whose target is the sublevel set of the final data.

Key Words: Geometric flows; Codimension k mean curvature flow; Inverse mean curvature flow; Stochastic target problem

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1. INTRODUCTION

A stochastic target problem is a non-classical control problem in which the controller tries to steer a controlled stochastic process into a given target set G by judicial choices of controls. The chief object of study is the set of all initial positions from which the controlled process can be steered into G with *probability one* in an allowed time interval. Clearly these *reachability sets* depend on the allowed time. Thus they can be characterized by an evolution equation which is the analoque of the dynamic programming, or equivalently the Bellman, equation of stochastic optimal control.

These geometric equations express the velocity of the boundary as a possibly nonlinear function of the normal and the curvature vectors. As a Cauchy problem these equations in general do not admit classical smooth solutions and a weak formulation is needed. Several such formulations were given starting with the pioneering work of Brakke.[5] Here we consider the viscosity formulation given independently by Chen, Giga and Goto[6] and by Evans and Spruck.[9] The main idea of this approach is to characterize the geometric solution as the zero level set of a continuous function. The level set approach in numerical studies was first introduced by Osher and Sethian[18] and in the physics literature by Okta et al.[17]

In our earlier work,[22,23] we have shown that smooth solutions of these geometric equations, when exist, are equal to the reachability sets. Also, under certain assumptions, the characteristic functions of the reachability sets are shown to be viscosity solutions of the geometric dynamic programming equations in the sense defined by the first author.[19] In particular, this result implies that the reachability set is included in the zero sub-level set of the solutions constructed in Refs. [6,9].

The chief goal of this paper is to give a stochastic characterization of the unique level set solutions of Refs. [6,9] in terms of the target problem. This is achieved by using the mentioned results of Ref. [22] and the techniques developed by Barles et al.[2] The main result in this direction is stated in Theorem 3.1. We give the proof of this representation in §5 and 6 by using a one parameter family of target problems whose targets are the sub-level sets of a given initial function. A restatement of the main theorem is given in Theorem 3.2 and the representation result is outline in Subsection 7.1.

These results can be interpreted in two ways. From a differential equations point of view it is a Feynman-Kac type of representation of level set solutions of the geometric equations. From the control point of view this gives a unique characterization of the reachability sets.

Let us mention that a similar representation theorem was recently obtained by Buckdahn et al.[4] by different techniques. However, their result is restricted to the level set equation of the codimension-1 mean curvature flow.

In this paper we first show that a function w defined in Eq. (3.4) is a viscosity solution of the corresponding geometric level set equation. In this construction, we consider a family of target problems whose targets are the sublevel sets of a given function g. If this equation has comparison as the large class of level set equations discussed in Ref. [6], then the above result shows that the reachability sets are the sublevel sets of the unique viscosity solution of the level set equation. This also provides a representation for the unique viscosity solution. These two results are proved in Theorems 4.2, and 3.2.

The paper is organized as follows. The target reachability problem is introduced in the next section. The statement of our main results is reported in §3. Section 4 discusses the dynamic programming principle and the induced class of geometric PDE's. The stochastic representation of this class of geometric PDE's in terms of the target problem is proved in §5. The level set characterization of the reachability sets is proved in §6. Examples are given in the final section.

2. TARGET REACHABILITY PROBLEM

In this section, we recall the target reachability problem introduced in Ref. [22] for diffusion processes.

We assume that the control set U is a compact subset of $\mathbb{R}^k$. The controlled process is a solution of the stochastic differential equation

$$dZ(s) = \mu(s, Z(s), u(s))\, ds + \sigma(s, Z(s), u(s))\, dW(s)\ , \tag{2.1}$$

where W is a d-dimensional standard Brownian motion and u is a U-valued progressively measurable map. As usual the drift μ is vector-valued and the diffusion coefficient is matrix-valued, i.e.,

$$\mu : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n \quad \text{and} \quad \sigma : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^{n\times d}.$$

We assume that both $\mu(t, z, u)$ and $\sigma(t, z, u)$ are bounded and continuous. For later use, we introduce the set

$$K(t, z) := \left\{(\mu(t, z, a), \sigma(t, z, a)) : a \in U\right\} \quad \text{for all} \quad (t, z) \in [0, T] \times \mathbb{R}^n. \tag{2.2}$$

In this paper, we relax the control problem slightly as in Ref. [11] by admitting all weak solutions of the stochastic differential equation (2.1). This forces us to consider all possible Brownian motions as part of the minimization process as well. This relaxation is needed only to ensure the existence of an optimal strategy and is not needed for the PDE results.

Mathematically, this is done as follows. For all initial data $(t, z) \in [0, T) \times \mathbb{R}^n$, let $\mathcal{U}(t, z)$ be the collection of all elements

$$\nu = \left(\Omega^\nu, \mathcal{F}^\nu, \mathbb{F}^\nu, P^\nu, \{W^\nu(s)\}_{s \geq t}, \{u^\nu(s)\}_{s \geq t}\right)$$

where $(\Omega^\nu, \mathcal{F}^\nu, \mathbb{F}^\nu, P^\nu)$ is an arbitrary filtered probability space, $\{W^\nu(s), s \geq t\}$ is a d dimensional standard Brownian motion, $\{u^\nu(s), s \geq t\}$ is a progressively measurable U-valued process on this space. For $\nu \in \mathcal{U}(t, z)$, let $\{Z^\nu_{t,z}(s)\}_{s \geq t}$ be the solution of Eq. (2.1) with (u^ν, W^ν) substituted for (u, W) and with initial condition $Z^\nu_{t,z}(t) = z$.

For a given Borel subset G of $\mathbb{R}^n$, the *target reachability set* is given by

$$V^G(t) := \left\{z \in \mathbb{R}^n : Z^\nu_{t,z}(T) \in G P^\nu - \text{a.s. for some } \nu \in \mathcal{U}(t, z)\right\}. \tag{2.3}$$

This set is the chief object of our study. A natural condition for $V^G(t)$ to be non-empty for any G is the following

$$\mathcal{N}(t, z, p) \neq \emptyset \quad \text{for all} \quad (t, z, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n,$$

where for $(t, z, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$

$$\begin{aligned} \mathcal{N}(t, z, p) &:= \{u \in U : \sigma(t, z, u)^* p = 0\} \\ &\text{for} \quad p \neq 0 \quad \text{and} \quad \mathcal{N}(t, z, 0) := U. \end{aligned} \tag{2.4}$$

In what follows, we always assume that this condition holds. As a corollary, if G is smooth, then $V^G(t)$ is non-empty at least for some $t > 0$. Although this is a natural general assumption, if we could apriori restrict the reachability sets into a smaller class such as graphs or epigraphs, then Assumption 4.1 can be relaxed: see Remark 4.3 below.

The stochastic target problem is introduced by the authors in Refs. [20,21] to study the problem of super-replication in mathematical finance. An application to stochastic volatility is given by the second author in Ref. [24], and jump-diffusion processes are discussed by Bouchard.[3] In addition to these examples, forward-backward stochastic differential equations (FBSDE) also can be seen as target reachability problems. We close this section by a brief discussion of these equations.

Example 2.1. (*unconstrained FBSDE's.*) The forward–backward stochastic differential equation is this. Given functions α, β, a, b, and γ (with appropriate domains and ranges) consider the problem of finding square integrable adapted processes $Z = (X, Y)$ valued in $\mathbb{R}^m \times \mathbb{R}^p$ and ν valued in $\mathbb{R}^d$ satisfying the differential equations

$$dX(s) = \alpha(s, Z(s), \nu(s))ds + \beta(s, Z(s), \nu(s))\, dW(s)$$
$$dY(s) = a(s, Z(s), \nu(s))ds + b(s, Z(s), \nu(s))\, dW(s)$$

together with the initial and final conditions

$$X(0) = x \quad \text{and} \quad Y(T) \;=\; \gamma(X(T)).$$

The main point here is that, unlike the deterministic framework, there is an important measurability problem : the processes $Z = (X, Y)$ and ν are required to be adapted to the given filtration $\mathbb{F}$. Note that an initial and a final condition is given and we could solve this only for certain values of x. The set of initial x for which a solution exists is indeed the projection on the first m coordinates of the target reachability problem for the process $Z = (X, Y)$ with target

$$G = \text{Graph}(\gamma) := \; \{(x, y) : y = \gamma(x)\}.$$

Further discussion of the connection between the target problems and FBSDE's is given in Remark 4.3 below.

The problem of FBSDE's has been motivated by applications in financial mathematics, namely the problem of hedging for a *large investor*. Loosely speaking, (i) the control ν is the investment strategy, i.e., the number of shares of risky assets to be held at each time, (ii) the dynamics of the process X, standing for the price process of m risky assets, is influenced by the investment strategy ν (large investor), (iii) and the Y component of the state process Z is the amount of wealth implied by the investment strategy ν; under the so-called self-financing condition, the dynamics of Y are given by $dY = \nu dX$.

For the existence of nontrivial solutions, certain restrictions on the coefficients, especially on b, are needed. We refer the reader to the recent lecture notes of Ma and Yong[16] and the references therein for information on FBSDE's. □

Example 2.2. (*constrained FBSDE's.*) Let $Z = (X, Y)$ with a scalar Y be as above and let A be a non-decreasing adapted process with $A(0) = 0$. Again we look for Z and ν in a certain convex set, satisfying the above differential equations together with the initial and final conditions

$$X(0) = x \quad \text{and} \quad Y(T) \;=\; \gamma(X(T)) + A(T).$$

This is again a target reachability problem with target

$$G = Epi(\gamma) := \{(x, y) : y \geq \gamma(x)\}.$$

The constraint that ν taking values in a convex set is the main difference between this and the unconstrained problem. For this reason the process A is introduced. A problem with constraints is considered by Cvitanić et al. in Ref. [8]. □

3. THE STOCHASTIC REPRESENTATION RESULT

The main result of this paper is the following representation formula for a partial differential equation. To state the theorem we need to define the nonlinear term in the equation. Let $\mathcal{S}^n$ be the set of all n by n symmetric matrices.

For $(t, z, p, A) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n$, define

$$F(t, z, p, A) := \sup_{\nu \in \mathcal{N}(t,z,p)} \left\{ -\mu(t, z, \nu)^* p - \frac{1}{2} \operatorname{trace}(\sigma\sigma^*(t, z, \nu)A) \right\}, \tag{3.1}$$

where $\mathcal{N}(t, z, p)$ is defined in Eq. (2.4).

Observe that $F(t, z, p, A)$ is singular at $p = 0$ because $\mathcal{N}(t, z, 0) = U$. In the sequel, we shall denote F_* and F^* the lower and the upper semicontinuous envelopes of F. Then the equation is

$$-w_t(t, z) + F(t, z, Dw(t, z), D^2 w(t, z)) = 0 \quad \text{on} \quad [0, T) \times \mathbb{R}^n. \tag{3.2}$$

We consider this equation together with the terminal condition

$$w(T, z) = g(z), \tag{3.3}$$

where g is a uniformly continuous function. Here we choose to study a terminal boundary value problem as they are more natural in optimal control. However, one could easily reverse time to obtain an initial value problem.

The main representation result is a consequence of the following theorem which requires a technical assumption, Assumption 4.1, that will be discussed in the next section.

Theorem 3.1. *Suppose Assumption* 4.1 *holds and that F is locally Lipschitz on* $\{p \neq 0\}$. *Then,*

$$w(t, z) := \inf_{\nu \in \mathcal{U}(t,z)} \operatorname{ess\,sup}_{\omega \in \Omega} g\big(Z^{\nu}_{t,z}(T, \omega)\big) \tag{3.4}$$

is a discontinuous viscosity solution of Eq. (3.2) *satisfying the terminal condition* (3.3) *pointwise*.

The proof of this theorem will be given in Sec. 6. If the solutions of these equations are unique, then the above theorem provides a stochastic representation formula for the unique solution.

Definition 3.1. We say that the Eq. (3.2) has comparison if for all functions $\underline{u}$, $\overline{u}$ satisfying

- $\underline{u}$ is an upper semicontinuous, bounded viscosity subsolution of Eq. (3.2) on $[0, T) \times \mathbb{R}^n$,
- $\overline{u}$ is a lower semicontinuous, bounded viscosity supersolution of Eq. (3.2) on $[0, T) \times \mathbb{R}^n$,
- $\underline{u}(T, \cdot) \leq h \leq \overline{u}(T, \cdot)$ for some uniformly continuous function $h : \mathbb{R}^n \to \mathbb{R}$,

we have $\underline{u} \leq \overline{u}$ on $[0, T] \times \mathbb{R}^n$.

In particular, if Eq. (3.2) has comparison and g is an uniformly continuous function on $\mathbb{R}^n$, then there exists at most one continuous viscosity solution to the Eq. (3.2) together with the terminal condition $u(T, \cdot) = g$. Notice that the requirement that h is uniformly continuous rules out the non-compact counterexamples to comparison constructed by Ilmanen.[12]

Comparison results for geometric equations have been first proved in Refs. [6,9] for the mean curvature flow. Also Ref. [6] provides a very general comparison result for a large class of geometric equations. In Sec. 7, we will give two examples of such flows.

With the assumption of comparison, Theorem 3.1 provides a stochastic representation formula for the unique solution of Eqs. (3.2)–(3.3). Our next result provides a characterization of the reachability set V^G as the zero sublevel set of the function w.

Theorem 3.2. *Let the conditions of Theorem* 3.1 *hold. Suppose that g is bounded and uniformly continuous, and Eq.* (3.2) *has comparison, so that w is the unique bounded continuous viscosity solution of Eqs.* (3.2)–(3.3).

Assume further that the set $K(t, z)$, defined in Eq. (2.2), *is closed and convex for all $(t, z) \in [0, T] \times \mathbb{R}^n$. Then,*

$$V^G(t) = \{z : w(t, z) \leq 0\}$$

with the target set

$$G := \{z \in \mathbb{R}^n : g(z) \leq 0\}.$$

Proof of this theorem is a straightforward application of Theorem 3.1 and Propositions 5.1–5.2. Observe that the boundedness of g is by no means a restricting condition, as one can replace g by $(1+\|g\|_\infty)^{-1}g$. Also, the boundedness of w is inherited from g, as it is immediately seen from its definition.

4. DYNAMIC PROGRAMMING

In this section, we recall several results of Ref. [22], which will be used in the proof of Theorem 3.1.

We first start by stating a geometric dynamic programming principle for the target reachability problem.

Theorem 4.1.[22] *Let G be a Borel subset of $\mathbb{R}^d$, and $t \in [0,T)$. For all stopping times $\theta \in [t,T]$,*

$$V^G(t) = \left\{ z \in \mathbb{R}^n : Z^\nu_{t,z}(\theta) \in V^G(\theta) P^\nu - a.s. \text{ for some } \nu \in \mathcal{U}(t,z) \right\}.$$

This principle is proved in Ref. [22] for general target reachability problems. While the inclusion of $V^G(t)$ in the right hand side of the above expression is obvious, the reverse inclusion is technical, and relies mainly on a measurable selection argument.

As in classical optimal control theory, the infinitesimal version of the dynamic programming principle yields a second order partial differential equation. This is also the case here. Indeed, in Ref. [22] it is proved that the characteristic function of the complement of the reachability sets

$$v^G(t,z) = 1 - \mathbf{1}_{V^G(t)}(z) = \begin{cases} 0 & \text{if } z \in V^G(t) \\ 1 & \text{otherwise} \end{cases} \tag{4.1}$$

is a viscosity solution of the geometric dynamic programming equation. This is proved under the following assumption.

Assumption 4.1. (Continuity of $\mathcal{N}(t,z,p)$.) Let $\mathcal{N}$ be as in Eq. (2.4). We assume that for any $(t_0,z_0,p_0) \in S \times \mathbb{R}^n$ and $u_0 \in \mathcal{N}(t_0,z_0,p_0)$, there exists a map $\hat{u} : S \times \mathbb{R}^n \to U$ satisfying,

$$\hat{u}(t_0,z_0,p_0) = u_0,$$
$$\hat{u}(t,z,p) \in \mathcal{N}(t,z,p) \quad \text{for all} \quad (t,z,p) \in S \times \mathbb{R}^n,$$

and that $\hat{u}$ is locally Lipschitz on $\{(t,z,p) : p \neq 0\}$.

A possible relaxation of this assumption is discussed in Remark 4.3 below. The following is proved in Ref. [22].

Theorem 4.2.[22] *Suppose that Assumption* 4.1 *holds and that* F *is locally Lipschitz on* $\{p \neq 0\}$. *Let* G *be a Borel subset of* $\mathbb{R}^n$. *Then, the support function of its reachability sets* v^G *is a discontinuous viscosity solution of the dynamic programming equation* (3.2).

We refer the reader to Refs. [7,11] for information on viscosity solutions.

By a discontinuous viscosity solution, we mean that the lower (resp. upper) semicontinuous envelope $(v^G)_*$ (resp. $(v^G)^*$) of v^G is a viscosity supersolution (resp. subsolution) of Eq. (3.2) with F^* (resp. F_*) substituted to F. While the proof of the supersolution property follows from judicial changes of measure, the subsolution property turns out to be a surprisingly technical proof. The complication is mainly related to the above-mentioned singularity of F at $p = 0$.

Remark 4.1. Although Eq. (3.2) is a second order partial differential equation, it admits a discontinuous function v^G as a solution. Uniqueness of discontinuous solutions to level set equations is not always expected due to the fattening phenomenon. This is studied extensively in the paper by Barles et al.[2] where the characteristic functions were first used as solutions of level set equations. Indeed when the target is a level set of a given function, then the reachability set $V^G(t)$ is a subset of this "fat" level-set. However, $V^G(t)$ is the equal to the whole level-set under mild assumptions as discussed in Proposition 5.2 and Theorem 3.2 provides an exact statement towards this problem. □

Remark 4.2. The nonlinearity F has the following two important properties

$$F(t, z, c_1 p, c_1 A + c_2 pp^*) = c_1 F(t, z, p, A) \qquad \forall\ c_1 > 0, c_2 \in \mathbb{R}, \tag{4.2}$$

$$F(t, z, p, A + B) \leq F(t, z, p, A), \qquad \forall\ B \geq 0. \tag{4.3}$$

The second property means that Eq. (3.2) is degenerate elliptic, while the first implies that it is *geometric*; see Ref. [2]. Note that the geometric property implies that Eq. (3.2) is degenerate along the gradient direction which is the normal direction to the level sets of v^G. □

The latter observation was the starting point of Ref. [23] where a stochastic representation of a class of *smooth* geometric flows in terms of target reachability problems is provided. In contrast with the technical proofs in Ref. [22], the stochastic representation of Ref. [23] relies on an easy application of Itô's lemma together with the use of the square distance function to the family of submanifolds.

Remark 4.3. Assumption 4.1 is restictive for the forward backward stochastic differential equations discussed in the previous section. Still our techniques apply to FBSDE's. Indeed the variable p in $\mathcal{N}$ stands for any possible normal vector of the reachability set, and in FBSDE's the reachability sets are either graphs or epigraphs of functions of the form $Y = \varphi(s, X)$. Therefore for these examples we need $\mathcal{N}(s, z, p)$ to be nonempty only for p's which are normals to graphs. To illustrate this point consider the Example 2.1 with $X \in \mathbb{R}^m$, $Y \in \mathbb{R}^1$, $\beta = \beta(s, z)$ and $b(s, z, \nu) = \nu \in \mathbb{R}^1$. Then the driving Brownian motion is one dimensional. Moreover, a normal p to the graph of any function $y = \varphi(x)$ has the form $p = \lambda\ (q, -1)$ for some scalar λ and $q \in \mathbb{R}^m$. For such a p, AQ1

$$\sigma^*(s, z, \nu)p = \lambda\ [\beta^*(s, z)q - \nu].$$

So $\nu = \beta^*(s, z)q$ belongs to $\mathcal{N}(s, z, p)$ whenever p is normal to a graph. In particular, $\mathcal{N}(s, z, p)$ is nonempty for normals. Although the Assumption 4.1 does not hold for every p, this is enough to use the techniques of the preceeding sections.

This example shows how to relax the Assumption 4.1 depending on the possible geometries of the reachable sets.

5. TARGETS AS LEVEL SETS

In this section, we provide a convenient alternative expression for the function w of Eq. (3.4). Then, as stated before, Theorem 3.2 follows from Theorem 3.1 and the results of this section.

In the context of this paper, we would like to see the target G as the zero sublevel set of some function $g : \mathbb{R}^n \to \mathbb{R}$, i.e.,

$$G = \{z \in \mathbb{R}^n : g(z) \leq 0\}.$$

where g is an uniformly continuous function on $\mathbb{R}^n$. This is not a restriction as we could always take g to be the signed distance to the boundary of G.

In order to prove Theorem 3.1, we need to derive an alternative expression of the function w defined in Eq. (3.4). For parameter $\alpha \in \mathbb{R}$, define the target

$$G_\alpha := \{z \in \mathbb{R}^n : g(z) \leq \alpha\},$$

together with the associated target reachability problem:

$$V^{G_\alpha}(t) := \{z \in \mathbb{R}^n : Z^\nu_{t,z}(T) \in G_\alpha P^\nu - \text{a.s. for some } \nu \in \mathcal{U}(t,z)\}.$$

Set

$$W(t,z) := \{\alpha \in \mathbb{R} : z \in V^{G_\alpha}(t)\}. \tag{5.1}$$

Then,

Lemma 5.1. *For all* $(t,z) \in [0,T] \times \mathbb{R}^n$*, we have*

$$w(t,z) = \inf W(t,z).$$

Proof.

(i) We first prove that $w(t,z) \leq \inf W(t,z)$. Take some arbitrary $\alpha > \inf W(t,z)$. By definition, this means that, for some $\nu \in \mathcal{U}(t,z)$, $g(Z^\nu_{t,z}(T)) \leq \alpha$ P–a.s. or equivalently $\operatorname{esssup}_{\omega \in \Omega} g\,(Z^\nu_{t,z}(T,\omega)) \leq \alpha$. Hence $w(t,z) := \inf_{\nu \in \mathcal{U}(t,z)} \operatorname{esssup}_{\omega \in \Omega}\, g(Z^\nu_{t,z}(T,\omega)) \leq \alpha$, and the required inequality follows by sending α to $\inf W(t,z)$.

(ii) To see that the reverse inequality holds, take an arbitrary $\alpha > w(t,z)$. Then, $g(Z^{\nu_n}_{t,z}(T)) \leq \alpha$ P–a.s. for some $\nu_n \in \mathcal{U}(t,z)$, or equivalently $z \in V^{G^\alpha}(t)$. Hence $\alpha \geq \inf W(t,z)$, and the required inequality follows by sending α to $w(t,z)$. □

Observe that $V^{G_\alpha}(t) \subset V^{G_\beta}(t)$ whenever $\alpha \leq \beta$. Hence

$$(w(t,z),\infty) \subset W(t,z) \subset [w(t,z),\infty) \quad \text{for all } (t,z) \in [0,T] \times \mathbb{R}^n.$$

The following result expresses the target reachability sets $V^G(\cdot)$ as the level sets of $w(\cdot,z)$. Its proof is straightforward and it is omitted.

Proposition 5.1. *For any* $t \in [0,T]$

$$\{z \in \mathbb{R}^n : w(t,z) < 0\} \subset V^G(t) \subset \{z \in \mathbb{R}^n : w(t,z) \leq 0\}.$$

If in addition $W(t,z)$ *is closed for all* $z \in \mathbb{R}^n$*, then*

$$V^G(t) = \{z \in \mathbb{R}^n : w(t,z) \leq 0\} \quad \textit{for all} \quad t \in [0,T].$$

Hence, in order to deduce Theorem 3.2 from Theorem 3.1, it remains to prove that W is a closed interval. This closedness property is the main reason for the relaxation of the stochastic reachability problem by means of weak solutions. In our previous paper,[22] the filtered probability space and the Brownian motions were fixed, and the controlled process Z^{ν} was defined as a strong solution of Eq. (2.1). However, such a setting requires stronger conditions in order to guarantee the closedness of $W(t, z)$. The following result is almost an immediate corollary of the results proved by Haussman,[14] and by ElKaroui et al.[10] AQ2

Proposition 5.2. *Fix a point $(t, z) \in [0, T) \times \mathbb{R}^n$ with $w(t, z) < \infty$, and suppose that the set $K(t, z)$, defined in Eq. (2.2), is closed and convex. Assume further that the function g, defining the target, is lower semicontinuous. Then, $W(t, z)$ is a closed interval, i.e., there exists a control $\hat{\nu} \in \mathcal{U}(t, z)$ such that*

$$g\left(Z^{\hat{\nu}}_{t,z}(T)\right) \leq w(t, z)P^{\hat{\nu}} - a.s..$$

Moreover, there exists a Borel measurable U-valued function $\bar{u}$ such that $u^{\hat{\nu}}(t) = \bar{u}(t, Z^{\hat{\nu}}_{t,z})(t)$, $P^{\hat{\nu}}-$ a.s.

Proof. We shall briefly recall the compactification method of Ref. [14]. Assertions of the Proposition follow easily from this compactification method.

1. We first rewrite the reachability set problem using the canonical space $\Omega = C([0, \infty), \mathbb{R}^d)$, $\mathcal{F}(t) = \sigma\{\omega(s), s \leq t\}$. Then we identify a weak solution of Eq. (2.1) with its induced measure; see Ref. [14]. With this identification, the set of (measure) controls is compact in the weak topology of Proposition 3.1 of Ref. [14].

2. Since $w(t, z) < \infty$ by assumption, the set of controls is non-empty. Now let $(\nu_n)_n$ be a minimizing sequence for the optimization problem $w(t, z)$, i.e.

$$g\left(Z^{\nu_n}_{t,z}(T)\right) \leq w(t, z) + 1/n \; P^{\nu_n} - \text{a.s.} \quad \text{for all} \quad n \geq 1.$$

Let P_n be the measure control associated to ν_n. Then, there is some (measure) control $\hat{P}$, identified to $\hat{\nu} \in \mathcal{U}(t, z)$, such that $P^n \longrightarrow \hat{P}$ weakly. By the definition of the weak convergence, this implies that $Z^{\nu_n}_{t,z}(T) \rightarrow Z^{\hat{\nu}}_{t,z}(T)P^{\hat{\nu}}-$ a.s. along some subsequence. Since g is lower semicontinuous, we pass to the limit in the above inequality, to obtain $g(Z^{\hat{\nu}}_{t,z}(T)) \leq w(t, z)P^{\hat{\nu}}-$a.s.

3. The final claim in Proposition 5.2 is proved in Lemmas 3.4, 3.5 and Proposition 3.2 of Ref. [14]. □

6. LEVEL SET EQUATION

In this section we prove Theorem 3.1 which states that the function w, defined in Eq. (3.4) (or Lemma 5.1), is a viscosity solution of the geometric dynamic programming equation (3.2) together with the terminal condition (3.3).

We start the proof with a straigtforward observation. Recall that v^G is defined in Eq. (4.1).

Lemma 6.1. *For any β, the semicontinuous envelopes of v^{G_β} satisfy*

(i) $(v^{G_\beta})_* \geq \mathbf{1}_{\{w_*>\beta\}}$, *and* $(v^{G_\beta})^* \leq \mathbf{1}_{\{w^*\geq\beta\}}$,
(ii) $w_*(t,z) < \beta \Longrightarrow (v^{G_\beta})_*(t,z) = 0$,
(iii) $w^*(t,z) > \beta \Longrightarrow (v^{G_\beta})^*(t,z) = 1$.

Proof. We shall only prove the statements concerning the lower semicontinuous envelopes. The statements concerning the upper semicontinuous envelopes is proved exactly the same way.

1. Let W be as in Eq. (5.1). Then,

$$v^{G_\beta}(t,z) = \mathbf{1}_{\{\beta\notin W(t,z)\}} \quad \text{for all} \quad (t,z) \in [0,T] \times \mathbb{R}^n. \tag{6.1}$$

Suppose that $\mathbf{1}_{\{w_*(t,z)>\beta\}} = 1$. Then, $w(t,z) \geq w_*(t,z) > \beta$. and this implies that $\beta \notin W(t,z)$. By Eq. (6.1), we conclude that $v^{G_\beta}(t,z) = 1$. Since $\mathbf{1}_{\{w_*>\beta\}}$ and v^{G_β} are valued in $\{0,1\}$, this proves that $\mathbf{1}_{\{w_*>\beta\}} \leq v^{G_\beta}$. Moreover, $\mathbf{1}_{\{w_*>\beta\}}$ is clearly lower-semicontinuous. Hence $\mathbf{1}_{\{w_*>\beta\}} \leq (v^{G_\beta})_*$.

2. Next, suppose that $(v^{G_\beta})_*(t,z) = 1$. Then $v^{G_\beta} = 1$ on some neighborhood B_0 of (t,z). By Eq. (6.1), $\beta \leq w$ and consequently $\beta \leq w_*$ on B_0. □

Remark 6.1. From the above lemma, it follows that:

$$(v^{G_\beta})_* = \mathbf{1}_{\{w_*>\beta\}} \text{ on } \{w_* \neq \beta\} \quad \text{and} \quad (v^{G_\beta})^* = \mathbf{1}_{\{w^*\geq\beta\}} \text{ on } \{w^* \neq \beta\}$$

Moreover, Part 2 of the proof provides that

$$(v^{G_\beta})_* > \mathbf{1}_{\{w_*>\beta\}} \Longrightarrow (t_0, z_0) \text{ is a point of local minimum of} w_*.$$

A similar statement holds for $(v^{G_\beta})^*$.

We first prove the w solves the geometric PDE.

Proposition 6.1. *Under the conditions of Theorem* 3.1, *w is a discontinuous viscosity solution of Eq.* (3.2).

Proof. We first prove that w^* is a viscosity subsolution of the dynamic programming Eq. (3.2) by applying Theorem 4.2 to $V^{G_{\alpha_n}}$ with a carefully chosen sequence of α_n.

1. Let $(t_0, z_0) \in [0, T) \times \mathbb{R}^n$ and $\varphi \in C^2([0, T] \times \mathbb{R}^n)$ be such that

$$0 = (w^* - \varphi)(t_0, z_0) > (w^* - \varphi)(t, z) \quad \text{for all } (t, z) \in [0, T) \times \mathbb{R}^n \backslash (t_0, z_0). \tag{6.2}$$

Note that $w^* \le \varphi$. We need to show that

$$-\varphi_t(t_0, z_0) + F_*\big(t_0, z_0, D\varphi(t_0, z_0), D^2\varphi(t_0, z_0)\big) \le 0. \tag{6.3}$$

Set

$$\alpha := w^*(t_0, z_0) = \varphi(t_0, z_0), \quad \text{and} \quad \alpha_n := \alpha - 1/n.$$

By Lemma 6.1 (i), we see that

$$\begin{aligned}((v^{G_{\alpha_n}})^* - \varphi)(t, z) &\le (\mathbf{1}_{\{w^* \ge \alpha_n\}} - \varphi)(t, z)\\ &\le (1 - \varphi)\mathbf{1}_{\{w^* \ge \alpha_n\}}(t, z) - \varphi\mathbf{1}_{\{w^* < \alpha_n\}}(t, z)\\ &\le (1 - w^*)\mathbf{1}_{\{w^* \ge \alpha_n\}}(t, z) - \varphi\mathbf{1}_{\{w^* < \alpha_n\}}(t, z)\\ &\le (1 - \alpha_n)\mathbf{1}_{\{w^* \ge \alpha_n\}}(t, z) - \varphi\mathbf{1}_{\{w^* < \alpha_n\}}(t, z).\end{aligned}$$

Now if $w^* < \alpha_n$, then the right hand side of the above inequality is equal to $-\varphi$ which is by the continuity of φ is less than $-\alpha_n$ on some bounded neighborhood B_0 of (t_0, z_0). In the opposite case, the right hand side is equal to $1 - \alpha_n$. So in any case, there exists a bounded neighborhood B_0 of (t_0, z_0) such that

$$((v^{G_{\alpha_n}})^* - \varphi)(t, z) \le (1 - \alpha_n) \quad \text{on} \quad B_0. \tag{6.4}$$

On the other hand, since $w^*(t_0, z_0) = \alpha > \alpha_n$, it follows from Lemma 6.1 (iii) that $\big(v^{G_{\alpha_n}}\big)^*(t_0, z_0) = 1$ for every n. Hence

$$((v^{G_{\alpha_n}})^* - \varphi)(t_0, z_0) = 1 - \alpha < 1 - \alpha_n. \tag{6.5}$$

2. Let (t_n, z_n) be a maximizer of $(v^{G_{\alpha_n}})^* - \varphi$ on $\mathrm{cl}(B_0)$, i.e.,

$$((v^{G_{\alpha_n}}‘)^* - \varphi)(t_n, z_n) = \sup_{B_0} ((v^{G_{\alpha_n}})^* - \varphi).$$

We claim that

$$(t_n, z_n) \longrightarrow (t_0, z_0) \quad \text{as} \quad n \to \infty. \tag{6.6}$$

Indeed, let $(\bar{t}, \bar{z})$ be the limit of some converging subsequence of $(t_n, z_n)_n$, that we rename (t_n, z_n). After possibly choosing a smaller neighborhood B_0,

$$\left(v^{G_{\alpha_n}}\right)^*(t_n, z_n) = 1 \quad \text{for all large } n. \tag{6.7}$$

This follows from the fact that $(v^{G_{\alpha_n}})^*(t_0, z_0) = 1$ together with the smoothness of φ. Hence,

$$\lim_{n\to\infty}((v^{G_{\alpha_n}})^* - \varphi)(t_n, z_n) = 1 - \varphi(\bar{t}, \bar{z}). \tag{6.8}$$

By Eqs. (6.4) and (6.5),

$$1 - \alpha = ((v^{G_{\alpha_n}})^* - \varphi)(t_0, z_0) \leq ((v^{G_{\alpha_n}})^* - \varphi)(t_n, z_n) \leq 1 - \alpha_n,$$

so that Eq. (6.8) yields $\varphi(\bar{t}, \bar{z}) = \alpha$. Using again Eq. (6.7) together with Lemma 6.1 (i), we conclude that $w^*(t_n, z_n) \geq \alpha_n$. Therefore,

$$\begin{aligned}
&(w^* - \varphi)(\bar{t}, \bar{z}) \\
&\quad = \limsup_{n\to\infty}(w^* - \varphi)(t_n, z_n) \\
&\quad \geq \limsup_{n\to\infty} \ \alpha_n - \varphi(t_n, z_n) = \alpha - \varphi(\bar{t}, \bar{z}) = 0.
\end{aligned}$$

In view of Eq. (6.2), this proves that $(\bar{t}, \bar{z}) = (t_0, z_0)$ and the proof of Eq. (6.6) is complete.

3. By Eq. (6.6), (t_n, z_n) is a local maximizer of $((v^{G_{\alpha_n}})^* - \varphi)$ on B_0, for sufficiently large n. Also, by Theorem 4.2, $v^{G_{\alpha_n}}$ is a discontinuous subsolution of the dynamic programming equation. Hence

$$-\varphi_t(t_n, z_n) + F\big(t_n, z_n, D\varphi(t_n, z_n), D^2\varphi(t_n, z_n)\big) \leq 0,$$

We now take liminf as n approaches to infinity to arrive at Eq. (6.3).

So w is a discontinuous viscosity subsolution of the dynamic programming equation.

(ii) It remains to prove that w_* is a viscosity supersolution of the dynamic programming equation. This part of the proof is very similar to (i).

4. Let $(t_0, z_0) \in [0, T) \times \mathbb{R}^n$ and $\varphi \in C^2([0, T] \times \mathbb{R}^n)$ be such that

$$\begin{aligned}
0 = (w_* - \varphi)(t_0, z_0) &< (w_* - \varphi)(t, z) \\
&\text{for all} \quad (t, z) \in [0, T) \times \mathbb{R}^n \backslash (t_0, z_0).
\end{aligned} \tag{6.9}$$

Observe that $w_* \geq \varphi$. Set $\beta_n = \alpha + 1/n$ where α is as in Step 1. Argueing as in Step 1,

$$\begin{aligned}\left(\left(v^{G_{\beta_n}}\right)_* - \varphi\right)(t,z) &\geq \left(\mathbf{1}_{\{w_*>\beta_n\}} - \varphi\right)(t,z)\\ &\geq (1-\varphi)\mathbf{1}_{\{w_*>\beta_n\}}(t,z) - \varphi\mathbf{1}_{\{w_*\leq\alpha_n\}}(t,z)\\ &\geq (1-\varphi)\mathbf{1}_{\{w_*>\beta_n\}}(t,z) - w_*\mathbf{1}_{\{w_*\leq\beta_n\}}(t,z)\\ &\geq (1-\varphi)\mathbf{1}_{\{w_*>\beta_n\}}(t,z) - \beta_n\mathbf{1}_{\{w_*\leq\beta_n\}}(t,z)\\ &\geq -\beta_n,\end{aligned} \tag{6.10}$$

on some bounded neighborhood B_0 of (t_0, z_0). On the other hand, since $w_*(t_0, z_0) = \alpha < \beta_n$, it follows from Lemma 6.1 (ii) that

$$\left(\left(v^{G_{\beta_n}}\right)_* - \varphi\right)(t_0, z_0) = -\alpha > -\beta_n. \tag{6.11}$$

5. Let (t_n, z_n) be a minimizer of $(v^{G_{\beta_n}})_* - \varphi$ on $\mathrm{cl}(B_0)$, i.e.

$$\left(\left(v^{G_{\beta_n}}\right)_* - \varphi\right)(t_n, z_n) = \inf_{B_0}\left(\left(v^{G_{\beta_n}}\right)_* - \varphi\right).$$

As in Step 2, we claim that

$$(t_n, z_n) \longrightarrow (t_0, z_0) \quad \text{as} \quad n \longrightarrow \infty. \tag{6.12}$$

We argue as before. Let $(\bar{t}, \bar{z})$ be the limit of some converging subsequence of $(t_n, z_n)_n$, that we rename (t_n, z_n). Observe that after possibly choosing a smaller neighborhood B_0,

$$\left(v^{G_{\beta_n}}\right)_*(t_n, z_n) = 0 \quad \text{for large } n. \tag{6.13}$$

This follows from the fact that $(v^{G_{\beta_n}})_*(t_0, z_0) = 0$ together with the smoothness of φ. Then,

$$\lim_{n\to\infty}\left(\left(v^{G_{\beta_n}}\right)_* - c\varphi\right)(t_n, z_n) = -\varphi(\bar{t}, \bar{z}). \tag{6.14}$$

We now use Eqs. (6.10) and (6.11) to conclude that

$$-\alpha = \left(\left(v^{G_{\beta_n}}\right)_* - \varphi\right)(t_0, z_0) \geq \left(\left(v^{G_{\beta_n}}\right)_* - \varphi\right)(t_n, z_n) \geq -\beta_n.$$

Hence by Eq. (6.14), $\varphi(\bar{t}, \bar{z}) = \alpha$. Using again Eq. (6.13) together with Lemma 6.1 (i), we see that $w_*(t_n, z_n) \leq \beta_n$. Therefore,

$$\begin{aligned}(w_* - \varphi)(\bar{t}, \bar{z}) &= \liminf_{n\to\infty}\,(w_* - \varphi)(t_n, z_n)\\ &\leq \liminf_{n\to\infty}\beta_n - \varphi(t_n, z_n) = \alpha - \varphi(\bar{t}, \bar{z}) = 0,\end{aligned}$$

which shows that $(\bar{t}, \bar{z}) = (t_0, z_0)$, in view of Eq. (6.9). This proves the claim.

6. By Eq. (6.12), (t_n, z_n) is a local minimizer of $((v^{G_{\beta_n}})_* - \varphi)$ on B_0, for large n. Since $v^{G_{\beta_n}}$ is a discontinuous subsolution of the dynamic programming equation, this proves that

$$-\varphi_t(t_n, z_n) + F\big(t_n, z_n, D\varphi(t_n, z_n), D^2\varphi(t_n, z_n)\big) \geq 0,$$

Letting n tend to infinity we obtain

$$-\varphi_t(t_0, z_0) + F^*\big(t_0, z_0, D\varphi(t_0, z_0), D^2\varphi(t_0, z_0)\big) \geq 0.$$

Hence w is a discontinuous viscosity supersolution of the dynamic programming equation. □

In order to conclude the proof of Theorem 3.1, it remains to show that w satisfies the terminal condition (3.3). In preparation of this, we start with

Lemma 6.2. *For any initial data $(t, z) \in [0, T) \times \mathbb{R}^n$, there exists $\tilde{\nu} \in \mathcal{U}(t, z)$ such that*

$$\left| Z^{\tilde{\nu}}_{t,z}(T) - z \right|^2 \leq C[(T-t)^2 + (T-t)]P^{\tilde{\nu}} - a.s.,$$

for some constant C depending on $\|\mu\|_\infty$ and $\|\sigma\|_\infty$.

Proof. Fix (t, z) and a small constant $\delta > 0$. Let u_0 be an arbitrary control in $\mathcal{U}$, and construct processes $\tilde{\nu}$ and $\tilde{Z} := Z^{\tilde{\nu}}_{t,z}$ so that for all $s \in [t, T]$,

$$\tilde{u}(s) := u^{\tilde{\nu}}(s) = u_0(s)\mathbf{1}_{\{|\tilde{Z}(s)-z|<\delta\}} + \hat{u}\big(s, \tilde{Z}(s), \tilde{Z}(s) - z\big)\mathbf{1}_{\{|\tilde{Z}(s)-z|\geq\delta\}},$$

where $\hat{u}$ is as defined in Assumption 4.1. Clearly for any arbitrary filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ equipped with an $\mathbb{R}^d$-valued Brownian motion W, $\tilde{\nu} := (\Omega, \mathcal{F}, \mathbb{F}, P, W, \tilde{Z}, \tilde{u}) \in \mathcal{U}(t, z)$.

Set $f(s) := \tilde{Z}(s) - z$, for $s \geq t$, and apply Itô's rule to $|f(s)|^2$,

$$d|f(s)|^2 = \big[2f(s)^*\mu(s, \tilde{Z}(s), \tilde{u}(s)) + \text{trace}\{\sigma\sigma^*(s, \tilde{Z}(s), \tilde{u}(s))\}\big]dt$$
$$+ 2f(s)^*\sigma(s, \tilde{Z}(s), \tilde{u}(s))\, dW(s).$$

Since $\tilde{u}(s) \in \mathcal{N}(s, \tilde{Z}(s), \tilde{u}(s))$ whenever $|f(s)| \geq \delta$, the stochastic term in the above equation is equal to zero. Hence, for $|f(s)| \geq \delta$,

$$d|f(s)|^2 \leq C(|f(s)| + 1)\, dt,$$

for some constant C, depending on the bounds of μ and σ. This proves that

$$|f(s)|^2 \le \delta^2 + C\int_t^s (1+|f(s)|)\mathbf{1}_{|f(s)|\ge\delta}\,ds$$
$$\le \delta^2 + C(s-t) + \frac{C}{\delta}\int_t^s |f(s)|^2\,ds.$$

We now use Gronwall's Lemma to arrive at

$$|f(s)|^2 \le \delta^2 e^{(C/\delta)(s-t)} + \delta\Big(e^{(C/\delta)(s-t)} - 1\Big) \quad \text{for} \quad s\in[t,T].$$

Choosing $\delta := T-t$ yields

$$|f(T)|^2 = |\tilde{Z}(T) - z|^2 \le e^C[(T-t)^2 + (T-t)].$$

The following result completes the proof of Theorem 3.1.

Proposition 6.2. *For all $z\in\mathbb{R}^n$, we have $w_*(T,z) = w^*(T,z) = g(z)$.*

Proof. We shall prove that $w_*(T,\cdot)\ge g$ and $w^*(T,\cdot)\le g$, then the required result follows from the trivial inequality $w^*\ge w_*$.

1. We first prove that $w_*(T,\cdot)\ge g$. Fix $z\in\mathbb{R}^n$ and consider a sequence $(t_n,z_n)_n$ such that

$$(t_n,z_n)\to(T,z) \quad \text{and} \quad w(t_n,z_n)\to w_*(T,z).$$

With $\beta_n := w(t_n,z_n)+1/n$, it follows from the definition of w that

$$g\big(Z^{\nu_n}_{t_n,z_n}(T)\big) \le \beta_n P^{\nu_n} - \text{a.s.} \quad \text{for some control} \quad \nu_n\in\mathcal{U}.$$

Since the functions μ and σ are bounded, it is easily seen that $Z^{\nu_n}_{t_n,z_n}(T)\to zP-$a.s. and therefore

$$g(z) = \lim_{n\to\infty} g\big(Z^{\nu_n}_{t_n,z_n}(T)\big) \le \lim_{n\to\infty}\beta_n = w_*(T,z)$$

by the continuity of g.

2. We now prove that $w^*(T,\cdot)\le g$. Let $\varepsilon>0$ be given. By Lemma 6.2, there exists $t_\varepsilon<T$ such that for any $z\in\mathbb{R}^n$, $t\in[t_\epsilon,T]$ there exists a control $\tilde\nu\in\mathcal{U}(t,z)$ satisfying

$$g(Z^{\tilde\nu}_{t,z}(T)) \le g(z)+\varepsilon \quad \text{for all} \quad t\in[t_\varepsilon,T]P^{\tilde\nu} - \text{a.s.}$$

By the definition of w, this yields

$$w(t,z)\le g(z)+\varepsilon \quad \text{for all} \quad t\in[t_\varepsilon,T].$$

Then we obtain the required inequality by taking limsup as (t, z) approaches to (T, z_0) and using the continuity of g. □

7. EXAMPLES

7.1. Stochastic Representation of Mean Curvature Type Geometric Flows

In this subsection we outline the above results with a view towards finding the stochastic representation.

1. Suppose that a level set PDE is given with a nonlinearity as in Eq. (3.1). Then Theorem 3.1 or Theorem 3.2 provide the desired representation. Of course, we need a uniqueness result for this equation together with the boundary condition.

For initial value problems, we need to reverse time in order to apply our results. Indeed if the coefficients μ, σ are independent of t, then this reversal is easy:

$$w(t, z) = \inf_{\nu \in \mathcal{U}(t, z)} \operatorname{ess\,sup}_{\omega \in \Omega} g\big(Z^{\nu}_{0, z}(t, \omega)\big).$$

2. Given a target problem we showed that the corresponding level set equation is Eq. (3.2). However, it is not always possible to find a corresponding geometric equation written purely in geometric quantities. The difficulty lies in the fact that the dimension of the reachability sets may change. Still formally, the geometric equation is

$$\vec{v}(t, x) = \inf\Big\{\mu(t, z, \nu) + \vec{H}_{a(t, z, \nu)} : \nu \in \mathcal{K}(t, z)\Big\} \quad \text{for} \quad z \in \Gamma(t), \tag{7.1}$$

where $\vec{v}$ is the normal velocity vector, and $\vec{H}_{a(t,z,\nu)}$ is the mean curvature vector at (t, z) using the metric generated by the quadratic form of the matrix $a(t, z, \nu) := \sigma\sigma^*(t, z, \nu)$, and

$$\mathcal{K}(t, z) := \big\{\nu \in U : \text{Normal space at } (t, z) \subset \text{Kernel } a(t, z)\big\}.$$

When the solution has co-dimension one, the normal space is one-dimensional. Then, assuming further that $\vec{v}$, μ and $\vec{H}$ are directed along the normal, the above infimum has to be understood as the infimum of scalar quantities obtained after taking the dot-product with the outward unit normal vector. However, in the general case, the above infimum is just a formal writing which needs a serious geometric study in order to be justified rigorously. In the subsequent section, we provide examples where the above geometric equation is fully justified.

3. Given a level set equation to obtain its stochastic representation, we need to express the equation as in Eqs. (3.1)–(3.2). Of course this is not always straightforward. When the coefficients μ and σ are linear functions of ν, it is possible to deduce μ and σ by the Fenchel transform. See Example 7.3 below.

7.2. Codimension-k Mean Curvature Flow

In this example we will show that with appropriate choices of μ and σ we can obtain the level set equation of the mean curvature flow in any codimension. The geometric equation for this flow is

$$\vec{v} = \vec{H},$$

where $\vec{v}$ is the normal velocity vector and $\vec{H}$ is the mean curvature vector. The corresponding level set equation in any codimension is obtained by Ambrosio and Soner.[1]

Let $\mathcal{U}_k$ be the set of all projections matrices onto a $n-k$ dimensional unoriented plane in R^n. Let the control set $U = \mathcal{U}_k$, and for $\nu \in \mathcal{U}_k$,

$$\mu \equiv 0, \qquad \sigma(s, z, \nu) = \sqrt{2}\,\nu.$$

Then the nonlinear term in the dynamic programming equation (3.2) is

$$F(p, A) = \inf\{\text{trace}\,[A\nu] : \nu \in \mathcal{U}_k,\ \nu p = 0\}.$$

In Ref. [22], it is shown that

$$F(p, A) = \sum_{i=1}^{n-k} \lambda_i(p, A),$$

where $\lambda_1(p, A) \le \cdots \le \lambda_{n-1}(p, A)$ are the eigenvalues of the matrix $[I - (pp^*)/|p|^2]\ A\ [I - (pp^*)/|p|^2]$ with eigenvectors orthogonal to p. This is exactly the nonlinearity in the level set equation of codimension k mean curvature flow as studied by Ambrosio and Soner.[1] The codimension one case is also included in the above formulation and agrees with level set equation of Refs. [6,9],

$$-w_t = \Delta w - (D^2 w Dw \cdot Dw)/|Dw|^2.$$

Note that we are considering this PDE in $[0, T) \times \mathbb{R}^n$ with final data at time T, and this accounts for the minus sign in front of w_t.

Comparison for the above codimension-k mean curvature flows falls in the generality of the comparison result established by Chen et al.[6] Hence, Theorem 3.2 applies and provides a representation of the flow as the target reachability set of $\{g(z) \leq 0\}$.

7.3. Inverse Mean Curvature Flow

The second example is a nonlinear function of the curvature. It provides a guideline how to construct the target problem starting from a geometric equation.

The geometric equation is only for codimension one, mean convex surfaces, i.e., for surface with positive mean curvature at every point. The equation is

$$v = -1/H,$$

where v is the normal velocity and H is the mean curvature. Note that we are requiring that the solution should have $H \geq 0$ everywhere. This equation is recently used by Huisken and Ilmanen[15] to prove the Riemanian positive mass conjecture of general relativity.

The staring point of the connection between the inverse mean curvature flow and the target problems is the Legendre transform of the concave function $-1/x$ restricted to positive x:

$$-1/x = \inf\{a^2 x - 2a : a \geq 0\}, \quad x > 0.$$

The level set equation for the mean curvature flow is

$$-\frac{w_t}{|Dw|} = -\frac{1}{D \cdot (Dw/|Dw|)} = \frac{|Dw|}{\Delta w - D^2 w Dw \cdot Dw/|Dw|^2}. \tag{7.2}$$

We now multiply the equation by $|Dw|$ and then use the expression for $-1/x$ to arrive at

$$-w_t = \inf_{a \geq 0} \{a^2[\Delta w - D^2 w Dw \cdot Dw/|Dw|^2] - 2a|Dw|\}.$$

We are now in a position to define the target problem. We first note that

$$[\Delta w - D^2 w Dw \cdot Dw/|Dw|^2] = \inf\{\text{trace } [A\nu] : \nu \in \mathcal{U}_1, \nu Dw = 0\},$$

and any $\nu \in \mathcal{U}_1$ is of the form $\nu = [I - \vec{n}\vec{n}^*]$ for some vector $\vec{n} \in S^{n-1}$. So instead of using projection matrices from $\mathcal{U}_1$, we could use S^{n-1}. With this identification, we set $U = S^{n-1} \times [0, \infty)$ and

$$\mu(\vec{n}, a) = -a\vec{n}, \qquad \sigma(\vec{n}, a) = \sqrt{2}\; a\; [I - \vec{n}\vec{n}^*].$$

By a direct calculation we can show that the nonlinear term F is given by

$$F(p, A) = \inf_{a \geq 0} \{a^2(\mathrm{trace}[A] - Ap \cdot p/|p|^2) - 2a|p|\}$$
$$= \frac{|p|^2}{(\mathrm{trace}[A] - Ap \cdot p/|p|^2)}.$$

Notice that Eq. (7.2) is exactly equal to the dynamic programming equation (3.2) with the above F.

In this example, the controls take values in unbounded set. Consequently, Theorems 4.2 and 3.2 do not apply to this context. The representation result needs to be proved for this specific case. Notice that a representation result for smooth inverse mean curvature flows is proved in Ref. [23].

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