

# Large liquidity expansion of super-hedging costs

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**Abstract.** We consider a financial market with liquidity cost as in [Çetin, Jarrow and Protter, *Finance and Stochastics* **8** (2004), 311–341], where the supply function  $S^\varepsilon(s, \nu)$  depends on a parameter  $\varepsilon \geq 0$  with  $S^0(s, \nu) = s$  corresponding to the perfect liquid situation. Using the PDE characterization of Çetin, Soner and Touzi [*Finance and Stochastics* **14**(3) (2010), 317–341], of the super-hedging cost of an option written on such a stock, we provide a Taylor expansion of the super-hedging cost in powers of  $\varepsilon$ . In particular, we explicitly compute the first term in the expansion for a European Call option and give bounds for the order of the expansion for a European Digital Option.

Keywords: super-replication, liquidity, viscosity solutions, asymptotic expansions

## 1. Introduction

The classical option pricing equation of Black and Scholes is derived under several simplifying assumptions. The “infinite” liquidity of the underlying stock process is one of them. In an attempt to understand the impact of liquidity, Çetin, Jarrow, Protter and collaborators [3–5] postulated the existence of a supply curve  $S(t, s, \nu)$  which is the price of a share of the stock when one wants to buy  $\nu$  shares at time  $t$ . In the Black and Scholes setting, this price function is taken to be independent of  $\nu$  corresponding to infinite amount of supply, hence infinite liquidity. In a recent paper, Çetin, Soner and Touzi [6] used this model and studied the liquidity premium in the minimum hedging cost of an option written on such a stock with less than infinite liquidity. They characterized the minimum hedging cost by a nonlinear Black and Scholes equation, given in (2.3). For convex options, it turns out that the minimum hedging cost inherits the convexity, and this equation takes the simple form

$$-V_t - \frac{1}{2}s^2\sigma^2(t, s)V_{ss} \left[ 1 + \frac{1}{4\ell(t, s)}V_{ss} \right] = 0 \quad \text{on } [0, T) \times (0, \infty),$$

satisfying the terminal condition  $V(T, \cdot) = g$ . Here the given function  $\ell$  is related to the supply curve through the equation,

$$\ell(t, s) := \left[ 4 \frac{\partial S}{\partial \nu}(t, s, 0) \right]^{-1}, \quad (t, s) \in [0, T] \times \mathbb{R}_+.$$

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Hence this *liquidity function*  $\ell$  measures the level of liquidity of the market. Namely, the larger values of  $\ell$  indicates a more liquid market. Indeed,  $\ell \equiv \infty$  corresponds to the classical Black–Scholes equation (with zero interest rate). However, for a finite  $\ell$ , the above equation is fully nonlinear and one may rewrite it as

$$-V_t - \frac{1}{2}s^2\hat{\sigma}^2(t, s, V_{ss})V_{ss} = 0, \quad \text{where } \hat{\sigma}(t, s, \gamma) := \sigma(t, s) \left[ 1 + \frac{1}{4\ell(t, s)}\gamma \right]^{1/2}$$

for the liquidity premium defined as the difference between the solution of Eq. (2.3) and the Black and Scholes price of the option. This representation shows that the illiquidity increases the volatility, in a nonlinear manner, through the so-called Gamma of the option. In the context of proportional transaction costs, this was observed by Barles and Soner [2]. The resulting equation in [2] has the same qualitative structure. Again for transaction costs, Leland [18] and then through a discrete approximation Kusuoka [16] also derived a similar volatility change. These papers, [2,6,16] characterize the resulting pricing function as the unique viscosity solution of a nonlinear Black–Scholes equation. However, this nonlinear equation can only be solved numerically as no explicit solutions are available. In the case of [6], it is clear that for large  $\ell$  values the pricing equation converges to the classical Black–Scholes equation. Therefore asymptotic expansions around this solution would provide valuable approximations. In this paper, we obtain rigorous asymptotic expansions for the *liquidity premium* defined as the difference between the solution of Eq. (2.3). For vanilla options with sufficiently regular pay-off, this expansion can be calculated explicitly giving further insight into the liquidity effects.

The power of asymptotic expansions is demonstrated for stochastic volatility models by Fouque, Papanicalou and Sircar [14]. They obtained expansions for a two-dimensional model with a fast variable by Black–Scholes equations with modified parameters. We refer to the very recent book by Fouque et al. [15] for more information on this deep theory and for further references. These asymptotic expansions have many applications such as easy calibration of the model and are widely used in the financial industry. In the context of liquidity, the limiting equation is the standard model without any modification of the parameters and the higher-order terms in the expansion give the information. Hence, the techniques used in this paper need to be different from the ones used in [15] as the equation we consider is nonlinear and the small parameter does not correspond to averaging.

As discussed above, the main objective of this paper is to analyze the large liquidity effect. Thus, we assume that the supply function depends on a small parameter  $\varepsilon$

$$\mathbf{S}^\varepsilon(t, s, \nu) := \mathbf{S}(t, s, \varepsilon\nu), \quad (t, s) \in [0, T] \times \mathbb{R}_+.$$

Then, the corresponding liquidity function is given by

$$\ell^\varepsilon(t, s) := \frac{1}{\varepsilon}\ell(t, s), \quad (t, s) \in [0, T] \times \mathbb{R}_+.$$

Hence, as  $\varepsilon$  tends to zero, the market becomes completely liquid. So we expect the price of an option  $V^\varepsilon$  to converge to the classical Black–Scholes price,  $v^{\text{BS}}$ , and we are interested in expansions of the form

$$V^\varepsilon = v^{\text{BS}} + \varepsilon v^{(1)} + \dots + \varepsilon^n v^{(n)} + \dots + o(\varepsilon^n). \quad (1.1)$$

In Theorem 4.1, we prove this expansion using viscosity solution techniques, under the assumption of “smoothness” defined precisely in Section 4. We also identify the functions  $v^{(n)}$ . In particular,

$$v^{(1)}(t, s) = \int_t^T \mathbb{E}_{t,s} \left[ \frac{S_u^2 \sigma^2(u, S_u)}{4\ell(u, S_u)} (v_{ss}^{\text{BS}}(u, S_u))^2 \right] du, \quad (1.2)$$

which is exactly the liquidity premium of the standard Black–Scholes hedge. By an approximation, the smooth result can be applied to nonsmooth options as well. Indeed, one may simply approximate any nonsmooth pay-off by smooth ones and then apply the expansion to this approximating sequence. This procedure provides a two parameter approximation which is quite useful in almost all interesting applications.

Moreover, nonsmooth pay-offs can also be analyzed directly. We carry out this analysis in the cases of call and digital options. For a call option the first-order term given in (1.2) is finite and we prove that the expansion (1.1) holds with  $n = 1$ . We also numerically demonstrate that this expansion provides a very good approximation for the solution of the nonlinear equation (2.3). The case of a digital option is quite different as the function  $v^{(1)}$  defined in (1.2) is infinite. Hence the expansion (1.1) is not available and a boundary layer is formed around the final time. This case is further studied in Section 7.

The paper is organized as follows. The problem is introduced in the next section and the approach is formally introduced in Section 3. Under a strong smoothness assumption, full expansion is obtained in Section 4. A simple convergence result is discussed in Section 5. The Call option is studied in Section 6 and the Digital option in the final section.

## 2. The general setting

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space endowed with a Brownian motion  $W$  with completed canonical filtration  $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$ , where  $T > 0$  is fixed maturity. The marginal price process  $S_t$  is defined by the stochastic differential equation

$$\frac{dS_t}{S_t} = \sigma(t, S_t) dW_t,$$

where  $\sigma$  is assumed to be bounded, Lipschitz-continuous and uniformly elliptic.

Given a continuous portfolio strategy  $Y$  with finite quadratic variation process  $\langle Y \rangle$ , the *small time liquidation value of the portfolio* is given by

$$dZ_t^{\varepsilon, Y} = Y_t dS_t - [4\ell^\varepsilon(t, S_t)]^{-1} d\langle Y \rangle_t = Y_t dS_t - \varepsilon [4\ell(t, S_t)]^{-1} d\langle Y \rangle_t.$$

The dependence of the process  $Z$  on its initial condition is suppressed for simplicity.

Given a function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfying

$$g \text{ is bounded from below} \quad \text{and} \quad \sup_{s>0} \frac{g(s)}{1+s} < \infty, \quad (2.1)$$

the super-hedging cost is defined by

$$V^\varepsilon(t, s) := \inf \{ z : Z_t^{\varepsilon, Y} = z \text{ and } Z_T^{\varepsilon, Y} \geq g(S_T) \text{ } \mathbb{P}\text{-a.s. for some } Y \in \mathcal{A}_{t,s} \}, \quad (2.2)$$

where the time origin is removed to  $t$  and the initial condition for the price process is  $S_t = s$ . We refer to [6] for the precise definition of the set of admissible strategies  $\mathcal{A}_{t,s}$ .

This problem is similar to the super-replication problem studied extensively in [7–9,21–24]. In the above setting, it is shown by Çetin, Soner and Touzi [6] that the value function of the super-hedging problem is the unique viscosity solution of the following nonlinear equation,

$$-V_t^\varepsilon + \hat{H}^\varepsilon(t, s, V_{ss}^\varepsilon) = 0 \quad \text{on } [0, T) \times (0, \infty), \quad (2.3)$$

satisfying the terminal condition  $V^\varepsilon(T, \cdot) = g$  and the growth condition

$$-C \leq V^\varepsilon(t, s) \leq C(1 + s), \quad (t, s) \in [0, T] \times \mathbb{R}_+, \text{ for some constant } C > 0. \quad (2.4)$$

Here,  $\hat{H}^\varepsilon$  denotes the elliptic majorant of the operator  $H^\varepsilon$ :

$$\begin{aligned} \hat{H}^\varepsilon(t, s, \gamma) &:= \sup_{\beta \geq 0} H^\varepsilon(t, s, \gamma + \beta), \\ H^\varepsilon(t, s, \gamma) &:= -\frac{1}{2}s^2\sigma^2(t, s)\gamma - \varepsilon[4\ell(t, s)]^{-1}s^2\sigma^2(t, s)\gamma^2. \end{aligned}$$

A direct calculation shows that

$$\hat{H}^\varepsilon(t, s, \gamma) = -\frac{1}{2}s^2\sigma^2(t, s) \left[ \gamma + \left( \gamma + \frac{\ell(t, s)}{\varepsilon} \right)^{-} + \frac{\varepsilon}{2\ell(t, s)} \left( \gamma + \left( \gamma + \frac{\ell(t, s)}{\varepsilon} \right)^{-} \right)^2 \right].$$

For  $\varepsilon = 0$ , both  $\hat{H}^\varepsilon$ ,  $H^\varepsilon$  coincide with the following standard elliptic operator,

$$\hat{H}^0(t, s, \gamma) = H^0(t, s, \gamma) = -\frac{1}{2}s^2\sigma^2(t, s)\gamma, \quad (t, s, \gamma) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}.$$

Hence, Eq. (2.3) reduces to the linear Black–Scholes equation

$$-\frac{\partial v^{\text{BS}}}{\partial t} - \frac{1}{2}s^2\sigma^2(t, s)v_{ss}^{\text{BS}} = 0. \quad (2.5)$$

We recall the well-known fact that its unique solution,  $v^{\text{BS}}$ , is the Black–Scholes price,

$$v^{\text{BS}}(t, s) = \mathbb{E}_{t,s}[g(S_T)], \quad (t, s) \in [0, T] \times \mathbb{R}_+,$$

where we used the notation  $\mathbb{E}_{t,s} = \mathbb{E}[\cdot | S_t = s]$ . We also recall that in our model the interest rate is zero and since the problem of super-replication depends only on the null sets of the probability measure, we use the risk-neutral one.

### 3. Formal calculations and assumptions

It is formally clear that as the market becomes more liquid,  $V^\varepsilon$  should converge to the Black–Scholes price  $v^{\text{BS}}$ . Indeed, this is proved in Section 5. As discussed in the Introduction, the main goal is the following Taylor expansion of  $V^\varepsilon$  in the parameter  $\varepsilon$ ,

$$V^\varepsilon(t, s) = v^{\text{BS}}(t, s) + \varepsilon v^{(1)}(t, s) + \varepsilon^2 v^{(2)}(t, s) + \cdots + \varepsilon^n v^{(n)}(t, s) + o(\varepsilon^n), \quad (3.1)$$

where  $o(\varepsilon^n)$  is the standard notation, indicating that  $o(\varepsilon^n)/\varepsilon^n$  converges to zero as  $\varepsilon$  tends to zero.

In this section, we provide a formal derivation of this expansion. In order to determine

$$v^{(n)}(t, s) = \frac{1}{n!} \left. \frac{\partial^n V^\varepsilon(t, s)}{\partial \varepsilon^n} \right|_{\varepsilon=0},$$

we formally differentiate Eq. (2.3)  $n$ -times with respect to  $\varepsilon$  and then set  $\varepsilon$  to zero.

The result is,

$$0 = -v_t^{(n)} - \frac{1}{2} s^2 \sigma^2(t, s) v_{ss}^{(n)} - F_n(t, s), \quad (3.2)$$

$$F_n(t, s) = \frac{s^2 \sigma^2(t, s)}{4\ell(t, s)} \sum_{k=0}^{n-1} [v_{ss}^{(k)}(t, s) v_{ss}^{(n-1-k)}(t, s)], \quad (3.3)$$

where we set  $v^{(0)} := v^{\text{BS}}$ . For all  $n \geq 1$ , the terminal data is  $v^{(n)}(T, \cdot) \equiv 0$ , so that the Feynman–Kac formula yields

$$v^{(n)}(t, s) = \sum_{k=0}^{n-1} \mathbb{E}_{t,s} \left[ \int_t^T \left( \frac{S_u^2 \sigma^2}{4\ell} v_{ss}^{(k)} v_{ss}^{(n-1-k)} \right) (u, S_u) du \right]. \quad (3.4)$$

In particular,  $v^{(1)}$  is given as in (1.2).

The above calculations, together with the theory of viscosity solutions, provide a rigorous proof for the expansion. On the other hand, for some discontinuous pay-offs the above functions may not be finite. For instance, for a digital option,  $v^{(1)} \equiv \infty$ . Indeed, consider the following specific example

$$g(s) := \mathbb{1}_{s \geq K}, \quad \sigma(t, s) \equiv \sigma \quad \text{and} \quad \ell(t, s) \equiv \ell.$$

We directly calculate that

$$\begin{aligned} v^{(1)}(t, s) &= \frac{1}{8\pi\ell\sigma^2} \int_t^T \frac{(u-t) e^{-\left(\frac{1}{\sigma\sqrt{T+u-2t}} \ln\left(\frac{s}{K}\right) + \frac{\sigma}{2} \frac{T-2u+t}{\sqrt{T+u-2t}}\right)^2}}{(T-u)^{3/2}(T+u-2t)^{3/2}} \\ &\quad + \frac{1}{8\pi\ell\sigma^2} \int_t^T \frac{e^{-\left(\frac{1}{\sigma\sqrt{T+u-2t}} \ln\left(\frac{s}{K}\right) + \frac{\sigma}{2} \frac{T-2u+t}{\sqrt{T+u-2t}}\right)^2}}{\sqrt{T-u}(T+u-2t)^{3/2}} \left( \frac{\ln\left(\frac{s}{K}\right)}{\sigma\sqrt{T+u-2t}} + \frac{\sigma}{2} \frac{T-2u+t}{\sqrt{T+u-2t}} \right)^2. \end{aligned}$$

The first term above is infinite due to the non-integrability of  $(T-u)^{-3/2}$  near  $T$ . Hence for this and other examples with infinite first-order term, the expansion is not valid and a careful study of the behavior of  $V^\varepsilon$  near the terminal data is needed. This will be done in Section 7 for the above example.

As discussed earlier, we first prove the full expansion in the “smooth” case. Then, in Section 6, we consider the Call option proving the expansion up to  $n = 2$ . Clearly, this later result extends to all Put options. Also, remarks on other pay-offs and higher expansions are given in Remarks 6.2 and 6.1.

#### 4. Expansion for smooth pay-offs

In this section, we prove the expansion under a smoothness assumption. Let  $v^{(n)}$  be the unique solution of (3.4). We assume that there is a constant  $\hat{C}$  so that

$$\begin{aligned} -\hat{C} &\leq v^{(n)}(t, s) \leq \hat{C}(1+s), \\ |(s^2+1)v_{ss}^{(n)}(t, s)| &\leq \hat{C}, \\ |F_n(t, s)| &\leq \hat{C}, \quad \forall (t, s) \in [0, T] \times \mathbb{R}_+, n = 1, 2, \dots \end{aligned} \quad (4.1)$$

Clearly, this is an implicit assumption on the pay-off  $g$ . Essentially, it holds for all smooth pay-offs growing at most linearly. In particular, (4.1) holds if  $\sigma(t, s) \equiv \sigma$ ,  $\ell(t, s) \equiv \ell$  and if there exists a constant  $C$  so that

$$-C \leq g(s) \leq C(1+s), \quad \left| (s^2+1) \frac{\partial^n}{\partial s^n} g(s) \right| \leq C, \quad \forall s \in \mathbb{R}_+, n = 2, 3, \dots$$

Under the above assumption on  $g$  and the market parameters, we prove (4.1) by using the homogeneity of the Black–Scholes equation and differentiating it repeatedly.

We prove the asymptotic result using the techniques developed in the papers [11,13,17,19,20]. In preparation for the proof, for an integer  $n \geq 0$  we define,

$$V^{\varepsilon, n}(t, s) := \frac{V^\varepsilon(t, s) - \sum_{k=0}^{n-1} \varepsilon^k v^{(k)}(t, s)}{\varepsilon^n}, \quad (4.2)$$

where as before we set  $v^{(0)} = v^{\text{BS}}$  and  $v^{(n)}$  is the unique solution (3.4).

**Theorem 4.1.** *Assume (4.1). Then, for every  $n = 1, 2, \dots$ , there are constants  $C_n$  and  $\varepsilon_0 > 0$  so that for every  $\varepsilon \in (0, \varepsilon_0]$ , and  $n = 1, 2, \dots$ ,*

$$v^{\text{BS}}(t, s) \leq V^\varepsilon(t, s) \leq v^{\varepsilon, n}(t, s) := \sum_{k=0}^{n-1} [\varepsilon^k v^{(k)}(t, s)] + \varepsilon^n C_n (T-t). \quad (4.3)$$

*In particular, as  $\varepsilon \downarrow 0$ ,  $V^\varepsilon$  converges to the Black–Scholes price  $v^{\text{BS}}$  uniformly on compact sets. Moreover, for every  $n \geq 1$ ,  $V^{\varepsilon, n}$  converges to  $v^{(n)}$ , again uniformly on compact sets.*

**Proof.** Clearly,  $v^{\text{BS}} \leq V^\varepsilon$ . We continue by proving the upper bound. Let  $v^{\varepsilon,n}$  be as in (4.3) with a constant  $C_n$  to be determined below. Using (3.2), we calculate that

$$\begin{aligned} -v_t^{\varepsilon,n}(t, s) + \hat{H}^\varepsilon(t, s, v_{ss}^{\varepsilon,n}(t, s)) &\geq -v_t^{\varepsilon,n}(t, s) + H^\varepsilon(t, s, v_{ss}^{\varepsilon,n}(t, s)) \\ &= -v_t^{\varepsilon,n} - \frac{1}{2}s^2\sigma^2 v_{ss}^{\varepsilon,n} - \frac{\varepsilon s^2\sigma^2}{4\ell(t, s)}(v_{ss}^{\varepsilon,n})^2 \\ &= \varepsilon^n C_n + \sum_{k=1}^{n-1} [\varepsilon^k F_k(t, s)] - \frac{\varepsilon s^2\sigma^2}{4\ell(t, s)}(v_{ss}^{\varepsilon,n})^2. \end{aligned}$$

In view of (3.3),

$$\frac{\varepsilon s^2\sigma^2}{4\ell(t, s)}(v_{ss}^{\varepsilon,n})^2 - \sum_{k=1}^{n-1} [\varepsilon^k F_k(t, s)] = \varepsilon^n F_n(t, s) + \varepsilon^{n+1} \frac{s^2\sigma^2}{4\ell(t, s)} g^\varepsilon(t, s),$$

where  $g^\varepsilon(t, s)$  is a quadratic function of  $v_{ss}^{(k)}(t, s)$  for  $k \leq n$  and possibly powers of  $\varepsilon$ . Hence by (4.1), there exists a constant  $C_n$  such that

$$\left| \sum_{k=1}^{n-1} [\varepsilon^k F_k(t, s)] - \frac{\varepsilon s^2\sigma^2}{4\ell(t, s)}(v_{ss}^{\varepsilon,n})^2 \right| \leq \varepsilon^n C_n.$$

Hence, we conclude that  $v^{\varepsilon,n}$  is a supersolution of (2.3). Moreover, by (4.1),  $-C \leq v^{\varepsilon,n}(t, s) \leq C(1+s)$ . Then, by the comparison theorem for (2.3), [6], Theorem 6.1, we conclude that  $V^\varepsilon(t, s) \leq v^{\varepsilon,n}(t, s)$ .

In particular, this estimate implies the convergence of  $V^\varepsilon$  to  $v^{\text{BS}}$ . To prove the convergence of  $V^{\varepsilon,n}$ , we first observe that

$$V^\varepsilon = \sum_{k=0}^n [\varepsilon^k v^{(n)}(t, s)] + \varepsilon^n V^{\varepsilon,n}.$$

Using Eqs (2.3) and (3.2), we conclude that  $V^{\varepsilon,n}$  is a viscosity solution of

$$-V_t^{\varepsilon,n} - \frac{1}{2}s^2\sigma^2(t, s)V_{ss}^{\varepsilon,n} + F^{\varepsilon,n}(t, s, V_{ss}^{\varepsilon,n}) = 0, \quad (t, s) \in [0, T] \times \mathbb{R}_+,$$

where

$$F^{\varepsilon,n}(t, s, \gamma) := \frac{1}{\varepsilon^n} \left[ \hat{H}^\varepsilon(t, s, v_{ss}^{\varepsilon,n}(t, s) + \varepsilon^n \gamma) + \frac{1}{2}s^2\sigma^2 v_{ss}^{\varepsilon,n} + \sum_{k=1}^{n-1} \varepsilon^k F^k(t, s) \right].$$

Tedious but a straightforward calculation shows that

$$\lim_{(t', s', \gamma', \varepsilon) \rightarrow (t, s, \gamma, 0)} F^{\varepsilon,n}(t', s', \gamma') = F^n(t, s),$$

where  $F_n$  is as in (3.3). Then, by the classical stability results of viscosity solutions [1,10,12], the Barles–Perthame semi-relaxed limits

$$\underline{v}^{(n)}(t, s) := \liminf_{(t', s', \varepsilon) \rightarrow (t, s, 0)} V^{\varepsilon, n}(t', s') \quad \text{and} \quad \bar{v}^{(n)}(t, s) := \limsup_{(t', s', \varepsilon) \rightarrow (t, s, 0)} V^{\varepsilon, n}(t', s'),$$

are, respectively, a viscosity supersolution and a subsolution of the equation (3.2) satisfied by  $v^{(n)}$ . Moreover it follows from (4.3) that

$$\underline{v}^{(n)}(T, \cdot) = \bar{v}^{(n)}(T, \cdot) = 0 = v^{(n)}(T, \cdot).$$

We now use the comparison result for the linear partial differential equation (3.2), and conclude that  $\underline{v}^{(n)} \geq \bar{v}^{(n)}$ . Since

$$\underline{v}^{(n)}(t, s) \leq \liminf_{\varepsilon \rightarrow 0} V^{\varepsilon, n}(t, s) \leq \limsup_{\varepsilon \rightarrow 0} V^{\varepsilon, n}(t, s) \leq \bar{v}^{(n)}(t, s)$$

on  $[0, T] \times \mathbb{R}_+$ , this proves that  $\underline{v}^{(n)} = \bar{v}^{(n)} = v^{(n)}$ . Hence,  $V^{\varepsilon, n}$  converges to the unique solution  $v^{(n)}$ , uniformly on compact sets.  $\square$

## 5. A general convergence result

In this section, we prove the convergence of solutions to the Black and Scholes price as the market becomes infinitely liquid (or equivalently, as  $\varepsilon$  tends to zero). This is an expected and an easy result. The only technical issue is the behavior of  $V^\varepsilon$  near the maturity. So we make the following assumption that is essentially satisfied by all polynomially growing options.

**Assumption 5.1.** *There is a decreasing sequence of smooth approximation  $g_m \geq g$  of the pay-off  $g$  satisfying (4.1) with  $n = 1, 2$ .*

Let  $v_m^{\text{BS}}$  be the Black and Scholes value for the pay-off  $g_m$  and let  $v_m^{(n)}$  and  $F_m^n$  be as in (3.4) and (3.3) for the terminal condition  $g_m$ . Then, the above assumption states that there is a constant  $c_m$  such that

$$F_m^1(t, s) \leq c_m. \tag{5.1}$$

This assumption is satisfied by all Lipschitz or for all bounded pay-offs.

**Theorem 5.1.** *Assume (2.1) and that Assumption 5.1 holds true. Then, as the market becomes more liquid, or equivalently as  $\varepsilon \downarrow 0$ ,  $V^\varepsilon$  converges to the Black–Scholes price  $v^{\text{BS}}$ .*

**Proof.** Let  $c_m$  be as above and set

$$u^\varepsilon(t, s) := v_m^{\text{BS}}(t, s) + \varepsilon c_m(T - t).$$



As in the proof of Theorem 4.1, we can show that  $u^\varepsilon$  is a super-solution of (2.3). Hence,  $V^\varepsilon \leq u^\varepsilon$ . Therefore,

$$\limsup_{\varepsilon \downarrow 0} V^\varepsilon(t, s) \leq v_m^{\text{BS}}(t, s).$$

By (2.1),  $v_m^{\text{BS}}(t, s)$  converges to  $v^{\text{BS}}(t, s)$ . Since  $V^\varepsilon \geq v^{\text{BS}}$ , this proves the convergence of  $V^\varepsilon$  to  $v^{\text{BS}}$ .  $\square$

## 6. First-order expansion for convex pay-offs

One important limitation of our previous result is that the Call pay-off does not satisfy assumption (4.1). Therefore, in this section, we prove the first term in the Taylor expansion (3.1), i.e.,

$$V^\varepsilon(t, s) = v^{\text{BS}}(t, s) + \varepsilon v^{(1)}(t, s) + o(\varepsilon), \quad (6.1)$$

for convex pay-offs satisfying weaker assumptions than (4.1). In particular, we will show that call options verify those assumptions.

### 6.1. The general result

In order to use the results we have already obtained for smooth pay-offs, we consider the following regularization of the problem,

$$\begin{aligned} -V_t^{\varepsilon, \alpha} + \widehat{H}^\varepsilon(t, s, V_{ss}^{\varepsilon, \alpha}) &= 0 \quad \text{for } (t, s) \in [0, T) \times \mathbb{R}_+, \\ V^{\varepsilon, \alpha}(T, s) &= \widehat{g}_\alpha(s), \end{aligned} \quad (6.2)$$

where  $\widehat{g}_\alpha(s) = \phi_\alpha * g(s)$  with  $\phi_\alpha(\cdot) := \frac{1}{\alpha} \phi(\frac{\cdot}{\alpha})$  and  $\phi$  is a positive, symmetric bump function on  $\mathbb{R}$ , compactly supported in  $[-1, 1]$  and satisfying

$$\int_{-1}^1 \phi(u) du = 1.$$

By the convexity of  $g$ , for all  $\alpha > 0$  we have  $\widehat{g}_\alpha \geq g$ , so that

$$V^\varepsilon \leq V^{\varepsilon, \alpha}.$$

The main idea of our proof is to find a super-solution of (2.3). Hence it suffices to find a super-solution of (6.2). Let  $v^{\text{BS}, \alpha}$  and  $v^{(1), \alpha}$ , respectively, be the Black–Scholes price and the first-order expansion term for the regularized option. We now state our assumptions.

#### Assumption 6.1.

- (i)  $v^{\text{BS}} + v^{\text{BS}, \alpha} + v^{(1)} + v^{(1), \alpha} < +\infty$ .

(ii) As  $\alpha$  tends to 0 we have

$$\begin{aligned} v^{\text{BS},\alpha}(t, s) &= v^{\text{BS}}(t, s) + \mathcal{O}(\alpha^2), \\ v^{(1),\alpha}(t, s) &= v^{(1)}(t, s) + \mathfrak{o}(1). \end{aligned}$$

(iii) There exists a constant  $c_*$  independent of  $s, T - t$  and  $\alpha$  and  $(\nu, \beta) \in [0, 1] \times [1/2, 1]$  such that  $1 < 2\beta + \nu < 2$  and

$$\frac{s^2 \sigma^2}{4\ell} (v_{ss}^{(1),\alpha}(t, s))^2 \leq \frac{c_*}{(T-t)^{1-\nu} \alpha^{2+2\nu}}, \quad s |v_{ss}^{\text{BS},\alpha}(t, s)| \leq \frac{c_*}{(T-t)^{1-\beta} \alpha^{2\beta-1}}.$$

This assumption is satisfied by Call options pay-offs as verified in the Section 6.2.

Let  $V^{\varepsilon,1}$  be as (4.2), i.e.

$$V^{\varepsilon,1}(t, s) := \frac{V^\varepsilon(t, s) - v^{\text{BS}}(t, s)}{\varepsilon}.$$

**Theorem 6.1.** *Let Assumption 6.1 hold true and let  $a \in (\frac{1}{2}, \frac{1}{2\beta+\nu})$ . Then for every  $(t, s) \in [0, T] \times \mathbb{R}_+$  we have,*

$$v^{\text{BS}} \leq V^\varepsilon \leq v^{\text{BS},\varepsilon^a} + \varepsilon v^{(1),\varepsilon^a} + c_*(T-t)^{\beta+(\nu-1)/2} \varepsilon^{2-a(\nu+2\beta)} + c_*(T-t)^\nu \varepsilon^{3-2a(1+\nu)}.$$

Moreover,  $V^\varepsilon \rightarrow v^{\text{BS}}, V^{\varepsilon,1} \rightarrow v^{(1)}$  uniformly on compact sets, and (6.1) holds true.

**Proof.** It is clear that  $V^\varepsilon \geq v^{\text{BS}}$ . To prove the reverse inequality, we start by following a technique similar to the one used in the proof of Theorem 4.1. Set

$$v^{\varepsilon,2} := v^{\text{BS},\varepsilon^a} + \varepsilon v^{(1),\varepsilon^a} + c_*(T-t)^{\beta+(\nu-1)/2} \varepsilon^{2-a(\nu+2\beta)} + c_*(T-t)^\nu \varepsilon^{3-2a(1+\nu)}.$$

We calculate that for  $(t, s) \in [0, T] \times \mathbb{R}_+$

$$\begin{aligned} & -v_t^{\varepsilon,2} + \hat{H}^\varepsilon(t, s, v_{ss}^{\varepsilon,2}) \\ & \geq -v_t^{\varepsilon,2} + H^\varepsilon(t, s, v_{ss}^{\varepsilon,2}) \\ & = \frac{c_* \varepsilon^{2-a(\nu+2\beta)}}{(T-t)^{1-\beta-(\nu-1)/2}} + \frac{c_* \varepsilon^{3-2a(1+\nu)}}{(T-t)^{1-\nu}} - v_t^{\text{BS},\varepsilon^a} - \varepsilon v_t^{(1),\varepsilon^a} - \frac{1}{2} s^2 \sigma^2 v_{ss}^{\varepsilon,2} - \frac{\varepsilon s^2 \sigma^2}{4\ell} (v_{ss}^{\varepsilon,2})^2 \\ & = \frac{c_* \varepsilon^{2-a(\nu+2\beta)}}{(T-t)^{1-\beta-(\nu-1)/2}} + \frac{c_* \varepsilon^{3-2a(1+\nu)}}{(T-t)^{1-\nu}} - \frac{s^2 \sigma^2}{4\ell} (v_{ss}^{(1),\varepsilon^a})^2 \varepsilon^3 - \frac{s^2 \sigma^2}{2\ell} v_{ss}^{\text{BS},\varepsilon^a} v_{ss}^{(1),\varepsilon^a} \varepsilon^2. \end{aligned}$$

In view of Assumption 6.1(iii), this quantity is always positive. We now analyze the terminal condition.

In view of the conditions imposed on  $a, \beta$  and  $\nu$

$$v^{\varepsilon,2}(T, s) = v^{\text{BS},\varepsilon^a}(T, s) = \hat{g}_{\varepsilon^a}(s).$$

Hence,  $v^{\varepsilon,2}$  is a super-solution of (6.2) and therefore of (2.3). Then, by the comparison theorem for (2.3) (proved in [6]), we conclude that  $V^\varepsilon(t, s) \leq v^{\varepsilon,2}(t, s)$ .

We now let  $\varepsilon$  go to 0 in the above inequalities. This proves that  $V^\varepsilon$  converges to  $v^{\text{BS}}$  uniformly on compact sets.

Finally, by Assumption 6.1(ii)

$$0 \leq V^{\varepsilon,1}(t, s) \leq v^{(1)}(t, s) + o(\varepsilon^{\min\{1-a(2\beta+\nu), 2-2a(1+\nu)\}}) + O(\varepsilon^{2a-1}),$$

where it is clear with our conditions on  $a, \beta$  and  $\nu$  that the  $o(\cdot)$  and  $O(\cdot)$  above go to 0 as  $\varepsilon$  tends to 0.

Using this estimate, we then prove the convergence of  $V^{\varepsilon,1}$  exactly as in Theorem 4.1.  $\square$

**Remark 6.1.** Higher expansions can be proved similarly, provided that we extend Assumption 6.1 for  $n \geq 2$ .

## 6.2. Expansion for the Call option

In this section, we take

$$g(s) = (s - K)^+, \quad \sigma(t, s) \equiv \sigma, \quad \ell(t, s) \equiv \ell,$$

and we verify that Assumptions 6.1(ii) and (iii) are satisfied, since Assumption 6.1(i) is trivial.

Straightforward but tedious calculations using the Feynman–Kac formula yield

$$v_{ss}^{\text{BS},\alpha}(t, s) = \frac{1}{\sigma s \sqrt{2\pi\tau}} \int_{-1}^1 \phi(u) \exp\left(-\frac{1}{2}d_1(s, K + \alpha u, \tau)^2\right) du,$$

$$v^{(1),\alpha}(t, s) = \frac{1}{8\ell\pi} \int_0^\tau \int_{-1}^1 \int_{-1}^1 \frac{\phi(x)\phi(y)h_\alpha(\tau, v, s, K, x, y)}{\sqrt{v(2\tau - v)}} dx dy dv,$$

where

$$\tau = T - t,$$

$$d_1(s, k, t) = \frac{1}{\sigma\sqrt{t}} \ln(s/k) + \frac{1}{2}\sigma\sqrt{t},$$

$$\delta(\tau, v, s, k) = \frac{1}{\sigma\sqrt{2\tau - v}} \ln(s/k) - \frac{\sigma}{2} \frac{\tau - 2v}{\sqrt{2\tau - v}},$$

$$h_\alpha(\tau, v, s, k, x, y) = \exp\left(-\delta(\tau, v, s, k)^2 + \frac{\delta(\tau, v, s, k)}{\sigma\sqrt{2\tau - v}} \left(\log\left(1 + \frac{\alpha x}{k}\right) + \log\left(1 + \frac{\alpha y}{k}\right)\right)\right)$$

$$\times \exp\left(-\frac{\tau}{2\sigma^2 v(2\tau - v)} \left(\log\left(1 + \frac{\alpha x}{k}\right) - \log\left(1 + \frac{\alpha y}{k}\right)\right)^2\right)$$

$$\times \exp\left(-\frac{1}{\sigma^2(2\tau - v)} \log\left(1 + \frac{\alpha x}{k}\right) \log\left(1 + \frac{\alpha y}{k}\right)\right).$$

The following two propositions, whose proof is relegated to the Appendix, ensure that Assumptions 6.1(ii) and (iii) are satisfied.

**Proposition 6.1.** *There exists a constant  $c_*$ , independent of  $s$ ,  $\tau$  and  $\alpha$  so that for all  $(\nu, \beta) \in [0, 1] \times [1/2, 1]$ :*

$$s|v_{ss}^{\text{BS},\alpha}(t, s)| \leq \frac{c_*}{\tau^{1-\beta}\alpha^{2\beta-1}}, \quad \frac{s^2\sigma^2}{4\ell}(v_{ss}^{(1),\alpha}(t, s))^2 \leq \frac{c_*}{\tau^{1-\nu}\alpha^{2+2\nu}}.$$

**Proposition 6.2.** *As  $\alpha$  tends to 0 we have the following expansions*

$$v^{\text{BS},\alpha}(t, s) = v^{\text{BS}}(t, s) + \alpha^2 \frac{e^{-1/2d_0(s,K,\tau)^2}}{2K\sigma\sqrt{2\pi\tau}} \int_{-1}^1 \phi(v)v^2 dv + \mathcal{O}(\alpha^4),$$

$$v^{(1),\alpha}(t, s) = v^{(1)}(t, s) - \alpha \frac{e^{-1/2d_0(s,K,\tau)^2}}{8K\sigma\ell\sqrt{2\pi\tau}} \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y)|x-y| dx dy + \mathcal{o}(\alpha),$$

where  $d_0(s, k, \tau) = \frac{1}{\sigma\sqrt{\tau}} \ln(s/k) - \frac{1}{2}\sigma\sqrt{\tau}$ .

**Remark 6.2.** It is clear that the results of Propositions 6.1 and 6.2 hold for all convex linear combination of call or put options. However, we cannot use the above proof for, say, a call spread option whose payoff is neither convex nor concave.

### 6.3. Numerical experiments

In order to have a better understanding of the liquidity effects, we also numerically solve (with simple finite difference methods) the PDE (2.3). We represent below the behaviour of the liquidity premium (that is to say  $V^\varepsilon - v^{\text{BS}}$ ) when the time to maturity  $t$  and the spot price vary.

In Fig. 1, the liquidity effect is strong for all at-the-money (ATM) options. Also, as expected, it disappears quickly for in-the-money (ITM) and out-of-the-money (OTM) options. Indeed, our calculations showed that the liquidity effect is, for the first order, driven by the  $\Gamma$  of the call option (see (A.1)), which explodes for ATM options near maturity. Moreover, with our set of parameters, the first-order correction is at most 0.06 for a BS price of 8.56, which means that the hedge against liquidity risk is not that expensive when the illiquidity is not too strong.

We now compare the actual liquidity premium with its first-order expansion term. Figure 2 shows that the first-order approximation remains excellent near the maturity  $T$  and  $s = K$ . In other regions, the first-order overvalues the liquidity premium.

## 7. Digital option

In this section, we analyze the specific example of a Digital option in the context of Black–Scholes model with constant liquidity parameter

$$g(s) := \mathbb{1}_{s \geq K} \quad \text{and} \quad \sigma(t, s) \equiv \sigma, \quad \ell(t, s) \equiv \ell.$$

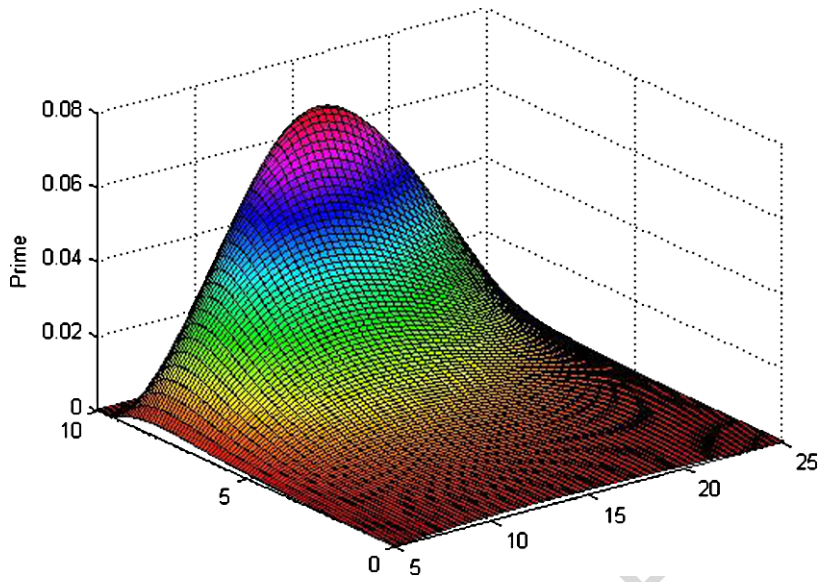


Fig. 1. Call liquidity premium –  $T = 10$ ,  $K = 15$ ,  $\sigma = 0.5$ ,  $\varepsilon = 0.1$ ,  $\ell = 1$ . (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/ASY-2011-1089>.)

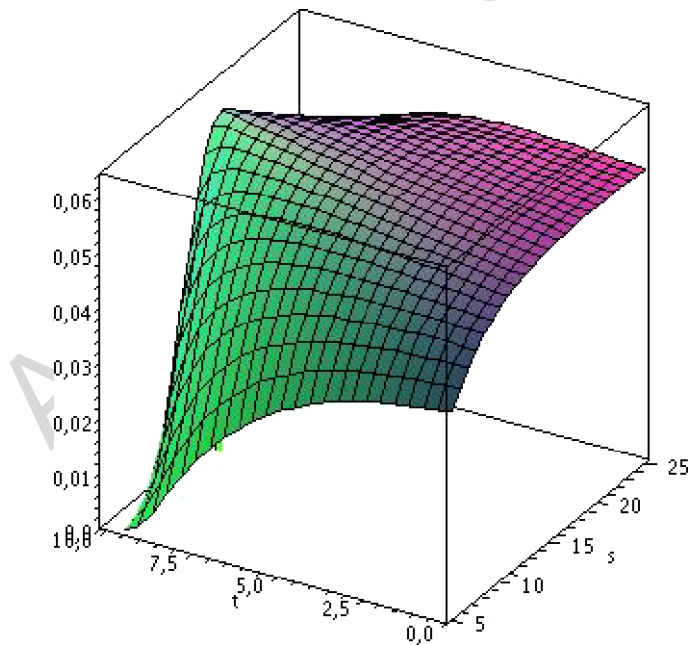


Fig. 2. Call first-order liquidity premium –  $T = 10$ ,  $K = 15$ ,  $\sigma = 0.5$ ,  $\varepsilon = 0.1$ ,  $\ell = 1$ . (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/ASY-2011-1089>.)

### 7.1. Theoretical bounds

As pointed out earlier, for the Digital option, the first-order term that we obtained formally is equal to  $+\infty$ . Thus, the expansion (3.1) is no longer valid and our aim in this section is to find bounds for the first-order of the expansion. We start by approximating the option by a sequence of regularized call spreads. Then the original problem (2.3) is replaced by

$$\begin{aligned} -V_t^{\varepsilon,\alpha} + \widehat{H}^\varepsilon(t, s, V_{ss}^{\varepsilon,\alpha}) &= 0 \quad \text{for } (t, s) \in [0, T] \times \mathbb{R}_+, \\ V^{\varepsilon,\alpha}(T, s) &= \widehat{g}_\alpha(s), \end{aligned} \quad (7.1)$$

where  $\widehat{g}_\alpha(s) = \phi_\alpha * g_\alpha(s)$  with  $g_\alpha(s) = \frac{(s-K+2\alpha)^+ - (s-K+\alpha)^+}{\alpha}$ .

Since  $\phi_\alpha$  has compact support in  $[-\alpha, \alpha]$ , notice that  $\widehat{g}_\alpha \geq g$ . Then, since the terminal condition is smooth, it follows from the comparison principle that

$$V^\varepsilon(t, s) \leq V^{\varepsilon,\alpha}(t, s) \quad \text{for } (t, s, \alpha) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+^*. \quad (7.2)$$

With the same notations as in the previous section, we directly calculate using again the Feynman–Kac formula that

$$\begin{aligned} v_{ss}^{\text{BS},\alpha}(t, s) &= \frac{1}{\sigma s \alpha \sqrt{2\pi\tau}} \int_{-1}^1 \phi(u) (e^{-1/2d_1(s, K+\alpha u-2\alpha, \tau)^2} - e^{-1/2d_1(s, K+\alpha u-\alpha, \tau)^2}) du, \\ v^{(1),\alpha}(t, s) &= \frac{1}{8\ell\pi\alpha^2} \int_0^\tau \int_{-1}^1 \int_{-1}^1 \frac{\phi(x)\phi(y)\widehat{h}_\alpha(\tau, v, s, K, x, y)}{\sqrt{v(2\tau-v)}} dx dy dv, \end{aligned}$$

where

$$\widehat{h}_\alpha(\tau, v, s, K, x, y) = \sum_{1 \leq i, j \leq 2} h_\alpha(\tau, v, s, K, x-i, y-j).$$

Then, we have the two following propositions which are proved exactly as in the call option case (since the functions involved here are essentially the same).

**Proposition 7.1.** *There exists a constant  $c_*$ , independent of  $s, \tau$  and  $\alpha$  so that for all  $(\nu, \beta) \in [0, 1] \times [1/2, 1]$*

$$s |v_{ss}^{\text{BS},\alpha}(t, s)| \leq \frac{c_*}{\tau^{1-\beta}\alpha^{2\beta}}, \quad \frac{s^2\sigma^2}{4\ell} (v_{ss}^{(1),\alpha}(t, s))^2 \leq \frac{c_*}{\tau^{1-\nu}\alpha^{6+2\nu}}.$$

**Proposition 7.2.** *As  $\alpha$  tends to 0 we have the following expansions:*

$$v^{\text{BS},\alpha}(t, s) = v^{\text{BS}}(t, s) + \frac{3}{2}\alpha \frac{e^{-1/2d_0(s, K, \tau)^2}}{K\sigma\sqrt{2\pi\tau}} + \mathcal{O}(\alpha^2),$$

and

$$v^{(1),\alpha}(t, s) = \alpha^{-1} \frac{e^{-1/2d_0(s,K,\tau)^2}}{8K\sigma\ell\sqrt{2\pi\tau}} \\ \times \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y)(|x-y-1| + |x-y+1| - 2|x-y|) dx dy + o(\alpha^{-1}).$$

Define  $V^{\varepsilon,1,c}$  by

$$V^{\varepsilon,1,c}(t, s) := \frac{V^\varepsilon(t, s) - v^{\text{BS}}(t, s)}{\varepsilon^c}.$$

**Theorem 7.1.** *Let  $(\beta, \nu) \in [1/2, 1] \times [0, 1]$  be such that  $\gamma := \frac{2\beta+\nu-1}{2\beta+\nu+4} \in (0, 1)$  and set  $a := \frac{2}{3}(1 - \gamma)$ . Then for all  $(t, s) \in [0, T] \times \mathbb{R}_+$ ,*

$$v^{\text{BS}} \leq V^\varepsilon \leq v^{\text{BS},\varepsilon^a} + \varepsilon v^{(1),\varepsilon^a} + c_*(T-t)^{\beta+(\nu-1)/2} \varepsilon^{2-3a-a(\nu+2\beta)} + c_*(T-t)^\nu \varepsilon^{3-2a(3+\nu)}.$$

In particular,  $V^\varepsilon$  converges to  $v^{\text{BS}}$ , uniformly on compact sets and

$$0 \leq \liminf_{(t',s',\varepsilon) \rightarrow (t,s,0)} V^{\varepsilon,1,a}(t', s', a) \\ \leq \limsup_{(t',s',\varepsilon) \rightarrow (t,s,0)} V^{\varepsilon,1,a}(t', s') \leq \frac{3}{2} \frac{e^{-1/2d_0(s,K,\tau)^2}}{K\sigma\sqrt{2\pi\tau}} + c_*(T-t)^{5\gamma/(2(1-\gamma))},$$

i.e., the order of the expansion is at least  $2/5$ .

**Proof.** It is clear that  $V^\varepsilon \geq v^{\text{BS}}$ . To prove the reverse inequality, we start by following a technique similar to the one used in the proof of Theorem 6.1. Set

$$v^{\varepsilon,2} := v^{\text{BS},\varepsilon^a} + \varepsilon v^{(1),\varepsilon^a} + c_*(T-t)^{\beta+(\nu-1)/2} \varepsilon^{2-3a-a(\nu+2\beta)} + c_*(T-t)^\nu \varepsilon^{3-2a(3+\nu)}.$$

We proceed exactly as in Theorem 6.1 using Proposition 7.1. The result is

$$-v_t^{\varepsilon,2}(t, s) + \hat{H}^\varepsilon(t, s, v_{ss}^{\varepsilon,2}(t, s)) \geq 0 \quad \text{for } (t, s) \in [0, T] \times \mathbb{R}_+.$$

We now analyze the terminal condition. Since  $2\beta + \nu > 1$ , we have

$$v^{\varepsilon,2}(T, s) = v^{\text{BS},\varepsilon^a}(T, s).$$

Hence,  $v^{\varepsilon,2}$  is a super-solution of (6.2) and therefore of (2.3). Then, by the comparison theorem for (2.3) (proved in [6]), we conclude that  $V^\varepsilon(t, s) \leq v^{\varepsilon,2}(t, s)$ .

Then, by Proposition 7.2 and the conditions imposed on  $a$ ,  $\beta$  and  $\nu$ , we obtain easily the uniform convergence on compact sets of  $V^\varepsilon$  to  $v^{\text{BS}}$  by letting  $\varepsilon$  go to 0.

For the first-order term, we would like to use our expansions and obtain a finite majorant for  $V^{\varepsilon,1,c}$  with the largest possible  $c$ . It is easy to argue that  $c = a$  is the best choice possible. This, in turn, imposes

the following condition

$$a \leq \min \left\{ \frac{1}{2}, \frac{2}{4 + 2\beta + \nu}, \frac{3}{7 + 2\nu} \right\} = \frac{2}{4 + 2\beta + \nu}.$$

Now it follows that, for all  $\gamma > 0$  small enough, there are  $\beta$  and  $\nu$  satisfying our conditions so that  $\frac{2}{4+2\beta+\nu} = \frac{2}{5}(1-\gamma)$ . It suffices then to take the lim inf and lim sup in the inequality to prove the result.  $\square$

## 7.2. Numerical results

*The digital option liquidity premium.* In this section, we provide numerical results for the case of the Digital option. As in the Section 6.3 the PDE (2.3) is solved with finite difference method. We represent in Fig. 3 the behavior of the liquidity premium when the time to maturity  $t$  and the spot price vary.

Qualitatively, the liquidity premium behaves as in the Call case. However, as expected the effects of illiquidity are even stronger for ATM options near maturity, since the  $\Gamma$  of a digital option explodes faster. Moreover, with our set of parameters, the first-order correction to the price is at most 0.04 for a BS price of 0.21, which means that the hedge against liquidity risk is much more expensive in the case of a digital option, for a same level of liquidity in the market.

*Numerical confirmation of the expansion order.* We represent in Fig. 4 the liquidity premium for a fixed value of the spot when the parameter  $\varepsilon$  varies with a logarithmic scale.

For small values of  $\varepsilon$  we observe the expected linear behaviour of  $\log(V^\varepsilon - v^{\text{BS}})$ . The slope of the above curve is roughly equal to  $1/2$  (the exact value here is 0.54), which is close to our minimal value of  $2/5$ . The numerical results suggest that the true expansion order lies in the interval  $[2/5, 1/2]$ .

It is also important to realize the financial implications of our results. We just have highlighted the fact that the first-order effect exhibits a phase transition for discontinuous pay-off, in the sense that derivative securities of the type of digital options induce a cost of illiquidity which vanishes at a significantly slower

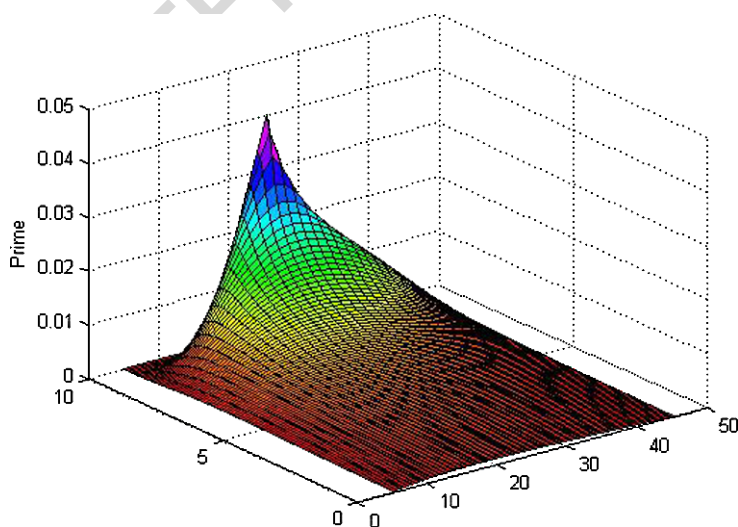


Fig. 3. Digital liquidity premium –  $T = 10$ ,  $K = 25$ ,  $\sigma = 0.5$ ,  $\varepsilon = 0.1$ ,  $\ell = 1$ . (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/ASY-2011-1089>.)



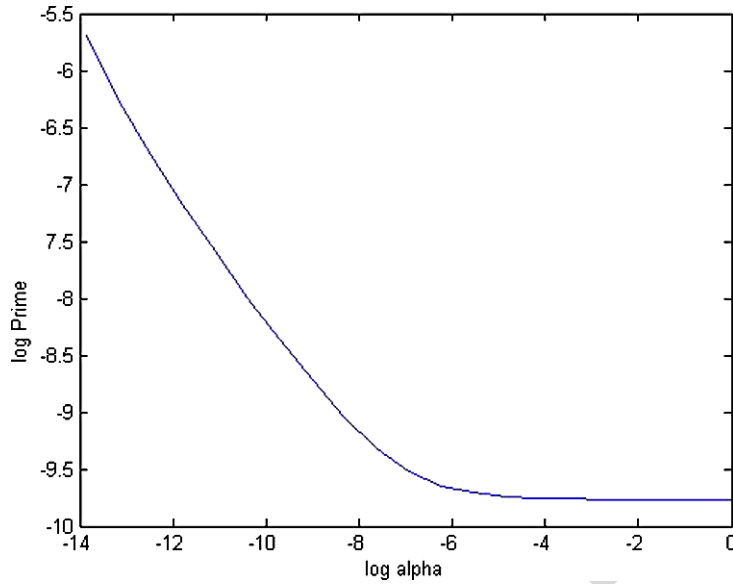


Fig. 4.  $\log(V^\varepsilon - v^{BS}) - T = 1, K = 25, s = 15, \sigma = 0.5, \varepsilon = 0.1, \ell = 1$ . (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/ASY-2011-1089>.)

rate than the continuous pay-off case. This means that derivative with discontinuous pay-off are more rapidly affected by the illiquidity cost.

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### Appendix: Technical proofs

**Proof of Proposition 6.1.** We start by proving the inequality for  $v_{ss}^{BS,\alpha}$ . By dominated convergence, it is clear that  $sv_{ss}^{BS,\alpha}$  goes to 0 when  $s$  approaches 0 or  $+\infty$ . Hence for  $\alpha \neq 0$ , it also converges to 0 when  $\tau$  tends to 0. Thus  $sv_{ss}^{BS,\alpha}$  is less than a constant  $C_\alpha$  independent of  $s$  and  $\tau$ . However, when  $\alpha$  tends to zero, we obtain the classical expression of the  $\Gamma$  of a call option

$$v_{ss}^{BS}(t, s) = \frac{e^{-1/2d_1(s,K,\tau)^2}}{s\sigma\sqrt{2\pi\tau}}, \tag{A.1}$$

which is known to explode only when  $s = K$  and  $\tau \rightarrow 0$ . Therefore, to understand the dependence in  $\alpha$  of  $C_\alpha$ , we only have to study the behaviour of  $sv_{ss}^{BS,\alpha}$  when  $s = K$  and when both  $\alpha$  and  $\tau$  go to 0.

Let us therefore take  $\alpha = \varepsilon^a$  and  $\tau = \varepsilon^b$  with  $a$  and  $b$  strictly positive numbers. For all  $\beta \in [1/2, 1]$  we have

$$\tau^{1-\beta} \alpha^{2\beta-1} s v_{ss}^{\text{BS}, \varepsilon^a} = \frac{\varepsilon^{(b/2-a)(1-2\beta)}}{\sigma \sqrt{2\pi}} \int_{-1}^1 \phi(u) e^{-1/2 \left( \frac{\sigma \varepsilon^{b/2}}{2} - \frac{\varepsilon^{-b/2}}{\sigma} \log(1 + \frac{\varepsilon^a u}{K}) \right)^2} du.$$

Therefore, if  $a < b/2$  (i.e., if  $\tau$  goes to 0 faster than  $\alpha$ ) the quantity above always goes to 0 when  $\varepsilon \rightarrow 0$  due to the exponential term. If  $a \geq b/2$ , the exponential term goes to 1, but since  $\beta \in [1/2, 1]$  the above expression has always a finite limit. Hence the inequality for  $s v_{ss}^{\text{BS}, \alpha}$ .

A change of variable and direct calculations imply that, for all  $\nu \in [0, 1]$ , we have

$$\begin{aligned} & \tau^{(1-\nu)/2} \alpha^{1+\nu} s v_{ss}^{(1), \alpha}(t, s) \\ &= \frac{\alpha^{1+\nu} \tau^{-(1+\nu)/2}}{8\ell\pi s} \int_0^1 \int_{(-1,1)^2} \frac{\phi(x)\phi(y)\tilde{h}_\alpha(\tau, \tau v, s, K, x, y)}{\sqrt{v}(2-v)^{3/2}} dx dy dv, \end{aligned} \quad (\text{A.2})$$

where

$$\begin{aligned} \frac{\tilde{h}_\alpha(\tau, v, s, K, x, y)}{h_\alpha(\tau, v, s, K, x, y)} &= 2 + \left( 2\delta(\tau, \tau v, s, K) - \frac{\log(1 + \frac{\alpha x}{K}) + \log(1 + \frac{\alpha y}{K})}{\sigma \sqrt{\tau(2-v)}} \right)^2 \\ &+ \left( 2\delta(\tau, \tau v, s, K) - \frac{\log(1 + \frac{\alpha x}{K}) + \log(1 + \frac{\alpha y}{K})}{\sigma \sqrt{\tau(2-v)}} \right) \sigma \sqrt{\tau(2-v)}. \end{aligned}$$

Using the same arguments as in the proof of the previous inequality, we can show again that the only problem corresponds to the case where  $s = K$  and  $\alpha$  and  $\tau$  go to 0. Using the same notations, we have

$$\begin{aligned} h_{\varepsilon^a}(\varepsilon^b, \varepsilon^b v, s, s, x, y) &= \exp\left(-\frac{\sigma^2 \varepsilon^b (1-2v)^2}{4(2-v)} + \frac{(1-2v)(\log(1 + \frac{\varepsilon^a x}{K}) + \log(1 + \frac{\varepsilon^a y}{K}))}{2(2-v)}\right) \\ &\times \exp\left(-\frac{\varepsilon^{-b}}{\sigma^2(2-v)} \log\left(1 + \frac{\varepsilon^a x}{K}\right) \log\left(1 + \frac{\varepsilon^a y}{K}\right)\right) \\ &\times \exp\left(-\frac{\varepsilon^{-b}(\log(1 + \frac{\alpha x}{K}) - \log(1 + \frac{\alpha y}{K}))^2}{2\sigma^2 v(2-v)}\right), \\ \frac{\tilde{h}_{\varepsilon^a}(\varepsilon^b, v, s, s, x, y)}{h_{\varepsilon^a}(\varepsilon^b, v, s, s, x, y)} &= 2 + \left( \frac{\sigma \varepsilon^{b/2}(1-2v)}{\sqrt{2-v}} + \varepsilon^{-b} \frac{\log(1 + \frac{\varepsilon^a x}{K}) + \log(1 + \frac{\varepsilon^a y}{K})}{\sigma \sqrt{2-v}} \right)^2 \\ &- \left( \frac{\sigma \varepsilon^{b/2}(1-2v)}{\sqrt{2-v}} + \varepsilon^{-b} \frac{\log(1 + \frac{\varepsilon^a x}{K}) + \log(1 + \frac{\varepsilon^a y}{K})}{\sigma \sqrt{2-v}} \right) \sigma \sqrt{2-v} \varepsilon^{b/2}. \end{aligned}$$

Therefore, if  $a < b/2$ ,  $\tilde{h}_{\varepsilon^a}$  always goes to 0. Otherwise, the integral has a finite limit but since  $\nu \in [0, 1]$  and  $a \geq b/2$ , the expression in (A.2) has a finite limit. This proves the second inequality.  $\square$

**Proof of Proposition 6.2.** The first result is straightforward and only uses the fact that the function  $\phi$  is symmetric, which allows us to get rid off the odd terms in the expansion. For the second one, we directly

calculate that

$$\begin{aligned}
v^{(1),\alpha} &= \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2v(1-v/(2\tau))} + o(\alpha^2)}}{8\pi\ell\sqrt{v(2\tau-v)}} dx dy dv \\
&+ \alpha \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2v(1-v/(2\tau))} + o(\alpha^2)}}{8\pi\ell K\sigma\sqrt{v(2\tau-v)}} \delta (x+y) dx dy dv \\
&+ \alpha^2 \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2v(1-v/(2\tau))} + o(\alpha^2)}}{16\pi\ell K^2\sigma^2\sqrt{v(2\tau-v)}^{3/2}} \\
&\quad \times (2(x+y)^2\delta^2 + \sigma\sqrt{2\tau-v}(x^2+y^2)\delta - 2xy) dx dy dv \\
&+ o\left(\alpha^2 \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2v(1-v/(2\tau))} + o(\alpha^2)}}{8\pi\ell\sqrt{v(2\tau-v)}} dx dy dv\right),
\end{aligned}$$

where we suppressed the arguments of the functions  $v^{(1),\alpha}$  and  $\delta$  for notational simplicity.

Note that all the above integrals are well-defined and finite. Then using dominated convergence and the fact that  $\phi$  is symmetric, it is easy to show that

$$\begin{aligned}
v^{(1),\alpha} &= \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2v(1-v/(2\tau))} + o(\alpha^2)}}{\sqrt{8\pi\ell v(2\tau-v)}} dx dy dv \\
&+ \alpha \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2} \delta}{8\pi\ell K\sigma\sqrt{v(2\tau-v)}} (x+y) dx dy dv \\
&+ \alpha^2 \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2} (2(x+y)^2\delta^2 + \sigma\sqrt{2\tau-v}(x^2+y^2)\delta - 2xy)}{16\pi\ell K^2\sigma^2\sqrt{v(2\tau-v)}^{3/2}} dx dy dv \\
&+ o(\alpha^2) \\
&= \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2v(1-v/(2\tau))} + o(\alpha^2)}}{8\pi\ell\sqrt{v(2\tau-v)}} dx dy dv + o(\alpha).
\end{aligned}$$

Now the first term in the expansion above goes clearly to  $v^{(1)}$  as  $\alpha$  tends to 0. Then we have

$$v^{(1),\alpha} - v^{(1)} = \int_0^\tau \int_{-1}^1 \int_{-1}^1 \frac{e^{-\delta(\tau, v, s, K)^2} \phi(x)\phi(y)}{8\pi\ell\sqrt{v(2\tau-v)}} \left( e^{-\frac{\alpha^2(x-y)^2}{4K^2\sigma^2v(1-v/(2\tau))} + o(\alpha^2)} - 1 \right) dx dy dv + o(\alpha).$$

Using the change of variable  $u = \frac{\alpha|x-y|}{2K\sigma\sqrt{v}}$ , the first term above can be rewritten as

$$\frac{\alpha}{8\pi\ell K\sigma} \int_{\frac{\alpha|x-y|}{2K\sigma\sqrt{\tau}}}^{+\infty} \int_{-1}^1 \int_{-1}^1 \frac{e^{-\delta(\tau, \frac{\alpha^2(x-y)^2}{4K^2\sigma^2u^2}, s, K)^2} \phi(x)\phi(y)|x-y| e^{-\frac{u^2}{1 - \frac{\alpha^2(x-y)^2}{8\tau K^2\sigma^2u^2}} + o(\alpha^2)}}{\sqrt{2\tau - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2u^2}} u^2} - 1 dx dy du.$$

A simple application of the dominated convergence and Fubini theorems shows that the above integral (without the  $\alpha$  factor) has a finite limit as  $\alpha$  approaches 0 and is given by

$$\frac{e^{-1/2d_0(s,K,\tau)^2}}{8\pi\ell K\sigma\sqrt{2\tau}} \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y)|x-y| dx dy \int_0^{+\infty} \frac{e^{-u^2} - 1}{u^2} du.$$

Since the last integral is equal to  $\sqrt{\pi}$ , we obtain the second expansion.  $\square$

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