

## LIQUIDITY IN A BINOMIAL MARKET

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We study the binomial version of the illiquid market model introduced by Çetin, Jarrow, and Protter for continuous time and develop efficient numerical methods for its analysis. In particular, we characterize the liquidity premium that results from the model. In Çetin, Jarrow, and Protter, the arbitrage free price of a European option traded in this illiquid market is equal to the classical value. However, the corresponding hedge does not exist and the price is obtained only in  $L^2$ -approximating sense. Çetin, Soner, and Touzi investigated the super-replication problem using the same supply curve model but under some restrictions on the trading strategies. They showed that the super-replicating cost differs from the Black–Scholes value of the claim, thus proving the existence of liquidity premium. In this paper, we study the super-replication problem in discrete time but with no assumptions on the portfolio process. We recover the same liquidity premium as in the continuous-time limit. This is an independent justification of the restrictions introduced in Çetin, Soner, and Touzi. Moreover, we also propose an algorithm to calculate the option's price for a binomial market.

KEY WORDS: super-replication, liquidity, binomial model, dynamic programming.

### 1. INTRODUCTION

It is well documented that the limited supply of a financial instrument introduces liquidity risk. This is due to the timing and the size of an order. Indeed, there have been numerous studies to incorporate the price impact of placing a sufficiently large order. One can classify these approaches in two main categories. The first one deals with feedback effects on dynamic portfolios on asset prices. In particular, in Platen and Schweizer (1998) it is demonstrated that if trading is carried out on a large scale, it has an effect on the asset price of the underlying in the form of an increase on the market volatility. This is especially important in an economy under the presence of a large trader. In Frey and Stremme (1997), the interaction of a program trader with reference traders has been studied. If the prices are modeled as a geometric Brownian motion for reference traders, the existence of a large trader changes the price process to an Ito process, where the volatility increases

Research partly supported by the European Research Council under the grant 228053-FiRM and by the Turkish Technical and Scientific Council, TÜBİTAK. Financial support from Credit Suisse through the ETH Foundation and by the National Centre of Competence in Research “Financial Valuation and Risk Management” (NCCR FINRISK) are also gratefully acknowledged.

*Manuscript received April 2009; final revision received March 2010.*

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DOI: 10.1111/j.1467-9965.2010.00462.x

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and becomes order size and time dependent. In Platen and Schweizer (1998), starting from a given volatility it is characterized how the volatility would transform under the price impact of the portfolio decision of the large trader. However, one is interested in the fixed point of this volatility transformation, because every time the large investor trades it is going to change the volatility of the diffusion process. This has been established in Frey (1998), where he exhibits a nonlinear partial differential equation (PDE) for the option replication problem under the setting of a large investor and investigates the existence and uniqueness of solutions as well as the trading strategies. Bank and Baum (2004) also studies the issue of feedback effect of the large trader when he makes his portfolio decision. In this work, the authors model prices as a family of semimartingales depending on a parameter. The parameter represents the large investor's position kept on a constant level. This approach generalizes the reaction-diffusion setting in Frey and Stremme (1997). In all these approaches, the effect of the order placed by the large investor persists until the next order makes the asset prices follow a different dynamics. Some other studies falling into this category are Jarrow (1992, 1994) and Papanicolaou and Sircar (1998).

In the second class of models, the size of the trade has also an instantaneous effect on the price of the asset; however, this effect is not permanent, it is a temporary impact. This is exactly the key difference between the two groups of the liquidity models. In particular, to model the impact of liquidity, Çetin, Jarrow, and Protter (2004) postulate the existence of an exogenously given supply curve to which any small trader in the market acts as a price taker. This supply curve produces a price for a given size and time of a trade. The trading history does not alter the shape of the supply curve so all small investors trading identical quantities at any time pay the same amount. In this article, we will consider the binomial version of the Çetin–Jarrow–Protter model.

Çetin, Jarrow, and Protter (2004) investigated the fundamental theorems of asset pricing under the existence of illiquidities with their supply curve model. In this framework, they showed that the market satisfies the No Free Lunch with Vanishing Risk property if and only if there exists an equivalent measure  $Q$  under which the marginal price process is a  $Q$ -local martingale. The marginal price process is the price paid per unit of extra infinitesimal stock. Under the existence of a unique equivalent measure  $Q$  that turns the marginal price process into a  $Q$ -local martingale, one can approximate any claim in  $L^2$ -sense with continuous and finite variation strategies. This is important because continuous and finite variation strategies incur no liquidity costs so that the price of a contingent claim is the Black-Scholes value. A similar argument appears in Bank and Baum (2004), where the large trader should use continuous and finite variation strategies to avoid transaction costs and the attainable claims under a small investor model become approximately attainable in the large trader setting.

Rogers and Singh (2010) consider also a temporary impact model. In their framework, illiquidity affects the wealth process due to the depth of the limit order book but not the price of the asset. This model, which uses only portfolio processes of finite variation, is obtained as the formal limit of a discrete time setup.

Later Çetin and Rogers (2007) considered the utility maximization problem in discrete time for an analog of the supply curve. In fact, this study leads to a nonzero liquidity premium. This suggests that the absence of liquidity premium is related to the portfolio processes considered in continuous time.

Çetin, Soner, and Touzi (2010) studied the same continuous-time model of Çetin, Jarrow, and Protter (2004). To obtain liquidity premium, they introduced additional conditions on the trading strategies similar to those in Soner and Touzi (2000)

and Cheridito et al. (2007). This class of optimal portfolio processes contains infinite variation strategies in contrast to Bank and Baum (2004) and Çetin, Jarrow, and Protter (2004). The infinite variation strategies are in the form of an integral of the gamma of the portfolio with respect to the marginal stock price process, whereas the finite variation part of the trading strategies involve an absolutely continuous process and a pure jump process. The integrand in the absolutely continuous part is interpreted as the rate of change of the portfolio with respect to time. On the other hand, the infinite variation part reflects the sensitivity of the portfolio with respect to stock changes. Instead of a  $L^2$ -sense replication, they consider a super-replication argument to price claims. They show that there exists a liquidity premium; a difference between the super-replicating cost and the Black–Scholes value. In fact, Çetin, Soner, and Touzi (2010), characterize the super-replicating cost of the option  $\phi(t, s)$  via the following partial differential equation:

$$(1.1) \quad -\phi_t - \inf_{\beta \geq 0} \left\{ \frac{1}{2} s^2 \sigma^2 (\phi_{ss} + \beta) + \Lambda(t, s) s^2 \sigma^2 (\phi_{ss}(t, s) + \beta)^2 \right\} = 0,$$

where the marginal price process evolves according to geometric Brownian motion and the interest rate is taken to be zero by discounting. For a convex payoff, this PDE has the form

$$(1.2) \quad 0 = -\phi_t - \frac{1}{2} s^2 \widehat{\sigma}^2(t, s) \phi_{ss},$$

$$(1.3) \quad \widehat{\sigma}^2(t, s) = \sigma^2(1 + 2\Lambda(t, s)\phi_{ss}(t, s)),$$

where  $\Lambda(t, s)$  is the liquidity parameter associated with the market. Observe that for  $\Lambda = 0$ , the above equation is simply the classical Black–Scholes equation. Moreover, the hedge for an option with convex payoff is given as in the usual Black–Scholes model, that is by  $\phi_s(t, s)$ . Also, in convex setting, (1.2) can be seen as the pricing in a model with increased volatility (1.3). This representation has connections to Barles and Soner (1998), Frey (1998), Frey (2000), Frey and Stremme (1997), Papanicolaou and Sircar (1998), and Platen and Schweizer (1998).

In this paper, we analyze the discrete time version of the supply curve model using the parameters derived in Çetin, Jarrow, and Protter (2004). We investigate the super-replication problem without imposing any conditions on the portfolio processes. Our first result, Theorem 3.1, is a solution technique through dynamic programming (3.1). The main observation for this simple result is to introduce the dependence of the minimal super-replication cost on the portfolio position as well as the current stock price and time to maturity. Without this extra state variable (namely the portfolio position) the dynamic programming is not valid. We use this method to develop an algorithm for the computation of the super-replication cost. Results of the implementations of this algorithm are reported in Section 6.

We then consider the continuous-time limit of our binomial model. We prove in Theorem 5.4 that the discrete minimal super-replication cost of the option agrees, in the continuous-time limit, with those obtained in the paper Çetin, Soner, and Touzi (2010). This is proved by the theory of viscosity solutions. In Section 7, we prove that lower-relaxed limit of the minimal super-replication cost is a viscosity super-solution of (1.1). Then in Section 8 the upper-relaxed limit is shown to be a viscosity sub-solution of the same equation. Then, the convergence is obtained by using a comparison result proved

in Çetin, Soner, and Touzi (2010). Although this is a standard technique in the theory of viscosity solutions, the sub- and super-solution properties of the limiting function are difficult to derive. The difficulty in this derivation is due to the additional dependence on the portfolio position. Furthermore, this dependence becomes irrelevant in the limit and exactly this fact renders the problem difficult. We overcome this by using an appropriate *corrector* function as in the applications of viscosity solutions to homogenization (Evans 1992). Our approach, although similar to Evans (1992), is further developed to analyze convergence results of this type. We believe that a probabilistic analysis of the corrector method in this context is also an interesting new question.

We also investigate the limit behavior of the super-replication cost numerically. Our numerical results, reported in Section 6 agree with our conclusion that there exists a nonzero liquidity premium. We also propose another algorithm for the continuous-time limit that is computationally faster. This second algorithm calculates the super-replication cost without introducing the additional variable. This reduction in the dimension enables us to gain considerable computational time. Numerically, we demonstrate that the results of both algorithms are very close for small time steps. Recently, a general numerical study for partial differential equations of the type (1.1) based on backward stochastic differential equations similar to our discrete model was obtained in Fahim, Touzi, and Warin (2009). However, since their approach relies on a continuous-time model, they do not need to increase the dimension by adding the portfolio variable in contrast to our approach. Hence, their algorithms are closer to our faster one. Also a continuous-time problem with large trader effects is analyzed both numerically and theoretically in Ly Vath, Minf, and Pham (2007).

The paper is organized as follows. After introducing the model in Section 2, we state the dynamic programming and the parameters of the problem in the next section. The interesting simple liquidation problem is briefly studied in Section 4. The main convergence result is stated in Section 5. Numerical methods and results are reported in Section 6. The super- and the sub-viscosity properties that imply the convergence result are proved in two following sections.

## 2. MODEL

We consider a market with one risk-free and one risky asset. By discounting, we take the interest rate  $r = 0$  and thus normalize the unit price of money market account to unity. The price of the risky asset follows the supply curve model introduced by Çetin, Jarrow, and Protter (2004) for a binomial market. In this setting, the price per share is given by  $S(t, S_t, \nu)$ , where  $\nu$  is the size of the transaction of the small investor. A positive order  $\nu > 0$  is a buy, whereas  $\nu < 0$  represents a sale order. For  $\nu = 0$  we capture the spot price  $S_t$ . We have a binomial tree structure for the marginal price process  $S_t$ . At any node of the recombinant tree it goes up by a factor of  $u$  and goes down by a factor of  $d$ . Clearly, to avoid arbitrage for the marginal price process, we need to assume that  $d < 1 < u$ .

We consider a European claim with *maturity*  $T$ . We assume that a Markovian claim with nonnegative pay-off  $g(S_T)$  and the time step is a given  $h > 0$ . The up and down factors  $u$  and  $d$  depend on  $h$ . All the processes are assumed to be constant on intervals of the form  $(nh, (n + 1)h]$  for an integer  $n$ .

We let  $Z_t$  be the number of *shares in the portfolio* and  $X_t$  be the *money invested in the money market account* at time  $t$ . We assume that the portfolio process  $Z_t$  is measurable with respect to the filtration  $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$  generated by the spot price process. Given such

a process  $Z$ , the marked to market value of this portfolio  $Y_t$  is given by

$$(2.1) \quad Y_t = X_t + Z_t S_t.$$

With an abuse of terminology we call  $Y_t$  the *wealth process*. Now suppose that an investor decides to make a transaction of  $\Delta Z_{t+h} = Z_{t+h} - Z_t$  shares at time  $t+h$ .

This transaction results in the following change in the money market account,

$$\Delta X_{t+h} := X_{t+h} - X_t = -\Delta Z_{t+h} \mathbf{S}(t+h, S_{t+h}, \Delta Z_{t+h}).$$

Under the usual self-financing assumption, the above implies the following system dynamics for the wealth process,

$$(2.2) \quad Y_{t+h} = Y_t + Z_t \Delta S_{t+h} - \Delta Z_{t+h} [\mathbf{S}(t+h, S_{t+h}, \Delta Z_{t+h}) - S_{t+h}],$$

where  $\Delta S_{t+h} = S_{t+h} - S_t$ . Observe that the liquidity effect appears as a positive penalty in the system dynamics for the wealth process. It is clear that a portfolio process  $Z$ , adapted to the filtration generated by the spot price process and an initial wealth  $y$  at time  $t$ , generates an adapted wealth process  $Y^{t,y,Z}$ . In particular,  $Y_t^{t,y,Z} = y$ .

We investigate the minimal super-replication cost of this European contingent claim with a given payoff  $g(\cdot)$  at the time horizon  $T$ . Then, the minimal super-replicating cost  $\phi^h(t, s)$  at time  $t$  and  $S_t = s$  is given by

$$(2.3) \quad \phi^h(t, s) := \inf \left\{ y \mid \exists \mathcal{F}\text{-adapted } \{Z\} \text{ so that } Y_T^{t,y,Z} \geq g(S_T^{t,s}) \text{ a.s.} \right\},$$

where  $S^{t,s}$  is the spot price process satisfying the initial condition  $S_t^{t,s} = s$ .

One may formulate the above problem as a convex program. However, since the number of constraints increase exponentially with the number of steps to maturity, it is quite hard to compute markets with many steps. To use dynamic programming, we need to introduce the dependence of the minimal super-replication cost on the initial portfolio position. Thus, we define

$$(2.4) \quad v^h(t, s, z) := \inf \left\{ y \mid \exists \mathcal{F}\text{-adapted } \{Z\} \text{ so that } Z_t = z, \text{ and } Y_T^{t,y,Z} \geq g(S_T^{t,s}) \text{ a.s.} \right\}.$$

Clearly,

$$\phi^h(t, s) = \inf_z v^h(t, s, z).$$

REMARK 2.1. In (2.4) we never take the infimum over an empty set, because we can always super-replicate by taking  $Z = z$  and choosing a sufficiently large  $y$ .

### 3. DYNAMIC PROGRAMMING

The following result can be easily proved by standard techniques in Fleming and Soner (1993) and Soner and Touzi (2002).

**THEOREM 3.1. (Dynamic Programming)** *For any  $t = nh < \tau = mh \leq T$ , the minimal super-replicating cost  $v^h(t, s, z)$  satisfies*

$$v^h(t, s, z) = \inf \{ y \mid \exists \mathcal{F}\text{-adapted } \{Z_t\} \text{ so that } Z_t = z, \text{ and } Y_\tau^{t,y,Z} \geq v^h(\tau, S_\tau^{t,s}, Z_\tau) \text{ a.s.} \}.$$

We take  $\tau = t + h$  in the above to conclude that  $v^h(t, s, z) = \min y$  among all  $y$  satisfying

$$Y_{t+h}^{t,y,Z} \geq v^h(t+h, S_{t+h}^{t,s}, Z_{t+h}),$$

both when the stock is up and down. We rewrite above inequality, using the wealth equation (2.2). The result is

$$y + Z_t \Delta S_{t+h} - \Delta Z_{t+h} \theta_{t+h}(\Delta Z_{t+h}) \geq v^h(t+h, S_{t+h}^{t,s}, Z_{t+h}),$$

where we use the notation

$$\theta_t(v) = \mathbf{S}(t, S_t, v) - S_t.$$

We choose  $Z_{t+h}$  to  $z_{\text{up}}$  or  $z_{\text{down}}$ , respectively, depending on whether the spot price is up or down. Then, we have the representation,

$$v^h(t, s, z) = \min y,$$

among all  $y$  satisfying

$$\begin{aligned} y + zs(u-1) - (z_{\text{up}} - z)\theta_{\text{up}}(z_{\text{up}} - z) &\geq v^h(t+h, su, z_{\text{up}}), \\ y + zs(d-1) - (z_{\text{down}} - z)\theta_{\text{down}}(z_{\text{down}} - z) &\geq v^h(t+h, sd, z_{\text{down}}), \end{aligned}$$

for some  $z_{\text{up}}$  and  $z_{\text{down}}$ . From now on we will suppress the dependence of the function  $\theta$  on the up or down state. We compactly rewrite the above equation by using the notation  $z_{\text{up}} = z + a$  and  $z_{\text{down}} = z + b$ . The result is the following difference equation which we call the *dynamic programming equation*.

$$(3.1) \quad v^h(t, s, z) = \max \left( \min_a \{v^h(t+h, su, a+z) - zs(u-1) + a\theta(a)\}, \right. \\ \left. \min_b \{v^h(t+h, sd, b+z) - zs(d-1) + b\theta(b)\} \right).$$

This equation is complemented by the terminal data

$$(3.2) \quad v^h(T, s, z) = g(s).$$

**REMARK 3.2.** We take our portfolio processes to be adapted to the filtration  $\mathcal{F}$  generated by the marginal price process  $S_t$ . If we were to take predictable portfolio processes, this would definitely increase the associated liquidity premium in discrete-time, since then in the dynamic programming we would choose only one control at time  $t$  for both the up and down case. Therefore, the liquidity premium for predictable portfolio processes in the continuous-time limit is at least as in the case of adapted trading strategies.

REMARK 3.3. In the terminal data, we implicitly ignore the liquidity cost at final time. One may also consider other ways of settlements for the option. For instance, if we assume that the option is settled in cash, then the investor has to liquidate her stock holdings at final time. This leads to the following formulation:

$$\bar{v}^h(t, s, z) := \inf \left\{ y \mid \exists \mathcal{F}\text{-adapted } \{Z\} \text{ so that } Z_t = z, \text{ and} \right. \\ \left. Y_T^{t, y, Z} - (-Z_T)\theta_T(-Z_T) \geq g(S_T) \text{ a.s.} \right\}.$$

Still the value function  $\bar{v}^h$  solves the same dynamic programming equation (3.1). But the terminal data changes to

$$\bar{v}^h(T, s, z) = g(s) + (-z)\theta_T(-z).$$

For a call option ( $g(s) = (s - K)^+$ ), another formulation used in the literature requires the investor to actually have physical delivery of the stock. This leads to a formulation with terminal data

$$(3.3) \quad \tilde{v}^h(T, s, z) = (s - K)^+ + (1 - z)\theta_T(1 - z)\chi_{s \geq K} + (-z)\theta_T(-z)\chi_{s < K}.$$

All these formulations lead to a larger liquidity premium. For that reason in our limit analysis we use the simplest boundary condition  $g(s)$ . However, numerically, all other terminal conditions can be studied easily. Indeed, in Section 6 we report some calculations with data given by (3.3).

### 3.1. Parameters

Although we can perform an analysis with a general loss function  $\theta_n$ , to simplify the presentation and the already technical proofs we make a specific choice. Also, our choice is motivated by earlier results of Çetin et al. (2004, 2006, 2010). In particular, analysis of Çetin, Soner, and Touzi (2010) shows that in continuous time the changes in optimal portfolios are small. In particular, only the partial derivative of the demand function  $\mathbf{S}(t, s, v)$  with respect to the  $v$ -variable at the origin (i.e.,  $\mathbf{S}_v(t, s, 0)$ ) is relevant for the continuous-time solution. Moreover, the empirical analysis of Çetin et al. (2006) indicates that this partial derivative is constant. This motivates the linear choice  $\theta_t(v) = \Lambda v$  for some *liquidity parameter*  $\Lambda > 0$ . The positivity requirement for the demand function forces us to modify this choice slightly and we choose

$$(3.4) \quad \mathbf{S}(t, s, v) = \begin{cases} s + \Lambda v & v \geq -\frac{s}{\Lambda}, \\ 0 & v \leq -\frac{s}{\Lambda}, \end{cases}$$

where  $s \in (0, \infty)$ . In this case

$$\theta_t(v) = (s + \Lambda v)^+ - s \geq \hat{\theta}(v) := \Lambda v.$$

Since we expect portfolio changes to be small, we will see that formulations with  $\theta$  and  $\hat{\theta}$  yield the same result.

Also for the up and down factors we make the standard choice,

$$(3.5) \quad u = 1 + \sigma\sqrt{h}, \quad d = 1 - \sigma\sqrt{h},$$

where  $\sigma > 0$  is the given volatility of the marginal price. Recall that  $h > 0$  is the time step.

Finally one can easily derive that with these choices the continuous-time equation (1.1) takes the form

$$(3.6) \quad -\phi_t - s^2\sigma^2 H(\phi_{ss}) = 0,$$

where

$$H(\gamma) = \begin{cases} \frac{1}{2}\gamma + \Lambda\gamma^2, & \text{for } \gamma \geq -\frac{1}{4\Lambda}, \\ -\frac{1}{16\Lambda}, & \text{for } \gamma \leq -\frac{1}{4\Lambda}, \end{cases}$$

where in (1.1) we take the derivative with respect to  $\beta$  and set it equal to zero and observe that the optimizer  $\beta^*$  has to be nonnegative.

#### 4. SIMPLE BOUNDS AND LIQUIDATION COST

In this section, we gather a few properties of the minimal super-replicating cost  $v^h$ .

Let  $l^h(t, s, z)$  be the minimal super-replication cost with zero pay-off, that is,  $g \equiv 0$ . Since the pay-off  $g$  is assumed to be nonnegative,  $v^h \geq l^h$ . Thus,  $l^h$  provides a lower bound for  $v^h$ . This lower bound will be useful to control the behavior of  $v^h(t, s, z)$  for large values of  $|z|$ . Also,  $l^h$  is an interesting object to analyze apart from this lower bound, since it is the minimal wealth required to be protected against liquidity losses and have nonnegative value at the terminal time. Note that the below analysis shows that  $l^h$  is positive for nonzero portfolio values  $z$ .

It is clear that both  $v^h$  and  $l^h$  are continuous in all variables and nonnegative.

LEMMA 4.1. *For all  $h \in (0, 1]$ ,  $t \leq T$ ,  $z \in \mathbb{R}^1$  and  $s > 0$ ,*

$$v^h(t, s, z) \geq l^h(t, s, z) \geq L^h(\lfloor(T-t)/h\rfloor, s, |z|),$$

where for  $z \geq 0$ ,

$$\begin{aligned} L^h(N+1, s, z) &:= \inf_{a=\{a_1, \dots, a_{N+1}\} \geq 0} \left\{ \sum_{n=0}^N [sd^n z_n^a \sigma \sqrt{h} - a_{n+1} \theta(-a_{n+1})] \right. \\ &= \left. |z_{n+1}^a = z_n^a - a_{n+1} \geq 0, z_0^a = z \right\} \end{aligned}$$

*Proof.* Clearly we have

$$L^h(N+1, s, z) = \inf_{0 \leq a \leq z} \{L^h(N, sd, z-a) - a\theta_{\text{down}}(-a)\} + sz\sigma\sqrt{h}, \quad \text{for } N \geq 0.$$

Set

$$\hat{L}^h(N, s, z) := l^h(T - Nh, s, z),$$

so that by dynamic programming

$$\begin{aligned} \hat{L}^h(N+1, s, z) &= \max \left\{ \inf_a \{ \hat{L}^h(N, sd, z-a) - a\theta(-a) \} + sz\sigma\sqrt{h}, \right. \\ &\quad \left. \inf_b \{ \hat{L}^h(N, su, z-b) - b\theta(-b) \} - sz\sigma\sqrt{h}, \right\}. \end{aligned}$$



First assume that  $z \geq 0$ . Then,

$$\hat{L}^h(N+1, s, z) \geq \inf_a \{ \hat{L}^h(N, sd, z-a) - a\theta(-a) \} + sz\sigma\sqrt{h}.$$

Since  $\hat{L}^h(N, s, z) \geq 0 = \hat{L}^h(N, s, 0)$ , we conclude that

$$\hat{L}^h(N+1, s, z) \geq \inf_{a \leq z} \{ \hat{L}^h(N, sd, z-a) - a\theta(-a) \} + sz\sigma\sqrt{h}.$$

Since in the definition of  $L^h$  we only consider the down movements of the stock process, it is clear that  $L^h(N, s, z') \geq L^h(N, s, z)$  for  $z' \geq z \geq 0$ . This implies that

$$\inf_{a \leq z} L^h(N, s, z-a) - a\theta(-a) = \inf_{0 \leq a \leq z} L^h(N, s, z-a) - a\theta(-a).$$

Hence,

$$L^h(N+1, s, z) = \inf_{a \leq z} \{ L^h(N, sd, z-a) - a\theta(-a) \} + sz\sigma\sqrt{h}.$$

By iterating the above inequalities for  $\hat{L}^h$  and  $L^h$ , we conclude that  $\hat{L}^h(N, s, z) \geq L^h(N, s, z)$ .

Now assume that  $z \leq 0$ . Then,

$$\begin{aligned} \hat{L}^h(N+1, s, z) &\geq \inf_b \{ \hat{L}^h(N, su, z-b) - b\theta(-b) \} - sz\sigma\sqrt{h} \\ &= \inf_{a \leq -z} \{ \hat{L}^h(N, su, z+a) + a\theta(a) \} - sz\sigma\sqrt{h}. \end{aligned}$$

By iterating the above inequality, we prove for  $z \leq 0$  that

$$\begin{aligned} &\hat{L}^h(N+1, s, z) \\ &\geq \inf_{a=\{a_1, \dots, a_{N+1}\} \geq 0} \left\{ \sum_{n=0}^N \left[ -su^n \bar{z}_n^a \sigma \sqrt{h} + a_{n+1} \theta_{\text{up}}(a_{n+1}) \right] \mid 0 \geq \bar{z}_{n+1}^a = \bar{z}_n^a + a_{n+1}, \bar{z}_0^a = z \right\} \\ &\geq \inf_{a=\{a_1, \dots, a_{N+1}\} \geq 0} \left\{ \sum_{n=0}^N \left[ -sd^n \bar{z}_n^a \sigma \sqrt{h} + a_{n+1} \theta_{\text{down}}(a_{n+1}) \right] \mid 0 \geq \bar{z}_{n+1}^a = \bar{z}_n^a + a_{n+1}, \bar{z}_0^a = z \right\} \\ &\geq \inf_{a=\{a_1, \dots, a_{N+1}\} \geq 0} \left\{ \sum_{n=0}^N \left[ -sd^n \bar{z}_n^a \sigma \sqrt{h} - a_{n+1} \theta_{\text{down}}(-a_{n+1}) \right] \mid 0 \geq \bar{z}_{n+1}^a = \bar{z}_n^a + a_{n+1}, \bar{z}_0^a = z \right\}, \end{aligned}$$

where in the last step we used to inequality  $a\theta(a) \geq -a\theta(-a)$  for  $a \geq 0$ . We now set  $z_n = -\bar{z}_n$ , we obtain,

$$\begin{aligned} \hat{L}^h(N+1, s, z) &\geq \inf_{a=\{a_1, \dots, a_{N+1}\} \geq 0} \left\{ \sum_{n=0}^N \left[ sd^n z_n^a \sigma \sqrt{h} - a_{n+1} \theta_{\text{down}}(-a_{n+1}) \right] \right. \\ &\quad \left. = \mid z_{n+1}^a = z_n^a - a_{n+1} \geq 0, z_0^a = -z \right\} \end{aligned}$$

□

Therefore,

$$\hat{L}^h(N+1, s, z) \geq L^h(N+1, s, -z) = L^h(N+1, s, |z|).$$

We can now state the lower bound that will be useful in the next sections.

PROPOSITION 4.2. *Given any  $0 < \alpha < 1$  there exists an integer  $N_\alpha > 0$  and  $h^*(\alpha)$ , so that for all  $h \in (0, h^*(\alpha)]$ ,  $t \leq T - N_\alpha h$  and  $s > 0$ ,*

$$\liminf_{|z| \rightarrow \infty, s' \rightarrow s} \frac{v^h(t, s', z)}{s'|z|} \geq \frac{\sigma\sqrt{h}}{2\sqrt{\alpha}}.$$

*Proof.* For  $s > 0$ ,  $N \geq 0$  set

$$K^h(N, s) := \liminf_{z \rightarrow \infty, s' \rightarrow s} \frac{L^h(N, s', z)}{s'z}.$$

By using dynamic programming,

$$\begin{aligned} K^h(N+1, s) &= \liminf_{z \rightarrow \infty, s' \rightarrow s} \inf_{0 \leq a \leq z} \left\{ \frac{L^h(N, ds', z-a)}{s'z} - \frac{a\theta(-a)}{s'z} \right\} + \sigma\sqrt{h} \\ &= \min\{ \liminf_{z \rightarrow \infty, s' \rightarrow s} A_c(N, s', z), \liminf_{z \rightarrow \infty, s' \rightarrow s} B_c(N, s', z) \} + \sigma\sqrt{h}, \end{aligned}$$

where  $0 < c < 1$  is arbitrary and

$$\begin{aligned} A_c(N, s, z) &:= \inf_{0 \leq a \leq cz} \left\{ \frac{L^h(N, sd, z-a)}{sz} - \frac{a\theta(-a)}{sz} \right\} \\ &\geq \inf_{0 \leq a \leq cz} \left\{ d(1-c) \frac{L^h(N, sd, z-a)}{(sd)(z-a)} \right\} \\ B_c(N, s, z) &:= \inf_{cz \leq a \leq z} \left\{ \frac{L^h(N, sd, z-a)}{sz} - \frac{a\theta(-a)}{sz} \right\} \geq -\frac{c\theta_{\text{down}}(-cz)}{s} = cd. \end{aligned}$$

In the last step, we used the fact that for large values of  $z$  and positive  $c$ ,  $\theta_{\text{down}}(-cz) = -sd$ . Also,

$$\begin{aligned} \liminf_{z \rightarrow \infty} A_c(N, s, z) &\geq \liminf_{z \rightarrow \infty} \inf_{0 \leq a \leq cz} \left\{ d(1-c) \frac{L^h(N, sd, z-a)}{(sd)(z-a)} \right\} \\ &\geq d(1-c)K^h(N, sd). \end{aligned}$$

Since the above holds for any  $c \in (0, 1)$ ,

$$\begin{aligned} K^h(N+1, s) &\geq \max_{c \in (0, 1)} \{ \min\{ d(1-c)K^h(N, sd), cd \} \} + \sigma\sqrt{h} \\ &\geq d \frac{K^h(N, sd)}{1 + K^h(N, sd)} + \sigma\sqrt{h}. \end{aligned}$$

Define the sequence  $k_\alpha(N)$  for  $0 < \alpha < 1$  by the difference equation

$$k_\alpha(N+1) = (1-\alpha) \frac{k_\alpha(N)}{1+k_\alpha(N)} + \alpha, \quad N \geq 0$$

with initial data  $k_\alpha(0) = 0$ . It follows by induction that  $K^h(N, s) \geq k_{\sigma\sqrt{h}}(N)$  for all  $s > 0$  and  $N \geq 0$ , since  $K^h(0, s) = 0$  for any  $s > 0$  and  $d = 1 - \sigma\sqrt{h}$ .

Because  $k_\alpha(N)$  is an increasing sequence, we can easily obtain

$$\lim_{N \uparrow \infty} \frac{1}{\alpha} k_\alpha(N) = \frac{1}{\sqrt{\alpha}}.$$

Therefore, given any  $\alpha \in (0, 1)$ , choose  $N_\alpha$  so that

$$\frac{1}{\alpha} k_\alpha(N) \geq \frac{1}{2\sqrt{\alpha}}, \quad \forall n \geq N_\alpha.$$

By induction on  $N$ , one can establish that  $k_\alpha(N)/\alpha$  is nonincreasing in  $\alpha$  for fixed  $N$ . Therefore, for any  $h$  satisfying  $\sigma\sqrt{h} \leq \alpha$

$$\frac{1}{\sigma\sqrt{h}} k_{\sigma\sqrt{h}}(N) \geq \frac{1}{\alpha} k_\alpha(N).$$

Therefore, for any  $0 < \alpha < 1$ , there exists an integer  $N_\alpha > 0$  and  $0 < h^*(\alpha) \leq 1 \wedge \frac{\alpha^2}{\sigma^2}$  so that for all  $N \geq N_\alpha$  and  $0 < h \leq h^*(\alpha)$

$$\frac{1}{\sigma\sqrt{h}} k_{\sigma\sqrt{h}}(N) \geq \frac{1}{2\sqrt{\alpha}}.$$

Multiplying both sides with  $\sigma\sqrt{h}$  and observing that for any  $z \in \mathbb{R}^1$

$$\frac{v^h(t, s, z)}{s|z|} \geq \frac{L^h \left( \left\lfloor \frac{T-t}{h} \right\rfloor, s, |z| \right)}{s|z|}$$

we prove the proposition.  $\square$

## 5. CONTINUOUS-TIME LIMIT

In this section, we state our main limit result. In the remainder of the paper, we use the parameter choices (3.4) and (3.5). Recall that the option is European with pay-off  $g(S_T)$  and the minimal discrete super-replicating cost  $v^h(t, s, z)$  is defined in (2.4). We assume that  $g$  is continuous and there is a constant  $C > 0$  so that

$$(5.1) \quad 0 \leq g(s) \leq C(1+s), \quad \forall s \geq 0.$$

The continuous-time minimal super-replicating cost  $\phi(t, s)$  is given as the unique solution of (3.6) with terminal data  $\phi(T, s) = g(s)$ .

We define the standard upper and lower relaxed limits in the theory of viscosity solutions of Barles and Perthame (1988); and Fleming and Soner (1993),

$$\begin{aligned} \phi^*(t, s, z) &:= \limsup_{h \downarrow 0, (t', s', z') \rightarrow (t, s, z)} v^h(t', s', z'), \\ \phi_*(t, s) &:= \inf_z \left\{ \liminf_{h \downarrow 0, (t', s', z') \rightarrow (t, s, z)} v^h(t', s', z') \right\}. \end{aligned}$$

**REMARK 5.1.** Note that in the definition of  $\phi_*$  we take the infimum over all initial portfolio values  $z$ . This is a technical choice consistent with the viscosity theory. This choice also preserves the lower semi-continuity with respect to  $(t, s)$ .

PROPOSITION 5.2.  $\phi^*$  is independent of  $z$ .

*Proof.* Fix  $t < T$ ,  $s > 0$ ,  $z, z'$  and an integer  $N$ . Assume that  $h$  is sufficiently small so that  $t + Nh < T$ . Use  $\Delta Z_n = (z' - z)/N$  for all  $n = t, \dots, t + (N - 1)h$  in the wealth equation (2.2). Then, we obtain the following upper bound,

$$v^h(t, s, z) \leq L^h(N, s, z, z') + \sup \{ v^h(t + Nh, s', z') : d^N \leq |s'/s| \leq u^N \},$$

where

$$L^h(N, s, z, z') := N \max\{|z|, |z'|\} u^N s \sigma h^{1/2} + \Lambda \frac{(z' - z)^2}{N}.$$

Now we use the fact that there exists  $(t_h, s_h, z_h) \rightarrow (t, s, z)$  as  $h \downarrow 0$  such that

$$(5.2) \quad \lim_{h \rightarrow 0} v^h(t_h, s_h, z_h) = \phi^*(t, s, z).$$

Moreover, if  $(t_h, s_h, z_h)$  is a sequence converging to  $(t, s, z)$  as  $h \rightarrow 0$ , then

$$\lim_{h \rightarrow 0} v^h(t_h, s_h, z_h) \leq \phi^*(t, s, z).$$

Observe that if  $z_h \rightarrow z$  then  $z'_h = z_h + (z' - z) \rightarrow z'$  so that

$$\phi^*(t, s, z) \leq \Lambda \frac{(z' - z)^2}{N} + \phi^*(t, s, z')$$

for any  $N$ . Thus,

$$\phi^*(t, s, z) \leq \phi^*(t, s, z'), \quad \forall z, z',$$

and consequently  $\phi^*$  is independent of  $z$ . □

REMARK 5.3. The above proof shows that for any sequence  $z_h \rightarrow z$ , there exists  $(t_h, s_h, z_h) \rightarrow (t, s, z)$  as  $h \downarrow 0$  such that (5.2) holds. In particular, we can take the constant approximating sequence  $z$  and can find appropriate  $(t_h, s_h, z) \rightarrow (t, s, z)$  with the property (5.2).

THEOREM 5.4. *Assume (5.1). As  $h \rightarrow 0$  the discrete minimal super-replicating cost  $v^h(t, s, z)$  converges locally uniformly to  $\phi(t, s)$ .*

We state two theorems that will be proved in the last two sections. Then we complete the proof of theorem using these results. In these proofs we will make extensive use of the theory of viscosity solutions. For information on viscosity solutions, we refer the reader to the seminal paper of Crandall, Ishii, and Lions (1992) or to the book Fleming and Soner (1993).

THEOREM 5.5. *The lower semi-continuous relaxed limit  $\phi_*$  is a viscosity super-solution of (3.6).*

We relegate the proof of this theorem to Section 7. We also have the dual result whose result is given in Section 8.

THEOREM 5.6. *The upper semi-continuous relaxed limit  $\phi^*$  is a viscosity sub-solution of (3.6).*

*Proof of Theorem 5.4.* A buy and hold strategy together with the estimate (5.1) show that

$$0 \leq \phi_*(t, s) \leq \phi^*(t, s) \leq C(1 + s).$$

Also, since  $g$  is continuous, we have that

$$\phi^*(T, s) = \phi_*(T, s) = g(s).$$

Hence the comparison theorem for (3.6) proved in Çetin, Soner, and Touzi (2010) implies that  $\phi^* \leq \phi_*$ . Since the opposite inequality follows from their definitions we immediately conclude that  $\phi^* = \phi_*$  and it is equal to the unique viscosity solution of (3.6). In view of the results of Çetin, Soner, and Touzi (2010), this unique solution is the minimal super-replication cost defined in that paper. Now the local uniform convergence of  $v^h(t, s, z)$  to  $\phi(t, s)$  will follow from the definitions of  $\phi_*$  and  $\phi^*$ .  $\square$

## 6. NUMERICAL METHODS

In this section we develop an algorithm that computes the discrete-time super-replicating cost  $v^h(t, s, z)$  and therefore

$$(6.1) \quad \phi^h(t, s) = \inf_z v^h(t, s, z).$$

This algorithm is based on the dynamic programming approach (3.1) and on the introduction of the extra state variable, namely the portfolio variable. One could approach solving  $\phi^h(t, s)$  by a convex program formulation. However, the disadvantage of this approach is that the number of constraints increases exponentially with the time step. Therefore, we believe that our method is more appropriate to compute large time steps.

The rest of this section is as follows. First we introduce an accurate algorithm, which we use to approximate the continuous-time super-replicating value of a European call option in the illiquid market setting. We exhibit with plots and data that liquidity premium does not vanish, as our theoretical results indicate. Although no explicit expression is known for the solution  $\phi(t, s)$  of the partial differential equation (1.1), an asymptotic expansion with respect to the liquidity parameter is obtained in a recent paper of Possamai, Soner, and Touzi (2010). So we compute the value function  $\phi^h(t, s)$  for a number of liquidity parameters  $\Lambda$  in the vicinity of  $\Lambda = 0$  and observe that the slope of the data is close to the value of first order expansion term. This illustrates that the liquidity premium exists in the limit and is equal to the solution  $\phi(t, s)$  of the (1.1). Furthermore, we propose another algorithm for the solution  $\phi(t, s)$  of the continuous-time equation (3.6) derived in Çetin, Soner, and Touzi (2010). This method has the advantage of not having the extra portfolio variable. Thus, it is much faster. We justify this second faster algorithm by comparing it to the slower but accurate numerical method based on dynamic programming.

The first method is to directly solve the dynamic programming equation (3.1). First we discretize the continuous variable  $z$ . Then, the chief step is to efficiently compute a minimization problem of the following type

$$(6.2) \quad F(\xi) = v^h(t + h, su, \xi) - zs\sigma\sqrt{h} + \Lambda(\xi - z)^2,$$

over the variable  $\xi$ . Once this is established, we compute  $v^h(t, s, z)$  for each  $z$  using the above minimization procedure and (3.1). Then,  $\phi^h(t, s)$  is computed by taking the infimum over  $z$ . As it is standard in dynamic programming, we work backwards in time. We first set  $v^h(T, s, z) = g(s)$  for all  $z$ , where  $g(\cdot)$  is the given payoff of the option. In the recursion step, we assume that  $v^h(t+h, s, z)$  for all  $z$  and  $s$  is computed. Then, we minimize the two problems of the type  $F(\xi)$  to calculate  $v^h(t, s, z)$ . In this method the only error is due to the discretization of  $z$ . Thus, by decreasing the grid width for  $\Delta z$ , we can increase the accuracy of the algorithm. To minimize  $F(\xi)$  we exploit the structure of the value function  $v^h(t, s, z)$ . It is interesting to note that  $v^h(t, s, z)$  is convex in the portfolio variable  $z$  for any contingent claim. This fact follows by the definition of  $v^h(t, s, z)$  and the convexity of  $a\theta(a)$  in  $a$ .

Therefore,  $F(\xi)$  is convex in  $\xi$ , and the minimizer  $\xi^*$  satisfies the first-order condition

$$z = f(\xi^*), \quad \text{where} \quad f(\xi) := \xi + \frac{v_\xi(t+h, su, \xi)}{2\Lambda}.$$

Therefore, the minimization problem reduces to finding  $\xi^*$  as a function of  $z$ . Observe that since  $F$  is convex in  $\xi$ , we have

$$f_\xi(\xi) = 1 + \frac{v_{\xi\xi}(t+h, su, \xi)}{2\Lambda} = \frac{F_{\xi\xi}(\xi)}{2\Lambda} \geq 0,$$

and hence  $f$  is nondecreasing. So we find  $\xi^*$  numerically such that  $f(\xi^*) \leq z < f(\xi^* + \Delta z)$ . To do so, we increase incrementally by  $\Delta z$  until the value of the function  $f$  exceeds  $z$ . The strength of the algorithm is that we do not have to start our search everytime from the lowest portfolio value. In fact, given the minimizer  $\xi^*(z)$  for  $z$  we can start the search for  $\xi^*(z + \Delta z)$  from  $\xi^*(z)$ , because  $f$  is nondecreasing. This is why this procedure provides a considerable decrease in the computational time.

We continue by summarizing several numerical results. In these experiments we use 150 time steps and the annual volatility is  $\sigma = 0.2$ . As in our analysis, up and down factors are  $1 \pm \sigma/\sqrt{150}$ , respectively, since we divide a year into 150 time steps. We compute  $\phi^{150}(t, s)$  given by (6.1) for a call option with 1-year maturity and strike  $K = 0.9$  for different liquidity parameters. All portfolio values lie in the interval  $[0, 1]$ , we partition this interval into three different intervals,  $[0, 0.8]$ ,  $[0.8, 0.95]$ , and  $[0.95, 1]$  and use 0.00025, 0.000025, and 0.00005 as the difference between any two consecutive  $z$  values in these intervals, respectively.

The results of this numerical experiment are given in the table below and plotted in Figure 6.1

$\Lambda$	0	0.1	0.2	0.3	0.4	0.5
$\phi^{150}(1, 1)$	0.135859	0.150109	0.159490	0.167005	0.173422	0.179097
$\Lambda$	0.6	0.7	0.8	0.9	1	
$\phi^{150}(1, 1)$	0.184206	0.188886	0.193218	0.197257	0.201068	

Notice that  $\Lambda = 0$  corresponds to the classical Black–Scholes problem. The continuous-time formula for a call option with strike  $K = 0.9$ ,  $\sigma = 0.2$  and 1 year to maturity is equal to 0.135891. The above result differs from this value only 0.023%.

We analyze the behavior of  $\phi^{150}(1, 1)$  for small values of  $\Lambda$  near zero. This is a perturbation around the Black–Scholes value. In a recent paper of Possamai, Soner,

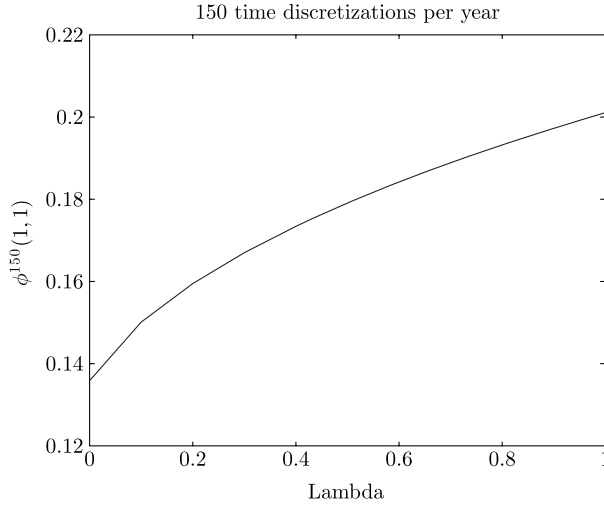


FIGURE 6.1. The dependence of  $\phi^{150}(1, 1)$  on the liquidity parameter  $\Lambda$ .

and Touzi (2010), an explicit expansion is obtained. They showed that for  $\mathbf{S}^\epsilon(t, s, \nu) = \mathbf{S}(t, s, \epsilon\nu)$ , the continuous-time super-replicating cost  $\phi^\epsilon$  has the following expansion.

$$\phi^\epsilon = \nu_{\text{BS}} + \epsilon\phi^{(1)} + \dots + \epsilon^n\phi^{(n)} + o(\epsilon^n),$$

where  $\nu_{\text{BS}}$  is the Black–Scholes value. In fact, for a call option with constant volatility  $\sigma$ ,

$$(6.3) \quad \phi^{(1)}(t, s) = \int_t^T \frac{1}{2\pi} \frac{1}{\sqrt{(T-x)(T+x-2t)}} \exp\left(-\frac{d(s, T-2x+t)^2}{\sigma^2(T+x-2t)}\right) dx,$$

where  $d(s, t) = \ln\left(\frac{s}{K}\right) + \frac{1}{2}\sigma^2 t$ .

We calculated the numerical value of the integral in (6.3) for a call option with  $K = 0.9$ ,  $T = 1$ ,  $t = 0$ ,  $s = 1$ ,  $\sigma = 0.2$ . This value is 0.21168.

Figure 6.2 illustrates the dependence of the discrete-time super-replicating cost  $\phi^{150}(1, 1)$  given by (6.1) on liquidity parameters  $\Lambda$  near zero. The data has an almost linear structure with slope 0.1939, which is a deviation from the theoretical value 0.21168 by 9%.

So far we exhibited our results with 150 time step discretizations a year. However, we are not limited with this step size. The reason we chose this number is that it is sufficient to obtain close results to continuous time. Moreover, we want to make the point that increasing step size does not necessarily mean more accurate results. There is actually a trade-off between the  $\Delta z$  and the time step size. For obtaining more accurate results, as the time step increases one should make the grid for portfolio values finer.

In Çetin, Soner, and Touzi (2010) it is established that for a convex payoff the optimal portfolio position is given by the delta-hedge  $\phi_s(t, s)$ . Our second algorithm is based on this observation to reduce the dimension by removing the dependence on the  $z$ -variable. We construct a function  $\hat{v}(t, s)$  again by backwards recursion. We start with  $\hat{v}(T, s, z) = g(s)$ . The next step is calculated by

$$\hat{v}(T-h, s, z) = \max(g(su) - zs\sigma\sqrt{h}, g(sd) + zs\sigma\sqrt{h}).$$

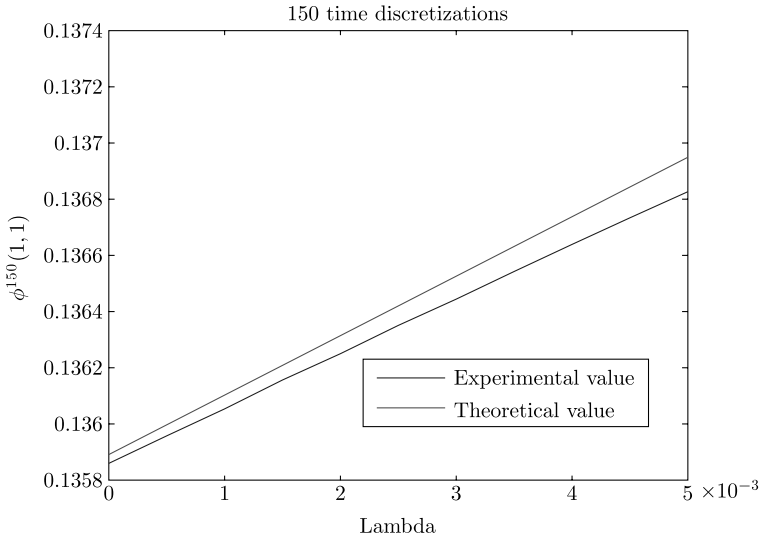


FIGURE 6.2. The behavior of  $\phi^{150}(1, 1)$  near  $\Lambda = 0$ .

We choose  $Z^*(T - h, s)$  that sets the two terms in the maximum equal to each other. The result is

$$Z^*(T - h, s) = \frac{g(su) - g(sd)}{s(u - d)}.$$

Then,

$$\hat{v}(T - h, s) = \hat{v}(T - h, s, Z^*(T - h, s)) = \frac{g(su) + g(sd)}{s(u - d)}.$$

We march backwards in this way. Namely, we define

$$\hat{v}(t, s, z) = \max\{\hat{v}(t + h, su) - zs\sigma\sqrt{h} + \Lambda(Z^*(t + h, su) - z)^2, \hat{v}(t + h, sd) + zs\sigma\sqrt{h} + \Lambda(Z^*(t + h, sd) - z)^2\}.$$

Again we choose  $Z^*(t, s)$  as the value that makes the two terms in the maximum equal to each other. This yields

$$(6.4) \quad Z^*(t, s) = \frac{\hat{v}(t + h, su) - \hat{v}(t + h, sd) + \Lambda(Z^*(t + h, su)^2 - Z^*(t + h, sd)^2)}{2s\sigma\sqrt{h} + 2\Lambda(Z^*(t + h, su) - Z^*(t + h, sd))}.$$

Notice that, formally  $\hat{v}(t + h, s) \approx \phi(t + h, s)$  and  $Z^*(t + h, s) \approx \phi_s(t + h, s)$ . Then, again formally, we arrive at

$$Z^*(t, s) \approx \frac{2\phi_s(t, s)s\sigma\sqrt{h} + 4\Lambda\phi_s(t, s)\phi_{ss}(t, s)s\sigma\sqrt{h}}{s\sigma\sqrt{h} + 4\Lambda\phi_{ss}(t, s)s\sigma\sqrt{h}} \approx \phi_s(t, s).$$

We formally expect that  $\hat{v}$  converges to  $\phi$ . Indeed, our numerical results support this fact. For numerical experimentation, we compare the two numerical values, previously



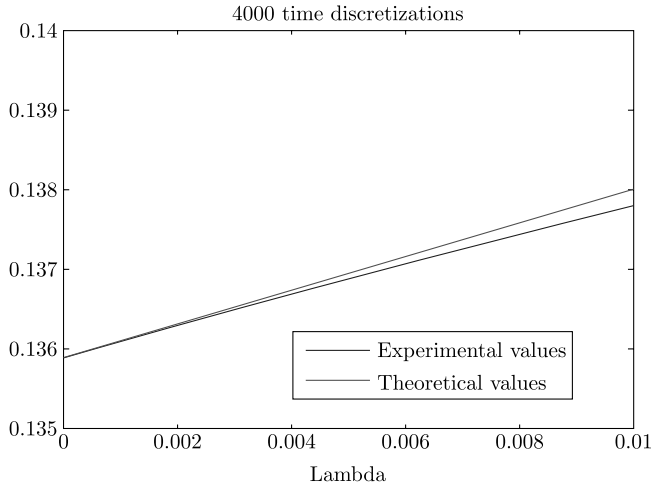


FIGURE 6.3. The behavior of  $\hat{v}(1, 1)$  near  $\Lambda$  near zero.

computed  $\phi^{150}(1, 1)$  and results from the second algorithm  $\hat{v}(1, 1)$  with  $h = 1/4000$  and  $\Lambda$  ranges between 0 to 1. Results are reported in the table below.

$\Lambda$	$\phi^{150}(1, 1)$	$\hat{v}(1, 1)$	Relative error
0	0.135859	0.135887016	0.0206%
0.1	0.150109	0.149922028	0.1245%
0.2	0.159490	0.159906968	0.2614%
0.3	0.167005	0.168681241	1.0037%
0.4	0.173422	0.177011752	2.0699%
0.5	0.179097	0.185250108	3.4356%
0.6	0.184206	0.193584309	5.0912%
0.7	0.188886	0.202119364	7.0060%
0.8	0.193218	0.210911485	9.1572%
0.9	0.197257	0.219985383	11.522%
1	0.201068	0.229344430	14.063%

We also test the second algorithm for small  $\Lambda$  values and compare with the theoretical value of the first-order term of the asymptotic expansion of  $\phi(t, s)$ . Again the plotted data has a form of an affine function with slope 0.19108, which deviates from the theoretical value 0.21168 by 9.73%. It is illustrated in Figure 6.3.

## 7. VISCOSITY SUPER-SOLUTION PROPERTY

In this section, we prove Theorem 5.5.

*Proof of Theorem 5.5.* We complete the proof in several steps.

1. Let a smooth function  $\varphi$  and the point  $(t_0, s_0) \in [0, T] \times (0, \infty)$  satisfy

$$0 = (\phi_* - \varphi)(t_0, s_0) = \min\{(\phi_* - \varphi)(t, s) \mid (t, s) \in [0, T] \times [0, \infty)\}.$$

In view of the definition of a viscosity super-solution, we need to show that

$$(7.1) \quad -\varphi_t(t_0, s_0) - s_0^2 \sigma^2 H(\varphi_{ss}(t_0, s_0)) \geq 0.$$

2. For  $h > 0$  set

$$\varphi^h(t, s, z) := \varphi(t, s) + s\sigma\sqrt{h} K(\varphi_{ss}(t, s)) |\varphi_s(t, s) - z|,$$

$$K(\gamma) := \begin{cases} 1 + 2\Lambda\gamma, & \text{for } \gamma \geq -1/(4\Lambda), \\ 1/2, & \text{for } \gamma \leq -1/(4\Lambda). \end{cases}$$

We claim that for sufficiently small  $h$ , the difference

$$U^h(t, s, z) := v^h(t, s, z) - \varphi^h(t, s, z)$$

attains its minimum at  $(t_h, s_h, z_h)$ . We showed that  $0 \leq \phi_*(t, s) \leq C(1 + s)$ . Because only local behavior of the test function around  $(t_0, s_0)$  is important, without loss of generality we can modify the test function as

$$(7.2) \quad \tilde{\varphi}(t, s) = \chi\varphi(t, s) - C(1 + s_0)(1 - \chi),$$

where  $\chi$  is a  $C^\infty$  function satisfying  $0 \leq \chi \leq 1$ ,  $\chi \equiv 1$  on some neighborhood  $\mathcal{N}$  of  $(t_0, s_0)$  and  $\chi \equiv 0$  off another neighborhood  $\tilde{\mathcal{N}}$  of  $(t_0, s_0)$  such that  $\mathcal{N} \subset \tilde{\mathcal{N}}$ . It is clear that

$$0 = (\phi_* - \tilde{\varphi})(t_0, s_0) = \min\{(\phi_* - \tilde{\varphi})(t, s) : (t, s) \in [0, T] \times [0, \infty)\}$$

and  $\tilde{\varphi}$  and  $\varphi$  have the same local behavior around  $(t_0, s_0)$ . By abuse of notation call  $\tilde{\varphi} = \varphi$ . Fix  $h > 0$  sufficiently small and set

$$L = \inf\{U^h(t, s, z) : (t, s) \in \tilde{\mathcal{N}}, z \in \mathbb{R}^1\}.$$

We will show that  $L$  is attained for some  $(t_h, s_h, z_h)$ . We can find  $(t_n, s_n, z_n)$  such that

$$(7.3) \quad L + \frac{1}{n} \geq v^h(t_n, s_n, z_n) - \varphi^h(t_n, s_n, z_n).$$

Since  $(t_n, s_n)$  belong to the compact interval  $\tilde{\mathcal{N}}$ , by passing to a subsequence if necessary  $(t_n, s_n) \rightarrow (t_h, s_h)$ . If we can show that  $|z_n|$  is uniformly bounded,  $z_n$  will have a convergent subsequence to  $z_h$ , and so the infimum will be attained. Hence, for a contradiction assume that  $|z_n| \rightarrow \infty$ . Now choose  $0 < \alpha < 1$  sufficiently small that

$$\frac{1}{4\sqrt{\alpha}} \geq \max_{\tilde{\mathcal{N}}} K(\varphi_{ss}(t, s)).$$

By Proposition 4.2 there exists  $N_\alpha \in \mathbb{N}$  and  $h^*(\alpha)$  such that for all  $h \leq h^*(\alpha)$  and  $t \leq T - N_\alpha h^*(\alpha)$ ,

$$\liminf_{|z| \rightarrow \infty, s' \rightarrow s_h} \frac{v^h(t, s', |z|)}{s'|z|} \geq \frac{\sigma \sqrt{h}}{2\sqrt{\alpha}}.$$

If necessary we can decrease  $h^*(\alpha)$  so that  $\{t : (t, s) \in \widetilde{\mathcal{N}} \text{ for some } s\} \subset [0, T - N_\alpha h^*(\alpha)]$ . Using these facts, if  $|z_n| \rightarrow \infty$  we reach the contradiction that the right-hand side of (7.3) becomes larger than the left-hand side. The next step is to show that

$$L = \inf \{U^h(t, s, z) : (t, s) \in [0, T] \times [0, \infty), z \in \mathbb{R}^1\}.$$

Clearly

$$L = U^h(t_h, s_h, z_h) \leq v^h(t_0, s_0, C) - \phi_*(t_0, s_0) \leq C(1 + s_0),$$

by a buy and hold argument. On the other side for  $(t, s) \notin \widetilde{\mathcal{N}}$

$$U^h(t, s, z) \geq l^h(t, s, z) + C(1 + s_0) - s|z|\sigma\sqrt{h} \geq C(1 + s_0)$$

so that the claim follows.

As it is standard in the theory of viscosity solutions (cf., Fleming and Soner 1993), without loss of generality we may assume that  $(t_0, s_0)$  is a *strict* minimum. It will be shown later in (7.7) that  $|z_h|$  remains uniformly bounded. Then it is easy to establish that as  $h \downarrow 0$ ,  $(t_h, s_h) \rightarrow (t_0, s_0)$  and the minimum value  $U^h(t_h, s_h, z_h)$  converges to the minimum value of the difference  $\phi_* - \varphi$ , which is equal to zero.

3. Set  $e^h := U^h(t_h, s_h, z_h)$ , so that

$$v^h(t, s, z) \geq \varphi^h(t, s, z) + e^h, \quad \forall (t, s, z), \quad \text{and} \quad v^h(t_h, s_h, z_h) = \varphi^h(t_h, s_h, z_h) + e^h.$$

Hence, in view of dynamic programming (3.1),

$$\begin{aligned} \varphi^h(t_h, s_h, z_h) &= \varphi(t_h, s_h) + s_h \sigma \sqrt{h} K(\varphi_{ss}(t_h, s_h)) |\varphi_s(t_h, s_h) - z_h| \\ &= -e_h + v^h(t_h, s_h, z_h) \\ &= -e_h + \max_a \{\min_a [v^h(t_h + h, us_h, z_h + a) + a\theta(a) + s_h z_h(1 - u)], \\ (7.4) \quad &\min_b [v^h(t_h + h, ds_h, z_h + b) + b\theta(b) + s_h z_h(1 - d)]\} \\ &\geq \max_a \{\min_a [\varphi^h(t_h + h, us_h, z_h + a) + a\theta(a) + s_h z_h(1 - u)], \\ &\min_b [\varphi^h(t_h + h, ds_h, z_h + b) + b\theta(b) + s_h z_h(1 - d)]\}. \end{aligned}$$

4. The following function will be used repeatedly in our analysis.

$$\min_a \{a^2 + B|\xi - a|\} = B^2 \psi \left( \frac{\xi}{B} \right),$$

where  $B > 0$  is an arbitrary constant and

$$\psi(r) = \begin{cases} r^2 & |r| \leq \frac{1}{2}, \\ |r| - \frac{1}{4} & |r| \geq \frac{1}{2}. \end{cases}$$

5. Let  $\Gamma^*$  be the upper bound of  $\varphi_{ss}$  in a neighborhood of  $(t_0, s_0)$ . Set

$$h^* := 4(\sigma K(\Gamma^*))^{-2},$$

so that for all  $h \leq h^*$  the following two minimizations are equivalent.

$$\min_a \{ \Lambda a^2 + |\xi - a|B \} = \min_a \{ a\theta_t(a) + |\xi - a|B \},$$

where

$$B = s\sigma\sqrt{h} K(\phi_{ss}(t, s)).$$

Indeed

$$\theta(a) = \begin{cases} \Lambda a & a \geq -\frac{s}{\Lambda} \\ -s & a \leq -\frac{s}{\Lambda} \end{cases} \quad \Rightarrow \quad a\theta(a) \leq \Lambda a^2,$$

and the equality holds for all  $a \geq -(s/\Lambda)$ . Moreover, the optimizer of the first term satisfies  $|a^*| \leq B/(2\Lambda)$ . The above definition of  $B$  implies that for  $h \leq h^*$ ,  $|a^*| \leq (s/\Lambda)$ . Therefore,  $a^*\theta(a) = \Lambda(a^*)^2$ .

In view of the definition of  $\varphi^h$ , we may apply this result in (7.4). We then conclude that the terms  $a\theta(a)$  and  $b\theta(b)$  in the minimizations can be replaced by  $\Lambda a^2$  and  $\Lambda b^2$ , respectively. The result is

$$(7.5) \quad \varphi^h(t_h, s_h, z_h) = \varphi(t_h, s_h) + \eta K(\varphi_{ss}(t_h, s_h)) |\varphi_s(t_h, s_h) - z_h| \geq \max\{J_1, J_2\},$$

where

$$\begin{aligned} \eta &:= s_h\sigma\sqrt{h}, \\ J_1 &= \varphi(t_h + h, us_h) - z_h\eta + \min_a \{ \Lambda a^2 + u\eta K(\varphi_{ss}(t_h + h, us_h)) |\varphi_s(t_h + h, us_h) - z_h - a| \}, \\ J_2 &= \varphi(t_h + h, ds_h) + z_h\eta + \min_b \{ \Lambda b^2 + d\eta K(\varphi_{ss}(t_h + h, ds_h)) |\varphi_s(t_h + h, ds_h) - z_h - b| \}. \end{aligned}$$

6. Next we use the Taylor expansion of the terms  $\varphi(t_h + h, us_h)$ ,  $\varphi(t_h + h, ds_h)$ , and their first and second space derivatives around  $x_h = (t_h, s_h)$ . In the sequel  $C$  denotes a generic constant depending on the local sup-norm of the derivatives of the test function  $\varphi$ . We introduce the notation

$$\gamma := \varphi_{ss}(t_h, s_h), \quad \xi := \varphi_s(t_h, s_h) - z_h, \quad x_h := (t_h, s_h), \quad \eta := s_h\sigma\sqrt{h}.$$

For the following computations we observe that for  $B > 0$  the minimization problem

$$\min_a \Lambda a^2 + |\xi - a|B$$

is monotone increasing in  $B$ . Furthermore, also by definition of  $K(\gamma)$  we have the inequality

$$K(\gamma - Ch^{1/2}) \geq K(\gamma) - 2\Lambda Ch^{1/2},$$

where in the argument below we denote  $2\Lambda C$  by  $C$ . These two statements along with the triangle inequality brings us to

$$\begin{aligned} J_1 &= \varphi(t_h + h, us_h) - z_h \eta + \min_a \{ \Lambda a^2 + u\eta K(\varphi_{ss}(t_h + h, us_h)) |\varphi_s(t_h + h, us_h) - z_h - a| \} \\ &\geq \varphi(x_h) + \varphi_t(x_h)h + \xi \eta + \frac{\eta^2}{2} \gamma + \min_a \{ \Lambda a^2 + \eta (K(\gamma) - Ch^{1/2}) |\xi + \eta\gamma - a| \} - Ch^{3/2} \\ &\geq \varphi(x_h) + \varphi_t(x_h)h + \xi \eta + \frac{\eta^2}{2} \gamma + \frac{\eta^2 (K(\gamma) - Ch^{1/2})}{\Lambda} \psi \left( \frac{\Lambda(\xi + \eta\gamma)}{\eta(K(\gamma) - Ch^{1/2})} \right) - Ch^{3/2}. \end{aligned}$$

In the last step we used the function  $\psi$  introduced in Step 4. We want to make a similar analysis for  $J_2$ ; however, since  $d < 1$  we cannot get rid of the  $d$  coefficient in front of  $K$  immediately. However, for some  $C'$

$$d\eta K(\gamma - Ch^{1/2}) \geq \eta(K(\gamma) - C'h^{1/2}),$$

where below we denote  $C'$  by  $C$ . Hence, we obtain

$$J_2 \geq \varphi(x_h) - \xi \eta + \varphi_t(x_h)h + \frac{\eta^2}{2} \gamma + \frac{\eta^2 (K(\gamma) - Ch^{1/2})}{\Lambda} \psi \left( \frac{\Lambda(\xi - \eta\gamma)}{\eta(K(\gamma) - Ch^{1/2})} \right) - Ch^{3/2}.$$

Using (7.5) and the above estimates we conclude that

$$(7.6) \quad 0 \geq \varphi_t(x_h)h - \eta K(\gamma)|\xi| + \frac{\eta^2}{2} \gamma + \max\{I_1, I_2\} - Ch^{3/2},$$

where

$$\begin{aligned} I_1 &= \frac{\eta^2 ((K(\gamma) - Ch^{1/2}))^2}{\Lambda} \psi \left( \frac{\Lambda(\xi + \eta\gamma)}{\eta(K(\gamma) - Ch^{1/2})} \right) + \xi \eta, \\ I_2 &= \frac{\eta^2 ((K(\gamma) - Ch^{1/2}))^2}{\Lambda} \psi \left( \frac{\Lambda(\xi - \eta\gamma)}{\eta(K(\gamma) - Ch^{1/2})} \right) - \xi \eta. \end{aligned}$$

7. In this step we show that

$$(7.7) \quad \limsup_{h \downarrow 0} |z_h| < \infty.$$

Indeed, if this is not the case, then  $|\xi|$  converges to infinity. Without loss of generality assume that this limit is  $+\infty$  then by (7.6),

$$\begin{aligned} 0 &\geq \varphi_t(x_h)h - \eta K(\gamma)|\xi| + \frac{\eta^2}{2} \gamma + \max\{I_1, I_2\} - Ch^{3/2} \\ &\geq \varphi_t(x_h)h - \eta K(\gamma)\xi + \frac{\eta^2}{2} \gamma + \eta (K(\gamma) - Ch^{1/2}) (\xi + \eta\gamma) \\ &\quad - \frac{\eta^2 ((K(\gamma) - Ch^{1/2}))^2}{4\Lambda} + \xi \eta - Ch^{3/2} \\ &\geq -C^*h + \xi \eta (1 - C\sqrt{h}), \end{aligned}$$

where  $C^*$  is a constant depending on  $\gamma$ ,  $\varphi_t$ , and others. Since  $\eta = s\sigma h^{1/2}$ , for small  $h$  the above inequality can not hold. Hence this proves (7.7).

8. In (7.6), since  $\psi$  is even,  $\max\{I_1, I_2\}$  is also symmetric in  $\xi$ . Therefore, it suffices to consider the case  $\xi \geq 0$ . Then, we may consider only  $I_1$  instead of  $\max\{I_1, I_2\}$ . Thus, to prove (7.1) it suffices to show that

$$(7.8) \quad I := \frac{\eta^2}{2}\gamma + \frac{\eta^2((K(\gamma) - Ch^{1/2}))^2}{\Lambda} \psi\left(\frac{\Lambda(\xi + \eta\gamma)}{\eta(K(\gamma) - C\sqrt{h})}\right) \\ + \xi\eta(1 - K(\gamma)) - \eta^2 H(\gamma) \geq -Ch^{3/2},$$

since by (7.6)

$$- \varphi_t(x_h)h \geq \frac{\eta^2}{2}\gamma + \frac{\eta^2((K(\gamma) - Ch^{1/2}))^2}{\Lambda} \psi\left(\frac{\Lambda(\xi + \eta\gamma)}{\eta(K(\gamma) - C\sqrt{h})}\right) \\ + \xi\eta(1 - K(\gamma)) - Ch^{3/2}.$$

We first consider the case

$$2\Lambda|\xi + \eta\gamma| \leq \eta(K(\gamma) - Ch^{1/2}).$$

Since  $\Lambda x^2 + Bx \geq -B^2/(4\Lambda)$  for all  $x$ ,

$$I = \Lambda(\xi + \eta\gamma)^2 + \xi\eta(1 - K(\gamma)) + \frac{\eta^2}{2}\gamma - \eta^2 H(\gamma) \\ = \Lambda(\xi + \eta\gamma)^2 + \eta(1 - K(\gamma))(\xi + \eta\gamma) - \eta^2\gamma(1 - K(\gamma)) + \frac{\eta^2}{2}\gamma - \eta^2 H(\gamma) \\ \geq \eta^2 \left[ -\frac{(1 - K(\gamma))^2}{4\Lambda} + \gamma \left( K(\gamma) - \frac{1}{2} \right) - H(\gamma) \right].$$

In the above, either  $\gamma \leq -1/(4\Lambda)$  and therefore  $K(\gamma) = 1/2$ ,  $H(\gamma) = -1/(16\Lambda)$ , or  $\gamma \geq -1/(4\Lambda)$  and therefore  $K(\gamma) = 1 + 2\Lambda\gamma$ ,  $H(\gamma) = \gamma/2 + \Lambda\gamma^2$ . In both cases, the right-hand side of the above inequality is exactly equal to zero.

In the following two cases, we assume that

$$(7.9) \quad 2\Lambda|\xi + \eta\gamma| \geq \eta(K(\gamma) - Ch^{1/2}).$$

9. We first consider the case  $\gamma \geq \bar{C}h^{1/2} - 1/(4\Lambda)$  where  $2\Lambda\bar{C} \geq C$ . Notice in this case

$$H(\gamma) = \frac{1}{2}\gamma + \Lambda\gamma^2, \quad K(\gamma) = 1 + 2\Lambda\gamma.$$

Since  $\xi \geq 0$ , the inequality (7.9) implies that  $\xi + \eta\gamma \geq 0$  and

$$\begin{aligned}
I &= \eta(K(\gamma) - Ch^{1/2})(\xi + \eta\gamma) - \frac{\eta^2(K(\gamma) - Ch^{1/2})^2}{4\Lambda} + \xi\eta(1 - K(\gamma)) + \frac{\eta^2}{2}\gamma - \eta^2 H(\gamma) \\
&= \eta(1 - C\sqrt{h})\xi + \eta^2\gamma(1 + 2\Lambda\gamma - C\sqrt{h}) - \frac{1}{4\Lambda}\eta^2(1 + 2\Lambda\gamma - C\sqrt{h})^2 - \Lambda\eta^2\gamma^2 \\
&= \eta(1 - C\sqrt{h})\xi + \frac{\eta^2}{4\Lambda}(1 + 2\Lambda\gamma - C\sqrt{h})(4\Lambda\gamma - 1 - 2\Lambda\gamma + C\sqrt{h}) - \Lambda\eta^2\gamma^2 \\
&= \eta(1 - C\sqrt{h})\xi + \frac{\eta^2}{4\Lambda}(2\Lambda\gamma + (1 - C\sqrt{h}))(2\Lambda\gamma - (1 - C\sqrt{h})) - \Lambda\eta^2\gamma^2 \\
&= \eta(1 - C\sqrt{h})\xi - \frac{1}{4\Lambda}\eta^2(1 - C\sqrt{h})^2.
\end{aligned}$$

Since in this subcase  $\xi \geq \frac{\eta}{2\Lambda}(1 - C\sqrt{h})$ ,

$$I \geq \frac{\eta^2(1 - C\sqrt{h})^2}{4\Lambda} \geq 0.$$

10. The only remaining case is  $\gamma \leq \bar{C}h^{1/2} - 1/(4\Lambda)$ .

First we assume that  $\xi + \eta\gamma \geq 0$  so that  $\xi + \eta\gamma \geq \frac{\eta}{2\Lambda}(K(\gamma) - C\sqrt{h})$

$$\begin{aligned}
I &= \eta(K(\gamma) - C\sqrt{h})(\xi + \eta\gamma) - \frac{\eta^2}{4\Lambda}(K(\gamma) - C\sqrt{h})^2 + \frac{\eta^2}{2}\gamma + (1 - K(\gamma))\xi\eta - \eta^2 H(\gamma) \\
&= \eta^2\gamma \left( \frac{1}{2} + K(\gamma) - C\sqrt{h} \right) + (1 - C\sqrt{h})\eta\xi - \eta^2 H(\gamma) - \frac{\eta^2}{4\Lambda}(K(\gamma) - C\sqrt{h})^2.
\end{aligned}$$

Since  $K(\gamma) \geq \frac{1}{2}$ , we have

$$\begin{aligned}
&\geq (1 - C\sqrt{h})\eta(\xi + \eta\gamma) - \eta^2 H(\gamma) - \frac{\eta^2}{4\Lambda}(K(\gamma) - C\sqrt{h})^2 \\
&\geq \frac{\eta^2}{2\Lambda}(1 - C\sqrt{h})(K(\gamma) - C\sqrt{h}) - \eta^2 H(\gamma) - \frac{\eta^2}{4\Lambda}(K(\gamma) - C\sqrt{h})^2 \\
&\geq \frac{\eta^2}{4\Lambda}(K(\gamma) - C\sqrt{h})(2 - C\sqrt{h} - K(\gamma)) - \eta^2 H(\gamma) \\
&\geq \frac{\eta^2}{4\Lambda}(1 - C\sqrt{h}) \geq 0.
\end{aligned}$$

In the last step, we used the fact that, since  $\gamma \leq -\frac{1}{4\Lambda} + \bar{C}\sqrt{h}$ , it follows

$$K(\gamma) \leq \frac{1}{2} + 2\Lambda\bar{C}\sqrt{h}, \quad H(\gamma) \leq -\frac{1}{16\Lambda} + \Lambda\bar{C}^2h.$$

Next suppose that  $-\xi - \eta\gamma \geq 0$ , then

$$-\xi - \eta\gamma \geq \frac{\eta}{2\Lambda}(K(\gamma) - C\sqrt{h}) \geq \frac{\eta}{2\Lambda} \left( \frac{1}{2} - C\sqrt{h} \right).$$

This implies

$$0 \leq \xi \leq -\frac{\eta}{4\Lambda}(1 + 4\Lambda\gamma - 2C\sqrt{h}).$$

Then using the same bounds for  $H(\gamma)$  and  $K(\gamma)$  as above, we obtain

$$\begin{aligned}
I &= \eta(K(\gamma) - C\sqrt{h})(-\xi - \eta\gamma) - \frac{1}{4\Lambda}\eta^2(K(\gamma) - C\sqrt{h})^2 + \frac{\eta^2\gamma}{2} \\
&\quad + (1 - K(\gamma))\eta\xi - \eta^2 H(\gamma) \\
&\geq \eta\left(\frac{1}{2} - C\sqrt{h}\right)(-\xi - \eta\gamma) - \frac{1}{4\Lambda}\eta^2(K(\gamma) - C\sqrt{h})^2 + \frac{\eta^2\gamma}{2} \\
&\quad + (1 - K(\gamma))\eta\xi - \eta^2 H(\gamma) \\
&\geq \eta\xi(C - 2\Lambda\bar{C})\sqrt{h} + \eta^2\gamma C\sqrt{h} - \eta^2\left(-\frac{1}{16\Lambda} + \Lambda\bar{C}^2h\right) \\
&\quad - \frac{\eta^2}{4\Lambda}\left(\frac{1}{2} + 2\Lambda\bar{C}\sqrt{h} - C\sqrt{h}\right)^2.
\end{aligned}$$

Since  $C - 2\Lambda\bar{C} \leq 0$ ,

$$\begin{aligned}
&\geq -\frac{\eta^2}{4\Lambda}(C - 2\Lambda\bar{C})(1 + 4\Lambda\gamma - 2C\sqrt{h})\sqrt{h} + \eta^2\gamma C\sqrt{h} \\
&\quad - \eta^2\left(-\frac{1}{16\Lambda} + \Lambda\bar{C}^2h\right) - \frac{\eta^2}{4\Lambda}\left(\frac{1}{2} + 2\Lambda\bar{C}\sqrt{h} - C\sqrt{h}\right)^2 \\
&\geq -Ch^{3/2}.
\end{aligned}$$

11. In steps 8, 9, and 10 we proved the claim (7.8). This proves that  $\phi_*$  is a viscosity super-solution of (3.6).  $\square$

## 8. VISCOSITY SUB-SOLUTION PROPERTY

In this section we prove Theorem 5.6.

*Proof of Theorem 5.6.* Again, we complete the proof in several steps.

1. Let a smooth function  $\varphi$  and the point  $(t_0, s_0) \in [0, T] \times (0, \infty)$  satisfy

$$0 = (\phi^* - \varphi)(t_0, s_0) = \max\{(\phi^* - \varphi)(t, s) \mid (t, s) \in [0, T] \times [0, \infty)\}.$$

In view of the definition of a viscosity sub-solution, we need to show that

$$(8.1) \quad -\varphi_t(t_0, s_0) - s_0^2\sigma^2 H(\varphi_{ss}(t_0, s_0)) \leq 0.$$

As in the super-solution argument without loss of generality we modify the test function as

$$\tilde{\varphi}(t, s) = \chi\varphi(t, s) + (1 - \chi)(C^*s + K),$$

where  $K$  is sufficiently large enough and  $C^* \geq C$ . As before,  $\chi$  is a  $C^\infty$  function satisfying  $0 \leq \chi \leq 1$ ,  $\chi \equiv 1$  on some neighborhood  $\mathcal{N}$  of  $(t_0, s_0)$  and  $\chi \equiv 0$  on a larger neighborhood  $\tilde{\mathcal{N}}$  of  $(t_0, s_0)$ . Since  $0 \leq \phi^*(t, s) \leq C(1 + s)$ ,

$$0 = (\phi^* - \tilde{\varphi})(t_0, s_0) = \max\{(\phi^* - \tilde{\varphi})(t, s) \mid (t, s) \in [0, T] \times [0, \infty)\},$$



and  $\tilde{\varphi}$  and  $\varphi$  has the same local behavior. Therefore, by abuse of notation, call  $\tilde{\varphi} = \varphi$ . Also again without loss of generality we may assume that  $(t_0, s_0)$  is a *strict* maximum.

2. By the proposition (5.1) and the remark after it we can find  $(t'_h, s'_h, \varphi_s(t'_h, s'_h)) \rightarrow (t_0, s_0, \varphi_s(t_0, s_0))$  as  $h \downarrow 0$  such that

$$0 = (\phi^* - \varphi)(t_0, s_0) = \lim_{h \downarrow 0} v^h(t'_h, s'_h, \varphi_s(t'_h, s'_h)) - \varphi(t'_h, s'_h).$$

Denote by  $\psi^h(t, s) = v^h(t, s, \varphi_s(t, s))$ . Then we claim

$$\max\{\psi^h(t, s) - \varphi(t, s) : (t, s) \in \tilde{\mathcal{N}}\} = \sup\{\psi^h(t, s) - \varphi(t, s) : (t, s) \in [0, T] \times [0, \infty)\}.$$

By compactness of  $\tilde{\mathcal{N}}$ ,  $\psi^h(t, s) - \varphi(t, s)$  attains its maximum at  $(t_h, s_h)$  and

$$\psi^h(t_h, s_h) - \varphi(t_h, s_h) \geq \psi^h(t_0, s_0) - \varphi(t_0, s_0) \geq -\phi^*(t_0, s_0) \geq C(1 + s_0).$$

On the other hand, for all  $(t, s) \notin \tilde{\mathcal{N}}$  we have

$$\psi^h(t, s) - \varphi(t, s) = v^h(t, s, C^*) - C^*s - K \leq C^*s + C - C^*s - K \leq -C(1 + s_0),$$

by the fact that  $v^h(t, s, z) \leq sz + C$  for  $z \geq C$ . Now by compactness of  $\tilde{\mathcal{N}}(t_h, s_h)$  converges to  $(\bar{t}, \bar{s})$  by passing to a subsequence if necessary so that

$$0 \leq \lim_{h \downarrow 0} \psi^h(t_h, s_h) - \varphi(t_h, s_h) = \phi^*(\bar{t}, \bar{s}) - \varphi(\bar{t}, \bar{s}).$$

Since the maximum is strict we can conclude that  $(\bar{t}, \bar{s}) = (t_0, s_0)$  and

$$e_h = \psi^h(t_h, s_h) - \varphi(t_h, s_h) \rightarrow 0.$$

Furthermore,

$$\psi^h(t, s) \leq \varphi(t, s) + e_h, \quad \forall (t, s), \quad \text{and} \quad \psi^h(t_h, s_h) = \varphi(t_h, s_h) + e_h.$$

In view of dynamic programming (3.1),

$$\begin{aligned} \varphi(t_h, s_h) &= -e_h + \psi^h(t_h, s_h) = -e_h + v^h(t_h, s_h, \varphi_s(t_h, s_h)) \\ &= -e_h + \max_a \{\min[v^h(t_h + h, us_h, \varphi_s(t_h, s_h) + a) + a\theta(a) + s_h \varphi_s(t_h, s_h)(1 - u)], \\ &\quad \min_b [v^h(t_h + h, ds_h, \varphi_s(t_h, s_h) + b) + b\theta(b) + s_h \varphi_s(t_h, s_h)(1 - d)]\}. \end{aligned}$$

Set

$$x_h := (t_h, s_h), \quad \eta := s_h \sigma h^{1/2}, \quad p := \varphi_s(x_h), \quad \gamma := \varphi_{ss}(x_h).$$

In dynamic programming, we choose

$$a := a_h = \varphi_s(t_h + h, us_h) - \varphi_s(t_h, s_h), \quad b := b_h = \varphi_s(t_h + h, ds_h) - \varphi_s(t_h, s_h),$$

so that

$$\begin{aligned} v^h(t_h + h, us_h, \varphi_s(t_h, s_h) + a_h) &= \psi^h(t_h + h, us_h) \leq \varphi(t_h + h, us_h) + e_h, \\ v^h(t_h + h, ds_h, \varphi_s(t_h, s_h) + b_h) &= \psi^h(t_h + h, ds_h) \leq \varphi(t_h + h, ds_h) + e_h. \end{aligned}$$

We use these choices in dynamic programming. The result is

$$(8.2) \quad \varphi(x_h) \leq \max \{(\varphi(t_h + h, us_h) + a_h \theta(a_h) - \eta p), (\varphi(t_h + h, ds_h) + b_h \theta(b_h) + \eta p)\}.$$

3. Since for an appropriate constant  $C$ ,

$$a_h \leq \gamma \eta + Ch, \quad \Rightarrow \quad a_h \theta(a_h) \leq \Lambda(a_h)^2 \leq \Lambda \gamma^2 \eta^2 + Ch^{3/2}.$$

Similarly,

$$b_h \theta(b_h) \leq \Lambda \gamma^2 \eta^2 + Ch^{3/2}.$$

These together with (8.2) imply that

$$\varphi(x_h) \leq \max \{(\varphi(t_h + h, us_h) - \eta p), (\varphi(t_h + h, ds_h) + \eta p)\} + \Lambda \eta^2 \gamma^2 + Ch^{3/2}.$$

We directly estimate that

$$\begin{aligned} \varphi(t_h + h, us_h) - \eta p &\leq \varphi_t(x_h)h + \frac{1}{2}\eta^2 \gamma + Ch^{3/2}, \\ \varphi(t_h + h, ds_h) + \eta p &\leq \varphi_t(x_h)h + \frac{1}{2}\eta^2 \gamma + Ch^{3/2}. \end{aligned}$$

We substitute these estimates into the previous inequality. The result is

$$0 \leq h [\varphi_t(x_h) + s_h^2 \sigma^2 \tilde{H}(\varphi_{ss}(x_h))] + Ch^{3/2} \quad \text{where} \quad \tilde{H}(\gamma) = \frac{1}{2}\gamma + \Lambda \gamma^2.$$

Hence,

$$-\varphi_t(t_0, s_0) - s_0^2 \sigma^2 \tilde{H}(\varphi_{ss}(t_0, s_0)) \leq 0.$$

For any  $\Gamma \geq 0$ , set  $\bar{\varphi}(t, s) := \varphi(t, s) + \Gamma(s - s_0)^2/2$ . Clearly,  $\phi^* - \bar{\varphi}$  attains its maximum at  $(t_0, s_0)$ . Our argument implies that

$$0 \geq -\bar{\varphi}_t(t_0, s_0) - s_0^2 \sigma^2 \tilde{H}(\bar{\varphi}_{ss}(t_0, s_0)) = \varphi_t(t_0, s_0) - s_0^2 \sigma^2 \tilde{H}(\varphi_{ss}(t_0, s_0) + \Gamma).$$

Since,

$$H(\gamma) = \inf_{\Gamma \geq 0} \tilde{H}(\gamma + \Gamma),$$

we conclude that (8.1) holds. Therefore,  $\phi^*$  is also a viscosity subsolution of (3.6).  $\square$

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