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On the Hamilton–Jacobi–Bellman Equations in Banach Spaces[1,2]

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Abstract. This paper is concerned with a certain class of distributed parameter control problems. The value function of these problems is shown to be the unique viscosity solution of the corresponding Hamiltonian-Jacobi-Bellman equation. The main assumption is the existence of an increasing sequence of compact invariant subsets of the state space. In particular, this assumption is satisfied by a class of controlled delay equations.

Key Words. Distributed control problems, viscosity solutions, controlled delay equations.

1. Introduction

The purpose of this paper is to characterize the value function of a certain class of infinite-dimensional deterministic control problems as the only solution of the corresponding Hamilton–Jacobi–Bellman equations. A more general class of equations (with nonconvex Hamiltonians) in infinite-dimensional spaces was studied extensively by Crandall and Lions (Refs. 1–2, see also Ref. 3). They have shown that, as in the finite-dimensional case (see Refs. 4–5), the notion of viscosity solutions provides a natural uniqueness class for these equations. But the definition of viscosity solutions needs to be modified to accommodate certain difficulties that arise only in the infinite-dimensional case. In this paper, by using the special structure of a class of control problems, we shall develop a notion of viscosity

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solutions, which is more restrictive than the one utilized by Crandall and Lions in Refs. 1-2.

We continue by defining a control problem, with a reflexive Banach space E as its state space. Let the control set $\mathscr{A}$ be a metric space, and let $\mathscr{A}_{\text{ad}}$ be the set of all piecewise constant maps $w(\cdot)$ of $[0, \infty)$ into $\mathscr{A}$. For each $(x, t) \in E \times [0, T]$, $w(\cdot) \in \mathscr{A}_{\text{ad}}$, controlled trajectories $y(s; x, t, w(\cdot))$ are given as the only solution of the following equation

$$(d/ds)y(s; x, t, w(\cdot)) = A(w(s))y(s; x, t, w(\cdot)) + f(y(s; x, t, w(\cdot)), s, w(s)), \qquad s \in (t, T), \tag{1a}$$

$$y(t; x, t, w(\cdot)) = x, \tag{1b}$$

where $A(w)$ is a (possibly unbounded) operator on $D(A(w)) \subset E$ and $f(x, t, w)$ is an E-valued function on $E \times [0, T] \times \mathscr{A}$. Under Assumptions (A2)-(A4), see Section 2, there is a unique mild solution to (1a)-(1b) (Theorem 1.2, page 184, Ref. 6). Then, the value function $u(x, t)$ is given by

$$u(x, t) = \inf_{w \in \mathscr{A}_{\text{ad}}} \left\{ \int_t^T \exp\left(\int_t^s -\lambda(y(\tau; x, t, w(\cdot)), \tau, w(\tau))\, d\tau \right) \times l(y(s; x, t, w(\cdot)), s, w(s))\, ds + \exp\left(\int_t^T -\lambda(y(\tau; x, t, w(\cdot)), \tau, w(\tau)) d\tau \right) \times g(y(T; x, t, w(\cdot))) \right\}, \tag{2}$$

where $l(x, t, w)$, $\lambda(x, t, w)$, $g(x)$ are real-valued functions. The corresponding Hamilton-Jacobi-Bellman equation has the following form:

$$-(\partial/\partial t)u(x, t) + H(x, t, u(x, t), D_x u(x, t)) = 0, \qquad (x, t) \in E \times [0, T), \tag{3}$$

$$u(x, T) = g(x), \qquad x \in E, \tag{4}$$

where H is given by

$$H(x, t, u, p) = \sup_{w \in \mathscr{A}} \{-l(x, t, w) + \lambda(x, t, w)u - \langle p, f(x, t, w) + A(w)x \rangle\}, \tag{5}$$

for $t \in [0, T]$, $u \in R$, $x \in \bigcap_{w \in \mathscr{A}} D(A(w))$, $p \in E^*$, the dual of E, and $\langle p, x \rangle$ is the value of p at $x \in E$.

When $A(w)$ is an unbounded operator, due to the term $\langle D_x u(x, t), A(w)x \rangle$, the notion of viscosity solutions needs modification. A resolution to this problem is given by Crandall and Lions (personal communication), if the unbounded operator is dissipative and is independent of w. Our approach differs from that of Crandall and Lions at this point. We make use of the following assumption together with the dissipativeness of $A(w)$.

Assumption (A1). (i) $E = F \times Z$, Z is finite dimensional; (ii) there is an increasing sequence of $E_n \subset E$ satisfying

$$E_n \subset \bigcap_{w \in \mathscr{A}} D(A(w)),$$

$$\forall x \in E_n,\ y(s; x, t, w(\cdot)) \in E_n,\ (s, t, w(\cdot)) \in [0, T]^2 \times \mathscr{A}_{\mathrm{ad}},$$

$$\text{closure}\left(\bigcup_n E_n\right) = E,$$

$$\{x = (\eta, z) \in E_n : |z| \leq K\} \text{ is compact for each } n,\ K > 0.$$

Basically, the motivation for the above condition comes from the optimal control of delay equations; see Section 3.

The organization of the paper is simple; in Section 2, the definition of viscosity solutions is given, and $u(x, t)$ is characterized as the only solution of (3) and (4); in Section 3, two examples are discussed.

2. Main Result

Let $\mathrm{BUC}(E \times [0, T])$, $C^1(E \times [0, T])$ denote the set of bounded uniformly continuous functions on $E \times [0, T]$ and continuously Frechét differentiable functions on $E \times [0, T]$, respectively.

Definition 2.1. (a) Any $u \in \mathrm{BUC}(E \times [0, T])$ is a viscosity subsolution to (3) if, for all $\varphi \in C^1(E \times [0, T])$,

$$-(\partial/\partial t)\varphi(x, t) + H(x, t, u(x, t), D_x\varphi(x, t)) \leq 0,$$

whenever

$$(u - \varphi)(x, t) = \max\{(u - \varphi)(y, x) \colon (y, s) \in E_n \times [0, T]\},$$

for some $n \geq 1$ and $(x, t) \in E_n \times [0, T)$.

(b) Any $u \in \mathrm{BUC}(E \times [0, T])$ is a viscosity supersolution to (3) if, for all $\varphi \in C^1(E \times [0, T])$,

$$-(\partial/\partial t)\varphi(x, t) + H(x, t, u(x, t), D_x\varphi(x, t)) \geq 0,$$

whenever

$$(u-\varphi)(x, t) = \min\{(u-\varphi)(y, x): (y, s) \in E_n \times [0, T]\},$$

for some $n \geq 1$ and $(x, t) \in E_n \times [0, T)$.

(c) u is a viscosity solution to (3) if it is both a subsolution and supersolution.

Note that $H(x, t, u(x, t), D_x\varphi(x, t))$ is defined for any $x \in E_n$ and $\varphi \in C^1(E \times [0, T])$, on account of (A1)(ii). At this point, we add that the above notion of viscosity solutions is more restrictive than the one used by Crandall and Lions in Refs. 1-2. Once again, we emphasize that this restriction is caused by the unboundedness of the Hamiltonian.

In addition (A1), we assume the following.

Assumption (A2). f, l, λ, g are Lipschitz continuous in x and t, uniformly with respect to w and $\lambda(x, t, w) \geq \lambda_0 > 0$.

Assumption (A3). For each $w \in \mathscr{A}$, $A(w)$ is the infinitesimal generator of a C_0-semigroup.

Assumption (A4). There is $\alpha \in R$ such that $A(w) - \alpha I$ is dissipative; i.e., for $x \in E$,

$$\langle x^*, A(w)x\rangle \leq \alpha \|x\|^2,$$

for some

$$x^* \in F(x) = \{x^* \in E^*: \langle x^*, x\rangle = \|x\|^2 = \|x^*\|_*^2\},$$

where $\|\cdot\|$ and $\|\cdot\|_*$ are the norms on E and E^*, respectively. Since E is reflexive, there is an equivalent norm on E such that $F(x)$ is a singleton for each $x \in E$. We assume that we are using that norm on E; i.e., see the assumption below.

Assumption (A5). $F(x)$ is a singleton for each $x \in E$.

Theorem 2.1. *The value function $u(x, t)$ is the only viscosity solution of* (3) *and* (4).

Proof. Observe that, due to Lipschitz continuity of $y(s; x, t, w(\cdot))$ in x uniformly with respect to other variables (Theorem 1.2, page 184, Ref. 6), $u \in \mathrm{BUC}(E \times [0, T])$. Also, the proof of the finite-dimensional analogue of the existence part of this theorem is applicable to the problem under investigation, due to (A1).

To prove the uniqueness of solutions to (3) and (4), assume that v is another viscosity solution. For $\epsilon > 0$, define ϕ^ϵ by

$$\phi^\epsilon(x, y, t, s) = u(x, t) - v(y, t) - \epsilon^{-1}[\|x-y\|^2 + |t-s|^2],$$

for $(x, y, t, x) \in E^2 \times [0, T]^2$. Suppose that ϕ^ϵ achieves its maximum on $E_n^2 \times [0, T]^2$ at $(x_n, y_n, t_n, s_n) \in E_n^2 \times [0, T]^2$. Since $E_n \cap (F \times B_k)$ is compact for every ball $B_k \subset X$, and since X is finite dimensional, we can always obtain such points by slightly perturbing the functions u and v, as in Theorem 4.1 in Ref. 4.

In view of (A5), the function $d(x) = \|x\|^2$ is Frechét differentiable on E, and $D_x d(x) \in F(x)$, for each $x \in E$. Now, the viscosity property of u and v implies the following, if $t_n, s_n < T$:

$$-2\epsilon^{-1}(t_n - s_n) + H(x_n, t_n, u(x_n, t_n), 2\epsilon^{-1} D_x d(x_n - y_n)) \leq 0,$$

$$-2\varepsilon^{-1}(t_n - s_n) + H(y_n, s_n, v(y_n, s_n), 2\epsilon^{-1} D_x d(x_n - y_n)) \geq 0.$$

Proceed by using (5) and (A2). We have

$$\begin{aligned} &\lambda_0(u(x_n, t_n) - v(y_n, s_n) \\ &\leq K(\epsilon^{-1}\|x_n - y_n\| + 1)(\|x_n - y_n\| + |t_n - s_n|) \\ &+ 2\epsilon^{-1} \sup_w \langle A(w)(x_n - y_n), D_x d(x_n - y_n)\rangle, \end{aligned} \tag{6}$$

where K is a suitable constant. Also, an account of (A4) and (A5),

$$\langle A(w)(x_n - y_n), D_x d(x_n - y_n)\rangle \leq \alpha \|x_n - y_n\|^2.$$

Hence, (6) yields

$$\begin{aligned} &\lambda_0(u(x_n, t_n) - v(y_n, s_n)) \\ &\leq K(\epsilon^{-1}\|x_n - y_n\| + 1)(\|x_n - y_n\| + |t_n - s_n|). \end{aligned} \tag{7}$$

Now, one finishes the proof as in the finite-dimensional case. But, for completeness, we give the proof.

Let $\bar{m}(\cdot)$ be a modulus of continuity for u and v. It is easy to verify the following:

$$\|x_n - y_n\| + |t_n - s_n| \leq K\epsilon^{1/2} \bar{m}(K\epsilon^{1/2}). \tag{8}$$

So, for each $(x, t) \in E_n \times [0, T]$,

$$\begin{aligned} u(x, t) - v(x, t) &\leq \phi^\epsilon(x_n, y_n, t_n, s_n) \\ &\leq K(\epsilon^{-1}\|x_n - y_n\| + 1)(\|x_n - y_n\| + |t_n - s_n|). \end{aligned}$$

Using (8) and then passing to the limit as ϵ tends to zero, we conclude that

$$u(x, t) \leq v(x, t), \qquad \forall (x, t) \in \bigcup_n E_n \times [0, T].$$

In view of (A1) (ii),

$$u \leq v, \qquad \text{on } E \times [0, T], \qquad \text{if } t_n, s_n < T.$$

But, if

$$\max(t_n, s_n) = T,$$

elementary considerations imply the same result. Hence, $u \leq v$. Since the argument is symmetric, the proof of the theorem is now complete. □

Remark 2.1. Suppose that $u \in C^1(E \times [0, T])$ and satisfies (3) at every $x \in \bigcap_w D(A(w))$. Then, we claim that it is a viscosity solution to (3). Let $\varphi \in C^1(E \times [0, T])$ and

$$(u - \varphi)(x, t) = \max\{(u - \varphi)(y, s): (y, s) \in E_n \times [0, t]\},$$

at some $(x, t) \in E_n \times [0, T)$. On account of the differentiability of u,

$$(\partial/\partial t)\varphi(x, t) \geq (\partial/\partial t)u(x, t),$$

$$(D_x\varphi(x, t), y) \geq (D_x u(x, t), y), \qquad \forall y \in E_n(x),$$

where

$$E_n(x) = \text{closure}\{y \in E: x + \epsilon y \in E_n \text{ for all } 0 \leq \epsilon \leq \epsilon_0(y)\}.$$

Since $y(s; x, t, w(\cdot)) \in E_n$, for all $s, t, w(\cdot)$, we conclude that

$$A(w)x + f(x, t, w) \in E_n(x).$$

Hence,

$$-(\partial/\partial t)\varphi(x, t) + H(x, t, u(x, t), D_x\varphi(x, t))$$
$$\leq -(\partial/\partial t)u(x, t) + H(x, t, u(x, t), D_x u(x, t)) = 0.$$

3. Examples

Example 3.1. *Optimal Control of Delay Equations.* Consider the following controlled functional differential equation

$$(d/ds)z(s) = g(\eta(s), z(s), s, w(s)), \qquad s \in (t, T], \tag{9}$$

where $z(s) \in R^n$, $w(s)$ is the control process taking values in a metric space $\mathscr{A}$, $\eta(s) \in L^2((-1, 0); R^n)$ represents the delay term, and it is defined by

$$\eta(s)(\tau) = z(s + \tau), \qquad \tau \in [-1, 0). \tag{10}$$

For a given control process and an initial condition

$$(\eta(t), z(t)) = (\eta, z) \in L^2((-1, 0); R^n) \times R^n, \tag{11}$$

Eqs. (9) and (10) have a unique solution, provided that the function $g(\eta, z, s, w)$ with domain $L^2((-1, 0); R^n) \times R^n \times [0, T] \times \mathscr{A}$ is Lipschitz continuous in η, z and is continuous in s, uniformly with respect to w (Ref. 7).

The finite-horizon optimal control of these equations was studied by several authors. The reader may refer to the survey of Banks and Manitius (Ref. 8) and to the references therein.

In this example, $E = L^2((-1, 0); R^n) \times R^n$; and, for $x = (\eta, z) \in E$, Ax is defined by

$$D(A) = \{x = (\eta, z) \in E: \eta \in H^1((-1, 0): R^n) \text{ and } \eta(0) = z\},$$

$$Ax = (\eta', 0), \qquad \text{for } x = (n, z) \in D(A),$$

where

$$\eta'(\tau) = (d/d\tau)\eta(\tau).$$

The operator A is the infinitesimal generator of the shift semigroup $T_t x = (\eta_t, z_t)$, given by

$$\eta_t(\tau) = \begin{cases} z, & \max\{-1, -t\} \leq \tau \leq 0, \\ \eta(\tau + t), & 0 \geq \tau \geq \max\{-1, -t\}, \end{cases}$$

$$z_t = z.$$

The norm

$$\|(\eta, z)\|^2 = \|\eta\|_{L^2}^2 + |z|^2$$

satisfies (A5) with $F(x) = \{x\}$, and (A4) holds due to the following inequality:

$$\begin{aligned} \langle x, Ax \rangle &= \int_{-1}^{0} \eta(\tau) \cdot (d/d\tau)\eta(\tau)\, d\tau \\ &= \tfrac{1}{2}|\eta(0)|^2 - \tfrac{1}{2}|\eta(-1)|^2 \\ &\leq \tfrac{1}{2}|z|^2 \leq \tfrac{1}{2}\|x\|^2. \end{aligned}$$

Finally, let

$$f(x, s, w) = (0, g(x, s, w)).$$

Define the invariant sets E_n as follows:

$$E_n = \{x = (\eta, z) \in D(A): \|\eta'\|_\infty \leq n\}.$$

For $n \geq \|g\|_\infty$, Assumption (A1) is satisfied.

Example 3.2. *Optimal Control of a Parabolic Equation.* In this example, the state variable satisfies a parabolic equation controlled through the diffusion constant. Again, this is a special case of an extensively studied problem. We refer to Ahmed and Teo (Ref. 9) and the references therein.

Let Ω be a bounded domain in R^n, $E = L^2(\Omega; R)$, and $\mathscr{A} = [m, M]$. Define $A(w)$ by

$$D(A(w)) = H_0^2(\Omega),$$

$$(A(w)x)(z) = w \sum_{i,j=1}^{n} (\partial/\partial x_i)(a_{ij}(\cdot)(\partial/\partial x_j)x(\cdot))(z), \qquad z \in \Omega,$$

where H_0^2 is the Sobolev space of functions which are L^2, along with their derivatives of order less than equal to two, and whose trace along $\partial\Omega$ is zero. If the matrix $a(z) = (a_{ij}(z))$ is uniformly elliptic and is continuous on $\bar{\Omega}$, $A(w)$ satisfies (A3)-(A4). Take $f \equiv 0$.

Now define E_n by

$$E_n = \left\{ z \in H_0^2 \colon \|x\|_{L^2}^2 \leq n, \int_\Omega (\nabla x(z) \cdot a(z) \nabla x(z))\, dz \leq n, \int_\Omega |\nabla \cdot (a(z) \nabla x(z)|^2\, dz \leq n \right\}.$$

Due to the energy estimates, (A1) is satisfied.

In this example, $\gamma(s, z) = y(s; x, t, w(\cdot))(z)$ solves

$$(d/dt)\gamma(s, z) = w(s)\nabla \cdot (a(z)\nabla \gamma(s, z)), \qquad s \in (t, T) z \in \Omega,$$

$$\gamma(t, z) = x(z), \qquad z \in \bar{\Omega},$$

$$\gamma(s, z) = 0, \qquad (s, z) \in [0, T] + \partial\Omega.$$

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