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Option hedging for small investors under liquidity costs

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Abstract Following the framework of Çetin et al. (Finance Stoch. 8:311–341, 2004), we study the problem of super-replication in the presence of liquidity costs under additional restrictions on the *gamma* of the hedging strategies in a generalized Black–Scholes economy. We find that the minimal super-replication price is different from the one suggested by the Black–Scholes formula and is the unique viscosity solution of the associated dynamic programming equation. This is in contrast with the results of Çetin et al. (Finance Stoch. 8:311–341, 2004), who find that the arbitrage-free price of a contingent claim coincides with the Black–Scholes price. However, in Çetin et al. (Finance Stoch. 8:311–341, 2004) a larger class of admissible portfolio processes is used, and the replication is achieved in the L^2 approximating sense.

Keywords Super-replication · Liquidity cost · Gamma process · Parabolic majorant · PDE valuation

Mathematics Subject Classification (2000) 91B28 · 35K55 · 60H30

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1 Introduction

The Black–Scholes methodology for the pricing and hedging of options requires the market to be frictionless and competitive. In other words, traders can trade any quantity of the asset without changing its price, and the trade is subject to no transaction costs and restrictions. There have been numerous works to relax these assumptions as it is now well known that the markets do not operate frictionlessly and are not perfectly competitive (see e.g. [2, 3, 11–14, 17], and [20]).

Relaxation of both the frictionless and competitive market hypotheses introduces the notion of liquidity risk. Roughly speaking the liquidity risk is the additional risk due to the timing and size of a trade. Recently, several authors have proposed a number of methods to incorporate the liquidity risk into asset pricing theory (see [1, 4–6], and [26]). The common characteristic of all these works is that the liquidity risk appears as some nonlinear transaction cost which appears due to the imbalance between supply and demand in the financial market, which is relevant if an agent is attempting to trade large volumes in a short time.

In the literature dealing with the modeling of "liquidity risk" one can clearly identify two different approaches. In the first approach, the modeler concentrates on the effects of the large trader's portfolio on the price of the stock (see, e.g., [17, 18, 25, 27, 28]). The authors postulate a *feedback function* that governs the dependence of the equilibrium stock price on the portfolio actions of the large trader. We call this class of models "models of feedback effects". The second class of models, e.g., [4–6, 26], ignores the feedback effects of the trades and concentrates on the equalization of supply and demand locally in time so that trade volume does not have a lasting impact on the asset price. Consequently the wealth process of a trader in an illiquid market can be decomposed into two components: One comes from the gains/losses due to the changes in the fundamental value of the stock, which does not depend on the history of the trades, and the other is the liquidity cost incurred over time due to the changes in the position. In other words, this class of models studies the behavior of price-taking traders in markets where the change of one's position has additional liquidity costs.

In this work we stay within the framework of the second class of models. In particular, we follow the model suggested by Çetin at al. [4], who introduced the so-called "supply curve" to model the asset price as a function of size and time. Starting with the given supply curve for, say, the stock, the authors show in [4] that the existence of liquidity costs makes trading strategies with infinite quadratic variation infeasible since they incur infinite liquidity costs. One important consequence of their modeling is that continuous trading strategies of finite variation incur no liquidity costs; thus, the market is approximately complete (in an L^2-sense) if there exists a unique equivalent martingale measure for the "marginal price process" (see [4] for details). In particular, they show that in a Black–Scholes type economy with liquidity costs, the price of an option is given by the standard Black–Scholes formula and the approximate hedging strategy can be obtained by some appropriate averaging of the Black–Scholes hedge (see [5] for some further results and numerical and empirical studies).

On the other hand, the liquidity model of [4] produces a nonzero liquidity premium for options when considered in discrete time (see [6] and results therein). If one looks at the self-financing condition in [4] in continuous time, which we recall in Sect. 2, one notices that the self-financing condition is defined to be the limit of the self-financing conditions in discrete time as the time step tends to zero. The discrete-time version of the self-financing condition of [4] is very natural since the only assumption on the liquidity cost, other than measurability, is that the bigger the position to liquidate, the larger is the liquidity cost. Therefore, one naturally wonders what happens to the liquidity premium when one passes to the continuous-time limit as it is shown by [4] that the pricing formulas for the contingent claims in their model coincide with those in the frictionless markets.

We see this situation as a paradox of the liquidity model of [4] and argue in this paper that the absence of the liquidity premium is linked to the choice of the set of admissible strategies and show that one can find a nonzero liquidity premium in continuous time for an appropriately defined set of admissible strategies.

The correct choice of an admissible set of strategies is crucial even in frictionless markets. Indeed, in models of markets with no friction in discrete time, no conditions, other than adaptability, need to be imposed on the trading strategies in order to solve the problems of pricing, hedging, and utility maximization. However, as soon as we consider the continuous-time limit of these models, the price of any contingent claim becomes unbounded from below (i.e., $-\infty$), and the value function of the utility maximization problem will typically be unbounded (i.e., ∞) as one can create arbitrage opportunities due to the existence of doubling strategies. It is now well known that this paradox can be solved by imposing certain restrictions on the trading strategies, such as certain integrability or lower bound assumptions. The notion of admissibility in frictionless markets is now well understood as an integral part of financial modeling. We believe that the apparent paradox in the model of [4] can be solved in a similar way by an appropriate choice of an admissibility condition in illiquid markets. The main purpose of this paper is to define a convenient set of admissible strategies so that the liquidity cost does not vanish in the continuous-time limit. This is achieved by placing constraints on the dynamics of the trading strategies and their corresponding gamma processes as in [7]. In a recent paper, Gökay and Soner [19] showed that the continuous-time limit of the corresponding binomial model yields exactly the same pricing equation as in this paper. Since the trading strategies in the binomial model are not restricted, the convergence result of [19] supports our choice of admissible strategies.

The restrictions that we place on the trading strategies in this paper can be seen as a relaxation of the restrictions in [21]. First of all, we allow a trading strategy to have infinite variation. More precisely, the admissible trading strategies form a larger subset of semimartingales (see (2.2)). As seen, the finite-variation part of a trading strategy consists of a pure-jump component and an absolutely continuous component. The remaining infinite-variation part is an integral with respect to the *marginal* price process of the stock, which is a martingale since we work under the unique risk-neutral measure for the marginal price process. The integrand in the absolutely continuous part of the trading strategy can be viewed as the rate of change of the trading strategy with respect to time, while the integrand in the infinite-variation part can be seen as the rate of change with respect to the changes in the stock price. As in [21], we assume that these "derivatives" are bounded (see Sect. 2 for the exact definitions). However, we do not impose uniform bounds over all admissible strategies. The price to pay for this relaxation is that we are no longer happy with the mere L^2-convergence but price contingent claims using super-replication arguments. We show that a trading strategy that attains the minimal super-replication cost is a perfect replicating strategy, and its cost of construction contains a liquidity premium, in contrast with the results of [4].

A related work on such trading restrictions can be found in Longstaff [21], who suggests a uniform bound on the time derivative of trading strategies to study the optimal portfolios in an illiquid market. More recently, Rogers and Singh [26] studied the Merton problem under liquidity costs where they placed similar restrictions on the trading strategies.

Under our admissibility condition, we show in Proposition 3.1 that strategies with jumps are not optimal so that the super-hedging problem can be restricted to continuous hedging strategies. This feature of our admissibility set is thus in agreement with the conventional wisdom according to which it is better to place consecutive small trades rather than a large one at once in illiquid markets. Our main result, Theorem 3.3, proves that the super-replication price V is the unique viscosity solution of the dynamic programming equation

$$-V_t - \frac{s^2\sigma^2}{4\ell}\left[-\ell^2 + \left((V_{ss}+\ell)^+\right)^2\right] = 0, \tag{1.1}$$

where the function $\ell > 0$ is the liquidity index of the market defined in (2.4) below. In fact, for more liquid markets, ℓ is larger, with $\ell = \infty$ corresponding to the complete Black–Scholes model. Using this equation, it is easy to show that, unless the payoff is affine, the solution to this equation is strictly larger than the Black–Scholes price. Hence there is a liquidity premium. This is proved in Corollary 3.4. Moreover, this liquidity premium can be calculated numerically using available methods for the solutions of PDEs of type (1.1).

These results are proved by the techniques developed in a series of papers by Soner and Touzi [29–32], Cheridito et al. [7, 8], and Cheridito et al. [9].

Although the set of admissible strategies that we consider is motivated by technical integrability conditions, our results are supported by a formal description of the corresponding hedging strategy which has a relevant financial interpretation. As we illustrate in Sect. 4, the optimal hedging strategy exhibits an asymmetry between claims with convex and concave payoffs. For derivatives with convex payoff, the hedging strategy is of dynamic Black–Scholes type. However, when the claim to be hedged has a concave payoff, there are two options for the trader: either employ a buy-and-hold strategy at a higher cost of construction but no further liquidity costs, or employ a perfect Black–Scholes-type replicating strategy but expect liquidity costs growing over time. Depending on the market conditions, it might be cheaper to use the buy-and-hold type hedge rather than the replicating strategy when the liquidity cost associated with the latter is expected to be high. In Sect. 4 we show that this decision should be based on the level of concavity of the value function for the minimal super-replication price and give a precise level below which it is cheaper to use a buy-and-hold strategy.

The outline of the paper is as follows. Section 2 formulates the problem. Section 3 presents the main results. Section 4 describes the formal hedging strategy. Section 5 finds the growth condition for the value function, while Sect. 6 shows the uniqueness of the solution. In the Appendix, we discuss the convexity properties and an illustrative example.

2 Problem formulation

Throughout this paper, we fix a finite time horizon $T > 0$, and we consider a one-dimensional Brownian motion $W = \{W(t),\ 0 \le t \le T\}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathbb{F} = \{\mathcal{F}(t),\ 0 \le t \le T\}$ the smallest filtration that contains the filtration generated by W and satisfies the usual conditions.

2.1 The financial market

The financial market consists of two assets, and the objective of the investor is to optimally allocate his wealth between these assets in order to hedge some contingent liability.

The first asset is nonrisky. Without loss of generality, we normalize its price to unity, which means that this asset is taken as the numéraire.

The risky asset is subject to liquidity costs. Following Çetin et al. [4], we account for the liquidity costs by modeling the price process of this asset as a function of the exchanged volume. We thus introduce a supply curve

$$\mathbf{S}\big(t, S(t), \nu\big),$$

where $\nu \in \mathbb{R}$ indicates the volume of the transaction, the process $S(t) = \mathbf{S}(t, S(t), 0)$ is the *marginal price process* defined by the stochastic differential equation

$$\frac{dS(r)}{S(r)} = \mu\big(r, S(r)\big)\,dr + \sigma\big(r, S(r)\big)\,dW(r) \tag{2.1}$$

and some given initial condition $S(0)$, and $\mathbf{S} : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a smooth function representing the price per share for some given volume of transaction and the marginal price. In addition to the technical conditions imposed in [4] on the supply curve, we assume for each t and s that

$$\frac{\partial \mathbf{S}}{\partial \nu}(t, s, 0) > 0.$$

In order to ensure that the stochastic differential equation (2.1) has a unique strong condition, we assume that the coefficient functions $\mu, \sigma : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ satisfy the usual local Lipschitz and linear growth conditions.

In order to exclude arbitrage opportunities, we assume the existence of an equivalent martingale measure $\mathbb{P}^0$, i.e.,

$$\frac{dS(r)}{S(r)} = \sigma\big(r, S(r)\big)\,dW^0(r),$$

where W^0 is a standard Brownian motion under $\mathbb{P}^0$, so that the process S is a martingale under $\mathbb{P}^0$.

We shall frequently move the time origin from zero to an arbitrary $t \in [0, T]$, and we then denote by $\{S_{t,s}(r), r \in [t, T]\}$ the process defined by (2.1) and the initial condition $S_{t,s}(t) = s$.

2.2 Trading strategies

A trading strategy is defined by a pair (X, Y), where $X(t)$ denotes the wealth in the bank, and $Y(t)$ is the number of shares held at each time t in the portfolio. For reasons which will become clear later, we restrict the process Y to be of the form

$$Y(r) = \sum_{n=0}^{N-1} y^n \mathbf{1}_{\{r \geq \tau_{n+1}\}} + \int_t^r \alpha(u)\, du + \int_t^r \Gamma(u)\, dS_{t,s}(u), \tag{2.2}$$

so that it has finite quadratic variation. Here, $t = \tau_0 < \tau_1 < \cdots$ is an increasing sequence of $[t, T]$-valued $\mathbb{F}$-stopping times, the random variable

$$N := \inf\{n \in \mathbb{N} \;:\; \tau_n = T\}$$

indicates the number of jumps, y^n is an $\mathbb{R}^d$-valued, $\mathcal{F}(\tau_n)$-measurable random variable satisfying $y^n \mathbf{1}_{\{\tau_n = T\}} = 0$, and α and Γ are two $\mathbb{F}$-progressively measurable real processes.

We shall show in the next section that it is not optimal to have jumps in Y.

We continue by deriving the continuous-time dynamics of our state variables. This derivation follows the discrete-time argument of [4]. Let $t = t_0 < \cdots < t_n = T$ be a partition of the interval $[0, T]$, and set $\delta\psi(t_i) := \psi(t_i) - \psi(t_{i-1})$ for any function ψ. By the self-financing condition, it follows that

$$\delta X(t_i) + \delta Y(t_i)\mathbf{S}\big(t_i, S(t_i), \delta Y(t_i)\big) = 0, \quad 1 \leq i \leq n.$$

Summing up these equalities, it follows from direct manipulations that

$$\begin{aligned} X(T) + Y(T)S(T) &= X(t) + Y(t)S(t) \\ &\quad - \sum_{i=1}^{n} \big[\delta Y(t_i)\mathbf{S}\big(t_i, S(t_i), \delta Y(t_i)\big) + \big(Y(t)S(t) - Y(T)S(T)\big)\big] \\ &= X(t) + Y(t)S(t) \\ &\quad - \sum_{i=1}^{n} \big[\delta Y(t_i)S(t_i) + \big(Y(t)S(t) - Y(T)S(T)\big)\big] \\ &\quad - \sum_{i=1}^{n} \delta Y(t_i)\big[\mathbf{S}\big(t_i, S(t_i), \delta Y(t_i)\big) - S(t_i)\big] \end{aligned}$$

$$= X(t) + Y(t)S(t) + \sum_{i=1}^{n} Y(t_{i-1})\delta S(t_i)$$

$$- \sum_{i=1}^{n} \delta Y(t_i)\big[\mathbf{S}\big(t_i, S(t_i), \delta Y(t_i)\big) - S(t_i)\big]. \tag{2.3}$$

The continuous-time dynamics of the process

$$Z := X + YS$$

are obtained by taking limits in (2.3) as the time step $\max\{(t_i - t_{i-1}), 1 \leq i \leq n\}$ of the partition shrinks to zero. The last sum term in (2.3) is the term due to the liquidity cost. Under the imposed smoothness assumption on $\nu \mapsto \mathbf{S}(t, s, \nu)$, it follows that

$$\sum_{i=1}^{n} \delta Y(t_i)\big[\mathbf{S}\big(t_i, S(t_i), \delta Y(t_i)\big) - \mathbf{S}\big(t_i, S(t_i), 0\big)\big]$$

$$\longrightarrow \int_t^T \frac{d[Y,Y]_r^c}{4\ell(r, S(r))} + \sum_{k=0}^{N-1} y^k\big[\mathbf{S}\big(\tau_k, S(\tau_k), y^k\big) - S(\tau_k)\big]$$

in probability, where

$$\ell(t, s) := \left[4\frac{\partial \mathbf{S}}{\partial \nu}(t, s, 0)\right]^{-1}. \tag{2.4}$$

In view of the form of the continuous-time process Y in (2.2), this provides

$$Z(r) = Z(t) + \int_t^r Y(u)\,dS(u) - \int_t^r \frac{1}{4\ell(r, S(r))}\,\Gamma(r)^2\sigma\big(r, S(r)\big)^2 S(r)^2\,dr$$

$$- \sum_{k=0}^{N-1} y^k\big[\mathbf{S}\big(\tau_k, S(\tau_k), y^k\big) - S(\tau_k)\big]\mathbf{1}_{\{r<\tau_{k+1}\}}. \tag{2.5}$$

In the absence of jumps in the portfolio process, the process Z approaches the classical wealth process in frictionless markets for large ℓ. Therefore, we refer to ℓ as the liquidity index of the market.

In the absence of liquidity costs, the process Z represents the total value of the portfolio of the investor. In the present setting, we assume that the investor is not subject to any liquidity cost at the final time T. Then, although the process Z has no direct financial interpretation, its final value $Z(T)$ is the total value of the investor portfolio at time T. A discussion of initial and terminal liquidity costs is given in Remark 3.2.

2.3 Admissible trading strategies and the hedging problem

The purpose of the investor is to hedge without risk some given contingent claim

$$G = g\big(S(T)\big) \quad \text{for some function } g : \mathbb{R}_+ \to \mathbb{R}.$$

In order to formulate the super-hedging problem in the context of our financial market with liquidity cost, we need to restrict further the trading strategies as in [32].

For $B, b \geq 0$, and for an $\mathbb{F}$-progressively measurable process $\{H(r),\ t \leq r \leq T\}$ taking values in $\mathbb{R}$, we define

$$\|H\|_{t,s}^{B,b} := \left\| \sup_{t \leq r \leq T} \frac{|H(r)|}{1 + |S_{t,s}(r)|^B} \right\|_{L^b(\Omega,\mathcal{F},\mathbb{P})}.$$

Throughout the paper, we fix $B > 0$. A trading strategy Y defined by (2.2) is said to be admissible if there is a parameter $b > 0$ such that

$$\|N\|_\infty < \infty, \qquad \|Y\|_{t,s}^{B,\infty} + \|\Gamma\|_{t,s}^{B,\infty} + \|\alpha\|_{t,s}^{B,b} < \infty,$$

and the process Γ is of the form

$$\Gamma(r) = \Gamma(t) + \int_t^r a(u)\,du + \int_t^r \xi(u)\,dW(u),$$

where a and ξ are two real-valued $\mathbb{F}$-progressively measurable processes satisfying

$$\|a\|_{t,s}^{B,b} + \|\xi\|_{t,s}^{B,2} < \infty.$$

Clearly a larger parameter B implies a larger admissible set. Hence, the parameter B can be viewed as an indicator of market depth. We refer to [8] and [32] for a justification of such restrictions. In addition, the already discussed convergence result of the binomial model [19] provides further support for this class of trading strategies.

Also, notice that we use the framework of [32], where the restrictions on the drift terms α and a are relaxed compared to [8]. This relaxation plays a crucial role in the present paper because, in contrast with our previous work [8], the state variable Z in (2.5) exhibits a jump term.

The collection of all admissible trading strategies $Y = \{Y(r), 0 \leq r \leq T\}$ is denoted by $\mathcal{A}_{t,s}$. For every $Y \in \mathcal{A}_{t,s}$, we denote by $Z_{t,z}^Y$ the process defined by (2.5). The purpose of this paper is to solve the super-hedging problem

$$V(t,s) := \inf\big\{z \in \mathbb{R} : Z_{t,z}^Y(T) \geq g\big(S_{t,s}(T)\big) \text{ for some } Y \in \mathcal{A}_{t,s}\big\}. \tag{2.6}$$

Notice that this formulation ignores the liquidity cost both at the time origin t and the final time T. As a consequence of Proposition 3.1 below, the elimination of the initial liquidity cost does not entail any loss of generality, see Remark 3.2. However, the absence of a liquidity cost at the final time is a standing assumption throughout the paper.

In the subsequent section, we shall prove that we can restrict the portfolios to be continuous, so that the above value function V coincides with

$$V^{\text{cont}}(t,s) := \inf\{z \in \mathbb{R} \;:\; Z^Y_{t,z}(T) \ge g\big(S_{t,s}(T)\big) \text{ for some } Y \in \mathcal{A}^{\text{cont}}_{t,s}\}, \tag{2.7}$$

where $\mathcal{A}^{\text{cont}}_{t,s}$ consists of all *continuous* portfolio processes in $\mathcal{A}_{t,s}$.

3 Main results

We need the following mild technical conditions. The first assumption is needed to ensure that the value function is locally bounded. It imposes that

$$g \text{ is bounded from below} \quad \text{and} \quad \sup_{s>0} \frac{g(s)}{1+s} < \infty. \tag{3.1}$$

Indeed, the lower bound on g is immediately inherited by V, and the affine growth condition guarantees the existence of a trivial buy-and-hold strategy which super-hedges the contingent claim $g(S(T))$, thus producing a locally bounded upper bound for g, see Proposition 5.1 below.

We place on the volatility function the standard condition that

$$\sigma \text{ is bounded and Lipschitz-continuous.} \tag{3.2}$$

The following condition on the liquidity function is needed for the comparison result of Sect. 6:

$$\begin{aligned} &\ell \text{ is locally Lipschitz-continuous,} \quad \text{and} \\ &\ell_\delta := \inf\{\ell(t,s) : \; \delta \le s \le \delta^{-1},\; t \in [0,T]\} > 0 \quad \text{for every } \delta > 0. \end{aligned} \tag{3.3}$$

3.1 Optimality of continuous portfolios

In this subsection, we first prove the optimality of continuous portfolio processes. Intuitively, it is clear that in an illiquid market it is better to make consecutive small trades instead of a large one. Then taking this idea to the limit, we formally see that jumps in the portfolio are not optimal. The following result proves this intuition. From the technical viewpoint, let us stress that the relaxation on the processes α and a plays a crucial role for the next result so that our definition of the set of admissible strategies allows one to preclude jumps from optimal strategies, thus agreeing with economic intuition.

Proposition 3.1 *Assume* (3.1), (3.2), *and* (3.3). *Then* $V^{\text{cont}} = V$.

Proof Fix (t,s) in $[0,T) \times (0,\infty)$. The inequality $V^{\text{cont}}(t,s) \ge V(t,s)$ is obvious as $\mathcal{A}^{\text{cont}}_{t,s} \subset \mathcal{A}_{t,s}$. To prove the reverse inequality, let $z > V(t,s)$ and $Y \in \mathcal{A}_{t,s}$ be such that $Z(T) \ge g(S(T))$ a.s. We denote by $\tau_1, \ldots, \tau_N$ the jump times of the portfolio process Y. From the definition of admissible strategies, recall that $\|N\|_\infty < \infty$.

Let $\varepsilon > 0$ be given. We first start by eliminating the final jump at time τ_N. Notice that $Z^Y_{t,z}(T) = Z^Y_{\tau_N, Z_{\tau_N}}(T) \geq g(S(T))$ a.s. Then, with $z_N := z + \varepsilon$, it follows from Lemmas 5.2 and 5.4 in [32] that $Z^{Y_N}_{t,z_N}(T) \geq g(S(T))$ a.s. for some $Y_N \in \mathcal{A}_{t,s}$ with $Y_N = Y$ on $[t, \tau_N)$ and such that (Y_N, Γ_N) is continuous on $(\tau_{N-1}, T]$.

Repeating the above procedure backward, we may eliminate all the jumps. Hence, with $z_0 := z + \varepsilon \|N\|_\infty$, there exists $Y_0 \in \mathcal{A}^{\text{cont}}_{t,s}$ such that $Y^{t,z_0,Y_0}(T) \geq g(S(T))$ a.s. Hence, $z_0 \geq V^{\text{cont}}(t,s)$. Since $\varepsilon > 0$ and $z > V(t,s)$ are arbitrary, we conclude that $V(t,s) \geq V^{\text{cont}}(t,s)$. □

We close this subsection by discussing the initial and final liquidation costs.

Remark 3.2 The previous result on the continuity of the optimal portfolio also proves that there is no initial liquidity cost. Indeed, suppose that we start with a portfolio value different than the optimal one. Then by shifting the initial wealth by ϵ, we can use Proposition 5.1 of [30]. This shows that we can construct a super-replicating portfolio with an arbitrary initial position in the risky asset as long as the initial wealth in the bank account is larger by ϵ.

The situation at the final time is different. At maturity, we are forced to liquidate. This results in a liquidation cost. We chose to ignore this in our analysis. Including this cost will make the liquidity premium even larger. Hence this assumption does not affect the main result of this paper, namely the existence of a liquidity cost. On the other hand, this final liquidity cost can be driven to zero if a nonzero amount of time is given for liquidation.

The liquidity premium that we prove to exist, however, is due to continuous-time trading. Moreover, this premium cannot be avoided by spreading our trades over time. This is the motivation behind ignoring the final liquidity cost.

3.2 The dynamic programming equation characterization

In this subsection, we prove the viscosity property of the minimal super-replication cost. Let V and V^{cont} be as in (2.6) and (2.7).

Theorem 3.3 *Assume* (3.1), (3.2), (3.3) *and that the payoff function g is continuous. Then* $V = V^{\text{cont}}$ *is the unique continuous viscosity solution of the dynamic programming equation*

$$-V_t + \hat{H}(t,s,V_{ss}) := \sup_{\beta \geq 0} \left(-V_t - \frac{1}{2} s^2 \sigma^2 (V_{ss} + \beta) - \frac{s^2 \sigma^2}{4\ell} (V_{ss} + \beta)^2 \right) = 0 \quad (3.4)$$

on $[0,T) \times (0,\infty)$, *satisfying the terminal condition* $V(T,\cdot) = g$ *and the growth condition*

$$-C \leq V(t,s) \leq C(1+s), \; (t,s) \in [0,T] \times \mathbb{R}_+, \quad \textit{for some constant } C > 0. \quad (3.5)$$

The proof of this theorem is completed in several steps. The viscosity property of the value function and the terminal data follows from a general result proved in Theorem 3.2 of the paper [32]. The growth condition (3.5) is derived in Sect. 5. Finally, the uniqueness result follows from the comparison result of Sect. 6.

We close this subsection by several observations on the structure of (3.4). First, observe that the dynamic programming equation (3.4) is parabolic, i.e., nonincreasing in V_{ss}, as all dynamic programming equations should be. Moreover, the differential operator appearing on the left-hand side of (3.4) is the parabolic envelope of the *first guess* operator

$$-V_t + H(t,s,V_{ss}) := -V_t - \frac{1}{2}s^2\sigma^2 V_{ss} - \frac{s^2\sigma^2}{4\ell}V_{ss}^2.$$

We refer to [7] for more details on the construction of parabolic majorants $\hat{H}$ of H.

Finally, by direct manipulation, we see that the maximizer in the dynamic programming equation (3.4) is given by

$$\hat{\beta}(t,s) := \big(V_{ss}(t,s) + \ell(t,s)\big)^-, \tag{3.6}$$

so that we can rewrite the dynamic programming equation as (1.1).

3.3 Liquidity premium

Let V_{BS} be the Black–Scholes price of the claim g. Clearly, $V \geq V_{\mathrm{BS}}$, and the liquidity premium is the difference. Our next result states that the liquidity premium is zero only for affine payoffs.

Corollary 3.4 *Assume that the hypotheses of Theorem 3.3 hold. Then $V = V_{\mathrm{BS}}$ if and only if g is an affine function. Hence the liquidity premium is nonzero for all nontrivial claims.*

Proof By the definition of $\hat{H}$, it is easily seen that

$$-V_t - H(t,s,V_{ss} + \hat{\beta}) = 0,$$

where $\hat{\beta} \geq 0$ is given by (3.6). If $V = V^{\mathrm{BS}}$, then V is smooth and is a classical solution both of the above equation and the Black–Scholes equation. This immediately implies that $\hat{\beta} = V_{ss} = 0$. Then g is affine. The reverse implication is trivial by verifying that affine functions satisfy the PDE of V. □

4 Formal description of an optimal hedging strategy

We now provide a formal description of an optimal hedging strategy for a payoff $g(S_T)$ under liquidity costs. An illustrative example is studied in Appendix A.2. The analysis of this section will be restricted to a formal discussion as we shall ignore some admissibility restrictions and regularity conditions.

For concreteness, we work in the context of the classical Black–Scholes model, i.e., $\sigma(t,s) \equiv \sigma$ for some positive constant σ. This would also enable us to compare our results with the classical Black–Scholes formula. We also assume that ℓ is independent of the t-variable.

4.1 Usual hedge

When the minimal super-replication cost V is a classical solution of (3.4) and if $\hat{H}(t,s,V_{ss}(t,s)) = H(t,s,V_{ss}(t,s))$ for all (t,s), then the usual hedge is replicating. We first state and prove this result. Then, in the Appendix, we provide sufficient conditions on ℓ.

First recall that $\hat{H}(t,s,V_{ss}(t,s)) = H(t,s,V_{ss}(t,s))$ if and only if we have $V_{ss}(t,s) \geq -\ell(t,s)$. This condition is equivalent to the convexity of

$$\hat{V}(t,s) := V(t,s) + \int_1^s \int_1^{s'} \ell(t,s'')\,ds''\,ds'.$$

Theorem 4.1 *Let V be the value function. Assume that $\hat{V}$ defined above is convex. Then V is smooth and is a classical solution of* (3.4). *Moreover, the classical hedge $Y(u) = V_s(u, S(u))$ is replicating.*

Proof We know that V is a viscosity solution of (3.4). Moreover, the convexity of $\hat{V}$ defined above implies that the optimizer β in (3.4) is zero and that (3.4) is locally uniformly elliptic with a convex nonlinearity. Then, one can use the celebrated regularity result of Evans [15] and Krylov and Safanov [22, 23] to conclude that V is smooth. Therefore, it is a classical solution of (3.4). Since (3.4) is a parabolic equation in one space dimension, one can also prove this regularity result directly without referring to the deep regularity theory of Evans and Krylov. Indeed, a fixed-point argument using the results and the techniques from the classical textbook of Ladyzhenskaya et al. [24] also yield this regularity.

Moreover,

$$V_{ss}(t,s) > -\ell(t,s) \quad \forall (t,s), \tag{4.1}$$

and (3.4) holds with $\hat{H}(t,s,V_{ss}(t,s)) = H(t,s,V_{ss}(t,s))$. Then, by a standard application of Itô calculus, we can show that the classical hedge $Y(u) = V_s(u,S(u))$ is replicating. □

In Appendix A.1, we discuss two sufficient conditions for (4.1).

4.2 Buy-and-hold versus dynamic hedging

In this subsection, we discuss the general structure of the hedge. An illustrative example with $g(s) = s \wedge 1$ will later be discussed in Appendix A.2. To simplify the discussion, we assume that the supply function is of the form

$$\mathbf{S}(t,s,\nu) := s e^{\alpha s \nu/4} \quad \text{so that } \ell(s) = \frac{1}{\alpha s^2}.$$

Set

$$\phi(t,s) := \frac{1}{4\alpha}\big[\sigma^2(t-T) + 4\ln s\big], \quad (t,s) \in [0,T] \times \mathbb{R}_+,$$

so that $\phi_{ss} + \ell = 0$.

Before turning to the description of a hedging strategy in the context of our financial market with liquidity costs, we should like to discuss the asymmetry between concavity and convexity from the point of view of superhedging. This will turn out to be the driving intuition for our hedging strategy.

The Black–Scholes hedging theory in a complete market says that the optimal superhedging strategy of some contingent claim is in fact a perfect replicating strategy and consists in the dynamic strategy of holding, at each time r, the number $\frac{\partial V}{\partial s}(r, S_r)$ of shares of the underlying risky asset. In our context, this strategy is more expensive than in the frictionless Black–Scholes model since it induces a nonzero gamma process, implying a penalization on the wealth process.

A buy-and-hold strategy on some time interval $[t,\tau)$ is defined by $Y(r) = Y(t)$ for every $r \in [t,\tau)$. In particular, $\Gamma = 0$ on $[t,\tau)$, the wealth process is not subject to the liquidity cost penalty, and it is given by the same expression as in the classical frictionless framework, namely

$$Z(r) = Z(t) + Y(t)\big(S(r) - S(t)\big) \quad \text{for } r \in [t,\tau).$$

For a concave payoff, a trivial static superhedging strategy is available. Indeed, performing the buy-and-hold strategy $Y(t) = \frac{\partial g}{\partial s}(S(t))$ on $[t,\tau)$ (for a nonsmooth g, let Y_t be a measurable selection in the supergradient of g at $S(t)$), and starting from the capital $Z_t = g(S(t))$, it follows from the concavity of the payoff function g that

$$Z(\tau) = Z(t) + Y(t)\big(S(\tau) - S(t)\big) \geq g\big(S(\tau)\big).$$

This discussion shows that, in our context of financial market with liquidity costs, when the super-replication value is concave, there is a trade-off between

– paying a higher cost for a buy-and-hold strategy, thus avoiding the liquidity costs,
– performing the Black–Scholes replicating strategy but paying the liquidity costs.

The hedging strategy which will be described in the next subsection provides a precise definition of the level of concavity below which the liquidity cost induced by a perfect hedging strategy is so significant that it is cheaper to use a buy-and-hold strategy. In the subsequent subsection we answer the question how a risk manager prefers a Black–Scholes-type replicating strategy over a buy-and-hold strategy by splitting the value of the option into two parts and replicating one part by the classical Black–Scholes hedge while hedging the other part by a combination of a buy-and-hold-strategy together with a classical hedge. The latter is achieved by optimally separating the state space into regions in which one or the other strategy is optimal.

4.3 Hedging under liquidity costs

In order to discuss the hedging strategy, we introduce the open set

$$\mathcal{C} := \big\{(t,s) \in [0,T) \times (0,\infty) : \ V_{ss}(t,s) < -\ell(t,s)\big\}.$$

Observe that on $\mathcal{C}$, (1.1) reduces to

$$-V_t + \frac{1}{4} s^2 \sigma^2(t,s)\ell(s) = 0. \tag{4.2}$$

Note that $(V-\phi)_{ss} < 0$ and $(V-\phi)_t = 0$ on $\mathcal{C}$. This implies that we have $(V-\phi)_{tt} = (V-\phi)_{ts} = 0$ on $\mathcal{C}$. Hence,

$$(t,s) \mapsto (V-\phi)(t,s) \text{ is concave}, \quad \text{and} \quad (V-\phi)_t = 0 \quad \text{on } \mathcal{C}. \tag{4.3}$$

Given an arbitrary initial position $(t,s) \in \mathcal{C}$, we define the exit time

$$\theta := \inf\{u > t : \big(u, S(u)\big) \notin \mathcal{C}\}.$$

We now consider the initial capital $V(t,s)$ at time t, together with the hedging strategy $\{Y(u),\ t \le u < \theta\}$ defined by

$$Y(t) := V_s(t,s), \qquad \Gamma(u) := \phi_{ss}\big(u, S(u)\big), \qquad \alpha(u) := \mathcal{L}\phi_s\big(u, S(u)\big).$$

In words, this hedging strategy consists in dynamically replicating the value function ϕ and performing a buy-and-hold strategy in order to super-hedge the remaining value $(V-\phi)$. Then, we directly calculate for $\tau \in (t,\theta)$ that

$$\begin{aligned}
Z(\tau) &= V(t,s) + \int_t^\tau Y(u)\,dS(u) - \frac{1}{4}\int_t^\tau \ell^{-1}\big(S(u)\big)\Gamma^2(u)\sigma^2(u,S(u))S^2(u)\,du \\
&= V(t,s) + V_s(t,s)(S(\tau)-s) \\
&\quad + \int_t^\tau \left(\int_t^u \mathcal{L}\phi_s\big(r,S(r)\big)\,dr + \phi_{ss}\big(r,S(r)\big)dS(r)\right) dS(u) \\
&\quad - \frac{1}{4}\int_t^\tau \ell\big(u,S(u)\big)\sigma^2 S^2(u)\,du \\
&= (V-\phi)(t,s) + (V-\phi)_s(t,s)\big[S(\tau)-s\big] + \phi\big(\tau,S(\tau)\big) \\
&\quad - \int_t^\tau \left(\mathcal{L}\phi\big(u,S(u)\big) + \frac{1}{4}\ell\big(u,S(u)\big)\sigma^2\big(u,S(u)\big)S^2(u)\right) du,
\end{aligned}$$

where we applied Itô's lemma twice to the process $\phi(u,S(u))$. We next observe that

$$\mathcal{L}\phi = \phi_t + \frac{1}{2}\sigma^2 s^2 \phi_{ss} = -\frac{1}{4}\ell(t,s)\sigma^2 s^2 \quad \text{on } \mathcal{C}.$$

Together with (4.3), this implies that

$$\begin{aligned}
Z(\tau) &= (V-\phi)(t,s) + (V-\phi)_s(t,s)\big[S(\tau)-s\big] + \phi\big(\tau,S(\tau)\big) \\
&\ge (V-\phi)\big(\tau,S(\tau)\big) + \phi\big(\tau,S(\tau)\big) = V\big(\tau,S(\tau)\big).
\end{aligned}$$

This shows that the above strategy is a super-hedging strategy in $\mathcal{C}$. Outside $\mathcal{C}$, one can show by Itô's lemma that the hedging strategy consists in performing a perfect replicating Black–Scholes strategy for the total value function V, i.e., $Y(u) = V_s(u,S(u))$.

In conclusion, the super-hedging strategy in our financial market with liquidity costs is formally described by applying successively a perfect dynamic replicating Black–Scholes strategy outside $\mathcal{C}$ and the above mixed strategy in $\mathcal{C}$ which consists in dynamically hedging ϕ and super-hedging the difference $(V-\phi)$ by means of a buy-and-hold strategy.

5 Growth condition

In this section, we prove that the growth condition (3.1) placed on the payoff g implies a similar growth condition on the minimal super-replication prices $\tilde{v}$ and v.

Proposition 5.1 *Assume* (3.1). *Then there is a constant C such that*

$$-C \le V(t,s) \le C(1+s) \quad \forall t \in [0,T],\ s \ge 0. \tag{5.1}$$

Proof Let $-C$ be a lower bound for g. Fix any initial point (t,s), and let Y be a super-replicating portfolio. Since the corresponding wealth process Z^Y is a super-martingale, we have the inequalities

$$Z(t) \ge \mathbb{E}\big[Z(T) \mid \mathcal{F}_t\big] \ge \mathbb{E}\big[g\big(S_{t,s}(T)\big) \mid \mathcal{F}_t\big] \ge -C.$$

Hence we have the lower bound.

To derive the upper bound, we use the bound $g(s) \le C(1+s)$, $s>0$, for some $C>0$, of (3.1). Since $V(T,s)=g(s)$, it only remains to derive this upper bound for $t<T$. Consider the buy-and-hold strategy consisting in holding an amount C in the bank and C shares of the risky asset until the maturity T, i.e., $Y(u)=C$. Clearly, Z^Y is super-replicating. Notice that this buy-and-hold strategy induces a liquidity cost at the initial time t. As in Proposition 3.1, this liquidity cost can be avoided within our set of admissible strategies, see also Remark 3.2. Hence, for every $\varepsilon>0$, one can find a super-replicating strategy with initial cost $\varepsilon + C(1+s)$, and therefore $V(t,s) \le C(1+s)$ for every $t \in [0,T)$. □

6 Uniqueness

To complete the proof of the main Theorem 3.3, we need to prove a comparison result for (3.4). Due to the quadratic nonlinearity in (3.4), standard results do not directly apply to this equation. Moreover, due to the lack of homogeneity in the s-variable, the techniques used in [2] do not apply either. However, we use the special structure of the equation coming from the fact that it is one-dimensional and consider the equivalent equation

$$-A(t,s)V_t + F(t,s,V_{ss}) = 0, \tag{6.1}$$

where

$$A(t,s) := \frac{4\ell(t,s)}{s^2\sigma^2(t,s)}, \qquad F(t,s,V_{ss}) := \big[\ell^2(t,s) - \big(\big(V_{ss}+\ell(t,s)\big)^+\big)^2\big]. \tag{6.2}$$

Proposition 6.1 *Let* (3.2) *and* (3.3) *hold, let* $\tilde{w}$ *be a lower semi-continuous supersolution of* (6.1), *and let* w *an upper semi-continuous subsolution of* (6.1). *Further assume that* $\tilde{w}$ *and* w *satisfy the growth condition* (5.1) *and the boundary conditions*

$$w(T,s) \le \tilde{w}(T,s), \quad s \ge 0, \tag{6.3}$$

$$w(t,0) \le \tilde{w}(t,0), \quad t \in [0,T]. \tag{6.4}$$

Then $w \le \tilde{w}$ *on* $[0,T] \times \mathbb{R}^+$.

Proof 1. Set

$$\psi(t,s) := w(t,s) - \tilde{w}(t,s).$$

The goal is to show that $\psi \le 0$ on $[0,T] \times \mathbb{R}^+$. We suppose the contrary and assume that there exists $(t_0, s_0) \in [0,T] \times \mathbb{R}^+$ such that $\psi(t_0,s_0) > 0$. Since $\psi \le 0$ on the parabolic boundary $(\{T\} \times \mathbb{R}^+) \cup ([0,T] \times \{0\})$ and ψ is upper semi-continuous, it is clear that $t_0 < T$ and $s_0 > 0$. Moreover, again by upper semi-continuity, for any compact subset $K \subset [0,T] \times \mathbb{R}^+$ containing (t_0,s_0), there exists $\delta > 0$ such that we have

$$\sup_K \psi = \sup_{\mathcal{N} \cap K} \psi, \quad \text{where } \mathcal{N} := [0, T-2\delta] \times \big[2\delta, \delta^{-1}\big).$$

2. Following the usual trick in the theory of viscosity solutions [10], we construct a strict supersolution to (6.1). In view of the previous step, we only need this property on the domain $\mathcal{N}$.

For $\gamma \ge 1$, we set

$$\eta(t,s) := \big[s \ln(s) + \gamma\big](T - t + 1),$$

so that, for $(t,s) \in \mathcal{N}$,

$$\begin{aligned}
I[\eta](t,s) &:= -A(s)\eta_t(t,s) + F\big(s, \eta_{ss}(t,s)\big) \\
&= A(s)\big[s\ln(s) + \gamma\big] - \frac{(T-t+1)^2}{s^2} - \frac{2\ell(s)(T-t+1)}{s} \\
&= \frac{1}{s^2}\left(\frac{4\ell(s)}{\sigma^2(t,s)}\big[s\ln(s)+\gamma\big] - (T-t+1)^2 - 2s\ell(s)(T-t+1)\right) \\
&= \frac{1}{s^2}\left(\frac{4\ell(s)}{\sigma^2(t,s)}\big[s\ln(s)+\gamma\big] - c - cs\ell(s)\right) \\
&\ge \frac{\ell(s)}{s^2\sigma^2(t,s)}\left(\left[2\gamma - \frac{\sigma^2(t,s)}{\ell(s)}\, c\right] + \big[4s\ln(s) + 2\gamma - c\sigma^2(t,s)s\big]\right).
\end{aligned}$$

By conditions (3.2) and (3.3),

$$\sup_{\mathcal{N}} \left\{\frac{\sigma^2(t,s)}{\ell(s)} + \sigma^2(t,s)\right\} < \infty.$$

Hence there is $\gamma \geq 1$ such that

$$c[\eta] := \inf_{\mathcal{N}} I[\eta] > 0.$$

Let C be the constant in (5.1). We can choose $\gamma \geq 1$ so that in addition to the above inequality, we also have

$$\eta(t,s) \geq C[1+s] \geq \max\left\{w(t,s);\ \tilde{w}(t,s)\right\}. \tag{6.5}$$

3. For $\mu \in [0,1]$ set

$$w^{\mu} := (1-\mu)\tilde{w} + \mu\eta.$$

Let $I[\cdot]$ be defined as in the previous step. Then, by the concavity of F, on $\mathcal{N}$,

$$I\left[w^{\mu}\right] \geq (1-\mu)I[\tilde{w}] + \mu I[\eta] \geq \mu c[\eta].$$

Hence, w^{μ} is a strict supersolution of (6.1) on $\mathcal{N}$.

4. Set

$$\psi^{\mu}(t,s) := w(t,s) - w^{\mu}(t,s).$$

In Step 1, we assumed that $\psi(t_0, s_0) > 0$. Hence, for μ sufficiently small, we also have $\psi^{\mu}(t_0, s_0) > 0$. In view of (6.4), (6.3), and (6.5), $\psi^{\mu} \leq 0$ on the parabolic boundary $(\{T\} \times \mathbb{R}^+)(\cup[0,T] \times \{0\})$. Also, for all $\mu > 0$, the growth of η is faster than linear, and by (5.1) and Step 1, we conclude that ψ^{μ} attains its maximum at some point $(t^{\mu}, s^{\mu}) \in [0, T-2\delta] \times [2\delta, (2\delta)^{-1}) \subset \mathcal{N}$, i.e.,

$$\psi^{\mu}\left(t^{\mu}, s^{\mu}\right) = \sup_{\mathcal{N}} \psi^{\mu} = \sup_{[0,T]\times[0,\infty)} \psi^{\mu}.$$

Fix $\mu > 0$ satisfying the above.

5. Let μ be as above and for $\alpha > 0$, consider the auxiliary function

$$\Phi^{\alpha,\mu}(t,s;\bar{t},\bar{s}) := w(t,s) - w^{\mu}(\bar{t},\bar{s}) - \frac{\alpha}{2}\left[|t-\bar{t}|^2 + |s-\bar{s}|^2\right].$$

In view of the previous step, for all small $\mu > 0$ and sufficiently large $\alpha \geq 1$, there is a maximizer $(t^{\alpha,\mu}, s^{\alpha,\mu}, \bar{t}^{\alpha,\mu}, \bar{s}^{\alpha,\mu})$ of $\Phi^{\alpha,\mu}$. Moreover, as α tends to infinity, $(t^{\alpha,\mu}, s^{\alpha,\mu}, \bar{t}^{\alpha,\mu}, \bar{s}^{\alpha,\mu})$ approaches to $(t^{\mu}, s^{\mu}, t^{\mu}, s^{\mu})$. Since

$$\left(t^{\mu}, s^{\mu}\right) \in [0, T-2\delta] \times \left[2\delta, (2\delta)^{-1}\right) \subset \mathcal{N},$$

for all large α, $(t^{\alpha,\mu}, s^{\alpha,\mu}, \bar{t}^{\alpha,\mu}, \bar{s}^{\alpha,\mu}) \in \mathcal{N} \times \mathcal{N}$. Also,

$$\begin{aligned} &\lim_{\alpha\to\infty} \alpha\left[\left|t^{\alpha,\mu} - \bar{t}^{\alpha,\mu}\right|^2 + \left|s^{\alpha,\mu} - \bar{s}^{\alpha,\mu}\right|^2\right] = 0, \\ &c_{\mu} := \sup\nolimits_{\alpha>1}\left[\left|s^{\alpha,\mu}\right| + \left|\bar{s}^{\alpha,\mu}\right|\right] < \infty. \end{aligned} \tag{6.6}$$

6. By the Crandall–Ishii lemma (see [10] or Sect. V.6 in [16]), there are $a^{\alpha,\mu} \le b^{\alpha,\mu}$ such that

$$\begin{aligned}&\big(q^\alpha, p^\alpha, a^{\alpha,\mu}\big) \in \mathcal{D}^{+(1,2)} w\big(t^{\alpha,\mu}, s^{\alpha,\mu}\big), \qquad \big(q^\alpha, p^\alpha, b^{\alpha,\mu}\big) \in \mathcal{D}^{-(1,2)} w^\mu\big(\bar{t}^{\alpha,\mu}, \bar{s}^{\alpha,\mu}\big),\\ &q^\alpha := \alpha\big[t^{\alpha,\mu} - \bar{t}^{\alpha,\mu}\big], \qquad p^\alpha := \alpha\big[s^{\alpha,\mu} - \bar{s}^{\alpha,\mu}\big],\end{aligned}$$

and the sets $\mathcal{D}^{-(1,2)}$, $\mathcal{D}^{+(1,2)}$ are defined in [10, 16]. Formally, q^α is the generalized time derivative, p^α is the generalized space derivative, and $a^{\alpha,\mu}, b^{\alpha,\mu}$ are the generalized second derivatives. We now use the viscosity property of w and w^μ to obtain

$$-A\big(t^{\alpha,\mu}, s^{\alpha,\mu}\big)q^\alpha + F\big(t^{\alpha,\mu}, s^{\alpha,\mu}, a^{\alpha,\mu}\big) \le 0, \tag{6.7}$$

$$-A\big(\bar{t}^{\alpha,\mu}, \bar{s}^{\alpha,\mu}\big)q^\alpha + F\big(\bar{t}^{\alpha,\mu}, \bar{s}^{\alpha,\mu}, b^{\alpha,\mu}\big) \ge \mu c[\eta]. \tag{6.8}$$

Moreover, as on p. 217 in [16], we can show that $a^{\alpha,\mu}$, $b^{\alpha,\mu}$ satisfy the inequalities $|a^{\alpha,\mu}| + |b^{\alpha,\mu}| \le \alpha$ and

$$\begin{bmatrix} a^{\alpha,\mu} & 0 \\ 0 & -b^{\alpha,\mu} \end{bmatrix} \le 3\alpha \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{6.9}$$

Using (6.6), (6.9), and the local Lipschitz property of the coefficients from (3.2), (3.3), we show in Lemma 6.2 below that there is a constant C_μ such that

$$\begin{aligned}&\big|A\big(t^{\alpha,\mu}, s^{\alpha,\mu}\big) - A\big(\bar{t}^{\alpha,\mu}, \bar{s}^{\alpha,\mu}\big)\big|\big|q^\alpha\big|\\ &\quad\le C_\mu \alpha\big(\big|t^{\alpha,\mu} - \bar{t}^{\alpha,\mu}\big|^2 + \big|s^{\alpha,\mu} - \bar{s}^{\alpha,\mu}\big|^2\big),\end{aligned} \tag{6.10}$$

$$\begin{aligned}&\quad - F\big(t^{\alpha,\mu}, s^{\alpha,\mu}, a^{\alpha,\mu}\big) + F\big(\bar{t}^{\alpha,\mu}, \bar{s}^{\alpha,\mu}, b^{\alpha,\mu}\big)\\ &\quad\le C_\mu\big(\alpha\big|t^{\alpha,\mu} - \bar{t}^{\alpha,\mu}\big|^2 + \alpha\big|s^{\alpha,\mu} - \bar{s}^{\alpha,\mu}\big|^2 + \big|t^{\alpha,\mu} - \bar{t}^{\alpha,\mu}\big|\\ &\qquad + \big|s^{\alpha,\mu} - \bar{s}^{\alpha,\mu}\big|\big).\end{aligned} \tag{6.11}$$

7. Subtract (6.7) from (6.8) and then use (6.10). The result is

$$\mu c[\eta] \le C_\mu \alpha\ \big[\big|t^{\alpha,\mu} - \bar{t}^{\alpha,\mu}\big|^2 + \big|s^{\alpha,\mu} - \bar{s}^{\alpha,\mu}\big|^2\big].$$

We let α tend to infinity and use (6.6). This implies that $\mu c[\eta] \le 0$. However, this contradicts the fact that μ and $c[\eta]$ are strictly positive. Hence, there is no (t_0, s_0) as in Step 1. Therefore, $\psi \le 0$ on $[0, T] \times \mathbb{R}^+$. □

We complete the above proof by proving the technical estimate (6.10).

Lemma 6.2 *Assume* (3.2) *and* (3.3). *Then* (6.10) *and* (6.11) *hold for all* $\alpha \ge 1$.

Proof 1. In view of (3.2) and (3.3), the coefficient A defined by (6.2) is locally Lipschitz on $\mathcal{N}$. Since by (6.6) $s^{\alpha,\mu}$ and $\bar{s}^{\alpha,\mu}$ are uniformly bounded in α, there exists a constant C_μ, possibly depending on μ, such that

$$
\begin{aligned}
&\left|A\left(t^{\alpha,\mu}, s^{\alpha,\mu}\right) - A\left(\bar{t}^{\alpha,\mu}, \bar{s}^{\alpha,\mu}\right)\right|\left|q^\alpha\right| \\
&\quad \le C_\mu\left[\left|t^{\alpha,\mu} - \bar{t}^{\alpha,\mu}\right| + \left|s^{\alpha,\mu} - \bar{s}^{\alpha,\mu}\right|\right]\left|q^\alpha\right| \\
&\quad = C_\mu \alpha \left[\left|t^{\alpha,\mu} - \bar{t}^{\alpha,\mu}\right|^2 + \left|s^{\alpha,\mu} - \bar{s}^{\alpha,\mu}\right|\left|t^{\alpha,\mu} - \bar{t}^{\alpha,\mu}\right|\right] \\
&\quad \le C_\mu \alpha\left[\left|t^{\alpha,\mu} - \bar{t}^{\alpha,\mu}\right|^2 + \left|s^{\alpha,\mu} - \bar{s}^{\alpha,\mu}\right|^2\right].
\end{aligned}
$$

2. We continue by proving inequality (6.11). To simplify the presentation, we suppress the superscripts in our notation, i.e., $s = s^{\alpha,\mu}$, $a = a^{\alpha,\mu}$, etc. By the definition of F in (6.2),

$$
\begin{aligned}
&-F\left(t^{\alpha,\mu}, s^{\alpha,\mu}, a^{\alpha,\mu}\right) + F\left(\bar{t}^{\alpha,\mu}, \bar{s}^{\alpha,\mu}, b^{\alpha,\mu}\right) \\
&\quad = -F(t, s, a) + F(\bar{t}, \bar{s}, b) \\
&\quad = \ell^2(\bar{t}, \bar{s}) - \ell^2(t, s) + \left(a + \ell(t, s)\right)^{+2} - \left(b + \ell(\bar{t}, \bar{s})\right)^{+2} \\
&\quad = C_\mu\left(\left|t^{\alpha,\mu} - \bar{t}^{\alpha,\mu}\right|\left|s^{\alpha,\mu} - \bar{s}^{\alpha,\mu}\right|\right) + \left(a + \ell(t, s)\right)^{+2} - \left(b + \ell(\bar{t}, \bar{s})\right)^{+2}.
\end{aligned}
$$

If $(a + \ell(t, s))^{+2} - (b + \ell(\bar{t}, \bar{s}))^{+2} \le 0$, then the proof of the required estimate is complete. We then continue to prove the estimate in the case

$$
\left(a + \ell(t, s)\right)^{+2} - \left(b + \ell(\bar{t}, \bar{s})\right)^{+2} > 0. \tag{6.12}
$$

Since the matrix inequality (6.9) implies that $a \le b$, it follows from the increase of the function $z \mapsto z^+$ that

$$
\begin{aligned}
&\left(a + \ell(t, s)\right)^{+2} - \left(b + \ell(\bar{t}, \bar{s})\right)^{+2} \\
&\quad = \left(\left(a + \ell(t, s)\right)^+ - \left(b + \ell(\bar{t}, \bar{s})\right)^+\right)\left(\left(a + \ell(t, s)\right)^+ + \left(b + \ell(\bar{t}, \bar{s})\right)^+\right) \\
&\quad \le \left(\left(a + \ell(t, s)\right)^+ - \left(b + \ell(\bar{t}, \bar{s})\right)^+\right)\left(\left(b + \ell(t, s)\right)^+ + \left(b + \ell(\bar{t}, \bar{s})\right)^+\right) \\
&\quad \le \left((a - b) + C_\mu\left(|s - \bar{s}| + |t - \bar{t}|\right)\right)\left(|b| + C_\mu\right).
\end{aligned}
$$

3. We now use again the restriction (6.9) to estimate the right-hand side of the final inequality in Step 2. We already know that (6.9) implies that $a \le b$, but it is stronger than that. Indeed, multiply (6.9) by a general vector (x, y) both from right and left. The result is

$$
ax^2 - by^2 \le 3\alpha(x - y)^2, \quad x, y \in \mathbb{R}.
$$

By choosing $x = y$, we obtain $a \le b$. But this choice need not be optimal, and by calculus we conclude that

$$
a - b \le -\frac{b^2}{3\alpha + b}.
$$

4. Observe that the above estimates imply that $\frac{b^2}{3\alpha+b} \geq C_\mu(|t-\bar{t}|+|s-\bar{s}|)$ contradicts (6.12). Hence $\frac{b^2}{3\alpha+b} \leq C_\mu(|t-\bar{t}|+|s-\bar{s}|)$. Since $|b| \leq \alpha$, this implies that

$$b^2 \leq C_\mu \alpha\big(|t-\bar{t}|+|s-\bar{s}|\big)$$

with a possibly different constant denoted by C_μ again. We substitute this into the estimate of Step 2. The result is

$$\begin{aligned}
&\big(a+\ell(t,s)\big)^{+2} - \big(b+\ell(\bar{t},\bar{s})\big)^{+2} \\
&\quad \leq C_\mu\big(|t-\bar{t}|+|s-\bar{s}|\big)\big(|b|+C_\mu\big) \\
&\quad \leq C_\mu\big(|t-\bar{t}|+|s-\bar{s}|\big) + C_\mu|b|\big(|t-\bar{t}|+|s-\bar{s}|\big) \\
&\quad \leq C_\mu\big(|t-\bar{t}|+|s-\bar{s}|\big) + C_\mu\big[|b|^2+1\big]\big(|t-\bar{t}|+|s-\bar{s}|\big) \\
&\quad \leq C_\mu\big(|t-\bar{t}|+|s-\bar{s}|\big) + C_\mu\big[C_\mu\alpha\ (|t-\bar{t}|+|s-\bar{s}|)+1\big]\big(|t-\bar{t}|+|s-\bar{s}|\big) \\
&\quad \leq C_\mu\big(|t-\bar{t}|+|s-\bar{s}|+\alpha|s-\bar{s}|^2+\alpha|t-\bar{t}|^2\big). \qquad \square
\end{aligned}$$

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Appendix

A.1 Sufficient conditions for (4.1)

In the following two remarks, we state conditions that imply (4.1). Set

$$L(s) := s\sqrt{\ell(s)}.$$

Our arguments below require polynomial-type growth conditions on the functions related to the second derivative of the value function. Here we avoid these technical discussions and simply assume the appropriate growth condition. However, they could be easily verified in all interesting examples.

Remark A.1 Set

$$h(t,s) := \frac{s\sigma}{2\sqrt{\ell(s)}}\ V_{ss}(t,s).$$

Suppose that $s^2 L_{ss}/L$ is bounded from above, the claim g is convex, and h is polynomially growing. Then the minimal super-replicating cost V is convex. In particular, (4.1) holds.

Proof Since g is convex, (4.1) holds on an open set including $\{T\} \times [0, \infty)$. On this subset,

$$0 = -V_t - \frac{\sigma^2 s^2}{4\ell(s)}(V_{ss})^2 - \frac{\sigma^2 s^2}{2} V_{ss}. \tag{A.1}$$

Then,

$$0 = -V_t - h^2(t, s) - \sigma L(s) h.$$

By a direct calculation,

$$\begin{aligned}
0 &= -h_t - \frac{s\sigma}{2\sqrt{\ell(s)}}\left(h^2\right)_{ss} - \frac{s\sigma^2}{2\sqrt{\ell(s)}}(Lh)_{ss} \\
&= -h_t - \frac{s\sigma}{\sqrt{\ell(s)}}\left[h h_{ss} + (h_s)^2\right] - \frac{s\sigma^2}{2\sqrt{\ell(s)}}[L h_{ss} + L_s h_s + L_{ss} h] \\
&= -h_t - \frac{s\sigma}{2\sqrt{\ell(s)}}[V_{ss} + \ell] h_{ss} - a(t, s) h_s + b(t, s) h,
\end{aligned}$$

where

$$a(t, s) = \frac{s\sigma}{2\sqrt{\ell(s)}}[2h_s + \sigma L_s], \qquad b(s) := -\frac{s\sigma^2}{2\sqrt{\ell(s)}} L_{ss} = -\frac{\sigma^2}{2} \frac{s^2 L_{ss}(s)}{L(s)}.$$

We assume that $h(T, \cdot) \geq 0$ and $b(s)$ is bounded from below. Hence, by the Feynman–Kac representation of linear equations (or equivalently by the classical maximum principle), we conclude that h is nonnegative. Hence the value function is convex on the open set where (A.1) holds. By iterating the procedure, we conclude that this open set is the whole space and V is convex. □

Remark A.2 Set

$$H(t, s) := \frac{s\sigma}{2\sqrt{\ell(s)}}\left[V_{ss}(t, s) + \ell(s)\right].$$

Suppose that $L^2(s) = s^2 \ell(s)$ is concave, the claim $g_{ss}(s) \geq -\ell(t, s)$, and H is polynomially growing. Then the minimal super-replicating cost V satisfies (4.1).

Proof Again (A.1) holds on an open set including $\{T\} \times [0, \infty)$. We rewrite it as

$$0 = -V_t + \frac{\sigma^2 L^2(s)}{4} - \frac{\sigma^2 s^2}{4\ell(s)}\left(V_{ss} + \ell(s)\right)^2.$$

Then,

$$0 = -H_t - \frac{s\sigma}{2\sqrt{\ell(s)}}\left[H^2 - \frac{\sigma^2 L^2}{4}\right]_{ss}$$
$$= -H_t - \frac{s\sigma}{2\sqrt{\ell(s)}}\left[2HH_{ss} + 2(H_s)^2 - \frac{\sigma^2}{4}\big(L^2\big)_{ss}\right]$$
$$\le -H_t - \frac{s\sigma}{2\sqrt{\ell(s)}}\big[2HH_{ss} + 2(H_s)^2\big].$$

Again, we conclude by using the maximum principle or equivalently by the Feynman–Kac representation. □

A.2 An example

In this subsection, we provide a simple example in which the buy-and-hold regime is nonempty. Since Theorem 4.1 and Remark A.1 imply that, for convex terminal data, hedging is classical, we study the simplest concave payoff, namely

$$g(s) := s \wedge 1,$$

with supply curve as in the previous section, i.e.,

$$\ell(s) = 1/\big(\alpha s^2\big), \qquad \phi(t,s) = \frac{1}{4\alpha}\big[\sigma^2(t-T) + 4\ln s\big].$$

For brevity, we do not provide all technical details of the subsequent arguments. However, we believe that these arguments can be turned into a rigorous proof with some care.

Since the claim g has a concave discontinuity at $s = 1$ and (3.4) is degenerate for sufficiently negative second derivatives, we expect that the value function inherits this property on some interval $(t^*, T]$. Then, $(t, 1) \in \mathcal{C}$ for $t \in (t^*, T)$. Since (1.1) has on $\mathcal{C}$ the form (4.2), a direct calculation implies that

$$V(t,1) = 1 + \phi(t,1), \quad t \in [t^*, T].$$

We continue by constructing the value function using this additional boundary condition. We also use the intuition that $\mathcal{C} = (t^*, T) \times \{1\}$. Indeed, with $t^* < T$ to be chosen below, let v^+ be the solution of the dynamic programming equation (3.4) on the domain $(t^*, T) \times (1, \infty)$ with boundary condition

$$v^+(T,s) = g(s) = 1, \quad s \ge 1, \qquad v^+(t,1) = V(t,1) = 1 + \phi(t,1), \quad t > t^*.$$

Observe that the boundary condition at $s = 1$ formally implies that $v^+_{ss}(t,1) = -\ell(1)$. This, together with the terminal condition and the arguments of Remark A.2, implies that $v^+_{ss}(t,s) \ge -\ell(s)$ and v^+ is smooth. Similarly, we define v^- as the solution of (3.4) on the domain $(t^*, T) \times (0,1)$ with boundary condition

$$v^-(T,s) = g(s) = s, \quad s \le 1, \qquad v^-(t,1) = V(t,1) = 1 + \phi(t,1), \quad t > t^*.$$

The same argument implies that $v^-_{ss}(t,s) \geq -\ell(s)$ as well. Set

$$v^+_s(1^+, t) := \lim_{s \downarrow 1} v^+_s(s,t), \qquad v^-_s(1^-, t) := \lim_{s \uparrow 1} v^-_s(s,t).$$

Observe that $v^+_s(1^+, T) = 0$ and $v^-_s(1^+, T) = 1$. Let t^* be the smallest time point such that

$$v^+_s(1^+, t) < v^-_s(1^-, t), \quad t \in (t^*, T].$$

We formally expect that $t^* < T$. Then

$$\bar{V}(t,s) := v^+(t,s)\mathbf{1}_{\{s \geq 1\}} + v^-(t,s)\mathbf{1}_{\{s \leq 1\}}$$

has a concave first-order discontinuity at $s = 1$. Using this fact, one may directly show that $\bar{V}$ is a viscosity solution of (3.4) on $(t^*, T) \times (0, \infty)$, although it is discontinuous at $s = 1$. Then, by the comparison result for (3.4), $\bar{V}$ is equal to the value function V on this region. For $t < t^*$, the value function satisfies condition (4.1), and it is a smooth solution.

Given the above structure of the value function, we can construct a hedge along the lines described in the previous section. However, in this example $\mathcal{C} = (t^*, T) \times \{1\}$, and this simplifies the construction. Indeed, with an initial point $t_0 > t^*$, $s_0 = 1$, choose $\delta \in (v^+_s(1^+, t_0), v^-_s(1^-, t_0))$. Set

$$\begin{aligned}\psi(t,s) &:= 1 + \big(\delta - \phi_s(t,1)\big)(s-1) + \phi(t,s) \\ &= 1 + (\delta - 1/\alpha)(s-1) + \frac{1}{4\alpha}\big[\sigma^2(t-T) + 4\ln s\big],\end{aligned}$$

so that ψ solves (A.1) and equivalently (3.4). Moreover, for each $t \geq t_0$, set

$$\begin{aligned} L(t) &:= \max\big\{s > 1 \;:\; \psi_s(t,s') \geq v^+_s(t,s'),\ \forall s' \in (1,s)\big\}, \\ R(t) &:= \min\big\{s > 1 \;:\; \psi_s(t,s') \leq v^-_s(t,s'),\ \forall s' \in (s,1)\big\}. \end{aligned}$$

Now define the hedge by $Y(u) = \psi_s(u, S(u))$ until the exit time τ_1 of the process $(u, S(u))$ from the domain $(R(t), L(t))$.

It is clear that $Y(\tau_1) = V_s(\tau_1, S(\tau_1))$, and by the calculations of Sect. 4.2,

$$Z(\tau_1) = \psi\big(\tau_1, S(\tau_1)\big) > V\big(\tau_1, S(\tau_1)\big).$$

After the exit time τ_1, set $Y(u) := V_s(u, S(u))$ until the next stopping time $S(\tau_2) = 1$.

We should like to continue this process. For that, we need $Y(\tau_2)$ to be strictly in the interval $(v^+_s(1^+, \tau_2), v^-_s(1^-, \tau_2))$, but in fact $Y(\tau_2)$ is equal to one of the end points. However,

$$Z(\tau_2) - V\big(\tau_2, S(\tau_2)\big) = Z(\tau_1) - V\big(\tau_1, S(\tau_1)\big) > Z(t_0) - V(t_0, 1) = 0.$$

Moreover, by the results of [32] (see also Remark 3.2), we may change the portfolio value $Y(\tau_2)$ to any value (in particular, to a point in the interval $(v^+_s(1^+, \tau_2), v^-_s(1^-, \tau_2)))$ with arbitrarily small cost which can be covered by the gains $Z(\tau_2) - V(\tau_2, S(\tau_2))$. Hence, we may reiterate the process to replicate this particular claim from any initial data $(t_0, 1)$. From any other initial data we simply follow the usual hedge until the stopping time $S(\tau) = 1$. Then, we use the above procedure.

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