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Limiting Behavior of the Ginzburg–Landau Functional

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We continue our study of the functional

$$\mathbb{E}_\varepsilon(u) := \int_U \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2\, dx,$$

for $u \in H^1(U;\mathbb{R}^2)$, where U is a bounded, open subset of $\mathbb{R}^2$. Compactness results for the scaled Jacobian of u^ε are proved under the assumption that $\mathbb{E}_\varepsilon(u^\varepsilon)$ is bounded uniformly by a function of ε. In addition, the Gamma limit of $\mathbb{E}_\varepsilon(u^\varepsilon)/(\ln \varepsilon)^2$ is shown to be

$$\mathbb{E}(v) := \tfrac{1}{2}\|v\|_2^2 + \|\nabla \times v\|_{\mathcal{M}},$$

where v is the limit of $j(u^\varepsilon)/|\ln \varepsilon|$, $j(u^\varepsilon) := u^\varepsilon \times Du^\varepsilon$, and $\|\cdot\|_{\mathcal{M}}$ is the total variation of a Radon measure. These results are applied to the Ginzburg–Landau functional

$$\mathbb{F}_\varepsilon(u, A; h_{\text{ext}}) := \int_U \frac{1}{2}|\nabla_A u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 + \frac{1}{2}|\nabla \times A - h_{\text{ext}}|\, dx,$$

with external magnetic field $h_{\text{ext}} \approx H|\ln \varepsilon|$. The Gamma limit of $\mathbb{F}_\varepsilon/(\ln \varepsilon)^2$ is calculated to be

$$\mathbb{F}(v, a; H) := \tfrac{1}{2}[\|v-a\|_2^2 + \|\nabla \times v\|_{\mathcal{M}} + \|\nabla \times a - H\|_2^2],$$

where v is as before, and a is the limit of $A^\varepsilon/|\ln \varepsilon|$. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

The functional

$$\mathbb{E}_\varepsilon(u) := \int_U e_\varepsilon(u)\,dx, \qquad e^\varepsilon(u) := \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2,$$

for $u \in H^1(U;\mathbb{R}^2)$, is an important model problem in the Calculus of Variations as it contains different length scales such as the vortex core, vortex spacing, and its solution has a rich topological structure. As pointed out by Bethuel *et al.* [4] this functional can be considered as a simplified version of a functional proposed by Ginzburg and Landau [29] as a phenomenological model for superconductivity.

In their book, Bethuel *et al.* [4] obtain a complete description of the asymptotic behavior of the minimizers of this functional with given Dirichlet data. They prove that asymptotically the minimizers have finitely many singularities called *vortices*. Each of these vortices carries $\pi|\ln\varepsilon|$ amount of energy and the number of the vortices is determined by the winding number of the Dirichlet data. These results indicate that a natural scaling for this functional is $|\ln\varepsilon|$. Motivated by this, in [17] the authors studied the Gamma limit of $\mathbb{E}_\varepsilon$ divided by the scaling factor $|\ln\varepsilon|$ and proved the following results for this scaling: suppose that for a sequence $\{u^\varepsilon\}$, $\mathbb{E}_\varepsilon(u^\varepsilon)/|\ln\varepsilon|$ is uniformly bounded in ε. Then, the Jacobian

$$Ju^\varepsilon = \nabla \times j(u^\varepsilon)/2, \qquad j(u^\varepsilon) := u^\varepsilon \times \nabla u^\varepsilon,$$

of these functions is precompact in the dual of Hölder continuous functions, and any limit J is an atomic Radon measure with weights equal to an integer multiple of π. The support of J is the asymptotic location of the vortices and the weights of J at these points are related to the limiting degree of u^ε. Moreover, for any sequence u^ε converging to u in $W^{1,1}$,

$$\liminf \frac{\mathbb{E}_\varepsilon(u^\varepsilon)}{|\ln\varepsilon|} \geqslant \|J\|_{\mathcal{M}},$$

where $\|J\|_{\mathcal{M}}$ is the total variation of the measure J and the energy concentrates on small balls around the vortices [17]. Also for any atomic measure J with weights equal to an integer multiple of π, there exists a sequence whose limit Jacobian is J and the above limit is achieved with an equality.

Important earlier work on Jacobians in a similar setting includes the paper of Brezis *et al.* [8] that demonstrated the relevance of Jacobians in studying harmonic maps with singularities. The Jacobian was subsequently used by Bethuel [2] to characterize the class of maps $B^3 \to S^2$ which can be approximated by smooth S^2-valued maps. More recently, the authors in [18, 19] proved, as a special case of more general results, that if $u \in W^{1,1} \cap L^\infty(\mathbb{R}^2, S^1)$ and the distributional Jacobian of u is a Radon measure, then this measure must be atomic. Similar results are found in the work of Giaquinta, Modica, and Soouček on Cartesian currents [15] and analog of this result in the space $H^{1/2}$ is proven by Bourgain *et al.* [7].

In this paper, we continue this analysis with different scalings. Suppose that for a sequence of functions $\{u^\varepsilon\}$,

$$\mathbb{E}_\varepsilon(u^\varepsilon) \leqslant K g_\varepsilon, \qquad |\ln \varepsilon| \leqslant g_\varepsilon \ll \varepsilon^{-2}. \tag{1.1}$$

Since $\varepsilon^2 g_\varepsilon$ tends to zero by assumption, the potential term in $\mathbb{E}_\varepsilon$ forces $|u^\varepsilon|$ to be close to one in most of the domain. However, $|u^\varepsilon|$ is still close to zero around vortices. In view of the results of [17], this contributes to the energy at least by an amount of $|\ln \varepsilon| \, \|J(u^\varepsilon)\|_{\mathcal{M}}$. Moreover, mentioned before, this "vortex energy" concentrates near the vortices: on the union of balls with small radii. We call this "vortex set" V_ε. Then, on this set, the energy $\mathbb{E}_\varepsilon(u^\varepsilon)$ is approximately bounded from below by

$$\int_{V_\varepsilon} e^\varepsilon(u^\varepsilon)\, dx \geqslant |\ln \varepsilon| \, \|J(u^\varepsilon)\|_{\mathcal{M}}.$$

A precise mathematical statement of this fact is demonstrated in Section 5 as a sharp lower bound of the energy in terms of the Jacobian.

Away from the vortices only the gradient term is active. Since for $u = \rho e^{i\varphi}$,

$$|\nabla u|^2 = |\nabla \rho|^2 + |\rho \nabla \varphi|^2, \qquad j(u) := u \times \nabla u = \rho^2 \nabla \varphi,$$

and since away from the vortices $|u^\varepsilon|$ is near one, in this region

$$|\nabla u^\varepsilon|^2 \approx |j(u^\varepsilon)|^2 = |u^\varepsilon \times \nabla u^\varepsilon|^2.$$

Hence, approximately

$$\int_{U \setminus V_\varepsilon} e^\varepsilon(u^\varepsilon)\, dx \geqslant \int_{U \setminus V_\varepsilon} |\nabla u^\varepsilon|^2 \, dx \geqslant \int_{U \setminus V_\varepsilon} |j(u^\varepsilon)|^2 \, dx.$$

This reasoning indicates that the functional $\mathbb{E}_\varepsilon(u^\varepsilon)$ is approximately bounded from below by

$$\tfrac{1}{2} \|j(u^\varepsilon) \chi_{U \setminus V_\varepsilon}\|_2^2 + |\ln \varepsilon| \, \|J(u^\varepsilon)\|_{\mathcal{M}}.$$

The excess energy between $\mathbb{E}_\varepsilon$ and the above expression is due to the extra winding around the vortices.

To examine possible scalings, set

$$v^\varepsilon := \frac{j(u^\varepsilon)}{\sqrt{g_\varepsilon}}, \qquad \hat{v}^\varepsilon := \frac{|\ln \varepsilon|}{g_\varepsilon} j(u^\varepsilon), \qquad w^\varepsilon := \frac{|\ln \varepsilon|}{g_\varepsilon} J(u^\varepsilon). \tag{1.2}$$

Then,

$$Ju^\varepsilon = \sqrt{g_\varepsilon} \nabla \times v^\varepsilon / 2 = \frac{g_\varepsilon}{|\ln \varepsilon|} \nabla \times \hat{v}^\varepsilon / 2$$

and

$$w^\varepsilon = \nabla \times \hat{v}^\varepsilon / 2, \qquad j(u^\varepsilon) = \frac{g_\varepsilon}{|\ln \varepsilon|} \hat{v}^\varepsilon.$$

So approximately,

$$\mathbb{E}_\varepsilon(u^\varepsilon) \geqslant \frac{g_\varepsilon}{2} \left[\| v^\varepsilon \chi_{U \setminus V_\varepsilon} \|_2^2 + \frac{|\ln \varepsilon|}{\sqrt{g_\varepsilon}} \| \nabla \times v^\varepsilon \|_{\mathcal{M}} \right]$$

or equivalently

$$\begin{aligned} \mathbb{E}_\varepsilon(u^\varepsilon) &\geqslant \frac{g_\varepsilon}{2} \left[\frac{g_\varepsilon}{|\ln \varepsilon|^2} \| \hat{v}^\varepsilon \chi_{U \setminus V_\varepsilon} \|_2^2 + \| \nabla \times \hat{v}^\varepsilon \|_{\mathcal{M}} \right] \\ &= g_\varepsilon \left[\frac{g_\varepsilon}{2 |\ln \varepsilon|^2} \| \hat{v}^\varepsilon \chi_{U \setminus V_\varepsilon} \|_2^2 + \| w^\varepsilon \|_{\mathcal{M}} \right]. \end{aligned}$$

Therefore, the critical scaling is $g_\varepsilon = (\ln \varepsilon)^2$. In this case, $v^\varepsilon = \hat{v}^\varepsilon$ and both expressions are identical. For $g_\varepsilon \gg (\ln \varepsilon)^2$, however, the first lower bound indicates that $\|v^\varepsilon\|_2^2$ is the dominating term. On the other hand, for $g_\varepsilon \ll (\ln \varepsilon)^2$, from the second estimate we see that the important term is $\|w^\varepsilon\|_{\mathcal{M}}$. Indeed, this is consistent with the results of [17] which studies the case $g_\varepsilon = |\ln \varepsilon|$ and proves that the Gamma limit is given by the limit of w^ε. Since in the critical case both terms appear in the limit behavior, we need to show that the contribution of $\|w^\varepsilon\|_{\mathcal{M}}$ is localized near the vortices. This fact is mathematically verified in a "sharp" Jacobian estimate in Section 5. This separation of energy renders the analysis of the critical case more difficult than the others and that is the only reason we study the case $g_\varepsilon = (\ln \varepsilon)^2$ in detail. All the ingredients of other cases are included in our analysis.

This formal expansion of $\mathbb{E}_\varepsilon$ together with (1.1) suggest that v^ε and w^ε are compact in appropriate spaces. Let v and w be the limits of v^ε w^ε, respectively.

In this paper, we prove the compactness of v^ε and w^ε, and study the Gamma limit in the critical case $g_\varepsilon = (\ln \varepsilon)^2$. Compactness results follow from the formal lower bounds of $\mathbb{E}_\varepsilon$ together with (1.1). Indeed, a compactness result for v^ε follows easily from the energy bound (1.1). For w^ε, we follow the techniques developed in [17].

THEOREM 1.1 (Compactness). *Assume* (1.1). *Then, the scaled Jacobians* w^ε *defined in* (1.2) *are precompact in* C^{0,α^*} *for all* $\alpha > 0$.

Further assume that $g_\varepsilon \leqslant \varepsilon^{-\gamma}$ *for some* $\gamma < 2$. *Then,* v^ε *defined in* (1.2) *satisfies*

$$\sup_\varepsilon \|v^\varepsilon/|u^\varepsilon|\,\|_2^2 < \infty.$$

Also, for every bounded open set $V \subset \mathbb{R}^2$ *and every* $1 \leqslant p < (2+\gamma)/(1+\gamma)$ *there exists some constant* $C = C(p, V, \gamma, K)$ *such that*

$$\|v^\varepsilon\|_{L^p(V)} \leqslant C \qquad \forall \varepsilon \in (0, 1].$$

Finally, $\{v^\varepsilon/|u^\varepsilon|\}$ *converges weakly to some limit in* L^2 *if and only if* v^ε *converges weakly in* L^p_{loc} *for all* p *as above, and the weak limits are equal.*

For $g_\varepsilon = |\ln \varepsilon|$, the first assertion in the above theorem is proved in [17].

In general, we do not expect the strong convergence of v^ε in L^2 even for the minimizers. Also, note that depending on the scaling either v or w is zero, except in the critical case $g_\varepsilon = (\ln \varepsilon)^2$.

We now specialize to the case $g_\varepsilon = (\ln \varepsilon)^2$. In view of the compactness result, on a subsequence, denoted by ε again, $(v^\varepsilon, w^\varepsilon)$ converge to some limit (v, w) in appropriate spaces. Moreover, due to the choice of the scaling

$$v^\varepsilon = \frac{j(u^\varepsilon)}{|\ln \varepsilon|}, \qquad w^\varepsilon = \nabla \times v^\varepsilon/2 = \frac{Ju^\varepsilon}{|\ln \varepsilon|}.$$

Hence $w = \nabla \times v/2$.

THEOREM 1.2 (Gamma Limit). *Assume* (1.1) *with* $g(\varepsilon) = (\ln \varepsilon)^2$. *Let* v *and* w *be as above. Then,* $v \in L^2$, $w = \nabla \times v/2$ *is a Radon measure and belongs to* H^{-1}. *Moreover,*

$$\liminf \frac{|\mathbb{E}_\varepsilon(u^\varepsilon)|}{(\ln \varepsilon)^2} \geqslant \mathbb{E}(v) := \frac{1}{2}[\|v\|_2^2 + \|\nabla \times v^\varepsilon\|_{\mathcal{M}}].$$

Finally, if U *is smooth and bounded, then for any given* $v \in L^2$ *such that* $w := \nabla \times v/2$ *is a Radon measure, there exists a sequence* $\{u^\varepsilon\}$ *in* $H^1(U)$ *such that* v^ε, w^ε *defined as in* (1.2) *converge to* v *and* w, *respectively, weakly in* L^2 *and in* C^{0,α^*} *for every* $\alpha > 0$, *and for this sequence the above limit is achieved with an equality.*

The above result says that the Gamma limit of $E_\varepsilon/g_\varepsilon$ is E. This theorem is proved in two steps: the lower bound is proved in Section 6, while the sequence achieving equality is constructed in Section 7.

We have only stated the result for the most interesting case $g_\varepsilon = (\ln \varepsilon)^2$, but we have given a nearly complete analysis in other scalings as well. Indeed, the Gamma limit for $g_\varepsilon \ll (\ln \varepsilon)^2$ is $\|w\|_{\mathcal{M}}$, while for $(\ln \varepsilon)^2 \ll g_\varepsilon \ll \varepsilon^{-2}$ the limit is $\|v\|_2^2/2$. The lower bound in the latter case is nearly trivial, and in the former case it follows from an easy modification of arguments in Sections 5 and 6, and the construction in Section 7 establishes the upper bound for general $g_\varepsilon \ll \varepsilon^{-2}$.

The above result is proved by a localization and a lower bound result of [17]. The main technical tool is a covering argument devised by the first author in [16] and by Sandier [23].

1.1. *Applications to Superconductivity*

A closely related functional is the Ginzburg–Landau functional for superconductivity. It is a phenomenological model for a complex-valued order parameter u and an $\mathbb{R}^2$-valued vector potential A. After an appropriate rescaling, the Ginzburg–Landau functional takes the form

$$\mathbb{F}_\varepsilon(u, A; h_{\text{ext}}) := \int_U \frac{1}{2}|\nabla_A u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2 + \frac{1}{2}|\nabla \times A - h_{\text{ext}}|^2 \, dx,$$

where h_{ext} is the applied magnetic field, and the covariant derivative $\nabla_A u := \nabla u - iAu$. We refer to the book by Tinkham [29] and the surveys of Rubinstein [22], and Chapman [10] for information.

Although the functional $\mathbb{E}_\varepsilon$ does not have some of the physical complexities of the Ginzburg–Landau functional, its analysis and the behavior of the minimizers are surprisingly similar. A complete analysis of the minimizers of this functional is recently carried out in a series of papers by Serfaty and Sandier–Serfaty; see [25–27] and the references therein. In particular, the vortex structure and the connection between the magnitude of the applied magnetic field and the number of vortices is proved through hard analysis. Compactness results for nonminimizers are also obtained in these papers.

In this paper, we obtain a Gamma limit for $\mathbb{F}_\varepsilon$ by using our results on $\mathbb{E}_\varepsilon$. The starting point of our analysis is the following decomposition of $\mathbb{F}_\varepsilon$ by Bethuel and Riviere [3]:

$$\mathbb{F}_\varepsilon(u, A; h_{\text{ext}}) = \mathbb{E}_\varepsilon(u) - \int_U j(u)A \, dx + \frac{1}{2}\int_U |A|^2|u|^2 + |\nabla \times A - h_{\text{ext}}|^2 \, dx, \qquad (1.3)$$

which clearly indicates the important role played by $\mathbb{E}_\varepsilon$ in the asymptotic behavior of $\mathbb{F}_\varepsilon$.

As in the analysis of $\mathbb{E}_\varepsilon$, suppose that a sequence $(u^\varepsilon, A^\varepsilon)$ and the external magnetic field h_{ext} satisfy

$$\mathbb{F}_\varepsilon(u^\varepsilon, A^\varepsilon; h_{\text{ext}}) \leqslant K g_\varepsilon, \qquad h_{\text{ext}} \approx K\sqrt{g_\varepsilon}, \tag{1.4}$$

for some function g_ε satisfying $|\ln \varepsilon| \leqslant g_\varepsilon \ll \varepsilon^{-2}$. Indeed, given the above behavior of the external magnetic field, the energy upper bound is satisfied by the minimizers. We simplify the presentation, we assume the following limit exists:

$$H := \lim \frac{h_{\text{ext}}}{\sqrt{g_\varepsilon}}. \tag{1.5}$$

Then, $|u^\varepsilon|$ tends to one and in view of our results on $\mathbb{E}_\varepsilon$ we have the formal approximate lower bound

$$\mathbb{F}_\varepsilon(u^\varepsilon, A^\varepsilon; h_{\text{ext}}) \geqslant \frac{g_\varepsilon}{2}\left[\|v^\varepsilon \chi_{U \setminus V_\varepsilon}\|_2^2 + \frac{|\ln \varepsilon|}{\sqrt{g_\varepsilon}} \|\nabla \times v^\varepsilon\|_{\mathcal{M}} - 2(v^\varepsilon, a^\varepsilon)_2 \right.$$
$$\left. + \|a^\varepsilon\|_2^2 + \|\nabla \times a^\varepsilon - H\|_2^2 \right],$$

where v^ε is as in (1.2), $(\cdot,\cdot)_2$ is the L^2 inner product and

$$a^\varepsilon := A^\varepsilon / \sqrt{g_\varepsilon}. \tag{1.6}$$

Again, the interesting case is $g_\varepsilon = (\ln \varepsilon)^2$ which corresponds to $h_{\text{ext}} \approx H|\ln(\varepsilon)|$. In this scaling, we rewrite the above approximate lower bound as

$$\mathbb{F}_\varepsilon(u^\varepsilon, A^\varepsilon; h_{\text{ext}}) \geqslant g_\varepsilon \mathbb{F}(v^\varepsilon, a^\varepsilon; H),$$

where

$$\begin{aligned}\mathbb{F}(v, a; H) &:= \tfrac{1}{2}[\|v\|_2^2 + \|\nabla \times v\|_{\mathcal{M}} - 2(v, a)_2 + \|a\|_2^2 + \|\nabla \times a - H\|_2^2] \\ &= \tfrac{1}{2}[\|v - a\|_2^2 + \|\nabla \times v\|_{\mathcal{M}} + \|\nabla \times a - H\|_2^2].\end{aligned} \tag{1.7}$$

As well known $\mathbb{F}_\varepsilon$ has a gauge invariance:

$$\mathbb{F}_\varepsilon(u, A; h_{\text{ext}}) = \mathbb{F}_\varepsilon(ue^{i\chi}, A + \nabla\chi; h_{\text{ext}}),$$

for any smooth χ; see [22, 29]. This invariance is inherited by $\mathbb{F}$ as well. Indeed,

$$\mathbb{F}(v, a; H) = \mathbb{F}(v + \nabla\chi, a + \nabla\chi; H).$$

This is natural as (v, a) is the scaled limit of $(j(u), A)$, and for a complex-valued function $u = \rho e^{i\varphi}$, $j(u) = \rho^2 \nabla \varphi$. Note that $v - a$ is the scaled limit of the superconducting current

$$j_{A^\varepsilon}(u^\varepsilon) := u^\varepsilon \times \nabla_{A^\varepsilon} u^\varepsilon = j(u^\varepsilon) - |u^\varepsilon|^2 A^\varepsilon.$$

Due to the gauge invariance, we expect compactness results only for the quantities that are gauge invariant. Important gauge invariant quantities are the superconducting current j_{A^ε} and the induced magnetic field $\nabla \times A^\varepsilon$. It turns out that the limit functional is described by the scaled limits of these quantities.

Theorem 1.3 (Gamma Limit). *Assume* (1.4) *with* $g_\varepsilon = [\ln \varepsilon]^2$. *Let* v^ε *be as in* (1.2) *and* a^ε *be as in* (1.6). *Further assume that* H *in* (1.5) *is finite. Then,* $\nabla \times a^\varepsilon$ *is weakly compact in* L^2 *and we have the same compactness for*

$$v^\varepsilon_{a^\varepsilon} := \frac{j_{A^\varepsilon}(u^\varepsilon)}{|\ln \varepsilon|} = v^\varepsilon - |u^\varepsilon|^2 a^\varepsilon$$

as in Theorem 1.1. *Any limit of* $(v^\varepsilon_{a^\varepsilon}, \nabla \times a^\varepsilon)$ *can be expressed as* $(v - a, \nabla \times a)$ *by a pair* $(v, a) \in L^2 \times H^1(U, \mathbb{R}^2)$. *Moreover,* $\nabla \times a \in L^2$, $\nabla \times v$ *is a Radon measure, and*

$$\liminf \frac{\mathbb{F}_\varepsilon(u^\varepsilon, A^\varepsilon; h_{\text{ext}})}{(\ln \varepsilon)^2} \geqslant \mathbb{F}(v, a; H).$$

Finally, given (v, a) *as above there exists a sequence* $(u^\varepsilon, A^\varepsilon)$ *so that the above limit is achieved with an equality, and* $v^\varepsilon_{a^\varepsilon}$, a^ε *are compact in the above spaces.*

Results of this type is already obtained by Serfaty and Sandier [26]. In particular, the upper bound and a compactness result is proved in [25, 26].

Several asymptotic results of interest can be obtained from the Gamma limit result. Here, we outline the derivation of the first critical threshold H_{c_1}: the largest value of H below.

Although the asymptotic formula for H_{c_1} is formally known for sometime, a rigorous derivation of it is only given recently by Serfaty [27] and Serfaty and Sandier [25]. Our derivation is similar to the proof given in these papers.

A quick formal calculation of H_{c_1} using the limit functional $\mathbb{F}$ in (1.7) is this. We expect for small H, the minimizers of $\mathbb{F}(v, a; H)$ to be gauge equivalent to (v^*, a^*) where $\nabla \times a^* \equiv 0$, and a^* is the minimizer of the last two terms in (1.7):

$$\mathscr{E}(a) := \tfrac{1}{2}[\|a\|_2^2 + \|\nabla \times a - H\|_2^2].$$

The question is how small H has to be for this pair to be the minimizer. This is a rather straightforward calculation.

We first observe that the minimizer a^* of $\mathscr{E}$ solves

$$\nabla \times [\nabla \times a^* - H] + a^* = 0 \quad \text{in } U, \qquad \nabla \times a^* - H = 0 \quad \text{on } \partial U. \tag{1.8}$$

By taking the curl of this equation we see that $[\nabla \times a^* - H] = H\hat{z}$, where $\hat{z}$ is the unique solution of

$$-\Delta \hat{z} + \hat{z} = -1 \quad \text{in } U, \qquad \hat{z} = 0 \quad \text{on } \partial U. \tag{1.9}$$

For any pair (v, a), set $b := a - a^*$ so that

$$\begin{aligned}\mathbb{F}(v, a; H) = \tfrac{1}{2}[&\|a^*\|_2^2 + \|b\|_2^2 + 2(a^*, b)_2 \\ &+ \|\nabla \times a^* - H\|_2^2 + \|\nabla \times b\|_2^2 + 2(\nabla \times a^* - H, \nabla \times b)_2 \\ &+ 2(v, a^*)_2 + 2(v, b)_2 + \|\nabla \times v\|_{\mathscr{M}} + \|v\|_2^2].\end{aligned}$$

Equations (1.8) and (1.9) imply that

$$(v, a^*)_2 = -(v, \nabla \times [\nabla \times a^* - H])_2 = -H(v, \nabla \times \hat{z})_2 = -H(\nabla \times v, \hat{z})_2$$

and

$$(a^*, b)_2 + (\nabla \times a^* - H, \nabla \times b)_2 = (a^* + \nabla \times [\nabla \times a^* - H], b)_2 = 0.$$

Hence, for any (v, a),

$$\mathbb{F}(v, a; H) = \mathbb{F}_1(v, b) + \mathbb{F}_2(v, a^*),$$

where

$$\begin{aligned}\mathbb{F}_1(v, b) &= \tfrac{1}{2}[\|\nabla \times b\|_2^2 + \|b\|_2^2 + 2(v, b)_2 + \|v\|_2^2] \\ &= \tfrac{1}{2}[\|\nabla \times b\|_2^2 + \|b - v\|_2^2] \geqslant 0\end{aligned}$$

and

$$\begin{aligned}\mathbb{F}_2(v, a^*) &= \mathscr{E}(a^*) - H(\nabla \times v, \hat{z})_2 + \tfrac{1}{2}\|\nabla \times v\|_{\mathscr{M}} \\ &\geqslant \mathscr{E}(a^*) + \tfrac{1}{2}\|\nabla \times v\|_{\mathscr{M}}[\tfrac{1}{2} - H \max_U |\hat{z}|].\end{aligned}$$

So the minimizer satisfies $\nabla \times v^* \equiv 0$, if and only if the term in the brackets is negative or equivalently when

$$H < H_{c_1} := \frac{1}{2 \max_U |\hat{z}|}. \tag{1.10}$$

Clearly, the above argument is a statement about the limit, but with some extra effort this can be used for the minimizers with small ε, showing that the minimizer u of $\mathbb{F}_\varepsilon$ never vanishes if and only if $H < H_{c_1}$.

In fact, this and more is proved by Serfaty and Sandier [26].

The second consequence of the Gamma limit result is the derivation of a mean field equation which is the Euler–Lagrange equation for $\mathbb{F}$. It turns out that this is a variational inequality closely related to London's equation. A rigorous derivation of this equation from the Ginzburg–Landau functional is first given by Sandier and Serfaty [25]. Later Brezis and Serfaty [6] used convex duality and earlier results of Brezis [5] on the convex dual of variational problems to obtain an alternate derivation.

A formal derivation of the dynamic version of this mean field equation is obtained by Chapman *et al.* [9].

We finish this introduction with related variational and dynamic problems. A related functional with the additional constraint $u = \nabla\varphi$ is an important model problem in phase transitions. Recently, compactness results for this functional is proved in [1, 12]. The properties of the Jacobians of S^1-valued functions is studied by the authors in [18, 19]. Related higher dimensional problems are studied in [20, 21, 24].

The paper is organized as follows. After recalling the notation and the results of [17], we first prove the compactness theorem in Section 4. A sharper Jacobian estimate is proved in Section 5, and the lower limit of $\mathbb{E}_\varepsilon$ is proved in Section 6. An upper bound of the energy is obtained in Section 7 by constructing a sequence of functions with certain asymptotic-properties. Last two sections are devoted to the derivation of the Gamma limit of $\mathbb{F}_\varepsilon$ and the mean field equations. Finally the proof of a technical lemma is given in the appendix.

2. NOTATION

We need to recall some of the notation of [17].

Set

$$e^\varepsilon(u) := \frac{1}{2}|Du|^2 + \frac{1}{\varepsilon^2}W(|u|^2),$$

so that

$$\mathbb{E}_\varepsilon(u) := \int_U e^\varepsilon(u)\,dx.$$

If $S \subset \mathbb{R}^2$, we will write χ_S to denote the characteristic function of S, so that

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S_y \\ 0 & \text{if } x \notin S. \end{cases}$$

Given $\phi \in C_c^{0,1}(U)$, we use the notation

$$\Omega(t) = \{x \in U \mid \phi(x) > t\}, \tag{2.1}$$

$$\begin{aligned}\mathrm{Reg}(\phi) := \{\, t \in [0,T] : \partial\Omega(t) = \phi^{-1}(t), \partial\Omega(t) \text{ rectifiable}, \\ \mathscr{H}^1(\partial\Omega(t)) < \infty\}.\end{aligned} \tag{2.2}$$

The co-area formula implies that $\mathrm{Reg}(\phi)$ is a set of full measure. For every $t \in \mathrm{Reg}(\phi)$, $\partial\Omega(t)$ is a union of finite Jordan curves $\Gamma_i(t)$, i.e.,

$$\partial\Omega(t) = \bigcup_i \Gamma_i(t), \qquad \forall t \in \mathrm{Reg}(\phi).$$

In particular, this holds for almost every t. For $t \in \mathrm{Reg}(\phi)$, we define

$$\Gamma(t) = \bigcup \left\{ \text{components } \Gamma_i(t) \text{ of } \partial\Omega(t) \mid \min_{x \in \Gamma_i(t)} |u(x)| > 1/2 \right\}. \tag{2.3}$$

Given any function $\phi \in C^{0,1}(U)$ such that $\phi = 0$ in ∂U, we define

$$D_d(\phi) := \{t \in \mathrm{Reg}(\phi) : \Gamma(t) \text{ is nonempty, and } |\deg(u; \Gamma(t))| \geqslant d\}, \tag{2.4}$$

$$D_d^{\varepsilon}(\phi) := D_d(\phi) \cap \{t : t \geqslant \varepsilon \|\nabla\phi\|_\infty\}. \tag{2.5}$$

We will normally write simply D_d when there is no possibility of confusion. Of course, $|D_d|$ depends on u as well as on ϕ.

Note that if $u \in H^1$ is fixed, then the ratio $|D_d(\phi)|/\|\phi\|_\infty$ is scale-invariant in that it is not changed when we multiply ϕ by a scalar, so that $|D_d(\phi)|/\|\phi\|_\infty = |D_d(\lambda\phi)|/\|\lambda\phi\|_\infty$. The same remark holds for $|D_d^{\varepsilon}(\phi)|/\|\nabla\phi\|$.

3. PREVIOUS RESULTS

In this section, we recall and restate two results of [17].

The first one is a localization and a lower energy bound result. It is essentially proved in [17] by a covering argument of the first author [16].

THEOREM 3.1. *Suppose that* $u \in H^1(U; \mathbb{R}^2)$ *and* $\varepsilon \in (0, 1]$. *Then, there exists* $\sigma^* = \sigma^*(u, \varepsilon) > 0$ *and* $C > 0$ (*independent of* ε *and* u) *such that for every* $\sigma \geqslant \sigma^*$ *there is a collection of pairwise disjoint closed balls* $\mathscr{B}(\sigma, u, \varepsilon) = \mathscr{B}^\sigma = \{B_i^\sigma\}_{i=1}^{k(\sigma)}$, *such that for any* $\phi \in C_0^{0,1}(U)$ *and for any positive* *integer d,*

$$\frac{|D_d^\varepsilon(\phi)|}{\|\nabla\phi\|_\infty} \leqslant 2R(\sigma), \tag{3.1}$$

whenever either $d\sigma > R(\sigma)$ *or*

$$\int_{\operatorname{spt}(\phi)\cap\left(\bigcup_i B_i^\sigma\right)} e^\varepsilon(u)\,dx < \pi d\left(\ln\left(\frac{R(\sigma)}{\varepsilon d}\right) + C\right). \tag{3.2}$$

Moreover,

$$\sigma \to R(\sigma) = \sum r_i^\sigma \text{ is continuous and nondecreasing on } [\sigma^*, \infty), \tag{3.3}$$

where r_i^σ *is the radius of* B_i^σ. *Finally,* σ^* *also satisfies*

$$R(\sigma^*) \leqslant \frac{\varepsilon}{C} E^\varepsilon(u). \tag{3.4}$$

This result is essentially a restatement of Proposition 6.4 and Remark 6.5 of [17]. For the reader's convenience, we give its short proof in the appendix. The second result is a Jacobian estimate proved in [17, Theorem 2.2]. The version stated below is slightly different than the statement of that theorem but the version below is actually proved in Step 5 of Theorem 2.2 in [17].

THEOREM 3.2. *Suppose that* $u \in H^1(U;\mathbb{R}^2)$. *There exists some constant* $C > 0$ *such that for any* $\phi \in C_0^{0,1}(U)$, *positive integer d and* $\varepsilon \in (0,1]$,

$$\left|\int_U \phi Ju\,dx\right| \leqslant \pi(d + C\sqrt{\varepsilon})\|\phi\|_\infty + |D_{d+1}|E^\varepsilon(u) + C\varepsilon^{1/3}\|\nabla\phi\|_\infty(E^\varepsilon(u) + (E^\varepsilon(u))^2). \tag{3.5}$$

Finally, we recall a compactness result which is proved by interpolation techniques; see [17, Remark 3.7].

THEOREM 3.3. *Suppose* v^ε *is any sequence of measures on a bounded open set* $U \subset \mathbb{R}^m$, *and there exists some* $\alpha > 0$ *such that*

$$|v^\varepsilon|(U) \leqslant K|\ln\varepsilon|, \qquad \int \phi\,dv^\varepsilon \leqslant C\|\phi\|_\infty + C\varepsilon^\alpha\|\nabla\phi\|_\infty$$

for all $\phi \in C_0^{0,1}(U)$. *Then,* $\{v^\varepsilon\}$ *is precompact in* $(C_0^{0,\beta}(U))^*$ *for all* $\beta \in (0,1]$.

4. COMPACTNESS

In this section, we prove Theorem 1.1. The compactness of the Jacobian is proved exactly as in [17] by using Theorem 3.1. The compactness of $j(u^\varepsilon)$ is more of a direct consequence of (1.1) and straigtforward estimates.

We start with v^ε.

THEOREM 4.1. *Suppose that* $u^\varepsilon \in U^1(U;\mathbb{R}^2)$ *satisfies* (1.1) *with* $g_\varepsilon \leqslant \varepsilon^{-\gamma}$ *for some* $0<\gamma<2$. *Let* v^ε *be as in* (1.2). *Then,*

$$\left\|\frac{v^\varepsilon}{|u^\varepsilon|}\right\|_2^2 \leqslant 2K \qquad \forall \varepsilon \in (0,1],$$

where K is the constant in (1.1). *Also, for every bounded open set* $V \subset \mathbb{R}^2$ *and every* $1 \leqslant p < (2+\gamma)/(1+\gamma)$ *there exists some constant* $C = C(p, V, \gamma, K)$ *such that*

$$\|v^\varepsilon\|_{L^p(V)} \leqslant C \qquad \forall \varepsilon \in (0,1]. \tag{4.1}$$

Finally, for any subsequence $\{\varepsilon_n\}$ *tending to zero,* $\{v^{\varepsilon_n}/|u^{\varepsilon_n}|\}$ *converges weakly to some limit in* L^2 *if and only if* v^{ε_n} *converges weakly in* L^p_{loc} *for all p as above, and the weak limits are equal.*

Remark 4.2. If we consider a nonlinearity of the form $W(|u|)$ such that $W(1)=0$, $W(s)>0$ for $s\neq 1$, and $W(s)\geqslant s^r - C$ for some $r\geqslant 2$, then similar results are true, for a different range of p.

Proof. Since $|Du|^2 = |D|u||^2 = |j(u)|^2/|u|^2$, the first conclusion of the theorem is obvious.

1. Because $v^\varepsilon \leqslant |u^\varepsilon|(|Du^\varepsilon|/\sqrt{g_\varepsilon})$, Hölder's inequality implies that

$$\begin{aligned}\int_V |v^\varepsilon|^p &\leqslant C\left(\frac{1}{g_\varepsilon}\int_V e^\varepsilon(u^\varepsilon)\right)^{p/2}\left(\int_V |u^\varepsilon|^{2p/(2-p)}\right)^{(2-p)/p}\\ &\leqslant C\left(\int_V |u^\varepsilon|^{2p/(2-p)}\right)^{(2-p)/p}.\end{aligned}$$

So to prove (4.1), it suffices to show that (1.1) implies that $\|u^\varepsilon\|_q \leqslant C$ for $q = 2p/(2-p)$, with $1\leqslant p<(2+\gamma)/(1+\gamma)$.

First, note that

$$|u^\varepsilon|^q \leqslant (|u^\varepsilon|-1)^q + C(q).$$

Because V is bounded, it suffices to prove that $\||u^\varepsilon| - 1\|_q$ is uniformly bounded for q as above. This is immediate if $q \leqslant 4$, since $(|u^\varepsilon| - 1)^4 \leqslant (|u^\varepsilon|^2 - 1)^2$, and so

$$\int_V (|u^\varepsilon| - 1)^4 \leqslant \int_V (|u^\varepsilon|^2 - 1)^2 \leqslant C\varepsilon^2 g_\varepsilon.$$

For $q > 4$, we interpolate as follows: writing $h^\varepsilon := |u^\varepsilon| - 1$, we see that $|Dh^\varepsilon| \leqslant |Du^\varepsilon|$, and so $\|Dh^\varepsilon\|_2^2 \leqslant Cg_\varepsilon$. This implies $\|Dh^\varepsilon\|^2_{L^r(V)} \leqslant C(V)g_\varepsilon$ for all $r < 2$. Using the Sobolev–Nirchberg–Gagliardo inequality

$$\|h^\varepsilon\|_q \leqslant C\|h^\varepsilon\|_4^\theta \|Dh^\varepsilon\|_r^{1-\theta} \leqslant C(\varepsilon g_\varepsilon)^{\theta/4} g_\varepsilon^{(1-\theta)/2} \leqslant C\varepsilon^{-\gamma/2+\theta/4(\gamma+1)}$$

where $\frac{1}{q} = \frac{\theta}{4} + (1-\theta)(\frac{1}{r} - \frac{1}{2})$, one finds by taking r arbitrarily close to 2 that in fact $\|h^\varepsilon\|_q \leqslant C$ for all q less than some number $q^*(\gamma) > 4$. A short computation shows that $q^*(\gamma) = 2(1+\gamma)/\gamma$, and after another short calculation one finds that (4.1) holds for all $1 \leqslant p < (\frac{1}{2} + \gamma)/(1+\gamma)$ as claimed.

2. Now suppose that $v^{\varepsilon_n}/|u^{\varepsilon_n}| \rightharpoonup v$ weakly in L^2. Note that $(|u^{\varepsilon_n}| - 1)^4 \leqslant (|u^{\varepsilon_n}|^2 - 1)^2$, so (1.1) implies that $|u^\varepsilon| \to 1$ strongly in L^4. Thus, the product $|u^{\varepsilon_n}|(v^{\varepsilon_n}/|u^{\varepsilon_n}|) = v^{\varepsilon_n}$ converges weakly in $L^{4/3}$ to the product of the strong limit of $|u^{\varepsilon_n}|$ and the weak limit of $j(u^{\varepsilon_n})/|u^{\varepsilon_n}|$ which is equal to v. It follows that, in fact, $v^{\varepsilon_n} \rightharpoonup v$ weakly in L^p_{loc}, for the entire range of p for which $\{v^{\varepsilon_n}\}$ is weakly precompact.

Finally, if $v^{\varepsilon_n} \rightharpoonup v$, then the above argument shows that any weakly convergent subsequence of $\{v^{\varepsilon_n}/|u^{\varepsilon_n}|\}$ must also converge to v. However, since $\{v^{\varepsilon_n}/|u^{\varepsilon_n}|\}$ is weakly precompact in L^2, in fact it must be the case that $v^{\varepsilon_n}/|u^{\varepsilon_n}| \rightharpoonup v$ in L^2. ∎

Proof of Theorem 1.1. The compactness of the rescaled Jacobian w^ε is the only part that remains to be proved. From the definition and the assumed energy bound (1.1) it is clear that $\|w^\varepsilon\|_1 \leqslant K$, so in view of Theorem 5.1, we only need to prove that $\int \phi(x) w^\varepsilon(x)\, dx \leqslant C\|\phi\|_\infty + C\varepsilon^\alpha \|\nabla\phi\|_\infty$ for Lipschitz test functions ϕ with compact support in U. This is proved for $g_\varepsilon = (\ln \varepsilon)^2$ in Theorem 5.1. The general case follows the argument of Step 3 of the proof of Theorem 5.1, by using Theorem 3.1 to show that if $d = d_\varepsilon$ is chosen to be sufficiently large (for example $d_\varepsilon \geqslant Kg_\varepsilon/|\ln \varepsilon|$ is good enough), then $|D^\varepsilon_{d_\varepsilon+1}| \leqslant C\varepsilon^\alpha$ for some a $\alpha \in (0, 1)$. The estimate then follows from Theorem 5.2. In applying Theorem 3.1 one can take $\sigma = 1$ say; the more careful choice of σ as in the proof of Theorem 5.1 is not necessary here. ∎

5. JACOBIAN ESTIMATE

The main result of this section is a sharp Jacobian estimate in terms of the Ginzburg–Landau energy. It also states that the vortex energy as measured by the Jacobian concentrates near the vortices. This allows us to separate the contributions of $j(u^\varepsilon)$ and Ju^ε to the energy. Since in the scaling $g_\varepsilon = (\ln \varepsilon)^2$ both of these contributions are of the same size, this decomposition is essential in the limit analysis.

THEOREM 5.1. *For any* $\alpha \in (0,1)$ *and* $K > 0$ *there exists* $\varepsilon_0(\alpha, K) > 0$ *such that* $u \in H^1(U;\mathbb{R}^2)$ *is any function satisfying*

$$\mathbb{E}_\varepsilon(u) \leqslant K(\ln \varepsilon)^2 \tag{5.1}$$

for some $\varepsilon \in (0, \varepsilon_0)$, *then there exists a collection of balls* $\tilde{\mathscr{B}}(\alpha, u, \varepsilon) = \tilde{\mathscr{B}} = \{\tilde{B}_i\}_{i=1}^{\tilde{k}}$ *such that*

$$\tilde{R} := \sum \tilde{r}_i \leqslant \varepsilon^\alpha, \tag{5.2}$$

where $\tilde{r}_i$ *is the radius of* $\tilde{B}_i$; *and such that for every nonnegative Lipschitz function* ϕ,

$$\left| \int_U \phi Ju \, dx \right| \leqslant \frac{\|\phi\|_\infty}{(1-\sqrt{\alpha})^2 |\ln \varepsilon|} \int_{\operatorname{spt}\phi \cap \left(\bigcup_i \tilde{B}_i\right)} e^\varepsilon(u^\varepsilon)\, dx + C\varepsilon^{\gamma(\alpha)/2} \|\phi\|_{C^{0,1}}, \tag{5.3}$$

where $\gamma(\alpha) = \min\{1/3, \alpha\}$.

Proof. (1) Fix some $\alpha \in (0,1)$ and $K > 0$, and suppose that $u \in H^1(U;\mathbb{R}^2)$ satisfies (5.1) for ε smaller than some small constant $\varepsilon_0(\alpha, K)$. We will give conditions on $\varepsilon_0(\alpha, K)$ in the course of the proof.

We first construct a collection of balls, and in later steps we will show that it has the desired properties.

Consider the collection of balls $\mathscr{B}(\sigma^*, u, \varepsilon) = \{B_i^{\sigma^*}\}_i$ produced in Theorem 3.1. Note that by (3.4) and (5.1),

$$R(\sigma^*) \leqslant \frac{\varepsilon}{C} E^\varepsilon(u) \leqslant \frac{\varepsilon}{C} K |\ln \varepsilon|^2.$$

By taking $\varepsilon_0(\alpha, K)$ sufficiently small, we can arrange that the right-hand side is less than ε^α. Because $\sigma \to R(\sigma) = \sum r_i^\sigma$ is continuous and nondecreasing on $[\sigma^*, \infty)$, it follows that for $\varepsilon < \varepsilon_0$, either

(i) there exists some $\tilde{\sigma} \geqslant \sigma^*$ such that $R(\tilde{\sigma}) = \varepsilon^\alpha$; or
(ii) $R(\sigma) \leqslant \varepsilon^\alpha$ for all $\sigma \geqslant \sigma^*$.

For the time being we assume that (i) holds, and we take our collection $\tilde{\mathscr{B}}$ to be the collection $\mathscr{B}(\tilde{\sigma}, u, \varepsilon) = \mathscr{B}^{\tilde{\sigma}}$ guaranteed by Theorem 3.1. Clearly, (5.2) is satisfied, so we only need to show that (5.3) holds.

Case (ii) is simpler and in some sense merely technical, and we will discuss it at the end of the proof.

(2) Now fix any function $\phi \in C_c^1(U)$ and set

$$d^\alpha = \left\lfloor \frac{1}{\pi|\ln \varepsilon|(1-\sqrt{\alpha})^2} \int_{\mathrm{spt}(\phi) \cap \left(\bigcup_i \tilde{B}_i\right)} e^\varepsilon(u^\varepsilon)\, dx \right\rfloor, \tag{5.4}$$

where $\lfloor a \rfloor$ is the integer part of a.

For any nonnegative integer d, define as before

$$D_d := \{t \in \mathrm{Reg}(\phi) : \Gamma(t) \text{ is nonempty, and } |\deg(u; \Gamma(t))| \geqslant d\}.$$

From (3.5), we have

$$\begin{aligned} \left| \int_U \phi J u\, dx \right| \leqslant \pi(d + C\sqrt{\varepsilon})\|\phi\|_\infty + |D_{d+1}| E^\varepsilon(u) \\ + C\varepsilon^{1/3} \|\nabla\phi\|_\infty (E^\varepsilon(u) + (E^\varepsilon(u))^2) \end{aligned} \tag{5.5}$$

for every d, and in particular for d^α. We will write $d^* = d^\alpha + 1$. Note from (5.4) that

$$\frac{1}{\pi|\ln \varepsilon|(1-\sqrt{\alpha})^2} \int_{\mathrm{spt}(\phi) \cap \left(\bigcup_i \tilde{B}_i\right)} e\varepsilon(u^\varepsilon)\, dx \leqslant d^* \leqslant K|\ln \varepsilon|. \tag{5.6}$$

Define as before $D_d^\varepsilon = D_d \cap \{t \mid t \geqslant t_\varepsilon\}$, where $t_\varepsilon := \varepsilon\|\nabla\phi\|_\infty$, so that

$$|D_d| \leqslant |D_d^\varepsilon| + \varepsilon\|\nabla\phi\|_\infty. \tag{5.7}$$

(3) From (5.6) and the choice of $\tilde{\sigma}$, it is clear that $R(\tilde{\sigma})/d^* = \varepsilon^\alpha/d^* \geqslant \varepsilon^{\sqrt{\alpha}-1}$, if $\varepsilon_0(\alpha, K)$ is chosen to be sufficiently small. So

$$\begin{aligned} \pi d^* \left(\ln\left(\frac{\tilde{R}}{\varepsilon d^*} \right) - C \right) &= \pi d^* (\ln(\varepsilon^{\sqrt{\alpha}-1}) - C) \\ &\geqslant \pi d^* ((1-\sqrt{\alpha})|\ln \varepsilon| - C). \end{aligned}$$

Again using (5.6), the right-hand side is greater than

$$\left(\int_{\mathrm{spt}(\phi) \cap \left(\bigcup_i \tilde{B}_i\right)} e^\varepsilon(u^\varepsilon)\, dx \right) \left(\frac{1}{(1-\sqrt{\alpha})} |\ln \varepsilon| - \frac{C}{(1-\sqrt{\alpha})^2} \right).$$

As a result,

$$\int_{\mathrm{spt}(\phi)\cap\left(\bigcup_i \tilde{B}_i\right)} e^{\varepsilon}(u^{\varepsilon})\,dx < \pi d^*\left(\ln\left(\frac{\tilde{R}}{\varepsilon d^*}\right) - C\right)$$

for ε sufficiently small (depending on K, α). Because the collection $\tilde{\mathcal{B}} = \mathcal{B}(\tilde{\sigma})$ of balls satisfies the conclusions of Theorem 3.1 for some $\tilde{\sigma} > 0$, and because the above inequality is exactly (3.2), we conclude that

$$\frac{|D^{\varepsilon}_{d^*}|}{\|\nabla\phi\|_{\infty}} \leqslant 2R(\tilde{\sigma}) = 2\varepsilon^{\alpha}. \tag{5.8}$$

Then (5.5), (5.7), and (5.8) imply that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\left|\int_U \phi Ju\,dx\right| \leqslant \pi(d^{\alpha} + C\sqrt{\varepsilon})\|\phi\|_{\infty} + C\varepsilon^{\gamma(\alpha)}\|\nabla\phi\|_{\infty}(E^{\varepsilon}(u) + (E^{\varepsilon}(u))^2),$$

where $\gamma(\alpha) = \min\{\alpha, 1/3\}$,

(4) Finally, suppose that (ii) holds, and consider the collection of balls $\mathcal{B}^{\sigma}$ given by Theorem 3.1 for some fairly large value of σ, say $\sigma = 1$ for example. Then for any positive integer d,

$$d\sigma = d > \varepsilon^{\alpha} \geqslant R(\sigma)$$

by the assumption of case (ii), and so Theorem 3.1 implies that $|D^{\varepsilon}_d| \leqslant 2R(\sigma)\|\nabla\phi\|_{\infty} \leqslant 2\varepsilon^{\alpha}\|\nabla\phi\|_{\infty}$, so (5.3) follows as before. In fact, more is true: since in particular $|D^{\varepsilon}_1| \leqslant 2\varepsilon^{\alpha}\|\nabla\phi\|_{\infty}$, taking $d = 0$ in (5.5), we find that if (ii) holds, then

$$\left|\int_U \phi Ju\right| \leqslant C\sqrt{\varepsilon}\|\phi\|_{\infty} + C\varepsilon^{\gamma(\alpha)}\|\nabla\phi\|_{\infty}(E^{\varepsilon}(u) + (E^{\varepsilon}(u))^2). \qquad \blacksquare$$

Remark 5.2. In the above proof, we actually proved a slightly stronger version of (5.3) than stated. Indeed, we proved that

$$\begin{aligned}\left|\int_U \phi Ju\,dx\right| &\leqslant \pi(d^{\alpha} + C\sqrt{\varepsilon})\|\phi\|_{\infty} + C\varepsilon^{\gamma(\alpha)}\|\nabla\phi\|_{\infty}(E^{\varepsilon}(u) + (E^{\varepsilon}(u))^2)\\ &\leqslant \pi(d^{\alpha} + C\sqrt{\varepsilon})\|\phi\|_{\infty} + C\varepsilon^{\gamma(\alpha)/2}\|\nabla\phi\|_{\infty},\end{aligned}$$

where d^{α} is as in (5.4).

6. LOWER BOUNDS

In this section, we prove the lower bound in the Gamma limit. The upper bound will be proved in the next section, completing the proof of Theorem 1.2.

We consider a sequence of functions $u^\varepsilon \in U^1(U;\mathbb{R}^2)$ satisfying (1.1) with $g_\varepsilon = [\ln \varepsilon]^2$. Then,

$$v^\varepsilon = \frac{j(u^\varepsilon)}{|\ln \varepsilon|}, \qquad w^\varepsilon = \frac{Ju^\varepsilon}{|\ln \varepsilon|} = \frac{1}{2}\nabla \times v^\varepsilon,$$

and by the compactness results of the previous sections, v^ε, $v^\varepsilon/|u^\varepsilon|$ and w^ε are compact in appropriate spaces. In the following theorem, we assume convergence in these spaces and prove a lower bound.

THEOREM 6.1. *Suppose that*

$$v^\varepsilon/|u^\varepsilon| \rightharpoonup v \quad L^2 \text{ weak}, \qquad v^\varepsilon \rightharpoonup v \quad L^p_{\text{loc}} \text{ weak} \quad \forall p<2. \tag{6.1}$$

Then, $w := \nabla \times v/2$ *is a measure*, *and*

$$\liminf_{\varepsilon\to 0} \frac{\mathbb{E}_\varepsilon(u^\varepsilon)}{|\ln \varepsilon|} \geqslant \frac{1}{2}[\|v\|_2^2 + \|\nabla \times v\|_{\mathcal{M}}]. \tag{6.2}$$

Proof. (1) Fix $\alpha \in (0,1)$, and for $\varepsilon < \varepsilon_0(\alpha, K)$ let $\tilde{\mathscr{B}}^\varepsilon = \tilde{\mathscr{B}}(\alpha, \varepsilon, u^\varepsilon)$ denote the collection of balls that is shown to exist in Theorem 5.1. We will write B_i^ε to denote a generic ball in $\tilde{\mathscr{B}}^\varepsilon$.

For each $\varepsilon \in (0, \varepsilon_0)$ define

$$\chi^\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \bigcup_i \tilde{B}_i, \\ 0 & \text{if not.} \end{cases}$$

Note that by Hölder's inequality, for any fixed $h \in L^2$,

$$\left(\int h \frac{v^\varepsilon}{|u^\varepsilon|}\chi^\varepsilon\right)^2 \leqslant \int |h|^2 \chi^\varepsilon \, dx \int \left|\frac{v^\varepsilon}{|u^\varepsilon|}\right|^2 dx.$$

The first integral on the right-hand side vanishes as $\varepsilon \to 0$ by the dominated convergence theorem, and the second is uniformly bounded. It follows that $(v^\varepsilon/|u^\varepsilon|)\chi^\varepsilon \rightharpoonup 0$ weakly in L^2 and hence that $(v^\varepsilon/|u^\varepsilon|)(1-\chi^\varepsilon) \rightharpoonup v$ weakly in L^2. As a result,

$$\begin{aligned}\liminf_{\varepsilon\to 0}\ |\ln\varepsilon|^{-2}\int_{U\setminus\left(\bigcup_i \tilde{B}_i^\varepsilon\right)} e^\varepsilon(u^\varepsilon)\,dx &\geqslant \liminf_{\varepsilon\to 0}\int_{U\setminus\left(\bigcup_i \tilde{B}_i^\varepsilon\right)}\frac{1}{2}\left|\frac{\nabla u^\varepsilon}{\ln\varepsilon}\right|^2 dx\\ &\geqslant \liminf_{\varepsilon\to 0}\int_{U\setminus\left(\bigcup_i \tilde{B}_i^\varepsilon\right)}\frac{1}{2}\left|\frac{v^\varepsilon}{|u^\varepsilon|}\right|^2 dx\\ &= \liminf \int_U \frac{1}{2}\left|\frac{v^\varepsilon}{|u^\varepsilon|}(1-\chi^\varepsilon)\right|^2 dx\\ &\geqslant \frac{1}{2}\|v\|_2^2\,dx. \end{aligned}\tag{6.3}$$

(2) Since $v^\varepsilon \rightharpoonup v$, it is clear that $w^\varepsilon = \nabla\times v^\varepsilon/2$ converges in the sense of distributions to $w=\nabla\times v/2$. For any $\phi\in C_c^\infty(U)$, Theorem 5.1 implies that

$$\begin{aligned}\left|\int_U \phi\nabla\times v\,dx\right| &= \left|\lim_{\varepsilon\to 0}\ |\ln\varepsilon|^{-1}\int_U \phi J u^\varepsilon\,dx\right|\\ &\leqslant \frac{\|\phi\|_\infty}{(1-\sqrt{\alpha})^2}\liminf_{\varepsilon\to 0}\ |\ln\varepsilon|^{-2}\int_{\operatorname{spt}(\phi)\cap\left(\bigcup_i \tilde{B}_i\right)} e^\varepsilon(u^\varepsilon)\,dx.\end{aligned}$$

By taking the supremum over all ϕ as above such that $\|\phi\|_\infty\leqslant 1$, we find that

$$\frac{1}{2}\|\nabla\times v\|_{\mathcal{M}} \leqslant \frac{1}{(1-\sqrt{\alpha})^2}\liminf_{\varepsilon\to 0}\ |\ln\varepsilon|^{-2}\int_{U\cap\left(\bigcup_i \tilde{B}_i^\varepsilon\right)} e^\varepsilon(u^\varepsilon)\,dx.$$

Adding this to (6.3), we find that

$$\liminf_{\varepsilon\to 0}\ |\ln\varepsilon|^{-2}\int_U e^\varepsilon(u^\varepsilon)\,dx \geqslant \frac{1}{2}(1-\sqrt{\alpha})^2\|\nabla\times v\|_{\mathcal{M}} + \frac{1}{2}\|v\|_2^2$$

for all $\alpha\in(0,1)$. Letting α tend to zero, we obtain (6.2). ∎

7. UPPER BOUNDS

In this section, we construct sequences of functions to prove that the lower bounds established earlier are essentially sharp. This construction is very similar to a construction given by Sandier and Serfaty [25] for the functional with the applied magnetic field. Here, we present this construction in a way that would be easier to generalize to higher dimensions.

We will prove:

PROPOSITION 7.1. *Suppose that U a bounded domain with a smooth boundary. Fix any $v \in C^\infty(U;\mathbb{R}^2)$. Let $\{d_\varepsilon\}_{\varepsilon\in(0,1]}$ be an increasing sequence such that $d_\varepsilon \to \infty$ and $\varepsilon d_\varepsilon \to 0$ as $\varepsilon \to 0$. Then, there exists a sequence of functions $\{u^\varepsilon\}_{\varepsilon\in(0,1]} \subset H^1(U;\mathbb{C})$ such that*

$$v^\varepsilon := j(u^\varepsilon)/d_\varepsilon \rightharpoonup v \quad \text{in } L^p \qquad \text{for all } p<2, \tag{7.1}$$

$$w^\varepsilon := Ju^\varepsilon/d_\varepsilon \rightharpoonup \nabla \times v/2 =: w \quad \text{in } W^{-1,p} \qquad \text{for all } p<2 \tag{7.2}$$

and

$$\mathbb{E}_\varepsilon(u^\varepsilon) \leqslant \frac{d_\varepsilon^2}{2}\|v\|_2^2 + d_\varepsilon|\ln\varepsilon|\,\|w\|_1 + o(d_\varepsilon^2) \tag{7.3}$$

as $\varepsilon \to 0$.

For the scaling $d_\varepsilon = O(1)$, essentially the same estimate is obtained in [17].

Note that (7.2) follows immediately from (7.1). Also, if $g_\varepsilon \leqslant |\ln\varepsilon|^2$, then, the compactness results and (7.3) imply that $w^\varepsilon \to w$ in C^{0,α^*} for all $\alpha > 0$. However, this does not add much since C^{0,α^*} is in a sense not much stronger than $W^{-1,p}$.

The upper bound in Theorem 1.2 follows from Proposition 7.1 (with $d_\varepsilon = |\ln\varepsilon|$) and an approximation argument. These sorts of approximation arguments are standard in the theory of Gamma convergence; see for example the book of Dal Maso [11]. To obtain a Gamma limit upper bound for the scaling $1 \ll g_\varepsilon \ll |\ln\varepsilon|^2$, one would use Proposition 7.1 with $d_\varepsilon = g_\varepsilon/|\ln\varepsilon|$, and for the scaling $|\ln\varepsilon|^2 \ll g_\varepsilon \ll \varepsilon^{-2}$, Proposition 7.1 with $d_\varepsilon = \sqrt{g_\varepsilon}$.

We can write $L^2(U;\mathbb{R}^2)$ as a direct sum

$$L^2(U;\mathbb{R}^2) := \mathscr{F} \oplus \mathscr{G} \oplus \mathscr{H},$$

where

$$\mathscr{F} := \{v : v = \nabla \times f,\ f \in H^1(U),\ f = 0 \text{ on } \partial U\},$$

$$\mathscr{G} := \{v : v = \nabla g,\ g \in H^1(U)\},$$

$$\mathscr{H} := \{v : \Delta v = 0 \text{ in } U,\ v \cdot \nu = 0 \text{ on } \partial U\}.$$

Note that as a consequence of our assumption on U, $\mathscr{H}$ is a finite-dimensional real vector space; see for instance, the lecture notes by Schwartz [28]. These also prove that if v is smooth, then its projections into $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ are also smooth.

We first consider the case $v \in \mathscr{F}(U)$. Note that for such functions, one can recover v from its curl via the formula

$$v = -\nabla \times \Delta_D^{-1} \nabla \times v,$$

where Δ_D denotes the Laplace operator with zero Dirichlet boundary data on ∂U.

We now prove

LEMMA 7.2. *If* $v \in \mathscr{F}$, *then there exists a sequence* u^ε *satisfying the conclusions of Proposition* 7.1. *In addition*, $v^\varepsilon = j(u^\varepsilon)/d_\varepsilon$ *has the form*

$$v^\varepsilon = (\rho^\varepsilon)^2 \hat{v}^\varepsilon \quad \textit{with } \hat{v}^\varepsilon \in \mathscr{F} \quad \textit{and} \quad \|(\rho^\varepsilon)^2 - 1\|_{L^q(U)} \to 0 \quad \forall 1 \leqslant q < \infty \tag{7.4}$$

as $\varepsilon \to 0$.

Proof 1. *Construction of auxilliary function*: We will use an auxilliary function q^ε that we define as follows.

First, fix a nonnegative smooth, rotationally symmetric function $\eta : \mathbb{R}^2 \to \mathbb{R}$ with support in the unit ball, such that $\int \eta \, dx = 1$. Define $\eta^\varepsilon(x) := \eta(\frac{x}{\varepsilon})/\varepsilon^2$, and note that the symmetry of η implies that

$$\text{if } \Delta H = 0 \text{ in } B_\varepsilon(x), \text{ then } \eta^\varepsilon * H(x) = H(x). \tag{7.5}$$

Define also $v(x) := \nabla \times \ln|x| = \frac{(x_2, -x_1)}{|x^2|}$, so that

$$\nabla \times v = -\Delta \ln|x| = 2\pi\delta_0, \qquad \nabla \cdot v = 0 \quad \text{on } \mathbb{R}^2.$$

In particular v, is harmonic away from the origin. We define q^ε by requiring that

$$q^\varepsilon(x) v(x) = \eta^\varepsilon * v(x) \tag{7.6}$$

for all x. We will need some properties of q^ε, summarized in the following lemma. The proof is deferred to the end of this section.

LEMMA 7.3. q^ε *is well-defined, smooth and radial and has the following properties*:

$$0 \leqslant q^\varepsilon \leqslant 1, \quad q^\varepsilon(x) = 1 \textit{ whenever } |x| \geqslant \varepsilon, \tag{7.7}$$

$$q^\varepsilon(x) = q^1\left(\frac{x}{\varepsilon}\right), \tag{7.8}$$

$$\|q^\varepsilon v\|_\infty \leqslant \frac{C}{\varepsilon}, \qquad \int_{B_\varepsilon} |q^\varepsilon(x)v(x)|^2\,dx \leqslant C \quad \textit{for } C \textit{ independent of } \varepsilon. \tag{7.9}$$

2. *Construction of* u^ε: We start by selecting points $\{a_i^\varepsilon\}_{i=1}^{N_\varepsilon}$ and integers $\sigma_i^\varepsilon = \pm 1$ such that

$$w^\varepsilon := \frac{\pi}{d_\varepsilon} \sum \sigma_i^\varepsilon \delta_{a_i^\varepsilon} \rightharpoonup w(x)\,dx \quad \text{weakly in } \mathscr{M} \text{ and strongly in } W^{-1,p} \quad \forall p<2, \tag{7.10}$$

$$|w^\varepsilon| := \frac{\pi}{d_\varepsilon} \sum \delta_{a_i^\varepsilon} \rightharpoonup |w(x)|\,dx \quad \text{weakly in } \mathscr{M} \text{ and strongly in } W^{-1,p} \quad \forall p<2, \tag{7.11}$$

$$|a_i^\varepsilon - a_j^\varepsilon| \geqslant c_0 d_\varepsilon^{-1/2} \quad \forall i \neq j, \qquad \operatorname{dist}(a_i^\varepsilon, \partial U) \geqslant c_0 d_\varepsilon^{-1/2} \quad \forall i, \tag{7.12}$$

where c_0 is some small constant that depends on $\|w\|_\infty$. We indicate in Lemma 7.5 how this can be done. Define $\hat{v}^\varepsilon \in W^{1,p}(U;\mathbb{R}^2)$ by

$$\hat{v}^\varepsilon = -2\nabla \times \Delta_D^{-1} w^\varepsilon.$$

Next define a function $\hat{u}^\varepsilon : U \to S^1$ satisfying

$$j(\hat{u}^\varepsilon)/d_\varepsilon = \hat{v}^\varepsilon, \tag{7.13}$$

and therefore

$$J\hat{u}^\varepsilon/d_\varepsilon = w^\varepsilon.$$

This is done by defining $\hat{u}^\varepsilon := e^{i\phi^\varepsilon}$, where ϕ^ε is a multivalued function satisfying $\nabla\phi^\varepsilon = d_\varepsilon \hat{v}^\varepsilon$. To fix an otherwise free constant we can select some point $x_0 \in U$ and specify that $\phi^\varepsilon(x_0) = 0$. The definition of $\hat{v}^\varepsilon$ implies that ϕ^ε is a well-defined modulo 2π, and thus that $\hat{u}^\varepsilon$ is well defined. The definitions also imply that $j(\hat{u}^\varepsilon) = \nabla\phi^\varepsilon$ and thus that (7.13) holds.

We finally define

$$u^\varepsilon := \rho^\varepsilon \hat{u}^\varepsilon, \qquad \rho^\varepsilon(x) := \prod q^\varepsilon(x - a_i^\varepsilon), \tag{7.14}$$

where q^ε is defined in (7.6). Thus, $v^\varepsilon := j(u^\varepsilon)/d_\varepsilon = (\rho^\varepsilon)^2\hat{v}^\varepsilon$. The definition of ρ^ε easily implies that

$$\|(\rho^\varepsilon)^p - 1\|_q \to 0 \qquad \text{as } \varepsilon \to 0 \text{ for all } i \leqslant p,\ q < \infty. \tag{7.15}$$

In particular, (7.4) holds.

3. *Convergence of* $j(u^\varepsilon)$ *and* Ju^ε: We use the notation $(\nabla\times)^{-1}\psi := -\nabla \times \Delta_D^{-1}\psi$. Then for every $p<2$

$$\begin{aligned}\|\hat{v}^\varepsilon - v\|_p &= 2\|(\nabla\times)^{-1}(w^\varepsilon - w)\|_p\\ &\leqslant C\|w^\varepsilon - w\|_{W^{-1,p}} \to 0 \qquad \text{as } \varepsilon \to 0\end{aligned}$$

by standard elliptic theory. Given $p<2$ fix some $\tilde{p} \in (p,2)$ and define $\tilde{q}$ by $\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = \frac{1}{p}$. Then,

$$\|\rho^\varepsilon\hat{v}^\varepsilon - \hat{v}^\varepsilon\|_p \leqslant \|\hat{v}^\varepsilon\|_{\tilde{p}}\|\rho^\varepsilon - 1\|_{\tilde{q}},$$

and (7.16) and (7.15) imply that the right-hand side tends to zero, and thus that

$$\|\rho^\varepsilon\hat{v}^\varepsilon - v\|_p \to 0$$

as $\varepsilon \to 0$, for all $p<2$. Exactly the same argument shows that

$$\|(\rho^\varepsilon)^2\hat{v}^\varepsilon - v\|_p = \left\|\frac{1}{d_\varepsilon}j(u^\varepsilon) - v\right\|_p \to 0 \qquad \text{as } \varepsilon \to 0$$

for every $p<2$. This is exactly (7.1).

4. *Decomposition of* $\mathbb{E}_\varepsilon(u^\varepsilon)$: Note that

$$|\nabla u^\varepsilon|^2 = |\nabla\rho^\varepsilon|^2 + (\rho^\varepsilon)^2|\nabla\hat{u}^\varepsilon|^2 = |\nabla\rho^\varepsilon|^2 + d_\varepsilon^2(\rho^\varepsilon)|\hat{v}^\varepsilon|^2.$$

One easily verifies that $|\nabla\rho^\varepsilon|^2 \leqslant \varepsilon^{-2}\chi_{\bigcup_j B_j^\varepsilon}$. It is similarly clear that $\varepsilon^{-2}W(|u^\varepsilon|^2) \leqslant C\varepsilon^{-2}\chi_{\bigcup_j B_j^\varepsilon}$. From (7.11), we see that

$$\frac{\pi}{d_\varepsilon}N_\varepsilon = \|w^\varepsilon\|_{\mathscr{M}} \to \|w\|_{\mathscr{M}}. \tag{7.16}$$

From these, we deduce that

$$\mathbb{E}_\varepsilon(u^\varepsilon) \leqslant CN_\varepsilon + \frac{d_\varepsilon^2}{2}\int_U |\rho^\varepsilon\hat{v}^\varepsilon|^2\,dx \leqslant Cd_\varepsilon\|w\|_{\mathscr{M}} + \frac{d_\varepsilon^2}{2}\int_U |\rho^\varepsilon\hat{v}^\varepsilon|^2\,dx.$$

And

$$\frac{1}{2}\int_U |\rho^\varepsilon \hat{v}^\varepsilon|^2 = \frac{1}{2}\int_U |v|^2\,dx + \int_U v\cdot(\rho^\varepsilon\hat{v}^\varepsilon - v)\,dx + \frac{1}{2}\int_U |\rho^\varepsilon\hat{v}^\varepsilon - v|^2\,dx$$
$$= \frac{1}{2}\int_U |v|^2\,dx + o(1) + \frac{1}{2}\int_U |\rho^\varepsilon\hat{v}^\varepsilon - v|^2\,dx.$$

So to complete the estimate of $\mathbb{E}_\varepsilon(u^\varepsilon)$, it suffices to show that

$$\|\rho^\varepsilon\hat{v}^\varepsilon - v\|_2^2 \leqslant \frac{|\ln\varepsilon|}{d_\varepsilon}\|\nabla\times v\|_{\mathcal{M}} + o(1).$$

It is clear that when $d_\varepsilon \gg |\ln\varepsilon|$, the right-hand side of the above estimate can be simply replaced by $o(1)$.

We define

$$\delta = \delta(\varepsilon) = c_0 d_\varepsilon^{-1/2}/3,$$

where c_0 is the constant in (7.12). For $r \leqslant \delta$, this choice of δ implies that $\cup B_r(a_i^\varepsilon)$ is a distance at least $2r$ from ∂U. Due to (7.5) we see that $\eta^r * v^\varepsilon = v^\varepsilon$ away from $\cup B_r(a_i^\varepsilon)$, and in particular in the set $\{x \in U : r \leqslant \mathrm{dist}(x, \partial U) < 2r\}$. Motivated by this, we use the convention that $\eta^r * v^\varepsilon(x) = v^\varepsilon(x)$ for $x \in \{x \in U : \mathrm{dist}(x, \partial U) \leqslant r\}$. If $r \leqslant \delta$ this makes $\eta^r * v^\varepsilon$ well defined and smooth in all of U, and indeed harmonic away from $\cup B_r(a_i^\varepsilon)$.

Using the triangle inequality,

$$\tfrac{1}{3}\|\rho^\varepsilon\hat{v}^\varepsilon - v\|_2^2 \leqslant \|\rho^\varepsilon\hat{v}^\varepsilon - \eta^\varepsilon * \hat{v}^\varepsilon\|_2^2 + \|\eta^\varepsilon * \hat{v}^\varepsilon - \eta^\delta * \hat{v}^\varepsilon\|_2^2 + \|\eta^\delta * \hat{v}^\varepsilon - v\|_2^2$$
$$= A^\varepsilon + B^\varepsilon + C^\varepsilon.$$

5. *Estimate of* C^ε: We first show that $C^\varepsilon \to 0$ as $\varepsilon \to 0$. Since (7.16) easily implies that $\eta^\delta * \hat{v}^\varepsilon \to v$ in $L^p(U)$ for all $p < 2$, it suffices to show that $\{\eta^\delta * \hat{v}^\varepsilon - v\}_{\varepsilon\in(0,1]}$ (or more simply $\{\eta^\delta * \hat{v}^\varepsilon\}_{\varepsilon\in(0,1]}$), is precompact in $L^2(U)$. We do this as follows:

From the definition of w^ε, in particular (7.12), and from the choice of δ one can see that at any $x \in U$, there is at most one point a_ε^i in $B_\delta(x)$, and so

$$\eta^\delta * w^\varepsilon(x) = \begin{cases} \pi\eta^\delta(x - a_i^\varepsilon)/d_\varepsilon & \text{if } \exists a_i^\varepsilon \in B_\delta(x), \\ 0 & \text{if not.} \end{cases}$$

(We are using essentially the same convention as above for extending the convolution near the boundary.) In particular, since $|\eta^\delta| - \eta^\delta \leqslant C/\delta^2 \leqslant Cd_\varepsilon$, this implies that $|\eta^\delta * w^\varepsilon| \leqslant C$ in U for C independent of ε. Interior regularity estimates then imply that

$$\|\Delta_D^{-1}\eta^\delta * w^\varepsilon\|_{W^{2,p}(U)} \leqslant C$$

for every $p < \infty$. So

$$\|\eta^\delta * \hat{v}^\varepsilon\|_{W^{1,p}(U)} = \|\nabla \times \Delta_D^{-1}\eta^\delta * w^\varepsilon\|_{W^{1,p}(U)} \leqslant C \tag{7.17}$$

for every $p < \infty$. This gives more than enough compactness to conclude that $C^\varepsilon \to 0$ as $\varepsilon \to 0$.

6. *Estimate of* B^ε: Recall that for any $r > 0, \eta^r * \hat{v}^\varepsilon(x) = \hat{v}^\varepsilon(x)$ unless $B_r(x) \cap \{a_i^\varepsilon\}_{i=1}^{N_\varepsilon}$ is nonempty. Thus $\eta^\varepsilon * \hat{v}^\varepsilon - \eta^\delta * \hat{v}^\varepsilon$ is supported in $\cup B_\delta(a_i^\varepsilon)$. Consider one such ball, say with center $a_{i_0}^\varepsilon$, which for simplicity we take to be the origin. Then in a neighborhood of the origin, we can write $\hat{v}^\varepsilon$ in the form $\hat{v}^\varepsilon = (v/d_\varepsilon) + H$ where H is a harmonic and as above $v(x) = (x_2, -x_1)/|x|^2$. This follows from the definitions of $\hat{v}^\varepsilon$ and v, which imply that $\hat{v}^\varepsilon - (v/d_\varepsilon)$ is harmonic away from $\{a_i^\varepsilon\}_{i=1,\dots,N_{\varepsilon,i\neq i_0}}$. In particular, this neighborhood contains the ball of radius 3δ, by our choice of δ. Thus,

$$(\eta^\varepsilon - \eta^\delta) * \hat{v}^\varepsilon = (\eta^\varepsilon - \eta^\delta) * \frac{v}{d_\varepsilon} + (\eta^\varepsilon - \eta^\delta) * H = (q^\varepsilon - q^\delta)\frac{v}{d_\varepsilon}$$

using the definition of q and (7.5). Since i_0 was arbitrary,

$$B^\varepsilon = \frac{1}{d_\varepsilon^2} N^\varepsilon \int_{B_\delta(0)} (q^\varepsilon - q^\delta)^2 |v|^2 \, dx = \frac{1}{d_\varepsilon^2} N^\varepsilon \int_{B_\delta(0)} (q^\varepsilon - q^\delta)^2 |x|^{-2} \, dx.$$

Lemma 7.3 implies that $0 \leqslant q^\varepsilon - q^\delta \leqslant 1$ when $\varepsilon \leqslant |x| \leqslant \delta$, and with (7.9), (7.16) this gives the estimate

$$B^\varepsilon \leqslant \pi \frac{N^\varepsilon}{d_\varepsilon^2}\left(\ln\left(\frac{\delta}{\varepsilon}\right) + C\right) \leqslant \frac{\|w^\varepsilon\|}{d_\varepsilon}(|\ln \varepsilon| + C).$$

7. *Estimate of* A^ε: Finally, note that $\rho^\varepsilon \hat{v}^\varepsilon - \eta^\varepsilon * \hat{v}^\varepsilon$ is supported in $\cup B_\varepsilon(a_i^\varepsilon)$. As above we fix some i_0, and we assume for simplicity that $a_{i_0}^\varepsilon$ is the origin. In this ball, we write as before $H = \hat{v}^\varepsilon - (v/d_\varepsilon)$, so that H is harmonic. In this ball $\rho^\varepsilon(x) = q^\varepsilon(x)$, so

$$\begin{aligned}\rho^\varepsilon \hat{v}^\varepsilon - \eta^\varepsilon * \hat{v}^\varepsilon &= q^\varepsilon\left(\frac{v}{d_\varepsilon} + H\right) - \eta^\varepsilon * \left(\frac{v}{d_\varepsilon} + H\right) \\ &= (\rho^\varepsilon - 1)H = (\rho^\varepsilon - 1)\eta^\delta * H.\end{aligned}$$

However, $\eta^\delta * H = \eta^\delta * \hat{v}^\varepsilon - \eta^\delta * (v/d_\varepsilon)$, so (7.9) implies that

$$|\rho^\varepsilon \hat{v}^\varepsilon - \eta^\varepsilon * \hat{v}^\varepsilon| \leqslant (1-\rho^\varepsilon)(|\eta^\delta * \hat{v}^\varepsilon| + C(\delta d_\varepsilon)^{-1}) \\ \leqslant (1-\rho^\varepsilon)(|\eta^\delta * \hat{v}^\varepsilon| + C d_\varepsilon^{-1/2}).$$

As a result,

$$A^\varepsilon \leqslant C \int_U (1-\rho^\varepsilon)^2 \left(|\eta^\delta * \hat{v}^\varepsilon|^2 + \frac{C}{d_\varepsilon} \right) dx.$$

Applying Hölder's inequality and using (7.15) and (7.17), we infer that $A^\varepsilon \to 0$ as $\varepsilon \to 0$. ∎

We next prove

LEMMA 7.4. *If $v \in C^\infty \cap (\mathcal{G} \oplus \mathcal{H})$, then there exists functions $\{u^\varepsilon\}_{\varepsilon\in(0,1]} \subset H^1(U; S^1)$ such that*

$$j(u^\varepsilon)/d_\varepsilon =: v^\varepsilon \to v \quad \text{in } C^k(U) \quad \text{for all } k, \qquad \mathbb{E}_\varepsilon(u^\varepsilon)/d_\varepsilon^2 \to \|v\|_2^2/2.$$

Also, $v^\varepsilon \in \mathcal{G} \oplus \mathcal{H}$ for every ε.

We remark that the assumptions imply that $\nabla \times v = 0$; this is why $\|\nabla \times v\|_1$ does not appear in the upper bound.

Proof. The functions we will construct satisfy $|u^\varepsilon| = 1$ a.e. As a result $|\nabla u^\varepsilon|^2 = |j(u^\varepsilon)|^2$ a.e., and so the stated convergence of $d_\varepsilon^{-2} E^\varepsilon(u^\varepsilon)$ will follow immediately once we establish the convergence of $j(u^\varepsilon)/d_\varepsilon$.

Recall that U has the form $G \backslash (\bigcup_{i=1}^m P_i)$, where $G, P_1, \ldots, P_m$ are open, connected and simply connected, and the P_i are pairwise disjoint subsets compactly contained in G. We assert that there exist functions H_i, $i = 1, \ldots, m$ in $\mathcal{H}$ characterized by

$$\int_{\partial P_j} H_i \cdot \tau = 2\pi \delta_{ij},$$

and moreover every function $H \in \mathcal{H}$ has the representation

$$H = \sum_{i=1}^m H_i \left(\frac{1}{2\pi} \int_{\partial P_i} H \cdot \tau \right).$$

The constant 2π is a convenient normalization. These claims follow from the Hodge theorem, see again Schwarz [28]. This can also be established by elementary arguments in this setting. An efficient way to do this is to use the fact that every function in $\mathscr{H}$ can be written as the curl of a scalar function ψ. A quick proof of this fact is sketched in [4, Lemma 1.1].

So if $v \in \mathscr{G} \oplus \mathscr{H}$, then it can be written in the form

$$v = \nabla g + c_i H_i$$

for certain constants c^i.

For each H_k, we define a functions $v_k : U \to S^1$ such that $j(v_k) = H_k$. To do this we define $v_k := e^{i\phi_k}$, where ϕ_k is a multivalued function satisfying $\nabla \phi_k = H_k$. As above, we fix an otherwise free constant by selecting some point $x_0 \in U$ and specifying that $\phi_k(x_0) = 0$. The definition of H_k implies that ϕ_k is well-defined modulo 2π, and thus that v_k is well defined. The definitions also imply that $j(v_k) = \nabla \phi_k = H_k$ as desired.

If we now define a function v by

$$u^\varepsilon = e^{id_\varepsilon g} \prod_{k=1}^{m} v_i^{p_i^\varepsilon}$$

for integers $p_1^\varepsilon, \ldots, p_m^\varepsilon$, then one checks that

$$v^\varepsilon := \frac{j(u^\varepsilon)}{d_\varepsilon} = \nabla g + \sum \frac{p_i^\varepsilon}{d_\varepsilon} H_i.$$

Taking p_i^ε such that $(d_\varepsilon / p_i^\varepsilon) \to c_i$, we immediately find that $v^\varepsilon \to v$ in $C^k(U)$ for all k. It is also clear that $v^\varepsilon \in \mathscr{G} \oplus \mathscr{H}$ for every ε. ∎

At the end of the section, we will prove the auxiliary lemmas used above. We first give the

Proof of Proposition 7.1. Suppose $v \in C^\infty(U; \mathbb{R}^2)$, and write $v = v_1 + v_2$, where $v_1 \in \mathscr{F}$ and $v_2 \in \mathscr{G} \oplus \mathscr{H}$. Let $\{u_i^\varepsilon\}_{\varepsilon \in (0,1]}$ and $\{u_2^\varepsilon\}_{\varepsilon \in (0,1]}$ be sequences satisfying the conclusions of Lemmas 7.2 and 7.4, respectively. Define u^ε to be the product, $u_1^\varepsilon u_2^\varepsilon$. We verify that $\{u^\varepsilon\}$ satisfies the conclusions of Proposition 7.1.

First, $v^\varepsilon = j(u^\varepsilon)/d_\varepsilon = |u_1^\varepsilon|^2 v_2^\varepsilon + |u_2^\varepsilon|^2 v_1^\varepsilon = v_1^\varepsilon + v_2^\varepsilon + (|u_1^\varepsilon|^2 - 1) v_2^\varepsilon$. From (7.15) we know that $\| |u_1^\varepsilon|^2 - 1 \|_q \to 0$ in l^q for all $q < \infty$, and this implies that (7.1) holds.

It follows that $w^\varepsilon = \nabla \times v^\varepsilon / 2 \to w$ in $W^{-1,p}$ for all $p < 2$.

Finally, to prove (7.3), we use the fact that $\|u_2^\varepsilon\| \equiv 1$ to compute

$$|Du^\varepsilon|^2 = |Du_1^\varepsilon|^2 + |u_1^\varepsilon|^2 |Du_2^\varepsilon|^2 + j(u_1^\varepsilon) \cdot j(u_2^\varepsilon) \leqslant |Du_1^\varepsilon|^2 + |Du_2^\varepsilon|^2 + j(u_1^\varepsilon) \cdot j(u_2^\varepsilon).$$

Again using the fact that $|u^\varepsilon| \equiv 1$, we infer $\mathbb{E}_\varepsilon(u^\varepsilon) \leqslant E_\varepsilon(u_1^\varepsilon) + E_\varepsilon(u_2^\varepsilon) + \int_U j(u_1^\varepsilon) \cdot j(u_2^\varepsilon)$. In view of Lemmas 7.2 and 7.4 it suffices to show that

$$\int_U v_1^\varepsilon \cdot v_2^\varepsilon \, dx \to 0 \qquad \text{as } \varepsilon \to 0.$$

To do this, we use (7.4) to write v_1^ε in the form $(\rho^\varepsilon)^2 \hat{v}_1^\varepsilon$, where $(\rho^\varepsilon)^2 - 1 \to 0$ in L^q for all $q < \infty$, and $\hat{v}_1^\varepsilon \in \mathscr{F}$. Then,

$$\int_U v_1^\varepsilon \cdot v_2^\varepsilon \, dx = \int_U \hat{v}_1^\varepsilon \cdot v_2^\varepsilon \, dx + \int_U ((p^c p)^2 - 1) v_1^\varepsilon \cdot v_2^\varepsilon \, dx.$$

But $\hat{v}_1^\varepsilon$ and v_2^ε are orthogonal in L^2, and since $\|v_2^\varepsilon\|_\infty \leqslant C$, $\|v_1^\varepsilon\|_p \leqslant C_p$ for all $p < 2$, we easily conclude that the right-hand side tends to zero as $\varepsilon \to 0$. ∎

LEMMA 7.5. *There exists families* $\{a_i^\varepsilon\}_{i=1}^{N_\varepsilon}$ *of points, satisfying* (7.10)–(7.12).

Proof. Write $U = \cup U_i^\varepsilon$, where for each i, U_i^ε is a set of the form $U \cap Q_i^\varepsilon$, and Q_i^ε is a cube of side length $d_\varepsilon^{-1/4}$. For each i, let $N_i^\varepsilon = \lfloor d_\varepsilon \int_{U_i^\varepsilon} |\omega| \, dx \rfloor$ if $\operatorname{dist}(U_i^\varepsilon, \partial U) > 0$, and let $N_i^\varepsilon = 0$ otherwise. Also let $\sigma_i^\varepsilon = \operatorname{sgn}(\int_{U_i^\varepsilon} \omega \, dx)$. In each U_i^ε select N_i^ε points $\{a_{ij}^\varepsilon\}_{j=1}^{N_i^\varepsilon}$ that are roughly equally distributed. Note that $N_i^\varepsilon \leqslant \|\omega\|_\infty \, d_\varepsilon^{1/2}$ for all i. This implies that the points can be chosen so that the distances are bounded below as in (7.12). Finally, define

$$w^\varepsilon := \sum_i \sum_{j=1}^{N_i^\varepsilon} \sigma_i^\varepsilon \delta_{a_{ij}^\varepsilon}.$$

Upon relabelling, this collection of points has the same form as in (7.10)–(7.12).

It is easy to see that this sequence of measures has uniformly bounded mass, so weak convergence in $\mathscr{M}$ will follow from strong convergence in $W^{-1,p}$, $p < 2$. For the latter, since functions in $W^{1,q}$, $q > 2$ are Hölder continuous, it suffices to verify that for every $\alpha > 0$,

$$\sup_{\|\phi\|_{C^{0,\alpha}} \leqslant 1} \left| \int_U \phi \, dw^\varepsilon - \int \phi(x) w(x) \, dx \right| \to 0$$

as $\varepsilon \to 0$. To verify this, note that if $\operatorname{dist}(U_i^\varepsilon, \partial U) > 0$ and $\|\phi\|_{C^{0,\alpha}} \leqslant 1$, then

$$\left| \int_{U_i^\varepsilon} \phi \, dw^\varepsilon - \int \phi(x) w(x) \, dx \right| \leqslant C d_\varepsilon^{-(1/2)-(\alpha/4)}.$$

Also, the number of sets U_i^ε satisfying such that $\mathrm{dist}(U_i^\varepsilon, \partial U) > 0$ is bounded by $Cd_\varepsilon^{1/2}$. A different but equally straightforward argument is needed to show that the error at the boundary vanishes in the limit. ∎

We complete the above proof by verifying the properties of the function q^ε stated in Step 1.

Proof of Lemma 7.3. Since η is rotationally symmetric

$$\frac{|x|^2}{x_1}\int \eta(x-y)\,\frac{y_1}{|y|^2}\,dy = \frac{|x|^2}{x_2}\int \eta(x-y)\,\frac{y_2}{|y|^2}\,dy.$$

Set $q^1(x)$ to be the above expression, and $q^\varepsilon(x) = q^1(x/\varepsilon)$. Then, (7.6) holds by definition. To prove the other properties, we first observe that for any $r > 0$ and $x \in \mathbb{R}^2$,

$$\int_{\partial B_r} \frac{x-y}{|x-y|^2}\,dy = \begin{cases} |\partial B_r|\frac{x}{|x|^2} & \text{if } |x| < r, \\ 0 & \text{if } |x| > r. \end{cases}$$

Therefore,

$$\begin{aligned} \eta * v(x) &= \int_{B_1} \eta(y)\,\frac{x-y}{|x-y|^2}\,dy \\ &= \int_0^1 \eta(r) \int_{\partial B_r} \eta(y)\,\frac{x-y}{|x-y|^2}\,dy\,dr \\ &= \frac{x}{|x|^2}\int_0^{1\wedge|x|} \eta(r)|\partial B_r|\,dr. \end{aligned}$$

Since

$$\int_0^1 \eta(r)|\partial B_r|\,dr = 1,$$

(7.7) follows. To obtain (7.9), observe that

$$\begin{aligned}
|q^\varepsilon(x)v(x)| &= |\eta^\varepsilon * v(x)| \\
&= \left| \int_{B_\varepsilon} \frac{1}{\varepsilon^2}\eta(y/\varepsilon)\frac{x-y}{|x-y|^2}\,dy \right| \\
&= \frac{1}{|x|}\int_0^{\varepsilon\wedge|x|} \frac{1}{\varepsilon^2}\eta(r/\varepsilon)|\partial B_r|\,dr \\
&= \frac{1}{|x|}\int_0^{1\wedge|x|/\varepsilon} \eta(\rho)|\partial B_\rho|\,d\rho \\
&\leqslant \frac{C}{\varepsilon}.
\end{aligned}$$

The L^2 inequality follows easily from the above and the fact that $|B_\varepsilon| = \pi\varepsilon^2$. ∎

8. GINZBURG–LANDAU FUNCTIONAL FOR SUPERCONDUCTIVITY

In this section, we examine the asymptotic behavior of the functional

$$\mathbb{F}_\varepsilon(u, A; h_{\text{ext}}) := \frac{1}{2}\int_U |\nabla_A u|^2 + |\nabla \times A - h_{\text{ext}}|^2 + \frac{(|u|^2-1)^2}{4\varepsilon^2}\,dx,$$

where the order parameter u is $\mathbb{C}$-valued, the magnetic potential A is $\mathbb{R}^2$-valued, and

$$\nabla_A u := \nabla u - iAu.$$

The applied magnetic field h_{ext} is assumed to be a constant that may depend on ε.

We will use the results of the previous sections to prove Theorem 1.3.

We recall that $\mathbb{F}_\varepsilon$ has a gauge invariance

$$\mathbb{F}_\varepsilon(u, A; h_{\text{ext}}) = \mathbb{F}_\varepsilon(ue^{i\chi}, A + \nabla\chi; h_{\text{ext}}),$$

for any smooth χ; see [22, 29]. Due to this invariance, and because the statement of Theorem 1.3 is gauge-invariant, it suffices to prove the theorem for a fixed gauge. We find it convenient to work with the Coloumb gauge: by an appropriate choice of χ and by relabelling $(ue^{i\chi}, A - \nabla\chi) \mapsto (u, A)$ we can arrange that

$$\nabla \cdot A = 0 \quad \text{in } U, \qquad A \cdot \nu = 0 \quad \text{on } \partial U, \qquad \int_U A\,dx = 0.$$

We will henceforth assume this to be the case. Our assumptions in Theorem 1.3 imply that $h_{\text{ext}} = H_\varepsilon |\ln \varepsilon|$, where H_ε converges to a finite limit H as $\varepsilon \to 0$. We assume for simplicity that $H_\varepsilon \equiv H$; this simplifies the notation a bit and otherwise does not affect the proof. Our method easily extends to cover h_{ext} of the form $p(x)|\ln \varepsilon|$, where $p(x)$ is some nonnegative square integrable function. We could also consider other scalings.

Proof. (1) First, let $(u^\varepsilon, A^\varepsilon)$ be a sequence such that

$$\mathbb{F}_\varepsilon(u^\varepsilon, A^\varepsilon; h_{\text{ext}}) \leqslant K |\ln \varepsilon|^2. \tag{8.1}$$

To establish compactness, note that

$$\|\nabla \times a^\varepsilon - H\|_2^2 = [\ln \varepsilon]^{-2} \|\nabla \times A^\varepsilon - h_{\text{ext}}\|^2 \leqslant [\ln \varepsilon]^{-2} \mathbb{F}_\varepsilon(u^\varepsilon, A^\varepsilon; h_{\text{ext}}) \leqslant K.$$

Also by the choice of the Coulomb gauge, $\nabla \cdot a^\varepsilon \equiv 0$. These imply that $\{a^\varepsilon\}$ is uniformly bounded in H^1, and we immediately get weak compactness in H^1.

We obtain compactness for u^ε by Theorem 1.1. For this, we need to verify (1.1). By (1.3)

$$\mathbb{E}_\varepsilon(u^\varepsilon) \leqslant \mathbb{F}_\varepsilon(u^\varepsilon, A^\varepsilon; h_{\text{ext}}) - (j(u^\varepsilon), A^\varepsilon)_2. \tag{8.2}$$

We estimate the unsigned term $(j(u^\varepsilon), A^\varepsilon)_2 = \int A^\varepsilon \cdot j(u^\varepsilon)\, dx$ by noting that

$$\begin{aligned} |A^\varepsilon \cdot j(u^\varepsilon)| &\leqslant \frac{1}{4} \frac{|j(u^\varepsilon)|^2}{|u^\varepsilon|^2} + |u^\varepsilon|^2 |A^\varepsilon|^2 \\ &\leqslant \frac{1}{4} |\nabla u^\varepsilon|^2 + (|u^\varepsilon|^2 - 1)|A^\varepsilon|^2 + |A^\varepsilon|^2 \\ &\leqslant \frac{1}{4} |\nabla u^\varepsilon|^2 + \frac{1}{8\varepsilon^2} (|u^\varepsilon|^2 - 1)^2 + 2\varepsilon^2 |A^\varepsilon|^4 + |A^\varepsilon|^2. \end{aligned}$$

Thus,

$$|(j(u^\varepsilon), A^\varepsilon)_2| \leqslant \tfrac{1}{2} \mathbb{E}_\varepsilon(u^\varepsilon) + 2\varepsilon^2 \|A^\varepsilon\|_4^4 + \|A^\varepsilon\|_2^2.$$

This together with (8.1) and (8.2) imply

$$\mathbb{E}_\varepsilon(u^\varepsilon) \leqslant C[(\ln \varepsilon)^2 + \|A^\varepsilon\|_4^4 + \|A^\varepsilon\|_2^2]. \tag{8.3}$$

But for any $p < \infty$, the Sobolev embedding theorem informs us that

$$\|A^\varepsilon\|_p = C_p \|A^\varepsilon\|_{H^1} \leqslant C |\ln \varepsilon|.$$

With (8.3) this implies the energy upper bound (1.1). The remaining compactness assertions for v^ε then follow from Theorem 1.1.

(2) Now suppose that $(v^\varepsilon, a^\varepsilon)$ are functions such that converge to (v, a) along the full sequence $\varepsilon \to 0$. We write the functional as a sum of terms,

$$\mathbb{F}_\varepsilon(u^\varepsilon, A^\varepsilon; h_{\text{ext}}) := \mathbb{F}_\varepsilon^1(u^\varepsilon, A^\varepsilon) + \mathbb{F}_\varepsilon^2(u^\varepsilon, A^\varepsilon) + \mathbb{F}_\varepsilon^3(u^\varepsilon, A^\varepsilon) + \mathbb{F}_\varepsilon^4(u^\varepsilon, A^\varepsilon),$$

where

$$\mathbb{F}_\varepsilon^1(u^\varepsilon, A^\varepsilon) := \mathbb{E}_\varepsilon(u^\varepsilon),$$

$$\mathbb{F}_\varepsilon^2(u^\varepsilon, A^\varepsilon) := \frac{(\ln \varepsilon)^2}{2}[\|\nabla \times a^\varepsilon - H\|_2^2 + \|a^\varepsilon\|_2^2],$$

$$\mathbb{F}_\varepsilon^3(u^\varepsilon, A^\varepsilon) := \frac{(\ln \varepsilon)^2}{2}\int_U (|u^\varepsilon|^2 - 1)|a^\varepsilon|^2\, dx,$$

$$\mathbb{F}_\varepsilon^4(u^\varepsilon, A^\varepsilon) := -(\ln \varepsilon)^2 \int_U a^\varepsilon \cdot v^\varepsilon\, dx.$$

It is an immediate consequence of our earlier results that

$$\liminf_{\varepsilon\to 0}\ |\ln \varepsilon|^{-2}\mathbb{F}_\varepsilon^{-1}(u^\varepsilon, A^\varepsilon) \geqslant \tfrac{1}{2}[\|v\|_2^2 + \|\nabla \times v\|_{\mathscr{M}}].$$

Also, the H^1 weak convergence of a^ε implies that

$$\liminf_{\varepsilon\to 0}\ |\ln \varepsilon|^{-2}\mathbb{F}_\varepsilon^2(u^\varepsilon, A^\varepsilon) \geqslant \tfrac{1}{2}[\|\nabla \times a - H\|_2^2 + |a\|_2^2].$$

The third term is estimated (similar to Step 2) by noting that

$$|\mathbb{F}_\varepsilon^3(u^\varepsilon, A^\varepsilon)| \leqslant \|\, |u^\varepsilon|^2 - 1\|_2 \|A^\varepsilon\|_4^2 \leqslant C\|\, |u^\varepsilon|^2 - 1\|_2 \|A^\varepsilon\|_{H^1}^2 \leqslant C\varepsilon |\ln \varepsilon|^3.$$

Finally, since a^ε converges to a weakly in H^1, it converges strongly in L^p for all $p < \infty$. The weak L^q convergence of $v^\varepsilon, q < 2$, is good enough to guarantee that

$$\liminf_{\varepsilon\to 0}\ |\ln \varepsilon|^{-2}\mathbb{F}_\varepsilon^4(u^\varepsilon, A^\varepsilon) = -(a, v)_2,$$

thus proving the Gamma limit lower bound

$$\liminf \frac{\mathbb{F}_\varepsilon(u^\varepsilon, A^\varepsilon; h_{\text{ext}})}{(\ln \varepsilon)^2} \geqslant \mathbb{F}(v, a; H). \tag{8.4}$$

(3) Finally, the upper bound is a very easy consequence of our earlier results. Given (v, a) as stated, we define $A^\varepsilon := |\ln \varepsilon| a$, and we let u^ε be the sequence constructed in the proof of Proposition 7.1 with $d_\varepsilon = |\ln \varepsilon|$. One then can easily check that in (8.4) equality holds for $(u^\varepsilon, A^\varepsilon)$. ∎

9. MEAN FIELD EQUATIONS AND THE FIRST CRITICAL THRESHOLD

In this section, we derive a variational inequality as the Euler–Lagrange equations for the functional $\mathbb{F}$, and obtain a formula for H_{c_1} as a corollary of it. These derivations were given in [25]. We provide the short computation for completeness.

The Euler–Lagrange equations of $\mathbb{F}$ is a variational inequality, (9.1) below, and they are interpreted as the mean field equations of superconductivity. Indeed, the variational problem (9.1) is derived by Serfaty and Sandier [25] as the equation satisfied by the limit of the minimizers of $\mathbb{F}_\varepsilon$. Equation (9.1) is related to the London-type evolution equations for superconductivity. A formal derivation of the time-dependent mean field equations is given in [9].

Since the functional

$$\mathbb{F}(v, a; H) := \tfrac{1}{2}[\|v - a\|_2^2 + \|\nabla \times v\|_{\mathcal{M}} + \|\nabla \times a - H\|_2^2]$$

is convex, the existence of a minimizer $v^* \in L^2$, $a^* \in H^1$ with $\nabla \times v^*$ a Radon measure is straightforward. Indeed, using the Coloumb gauge $\nabla \cdot a = 0$, a minimizer is easily constructed by lower semicontinuity arguments.

Next theorem gives a characterization of the minimizers as solutions of a variational inequality.

THEOREM 9.1 (Sandier–Serfaty [25]). *Let a^*, v^* be a minimizer of $\mathbb{F}$. Then, $z^* = [\nabla \times a^* - H]$ is the unique minimizer of the functional*

$$\inf_{\mathcal{K}} \mathbb{D}(z; H), \qquad \mathbb{D}(z; H) := \tfrac{1}{2}[\|\nabla z\|_2^2 + \|z\|_2^2| + (z, H)_2, \tag{9.1}$$

where

$$\mathcal{K} := \{z \in H_0^1 : z \geqslant -1/2 \ a.e.\},$$

and v^* is computed by the equation

$$\nabla \times [\nabla \times a^* - H] + a^* = v^* \quad in\ U, \qquad \nabla \times a^* - H = 0 \quad on\ \partial U. \tag{9.2}$$

Moreover, $-\frac{1}{2} \leqslant z^* \leqslant 0$,

$$\mu^* := \nabla \times v^* \geqslant 0 \quad and \quad \text{support } \mu^* \subset \{z^* = -1/2\}.$$

This proof is a combination of Lemmas III.3 and III.4 in [25]. Also, see a recent paper of Brezis and Serfaty [6] and a paper by Brezis [5] for the use of convex duality in this context.

Proof. A direct variation in a yields (9.2).

1. To obtain a second equation, we need to do variations in v. For this, we find it convenient to vary the curl of v, instead of varying v. Indeed, for any Radon measure $\mu \in H^{-1}$, the vector field $v_\mu := -\nabla \times \Delta_D^{-1}\mu$ is in L^2 and $\nabla \times v_\mu = \mu$.

We write $\mu^{ac} + \mu^{sing} = \mu$ for, respectively, the absolutely continuous and the singular parts of μ with respect to μ^*, and for any Radon measure $\mathrm{sign}(\mu)$ gives Hahn decomposition of μ into its positive and negative parts by

$$\mathrm{sign}(\mu) = \pm 1, \quad \mu - \text{a.e.}, \qquad \mathrm{sign}(\mu)\, d\mu = d\|\mu\|.$$

See [13] for an introduction.

Recall that $\mu^* := \nabla \times v^*$, and for any Radon measure μ, set $f(t;\mu) = \mathbb{F}(v^* + t_{v_\mu}, a^*; H)$.

2. Using (9.2), definitions of z^*, v_μ, and integration by parts we see that $(v^* - a^*, v_\mu)_2 = (z^*, \mu)_2$. Therefore,

$$\begin{aligned}
0 \leqslant D^+ f(0;\mu) &:= \lim_{t \downarrow 0} \frac{f(t;\mu) - f(0;\mu)}{t} \\
&= (v^* - a^*, v_\mu)_2 + \frac{1}{2}\int \mathrm{sign}(\mu^*)\, d\mu^{ac} + \frac{1}{2}\|\mu^{sing}\|(U) \\
&= \int \left(z^* + \frac{1}{2}\mathrm{sign}(\mu^*)\right) d\mu^{ac} + \int \left(z^* + \frac{1}{2}\mathrm{sign}(\mu)\right) d\mu^{sing}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
0 \geqslant D^- f(0;\mu) &:= \lim_{t \uparrow 0} \frac{f(t;\mu) - f(0;\mu)}{t} \\
&= (v^* - a^*, v_\mu)_2 + \frac{1}{2}\int \mathrm{sign}(\mu^*)\, d\mu^{ac} - \frac{1}{2}\|\mu^{sing}\|(U) \\
&= \int \left(z^* + \frac{1}{2}\mathrm{sign}(\mu^*)\right) d\mu^{ac} + \int \left(z^* - \frac{1}{2}\mathrm{sign}(\mu)\right) d\mu^{sing}.
\end{aligned}$$

Since we can choose μ^{ac} and μ^{sing} independently, we immediately conclude that

$$|z^*| \leqslant \tfrac{1}{2}, \qquad -2z^*\, d\mu^* = d\|\mu^*\|.$$

The second identity is equivalent to $z^* = -1/2\,\mathrm{sign}(\mu^*)$ on the support of μ^*.

3. In this step, we show that μ^* is nonnegative. Set

$$\mu^*_- := (\mathrm{sign}(\mu^*))^- \mu^*$$

to be the negative part of μ^*. Then by the previous step, for any function f, with $f(z) = 0$ for $z \leqslant 0$, we have

$$\int f(z^*)\, d\mu^* = -f(1/2)\mu^*_-(U).$$

Let f be such a function, which is smooth and nondecreasing. By taking the curl of (9.2), we see that $\mu^* = -\Delta z^* + z^* + H$. Hence, noting that $f(z^*) = 0$ on ∂U,

$$\int f(z^*)\, d\mu^* = \int f'(z^*)|\nabla z^*|^2 + \int f(z^*)[z^* + H] \geqslant 0.$$

Since we could take $f(1/2) > 0$, we conclude that $\mu^*_-(U) = 0$. Hence μ^* is nonnegative.

4. We next verify (9.1). Indeed, by the inequality $|v|^2 - |w|^2 \geqslant 2(v - w) \cdot w$, we have the following for any $z \in \mathscr{K}$:

$$\begin{aligned}\mathbb{D}(z; H) - \mathbb{D}(z^*; H) &\geqslant (\nabla[z - z^*], \nabla z^*)_2 + ([z - z^*], z^*)_2 + ([z - z^*], H)_2 \\ &= ([z - z^*], [-\Delta z^* + z^* + H)]_2 \\ &= ([z - z^*], \mu^*)_2.\end{aligned}$$

In the final step, we used the curl of (9.2).

Since μ^* is a nonnegative measure whose support is included in $\{z^* = -1/2\}$, we conclude that for any $z \geqslant -1/2, ([z - z^*], \mu^*)_2 \geqslant 0$.

5. By the theory of variational problems like (9.1), z^* is the unique solution of the variational inequality

$$-\Delta z^* + z^* + H \geqslant 0, \quad z^* \geqslant -1/2,$$

$$(-\Delta z^* + z^* + H)(z^* + 1/2) = 0 \quad \text{in } U, \tag{9.3}$$

with zero boundary conditions; see for instance [14]. Then, by maximum principle $z^* \leqslant 0$. ∎

We obtain a quick formulation of H_{c_1} as a corollary of the variational formulation of z^*.

Corollary 9.2. *Let* (v^*, a^*) *be a minimizer of* $\mathbb{F}(\cdot, \cdot, H)$. *Then the limiting vorticity* $\nabla \times v^*$ *is identically equal to zero, if* $H < H_{c_1}$, *where* H_{c_1} *is as in* (1.10). *Moreover,* $\nabla \times v^*$ *is nonzero for* $H > H_{c_1}$.

Proof. First suppose that $H < H_{c_1}$. Let $\hat{z}$ be the solution of (1.9). Then, $H\hat{z}$ is a solution of (9.3), and therefore $z^* = H\hat{z}$. Moreover, $z^* > -1/2$, and the support of $\nabla \times v^*$ is empty.

For $H > H_{c_1}$. The obstacle is active in the variational inequality (9.3), and by the theory of obstacle problems we see that $-\Delta z^* + z^* + H = \nabla \times v^*$ is a nonzero measure. ∎

APPENDIX

In the section, we provide the proof of Theorem 3.1.

Proof of Theorem 3.1. In Proposition 6.4 of [17], a collection of disjoint, closed balls $\mathscr{B}(\sigma) = \{B_k^\sigma\}_{k=1}^{k(\sigma)}$ satisfying $r_k^\sigma \geqslant \varepsilon$,

$$\int_{U \cap B_k^\sigma} E^\varepsilon(u)\, dx \geqslant \frac{r_k^\sigma}{\sigma} \Lambda^\varepsilon(\sigma), \tag{10.1}$$

$$r_k^\sigma \geqslant \sigma |d_k^\sigma| \qquad \text{whenever } B_k^\sigma \cap \partial U = \emptyset, \tag{10.2}$$

where d_k^σ is the essential degree as defined in [17] and it has the same additive properties as the usual degree. In particular, for $t \in \text{Reg}(\phi)$ $dg(u, \Gamma(t)) = \deg(u; \Gamma(t))$. Λ^ε is an additive function satisfying

$$s \mapsto \Lambda^\varepsilon(s)/s \text{ is nonincreasing} \tag{10.3}$$

and

$$\Lambda^\varepsilon(s) \geqslant \pi \ln(s/ep) + c_0, \qquad \text{for } s \geqslant \varepsilon,$$

for some constant c_0. Define

$$C := \left\{ t \in (0, \|\phi\|_\infty) \mid \Gamma(t) \cap \left[\bigcup_k B_k^{\bar{\sigma}} \right] \neq \emptyset \right\}.$$

The definition implies that $C \subset \bigcup_k \phi(B_k^{\bar{\sigma}})$, and as a consequence

$$|C| \leqslant 2\|\nabla\phi\|_\infty \sum_k r_k^{\bar{\sigma}} = 2\|\nabla\phi\|_\infty R(\sigma).$$

Thus if $|D_d^\varepsilon| > 2R(\sigma)\|\nabla\phi\|_\infty$, then $D_d^\varepsilon \backslash C \neq \emptyset$, and we may select some $t_0 \in D_d^\varepsilon \backslash C$. The definition of D_d^ε and the essential degree imply that $|\deg(u; \Gamma(t_0))| = |\deg(u; \Gamma(t_0))| \geqslant d$. On the other hand, the definition of C implies that $\Gamma(t_0) \cap (\bigcup_k B_k^{\bar{\sigma}}) = \emptyset$. Since the balls covers essential zero set of

u, the additivity of the degree yield

$$d \leqslant |\mathrm{degree}(u; \Gamma(t_0))| \leqslant \sum_{\{k: B_k^{\bar{\sigma}} \subset \Omega(t_0)\}} |d_k^{\bar{\sigma}}| \leqslant \sum_{\{k: B_k^{\bar{\sigma}} \cap \partial U = \emptyset\}} |d_k^{\bar{\sigma}}|.$$

So (10.2) implies that $d\sigma < R(\sigma)$.

Since $\Omega(t_0) \subset \mathrm{spt}\,\phi$, the negation of (3.2) follows directly from (10.1) and (10.3).

Final inequality is obvious in the construction, and the continuity assertion is made in Remark 6.5 of [17]. ∎

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