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Vortex Density Models for Superconductivity and Superfluidity

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Abstract: We study some functionals that describe the density of vortex lines in superconductors subject to an applied magnetic field, and in Bose-Einstein condensates subject to rotational forcing, in quite general domains in 3 dimensions. These functionals are derived from more basic models via Gamma-convergence, here and in the companion paper (Baldo et al. in Arch Rat Mech Anal 205(3):699–752, 2012). In our main results, we use these functionals to obtain leading order descriptions of the first critical applied magnetic field (for superconductors) and forcing (for Bose-Einstein), above which ground states exhibit nontrivial vorticity, as well as a characterization of the vortex density in terms of a non local vector-valued generalization of the classical obstacle problem.

1. Introduction

In this paper we study certain limits of the Ginzburg-Landau model, which describes a superconducting object in an external magnetic field, and the Gross-Pitaevskii functional, which describes a Bose-Einstein condensate confined in a trap and subject to rotational forcing.

Most prior mathematical work on these sorts of problems has been limited to 2-dimensional models that are good descriptions, in various regimes, either of very flat, thin objects (superconductors or condensates), or of objects that are translation-invariant, or very nearly so, in one direction. Important results about the 2-dimensional Ginzburg-Landau model, obtained by Sandier and Serfaty in [30,31] (see also the book [32]) include the characterization of the applied critical magnetic field, below which the ground state of a superconductor expels the magnetic field, and above which the superconductor in the ground state is penetrated by magnetic vortices; and a description, in terms of an obstacle problem solved by the magnetic field, of the limiting density of magnetic vortices above the critical applied field. Similar descriptions of 2d Bose-Einstein condensates hold, though they are somewhat less well-documented in the literature.

In this paper we prove analogous results for the full physical problems of 3-dimensional superconductors and condensates. In particular, we determine the leading-order first critical applied field for superconductors, characterized by essentially the same dichotomy as in the 2d case; and we obtain a description, in the supercritical case, of the limiting vortex density in terms of a constrained minimization problem solved by the magnetic field. This problem is *not* a classical obstacle problem, but it is a kind of nonlocal, vector-valued obstacle problem with an interesting structure. We also establish corresponding results about vortices in Bose-Einstein condensate wave functions, that is, ground states of the Gross-Pitaevskii functional with rotational forcing. These results include the determination of the leading-order critical rotational velocity, and a characterization, in terms of a nonlocal generalization of an obstacle problem, of the limiting vortex density for rotations above this critical value.

We obtain these results from the study of certain functionals whose ground states characterize vortex density, and other associated quantities, in limits of sequence of minimizers of (suitably scaled) Ginzburg-Landau or Gross-Pitaevskii functionals. In the case of Ginzburg-Landau, this limiting functional was derived in a companion paper, see [7], as a corollary of a general result proved there about the asymptotic behavior of a relatively simple model functional. In the case of Gross-Pitaevskii, the derivation of the limiting functional, using results of [7], is given in Sect. 4.1.

The functionals that we derive and study are in fact expressed as functions of the limiting (rescaled) current v rather than the limiting vorticity dv. In this we follow the approach developed by two of the authors in [23] in the setting of 2d superconductivity. Having found the energy-minimizing current v_0, one can then decompose it into components of different physical origin: the currents generated by the bulk vorticity, permanent currents (possible in a multiply-connected domain), and a potential flow (typically present for Bose-Einstein condensates and not for homogeneous superconductors). Mathematically, this corresponds to a Hodge decomposition of v_0 into harmonic, curl, and gradient parts, for example $v_0 = \gamma_0 + d^*\beta_0 + d\Phi_0$ for superconductivity, or an analogous weighted Hodge decomposition for superfluids.

We do not impose any topological restrictions on the domains that we study. Indeed, it is an advantage of the formulation in terms of currents that it allows for a simple and unified treatment of domains of varying topological type. By contrast, if the limiting free energy is formulated as a function of the vorticity dv, then (since for example any possible permanent currents, corresponding to the harmonic part of v, cannot be recovered from the bulk vorticity dv) multiply-connected domains become somewhat harder to analyze. This is also the case in 2 dimensions, where the first results to treat domains of general topological type were those of [23], using the current formulation adopted here. It was later shown in [4,5] that one can extend the vorticity formulation of [30,31] to the case of multiply-connected domains, but doing so requires additional detailed consideration of permanent currents as well as of bulk vorticity.

Although we mostly emphasize the analogy between the problems we study here and obstacle problems, there are also close connections between our vortex density models and total variation models in image processing as introduced by Rudin, Osher, and Fatemi [29]. (See [10] for a survey of related mathematical results.) In particular, the functional $\mathcal{G}$ derived in Proposition 5, see (1.28), can be viewed as a generalization of the Rudin-Osher-Fatemi model, and in situations with rotational symmetry, it reduces to exactly a (weighted) Rudin-Osher-Fatemi functional. The paper concludes in Sect. 5 with a discussion of this and some related issues.

1.1. The Ginzburg-Landau functional. Let Ω be a bounded open subset of $\mathbb{R}^3$. A superconducting sample occupying the region Ω may be described by a pair (u, A), where u is a complex-valued function on Ω and A is a 1-form on $\mathbb{R}^3$, that encodes various physical attributes of the superconductor. For example, $|u|^2$ corresponds to the density of Cooper pairs of superconducting electrons; dA can be identified with the magnetic field; and the superconducting current is given by

$$j_A u := \frac{i}{2}(u\overline{d_A u} - \bar{u} d_A u), \quad \text{where } d_A u := du - iAu. \tag{1.1}$$

As in the 2d case, the right notion of vorticity here is given by

$$\text{vorticity} = J_A u = \frac{1}{2} d(j_A u + A).$$

In 3d, this is a vector-valued quantity (which we choose to realize as a 2-form) that measures both the location and topological degree of vortex filaments arising in the mixed phase of type II superconductors.

Stable states of a superconductor in an external magnetic field $H_{\epsilon,ex} = dA_{\epsilon,ex}$, with $A_{\epsilon,ex} \in H^1_{loc}(\mathbb{R}^3; \Lambda^1\mathbb{R}^3)$, correspond to minimizers (or local minimizers) of the Ginzburg-Landau functional:

$$\mathcal{F}_\epsilon(u, A) = \int_\Omega \frac{|d_A u|^2}{2} + \frac{(|u|^2 - 1)^2}{4\epsilon^2}\, dx + \int_{\mathbb{R}^3} \frac{|dA - H_{\epsilon,ex}|^2}{2}\, dx. \tag{1.2}$$

Here the parameter ϵ is related to physical properties of the superconducting sample. We will study the limit $\epsilon \to 0$, with $A_{\epsilon.ex}$ scaling so that it will turn out to be comparable to the critical value mentioned above. For a discussion of the physical relevance of this scaling, see for example [32].

The model case is a constant external magnetic field, for which we may take $A_{\epsilon,ex} = \frac{1}{2} c_\epsilon (x_1 dx^2 - x_2 dx^1)$ for some real-valued scaling factor c_ϵ, corresponding to a spatially constant external field $H_{\epsilon,ex} = c_\epsilon dx^1 \wedge dx^2$, which in this example points in the e_3 direction.

The functional $\mathcal{F}_\epsilon$ makes sense for $u \in H^1(\Omega; \mathbb{C})$ and A such that $A - A_{\epsilon,ex} \in \dot{H}^1(\mathbb{R}^3; \Lambda^1\mathbb{R}^3)$. As is well known, the functional is gauge-invariant in the sense that for any such (u, A) and for any function ϕ such that $d\phi \in \dot{H}^1(\mathbb{R}^3)$, the identity $\mathcal{F}_\epsilon(u, A) = \mathcal{F}_\epsilon(e^{i\phi}u, A + d\phi)$ holds. Moreover, (u, A) and $(e^{i\phi}u, A + d\phi)$ correspond to *exactly* the same physical state, in the sense that all physically observable quantities are pointwise equal for the two pairs.

Our starting point is the following, which is an immediate consequence of [7], Thm. 4. We use the notation

$$\dot{H}^1_*(\Lambda^1\mathbb{R}^3) := \{\text{1-forms } A \text{ in } \dot{H}^1(\mathbb{R}^3) : d^*A = 0\}$$

which is a Hilbert space with the inner product $(A, B)_{\dot{H}^1_*} := \int_{\mathbb{R}^3} dA \cdot dB$. We will often write H^1_* for short. We also write $H^k(\Lambda^p U)$ to denote the space of p-forms on U with coefficients in the Sobolev space H^k.

Proposition 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with C^1 boundary. Assume that $A_{\epsilon,ex} \in H^1_{loc}(\Lambda^1 R^3)$ and that there exists $A_{ex} \in H^1_{loc}(\Lambda^1\mathbb{R}^3)$ such that*

$$\frac{A_{\epsilon,ex}}{|\log \epsilon|} - A_{ex} \to 0 \quad \text{in } \dot{H}^1_*(\Lambda^1\mathbb{R}^3) := \{A \in \dot{H}^1(\Lambda^1\mathbb{R}^3) : d^*A = 0\}. \tag{1.3}$$

Let (u_ϵ, A_ϵ) *minimize* $\mathcal{F}_\epsilon$ *in* $H^1(\Omega;\mathbb{C}) \times [A_{\epsilon,ex} + \dot{H}^1_*(\Lambda^1\mathbb{R}^3)]$. *Then there exists some* $A_0 \in [A_{ex} + \dot{H}^1_*]$ *and* $v_0 \in L^2(\Omega;\Lambda^1\mathbb{R}^3)$ *such that* dv_0 *is a measure, and such that*

$$\frac{A_\epsilon}{|\log\epsilon|} - A_0 \rightharpoonup 0 \quad \textit{weakly in } \dot{H}^1_*(\Lambda^1\mathbb{R}^3), \tag{1.4}$$

$$\frac{j_{A_\epsilon}u_\epsilon}{|\log\epsilon|} \rightharpoonup v_0 - A_0 \quad \textit{weakly in } L^2(\Lambda^1\Omega)\,, \tag{1.5}$$

and

$$\frac{1}{|\log\epsilon|}J_{A_\epsilon}u_\epsilon := \frac{1}{2|\log\epsilon|}d(j_{A_\epsilon}u + A_\epsilon) \to \frac{1}{2}dv_0 \qquad \textit{in } W^{-1,p}(\Lambda^2\Omega) \quad \forall\, p \le 3/2. \tag{1.6}$$

Moreover, (v_0, A_0) *minimizes the functional*

$$\mathcal{F}(v,A) = \frac{1}{2}|dv|(\Omega) + \frac{1}{2}||v-A||^2_{L^2(\Lambda^1\Omega)} + \frac{1}{2}||dA - H_{ex}||^2_{L^2(\Lambda^2\mathbb{R}^3)} \tag{1.7}$$

in $L^2(\Omega;\Lambda^1\mathbb{R}^3) \times [A_{ex} + \dot{H}^1_*]$, *where* $H_{ex} = dA_{ex}$. *Here* $|dv|$ *denotes the total variation measure associated with* dv. *(We understand* $\mathcal{F}(v,A)$ *to equal* $+\infty$ *if* dv *is not a measure.)*

Note that $\mathcal{F}$ is gauge-invariant in the sense that if $\gamma \in \dot{H}^2(\Lambda^1\mathbb{R}^3)$, then

$$\mathcal{F}(v - d\gamma|_\Omega, A + d\gamma) = \mathcal{F}(v,A). \tag{1.8}$$

Our new results about superconductivity in this paper are derived entirely by studying properties of the limiting functional $\mathcal{F}$; the connection to the more basic Ginzburg-Landau model is provided by the above Proposition 1.

Our first main result reformulates the problem of minimizing $\mathcal{F}$ through convex duality, the relevance of which in these settings was first pointed out in [9].

Theorem 2. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded open set with* C^1 *boundary, and assume that* $A_{ex} \in H^1_{loc}(\Lambda^1\mathbb{R}^3)$. *A pair* (v_0, A_0) *minimizes* $\mathcal{F}$ *in* $L^2(\Omega;\Lambda^1\mathbb{R}^3) \times [A_{ex} + \dot{H}^1(\Lambda^1\mathbb{R}^3)]$ *if and only if the following two conditions are satisfied:*

1. *The* 2*-form* $B_0 = d(A_0 - A_{ex})$ *belongs to the constraint set*

$$\mathcal{C} := \left\{B \in H^1(\Lambda^2\mathbb{R}^3) \cap d\dot{H}^1(\Lambda^1\mathbb{R}^3),\ \operatorname{supp}(d^*B) \subset \bar{\Omega},\ \ \|B\|_* \le \frac{1}{2}\right\}, \tag{1.9}$$

where

$$\|B\|_* := \sup\left\{\int_{\mathbb{R}^3} B \cdot d\alpha \ :\ \alpha \in H^1(\Lambda^1\mathbb{R}^3), \int_\Omega |d\alpha| \le 1\right\}. \tag{1.10}$$

In addition,

$$d^*B_0 + \mathbf{1}_\Omega(A_0 - v_0) = 0 \qquad \textit{in } \mathbb{R}^3, \tag{1.11}$$

and if $\bar{v}_0 \in L^2(\Lambda^1\mathbb{R}^3)$ *is any* 1*-form such that* $\bar{v}_0|_\Omega = v_0$, *then*

$$\int_{\mathbb{R}^3} B_0 \cdot d\bar{v}_0 = -\frac{1}{2}\int_\Omega |d\bar{v}_0|. \tag{1.12}$$

2. *B_0 is the unique minimizer in $\mathcal{C}$ of the functional*

$$B \mapsto \mathcal{E}_0(B; A_{ex}) := \frac{1}{2}\int_{\mathbb{R}^3} |B|^2 + \frac{1}{2}\int_{\Omega} |d^*B + A_{ex}|^2. \tag{1.13}$$

It is clear that if $\|B\|_* < \infty$ then

$$\int_{\mathbb{R}^3} B \cdot dv = 0 \qquad \text{for all } v \in H^1(\Lambda^1\mathbb{R}^3) \text{ such that } dv = 0 \text{ in } \Omega. \tag{1.14}$$

Indeed, if $v \in H^2(\Lambda^1\mathbb{R}^3)$ and $dv = 0$, then the definition of the norm $\|\cdot\|_*$ implies that $\lambda \int_{\mathbb{R}^3} B \cdot dv \le \|B\|_*$ for any $\lambda \in \mathbb{R}$, and this implies (1.14). For $v \in H^1(\Lambda^1\mathbb{R}^3)$, the same holds by an approximation argument.

Remark also that (1.12) implies that $\|B_0\|_* = \frac{1}{2}$ if the limiting (rescaled) vorticity $dv_0 \neq 0$. This is related to the following necessary and sufficient condition for the limiting vorticity to vanish.

Theorem 3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with C^1 boundary, and assume that $A_{ex} \in H^1_{loc}(\Lambda^1\mathbb{R}^3)$. Let (v_0, A_0) minimize $\mathcal{F}$ in $L^2(\Omega; \Lambda^1\mathbb{R}^3) \times [A_{ex} + \dot{H}^1(\Lambda^1\mathbb{R}^3)]$, and let B_* denote the unique minimizer of $\mathcal{E}_0(\,\cdot\,; A_{ex})$ in the set*

$$\mathcal{C}' := \Big\{B \in H^1(\Lambda^2\mathbb{R}^3) \cap d\dot{H}^1(\Lambda^1\mathbb{R}^3) : B \text{ satisfies } (1.14)\}, \tag{1.15}$$

where $\mathcal{E}_0$ is defined in (1.13).

Then $dv_0 = 0$ if and only if $\|B_*\|_* \le \frac{1}{2}$, i.e. if and only if $B_* = B_0$, where B_0 is the constrained minimizer in Theorem 2.

Remark 1. We give a different (but necessarily equivalent) characterization of when v_0 is vortex-free, and a different dual problem, see Theorem 11 and Lemma 12 in Sect. 3.3.

Remark 2. Condition (1.14) is easily seen to imply that $\mathrm{supp}(d^*B) \subset \bar{\Omega}$. The converse holds if and only if Ω is simply connected. We do not know whether the minimizer of $\mathcal{E}_0$ in the space $\mathcal{C}'' := \{B \in H^1(\Lambda^2\mathbb{R}^3) \cap d\dot{H}^1(\Lambda^1\mathbb{R}^3) : \mathrm{supp}(d^*B) \subset \bar{\Omega}\}$ coincides with B_* when Ω fails to be simply connected.

Remark 3. It may appear unsettling that the functional $\mathcal{E}_0$ contains the non-gauge-invariant quantity A_{ex}. This can be effectively eliminated, however, by decomposing $A_{ex}|_\Omega = A^1_{ex} + A^2_{ex}$, where $A^1_{ex} \in (\ker d)^\perp$ and $A^2_{ex} \in \ker d$ (see (2.5), (2.6)). Then A^1_{ex} is gauge-invariant, and $\int_\Omega |d^*B + A_{ex}|^2 = \int_\Omega |d^*B + A^1_{ex}|^2 + |A^2_{ex}|^2$.

Remark 4. As a consequence of Theorem 3, if (u_ϵ, A_ϵ) minimizes $\mathcal{F}_\epsilon$ in $H^1(\Omega; \mathbb{C}) \times [A_{\epsilon,ex} + \dot{H}^1_*(\Lambda^1\mathbb{R}^3)]$, then the vorticity vanishes to leading order, i.e.

$$|\log\epsilon|^{-1} J_{A_\epsilon} u_\epsilon \to 0 \quad \text{as } \epsilon \to 0 \quad \text{in } W^{-1,p}, \quad p < 3/2, \tag{1.16}$$

if and only if $\|B_*\|_* \le \frac{1}{2}$.

In particular, in the situation corresponding to a uniform magnetic field parallel to the vertical axis, we have $H_{\epsilon,ex} = \lambda|\log\epsilon| dx^1 \wedge dx^2 = dA_{ex}|\log\epsilon|$ with $A_{ex} = \frac{\lambda}{2}(x^1dx^2 - x^2dx^1)$. Let B^λ_* denote the minimizer of $\mathcal{E}_0(\,\cdot\,, A_{ex})$ in $\mathcal{C}'$. Then it is clear that $B^\lambda_* = \lambda B^1_*$, and it follows that the leading-order first critical field is given by

$$H_{\epsilon,c_1} = (\lambda_{c_1} + o(1))|\log\epsilon| dx^1 \wedge dx^2, \qquad \lambda_{c_1} = \frac{1}{2\|B^1_*\|_*} \tag{1.17}$$

in the sense that for $H_\epsilon = \lambda|\log\epsilon|$, the vorticity vanishes to leading order as $\epsilon \to 0$ if $\lambda < \lambda_{c_1}$ and does not do so for $\lambda > \lambda_{c_1}$.

Note that we do *not* prove that if $H_\epsilon = \lambda|\log\epsilon|$ for $\lambda < \lambda_{c_1}$, then the vorticity vanishes in the stronger sense that $J_{A_\epsilon} u_\epsilon \to 0$ in any norm, or in the still stronger sense that $|u_\epsilon| \geq \frac{1}{2}$ in Ω for all sufficiently small ϵ. This is why we only say that we have found the *leading-order* first critical field. We very much expect that these stronger conclusions in fact hold or equivalently, as one might say, that the leading-order first critical field agrees (to leading order) with the actual first critical field. This is shown to hold in the 2d case in [33].

The formula (1.17) has been previously identified by quite different arguments in [6], in the special case when Ω is a ball, as a candidate for the first critical field. The same paper also shows that if $H_\epsilon = \lambda|\log\epsilon|$ for $\lambda < \lambda_m^* = \frac{1}{2\|B_*^1\|_\infty} < \lambda_{c_1}$, then the vorticity vanishes as $\epsilon \to 0$ in the sense (stronger than (1.16)) that $J_{A_\epsilon} u_\epsilon \to 0$ in $W^{-1,p}$, $p < 3/2$.

Remark 5. Unlike the 2d case, to deduce information on the critical field in the 3d case one has to solve a variational problem involving the L^∞ norm: observe namely that for $B \in \mathcal{C}'$ we have, by virtue of Hahn-Banach theorem,

$$\|B\|_* = \inf\{\|\beta\|_{L^\infty(\Omega)}, \quad \beta \in H^1(\Lambda^2\mathbb{R}^3), \quad d^*\beta = d^*B\},$$

so that $\|B\|_*$ can be interpreted as a nonlocal L^∞ norm of $B \in \mathcal{C}'$ for $\Omega \subset \mathbb{R}^N$ with $N \geq 3$, while for $\Omega \subset \mathbb{R}^2$ we have $\|B\|_* = \|B\|_{L^\infty(\Omega)}$, since in that case in (1.10) one can test with forms $\alpha_j \in H^1(\Lambda^1\mathbb{R}^2)$ with $d\alpha_j = f_j(x)dx_1 \wedge dx_2$ such that the scalar functions $f_j(x)$ converge to a Dirac mass δ_{x_0} for arbitrarily fixed $x_0 \in \Omega$.

Hence in the 2-dimensional case $\Omega \subset \mathbb{R}^2$ the constrained variational problem (1.9), (1.13) of Theorem 2 corresponds to a classical obstacle problem (as is well-known), while in three (or higher) dimensions one may interpret it as a generalized, nonlocal, vectorial obstacle problem. Remark also that norms related to $\|\cdot\|_*$ have been studied in the context of critical Sobolev spaces (see [8,35]).

Remark 6. Note that the formula (1.17) for the leading-order critical field is far from explicit, in that to determine a numerical value for it, one would need first to find B_*^1, which depends on the domain Ω and then compute the norm $\|B_*^1\|_*$, which involves solving the variational problem discussed in Remark 5 above. The latter problem is not expected to be explicitly solvable for general domains Ω.

In the case of a superconducting ball, however, $\lambda_{c_1} = \frac{1}{2\|B_*^1\|_*}$ is found in Proposition 4.2 of [6], where it is noted that in this geometry, B_*^1 is known from classical work of London [26], and moreover that $\|B_1^*\|_*$ can be computed explicitly. Indeed, the problem of determining the norm $\|B_*^1\|_*$ can be rephrased as one of finding a curve γ through Ω that minimizes the $\frac{1}{|\gamma|}\int_\gamma B_*^1$, and in the case of a ball, it is shown in [6] that the optimal curve γ is precisely a diameter of the ball parallel to the applied magnetic field.

Based on experience with Bose-Einstein condensates (see Remark 11 below), we expect that even in the family of rotationally symmetric convex domains, there should be some for which no explicit formula for λ_{c_1} exists.

As mentioned before, a rather complete analysis of the asymptotic behavior of the Ginzburg-Landau functional for superconductivity in 2d and the corresponding critical fields can be found in [32]. In the 3d case, some results related to critical fields in agreement with Proposition 1 have been obtained by formal arguments, as in [11], and rigorously in the mentioned work [6] for the case of the ball, using some arguments of [22]. Critical fields on thin superconducting shells, among other results, have been derived in [12, 13] via a reduction to a limiting problem on a 2d manifold.

In the 2d asymptotic analysis of the Ginzburg-Landau functional $\mathcal{F}_\epsilon$ also higher applied fields $|H_{\epsilon,ex}| \gg |\log\epsilon|$ have been considered, corresponding to energy regimes $\mathcal{F}_\epsilon(u_\epsilon, A_\epsilon) = O(|H_{\epsilon,ex}|^2)$, see e.g. [32]. In the range $|\log\epsilon| \ll |H_{\epsilon,ex}| \ll \epsilon^{-1}$, adapting to this situation the proof of Proposition 1 given in [7], one immediately obtains the following

Proposition 4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with C^1 boundary. Assume $|\log\epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$, $A_{\epsilon,ex} \in H^1_{loc}(\Lambda^1 R^3)$ such that there exists $A_{ex} \in H^1_{loc}(\Lambda^1\mathbb{R}^3)$ such that*

$$\frac{A_{\epsilon,ex}}{\sqrt{g_\epsilon}} - A_{ex} \to 0 \quad in\ \dot{H}^1_*(\Lambda^1\mathbb{R}^3) := \{A \in \dot{H}^1(\Lambda^1\mathbb{R}^3) : d^*A = 0\}. \tag{1.18}$$

Let (u_ϵ, A_ϵ) minimize $\mathcal{F}_\epsilon$ in $H^1(\Omega;\mathbb{C}) \times [A_{\epsilon,ex} + \dot{H}^1_*(\Lambda^1\mathbb{R}^3)]$. Then there exists some $A_0 \in [A_{ex} + \dot{H}^1_*]$ and $v_0 \in L^2(\Omega;\Lambda^1\mathbb{R}^3)$ such that

$$\frac{A_\epsilon}{\sqrt{g_\epsilon}} - A_0 \rightharpoonup 0 \quad weakly\ in\ \dot{H}^1_*(\Lambda^1\mathbb{R}^3), \tag{1.19}$$

$$\frac{j_{A_\epsilon}u_\epsilon}{\sqrt{g_\epsilon}} \rightharpoonup v_0 - A_0 \quad weakly\ in\ L^2(\Lambda^1\Omega). \tag{1.20}$$

Moreover, (v_0, A_0) minimizes the functional

$$\tilde{\mathcal{F}}(v, A) = \frac{1}{2}||v - A||^2_{L^2(\Lambda^1\Omega)} + \frac{1}{2}||dA - H_{ex}||^2_{L^2(\Lambda^2\mathbb{R}^3)} \tag{1.21}$$

in $L^2(\Omega;\Lambda^1\mathbb{R}^3) \times [A_{ex} + \dot{H}^1_*]$.

It is clear that in this case $(v_0, A_0) = (A_{ex}|_\Omega, A_{ex})$ is the unique minimizer of $\tilde{\mathcal{F}}$. In particular, this implies that for a uniform applied field $H_{\epsilon,ex} = c_0\sqrt{g_\epsilon}dx^1 \wedge dx^2$ corresponding to $A_{\epsilon,ex} = \sqrt{g_\epsilon}\frac{c_0}{2}(x_1dx_2 - x_2dx_1)$ with g_ϵ as above, the limiting vorticity is given by $dv_0 = c_0dx_1 \wedge dx_2$, corresponding to an asymptotically uniform distribution of vortex lines throughout the sample, regardless of its geometry or topology. Analogous results have been obtained in [24] by different methods.

Remark 7. In higher energy regimes $g_\epsilon \gtrsim \epsilon^{-2}$ corresponding to further critical fields, as established in the 2d case (see [17]), different phenomena are expected to take place (for example surface superconductivity), so that the reduced functional $\tilde{\mathcal{F}}$ is no longer expected to give information in this case.

1.2. The Gross-Pitaevskii functional. The second main object of study in this paper is a variational problem that describes a Bose-Einstein condensate with mass m, confined by a smooth potential $a : \mathbb{R}^3 \to [0, \infty)$ such that

$$a \in C^\infty(\mathbb{R}^3), \quad a(x) \to +\infty \quad \text{as } |x| \to +\infty, \tag{1.22}$$

and subjected to forcing Φ_ϵ that in general depends on a scaling parameter ϵ. In the model case corresponding to rotation about the z-axis, $\Phi_\epsilon := \frac{1}{2}c_\epsilon(x_1dx^2 - x_2dx^1)$, and $a(x)$ grows quadratically or faster.

We will study the functional in the scaling regime

$$G_\epsilon(u) := \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 - \Phi_\epsilon \cdot ju + \frac{1}{\epsilon^2}(\frac{|u|^4}{4} + a(x)\frac{|u|^2}{2}),$$

where $\Phi_\epsilon = |\log\epsilon|\Phi$ for some fixed Φ and

$$j(u) = \frac{i}{2}(ud\bar{u} - \bar{u}du).$$

We introduce the function space

$$H^1_a(\mathbb{R}^3;\mathbb{C}) := H^1_a := \text{ completion of } C^\infty_c(\mathbb{R}^3;\mathbb{C}) \text{ with respect to } \|\cdot\|_a, \quad (1.23)$$

where the norm $\|\cdot\|_a$ is defined by $\|u\|_a^2 := \int_{\mathbb{R}^3}|du|^2 + (1+a)|u|^2$. We also define

$$H^1_{a,m}(\mathbb{R}^3;\mathbb{C}) := H^1_{a,m} := \{u \in H^1_a \ : \ \int |u|^2 = m\}.$$

We will study the behavior of minimizers of G_ϵ in $H^1_{a,m}$, which describe condensates in the ground state. We start by rewriting the functional. Define

$$\rho(x) := (\lambda - a(x))^+, \quad w(x) := (\lambda - a(x))^-, \quad \text{for } \lambda \text{ such that } \int_{\mathbb{R}^3} \rho\, dx = m. \quad (1.24)$$

The last condition clearly determines λ uniquely. The function ρ is called the *Thomas-Fermi density* in the physics literature, and gives to the leading-order condensate density in the limit $\epsilon \to 0$.

By completing the square we find that

$$\frac{|u|^4}{4} + a\frac{|u|^2}{2} = \frac{1}{4}(\rho - |u|^2)^2 \ - \ \frac{1}{4}\rho^2 + \frac{1}{2}w|u|^2 + \frac{\lambda}{2}|u|^2.$$

Since $\int \lambda|u|^2 = \lambda m$ for all $u \in H^1_{a,m}$, it follows that u minimizes G_ϵ in $H^1_{a,m}$ if and only if u minimizes

$$\mathcal{G}_\epsilon(u) \ := \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 - \Phi_\epsilon \cdot ju + \frac{1}{4\epsilon^2}(\rho - |u|^2)^2 + \frac{w}{2\epsilon^2}|u|^2 \quad (1.25)$$

in $H^1_{a,m}$. We will henceforth write the Gross-Pitaevskii functional in the form (1.25), which is more convenient for our analysis.

Throughout our discussion of the Gross-Pitaevskii functional we will use the notation

$$\Omega = \{x \in \mathbb{R}^3 : \rho(x) > 0\}. \quad (1.26)$$

We will always assume that λ is a regular value of a, so that $|Da| \geq c > 0$ on $\partial\Omega$, and hence $w > 0$ in $\mathbb{R}^3\backslash\bar{\Omega}$, and

$$|\nabla\rho(x)|^2 + \rho(x) \geq c > 0, \qquad \rho(x) \geq c\,\mathrm{dist}(x,\partial\Omega), \qquad \text{for all } x \in \Omega. \quad (1.27)$$

1.2.1. Results: Bose-Einstein condensates Our results for the Gross-Pitaevskii functional parallel those we obtain for the Ginzburg-Landau functional: we identify a limiting variational problem, see (1.28), (1.29) below, characterize when minimizers of the limiting problem are vortex-free to leading order, and obtain a description of minimizers of the limiting problem as solutions of a sort of nonlocal vector-valued obstacle problem.

We start by proving a theorem that characterizes Γ-limits of the Gross-Pitaevskii functional, see Theorem 14 in Sect. 4. This is parallel to Theorem 4 from [7] for the Ginzburg-Landau functional, and the proof relies on results from [7] on the reduced GL functional (i.e. without magnetic field). An immediate consequence of Theorem 14 is the following.

Proposition 5. *Assume that* $\Phi_\epsilon = |\log\epsilon|\Phi$, *with* $\Phi \in L^4_{loc}(\Lambda^1\mathbb{R}^3)$ *and that* $|\Phi(x)|^2 \le Ca(x)$ *outside some compact set* K.

Assume that u_ϵ *minimizes* $\mathcal{G}_\epsilon$ *in* $H^1_{a,m}$. *Then*

$$|u_\epsilon| \to \rho \quad \text{in } L^4(\mathbb{R}^3)$$

for ρ *defined in* (1.24), *and there exists* $j_0 \in L^{4/3}(\Lambda^1\Omega)$ *such that*

$$|\log\epsilon|^{-1} ju_\epsilon \rightharpoonup j_0 \text{ weakly in } L^{4/3}(\mathbb{R}^3).$$

Moreover, $j_0 = \rho v_0$, *where* v_0 *is the unique minimizer of*

$$\mathcal{G}(v) := \int_\Omega \rho\left(\frac{|v|^2}{2} - v\cdot\Phi + \frac{1}{2}|dv|\right) \tag{1.28}$$

in the space

$$L^2_\rho(\Lambda^1\Omega) := \left\{ v \in L^1_{loc}(\Lambda^1\Omega) \ : \ \int_\Omega \rho|v|^2\,dx < \infty \right\}. \tag{1.29}$$

(We set $\mathcal{G}(v) = +\infty$ *if* dv *is not a Radon measure or if* ρ *is not* $|dv|$*-integrable.)*

For a range of higher rotations, we obtain a limiting functional $\tilde{\mathcal{G}}$, in a sense similar to Proposition 5, where $\tilde{\mathcal{G}}(v) = \frac{1}{2}\int_\Omega \rho|v-\Phi|^2$. Thus it is immediate that the unique minimizer of $\tilde{\mathcal{G}}$ is Φ. Arguing along these lines, we will prove

Proposition 6. *Assume that* $\Phi_\epsilon = \sqrt{g_\epsilon}\Phi$ *for* $|\log\epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$, *and assume that* Φ, a *satisfy the assumptions of Proposition 5. Assume also that* u_ϵ *minimizes* $\mathcal{G}_\epsilon$ *in* $H^1_{a,m}$.

Then $|u_\epsilon| \to \rho$ *in* $L^4(\mathbb{R}^3)$ *for* ρ *defined in* (1.24), *and* $g_\epsilon^{-1} ju_\epsilon \rightharpoonup \rho\Phi$ *weakly in* $L^{4/3}(\mathbb{R}^3)$.

Remark 8. In particular, Proposition 6 implies that for a supercritical rotation around the (vertical) x_3 axis, corresponding to $\Phi_\epsilon = \sqrt{g_\epsilon}\frac{c_0}{2}(x_1dx_2 - x_2dx_1)$ with g_ϵ as above, the limiting rescaled ground-state vorticity is given by $dv_0 = c_0 dx_1 \wedge dx_2$, corresponding to an asymptotically uniform distribution of vortex lines throughout the condensate, regardless of its geometry or topology. This generalizes to 3 dimensions results obtained in [15] in the 2d case.

For even higher forcing regimes $g_\epsilon \ge \epsilon^{-2}$, as in the case of superconductivity, this picture is expected to be no longer true, as suggested by the 2d phenomenology (see Sect. 1.2.2).

We next identify a necessary and sufficient condition on Φ and ρ for minimizers of the limiting functional $\mathcal{G}$ to be vortex-free, by which we mean that $dv_0 = 0$ in Ω. For this result, it is useful to note that the space $L^2_\rho(\Lambda^1\Omega)$ defined above is a Hilbert space with an inner product and norm that we will write as

$$(v, w)_\rho := \int_\Omega \rho\, v \cdot w\, dx, \qquad \|v\|_\rho := (v, v)_\rho^{1/2}.$$

We will sometimes use the same notation to denote the ρ-weighted L^2 inner product or norm for k-forms with values of k other than 1; the meaning should always be clear from the context. We let P_ρ denote the orthogonal projection with respect to the L^2_ρ inner product, onto $(\ker d)_\rho$, where

$$(\ker d)_\rho := L^2_\rho\text{-closure of } \{\phi \in C^\infty(\Lambda^1\Omega) : d\phi = 0,\ \|\phi\|_\rho < \infty\}. \tag{1.30}$$

We will also write $P_\rho^\perp$ for the complementary orthogonal projection. Note that if $w \in \text{Image}(P_\rho^\perp) = (\ker d)_\rho^\perp$, then $\int(\rho w)\cdot\phi = 0$ for all $\phi \in (\ker d)_\rho \supset \ker d$. Thus $\rho w \in (\ker d)^\perp$, and so it follows from the standard unweighted Hodge decomposition (see Sect. 2.2, and in particular (2.7)) that

$$\forall w \in (\ker d)_\rho^\perp,\ \exists \beta \in H^1_N(\Lambda^2\Omega) \text{ such that } w = \frac{d^*\beta}{\rho} \text{ and } \int_\Omega \frac{|d^*\beta|^2}{\rho} = \|w\|^2_\rho. \tag{1.31}$$

Thus if $\Phi \in L^2_\rho$, there exists $\beta_\Phi \in H^1_N$ such that $d^*\beta_\Phi \in L^2_\rho$ and

$$\Phi = P_\rho\Phi + \frac{d^*\beta_\Phi}{\rho}. \tag{1.32}$$

We now state

Theorem 7. *Suppose that Ω is a bounded, open subset of $\mathbb{R}^3$ and that $\rho \in C^1(\Omega)$ and $\Phi \in L^4_{loc}(\Lambda^1\mathbb{R}^3) \cap L^2_\rho(\Lambda^1\Omega)$ are given, with ρ satisfying (1.27).*

Let $\beta_\Phi \in H^1_N(\Lambda^2\Omega)$ be such that $P^\perp_\rho\Phi = \frac{d^*\beta_\Phi}{\rho}$, and let β_0 minimize the functional

$$\beta \mapsto \frac{1}{2}\int_\Omega \frac{|d^*\beta|^2}{\rho} \tag{1.33}$$

in the space

$$\left\{\beta \in H^1_N(\Lambda^2\Omega)\ :\ \frac{d^*\beta}{\rho} \in L^2_\rho(\Lambda^1\Omega),\ \ \|\beta - \beta_\Phi\|_{\rho*} \le \frac{1}{2}\right\}, \tag{1.34}$$

where

$$\|\beta\|_{\rho*} := \sup\left\{\int_\Omega \beta\cdot dw :\ w \in C^\infty(\Lambda^1\bar{\Omega}),\ \int_\Omega \rho|dw| \le 1\right\}. \tag{1.35}$$

Then $v_0 = P_\rho\Phi + \frac{d^*\beta_0}{\rho}$ is the unique minimizer of $\mathcal{G}(\cdot)$ in $L^2_\rho(\Lambda^1\Omega)$.
Moreover,

$$\int_\Omega (\beta_\Phi - \beta_0)\cdot dv_0 = \frac{1}{2}\int_\Omega \rho|dv_0|. \tag{1.36}$$

Finally, $dv_0 = 0$ if and only if $\|\beta_\Phi\|_{\rho*} \le \frac{1}{2}$.

Note that (1.36) states that the action of the vorticity distribution dv_0 on the potential $\beta_0 - \beta_\Phi$ is the largest possible given the constraint (1.34). Similar considerations apply to (1.12) and (1.9) in the case of superconductivity.

Remark 9. Since $P_\rho \Phi \in (\ker d)_\rho$, it can be further decomposed into a harmonic part, corresponding to permanent currents possibly present for multiply-connected domains, and a global phase, or equivalently a gradient part. In the 2d case the relevance of those different contributions has been pointed out in [1,18,19]. Our formulation has the virtue of automatically incorporating permanent currents and a global phase, without the need for any special consideration.

Remark 10. It is known from earlier work on the 3d Gross-Pitaevskii functional, see [3,21], that the bulk vorticity associated to a wave function u_ϵ is naturally identified with dv_ϵ, where $v_\epsilon = u_\epsilon / f_\epsilon$ and f_ϵ is a vortex-free minimizer of $\mathcal{G}_\epsilon$, see Step 3 of the proof of Theorem 14 for the definition and a discussion. Since $f_\epsilon \to \sqrt{\rho}$ uniformly on $\mathbb{R}^3$ ([27] Lem. B.1), it follows from Proposition 4.2 and Theorem 7 that the leading-order vorticity vanishes, that is,

$$|\log \epsilon|^{-1} dv_\epsilon \to 0 \quad \text{in } W^{-1,p}(K) \text{ for all } p < 4/3,\ K \subset\subset \Omega \tag{1.37}$$

if and only if $\|\beta_\Phi\|_{\rho *} \le \frac{1}{2}$. (One could also formulate inhomogeneous $W^{-1,p}$ norms, incorporating f_ϵ as a weight, such that the above convergence holds on all of $\mathbb{R}^3$.)

In particular, parallel to Remark 4, for a uniform rotation about the x_3 axis, one easily checks that the *leading-order* critical rotation is given by

$$\Phi_{\epsilon,c_1} = (\lambda_{c_1} + o(1))|\log \epsilon|(x_1 dx_2 - x_2 dx_1), \qquad \lambda_{c_1} := \frac{1}{2\|\beta_{\Phi^1}\|_{\rho *}}. \tag{1.38}$$

As in Remark 4, it is expected that the leading-order critical rotation agrees (to leading order) with the actual critical rotation. For a special class of potentials, a form of this assertion follows by combining our results with those of [3,21]. Indeed, these authors show (formally in [3], rigorously in [21]) that for

$$a(x) = \sum_{i=1}^{3} \omega_i x_i^2 \quad \text{with } \omega_i > 0, \qquad \Phi_\epsilon = \lambda |\log \epsilon|(x_1 dx_2 - x_2 dx_1), \tag{1.39}$$

the vorticity vanishes when $\lambda < \lambda_{c_1}$ in the sense that $dv_\epsilon \to 0$ in various weak norms[1] as $\epsilon \to 0$. The techniques used in [21] give essentially no information about minimizers for $\lambda > \lambda_{c_1}$. A paper of Montero [27] extends some results of [21] to the larger class of trapping potentials a and forcing terms Φ_ϵ considered here. Although it was not done in [27], these results could in principle be used to prove that $dv_\epsilon \to 0$ for subcritical rotations in this generality, strengthening (1.37).

Remark 11. Parallel to Remark 6, note that determining a numerical value for λ_{c_1} as defined in (1.38) requires solving the variational problem implicit in the definition of the $\|\cdot\|_{\rho *}$ norm, and this problem does not in general admit an explicit solution. The variational problem in question can be rephrased as one of finding a relatively closed curve γ_0 in Ω that minimizes $\gamma \mapsto (\int_\gamma \rho\, ds)^{-1} \int_\gamma \star\beta$. For the model case (1.39), in which Ω is an ellipse, this equivalent problem is studied in [2]. In particular, it is shown that

[1] A still more satisfactory statement that "the vorticity vanishes when $\lambda < \lambda_{c_1}$" would show that $|v_\epsilon|$ is bounded away from 0 for this range of rotations, at least in the interior of Ω. This would yield a stronger characterization of λ_{c_1} as the first critical rotation.

- If $\omega_3 \geq \max(\omega_1, \omega_2)$, so that Ω is a "pancake-shaped" ellipsoid, then the optimal curve γ_0 is a vertical diameter of the Ω. Then $\|\beta_\Phi\|_{\rho*}$, and hence λ_{c_1} can be computed explicitly.
- If $\omega_3 < \sqrt{2/13}\max(\omega_1, \omega_2)$ (which occurs for example if Ω is sufficiently "cigar-shaped"), then the the vertical diameter is not an optimal curve, and there is presumably no explicit formula for λ_{c_1}.

Remark 12. There can exist $\beta \in H^1_N(\Lambda^2\Omega)$ satisfying $\frac{d^*\beta}{\rho} \in L^2_\rho(\Lambda^1\Omega)$ and $\|\beta\|_{\rho*} < \infty$, but such that β is not $|dw|$-integrable for some $w \in L^2_\rho(\Lambda^1\Omega)$ such that $\int \rho|dw| < 1$. Hence the restriction to smooth 1-forms w in the supremum that appears in the definition of the $\|\cdot\|_{\rho*}$ norm.

1.2.2. Related results about 2d BEC. We do not know of any source in the literature that establishes 2d results analogous to Proposition 5 and Theorem 7. Such results however can be established by arguing *exactly* as in the proofs we supply here in the 3d case, but taking as a starting-point results from [23] about Γ-limits of the reduced Ginzburg-Landau functional in 2d, rather than the analogous results about the same problem in 3d from [7], which (together with very general convex duality arguments) are the chief input in the relevant proofs.

In particular, limits of sequences of minimizers in 2d are described, in the same sense as in Proposition 5, by a functional $\mathcal{G}$ on $L^2_\rho(\Omega')$ of *exactly* the same form as in (1.28), for a suitable $\Omega' \subset \mathbb{R}^2$. More generally, this functional can be obtained as a Γ-limit of the scaled 2d Gross-Pitaevskii energy, completely parallel to Theorem 14. Moreover, this limiting functional admits a dual formulation as functional with constraints, parallel to that in Theorem 7, and from this one can easily determine a necessary and sufficient condition for the limiting vorticity to vanish.

On the other hand, for more extreme rotation regimes in anharmonic trapping potentials in 2d, a quite detailed analysis has been carried out recently in [14,15,28].

In a different direction, the critical rotation has been derived in certain highly symmetric domains in for example [1,18,19]. These references also examine the behavior of minimizers for *slightly* supercritical rotations.

The main difference between 2 and 3 dimensions is the form of the constraint in the limiting variational problem. In particular, in 2d, as in 3d, it is the case that if v_0 minimizes $\mathcal{G}$, then $dv_0 = d(\frac{d^*\beta_0}{\rho})$, where the potential β_0 minimizes the functional (1.33), subject to the constraint (1.34), where the norm in the constraint is defined as in (1.35). The difference is that in 2d, the potentials β are 2-forms on $\mathbb{R}^2$, and so can be identified with functions. And since it is not hard to check that $\{d\omega : \int_\Omega \rho|d\omega| \leq 1\}$ is weakly dense in the set of signed measures μ such that $\int_\Omega \rho d|\mu| \leq 1\}$, the 2d constrained problem reduces to minimizing (1.33) in the set

$$\left\{\beta \in H^1(\Lambda^2\Omega) : \|\frac{1}{\rho}(\beta - \beta_\Phi)\|_{L^\infty} \leq \frac{1}{2}\right\}. \tag{1.40}$$

This is a classical (weighted) 2-sided obstacle problem; for many Φ, using the maximum principle it in fact reduces to a one-sided obstacle problem.

Thus we view the problem in Theorem 7 as a nonlocal, vector-valued analog of the classical obstacle problem.

2. Background and Notation

2.1. Differential forms. If U is an open subset of $\mathbb{R}^n$, we will use the notation $W^{1,p}(\Lambda^k U)$ to denote the space of maps $U \to \Lambda^k\mathbb{R}^n$ (that is, k-forms on U) that belong to the Sobolev space $W^{1,p}$. A generic element $\omega \in W^{1,p}(\Lambda^k U)$ thus has the form

$$\sum_{\{\alpha:1\leq\alpha_1<\cdots<\alpha_k\leq n\}} \omega_\alpha dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k} \tag{2.1}$$

with $\omega_\alpha \in W^{1,p}(\Omega;\mathbb{R})$ for every multiindex α. We use the notation $L^p(\Lambda^k U)$, $C^\infty(\Lambda^k U)$, and so on in a parallel way.

For an open set Ω with nonempty boundary and $\omega \in C^0(\Lambda^k\bar{\Omega})$, we define $\omega_\top$ and ω_N in $C^0(\Lambda^k\partial\Omega)$ by

$$\omega_\top = i^*\omega, \text{ where } i : \partial\Omega \to \bar{\Omega} \text{ is the inclusion;} \qquad \omega_N = \omega|_{\partial\Omega} - \omega_\top.$$

One refers to $\omega_\top$ and ω_N as the tangential and normal parts of ω on $\partial\Omega$. We will use the same notation $\omega_\top, \omega_N$ to refer to the tangential and normal parts of (the trace of) a form $\omega \in W^{1,p}(\Lambda^k\Omega)$, which one can define by noting that for example the map $\omega \mapsto \omega_\top$, well-defined on a dense subset of $W^{1,p}(\Lambda^k\Omega)$, extends to a bounded linear map $W^{1,p}(\Lambda^k\Omega) \to L^p(\Lambda^k\partial\Omega)$, or equivalently by applying the pointwise definition of $\omega_\top$, say, to the trace of ω at a.e. point of $\partial\Omega$.

If ω, ϕ are elements of $L^2(\Omega; \Lambda^k\mathbb{R}^n)$, written as in (2.1), we will write $\omega\cdot\phi$ to denote the integrable function defined by

$$\omega \cdot \phi = \sum_{\{\alpha:1\leq\alpha_1<\cdots<\alpha_k\leq n\}} \omega_\alpha\phi_\alpha.$$

This allows us to define an L^2 inner product on spaces of differential forms in the obvious way. We write d^* to denote the formal adjoint of d, so that $\int d\omega\cdot\phi = \int \omega\cdot d^*\phi$ when ω is a smooth $k-1$-form and ϕ a smooth k-form for some k, and at least one of them has compact support. Then

$$d^*\phi = (-1)^k \star d \star \phi \quad \text{if } \phi \text{ is a } k\text{-form},$$

where in $\mathbb{R}^3$, the $\star$ operator, mapping k-forms to $(3-k)$-forms, is characterized by

$$\omega \wedge \star\phi = \star\omega \wedge \phi = \omega\cdot\phi\, dx^1 \wedge dx^2 \wedge dx^3.$$

(In even dimensions one must be more careful about signs.)

We will use the notation

$$W^{1,p}_\top(\Lambda^k\Omega) := \{\omega \in W^{1,p}(\Lambda^k\Omega) : \omega_\top = 0\},$$
$$W^{1,p}_N(\Lambda^k\Omega) := \{\omega \in W^{1,p}(\Lambda^k\Omega) : \omega_N = 0\},$$

and

$$\mathcal{H}_\top(\Lambda^k\Omega) := \{\omega \in W^{1,p}(\Lambda^k\Omega) : \omega_\top = 0, d\omega = d^*\omega = 0\},$$
$$\mathcal{H}_N(\Lambda^k\Omega) := \{\omega \in W^{1,p}(\Lambda^k\Omega) : \omega_N = 0, d\omega = d^*\omega = 0\}.$$

In fact forms in $\mathcal{H}^k_\top$ and $\mathcal{H}^k_N$ are known to be smooth.

Gauge-invariance implies that the set of minimizers of $\mathcal{F}_\epsilon$ is noncompact in $H^1(\Omega;\mathbb{C}) \times [A_{ex} + H^1(\Lambda^1\mathbb{R}^3)]$. In order to remedy this, we will often restrict $\mathcal{F}_\epsilon$ to a smaller space. Thus we introduce

$$\dot{H}^1_*(\mathbb{R}^3;\Lambda^1\mathbb{R}^3) = \dot{H}^1_* := \{A \in \dot{H}^1(\mathbb{R}^3;\Lambda^1\mathbb{R}^3) : d^*A = 0\} \tag{2.2}$$

with the inner product $(A,B)_{\dot{H}^1_*(\Lambda^1\mathbb{R}^3)} = (A,B)_* := (dA,dB)_{L^2(\Lambda^2\mathbb{R}^3)}$. This makes $\dot{H}^1_*(\Lambda^1\mathbb{R}^3)$ into a Hilbert space, satisfying in addition the Sobolev inequality

$$\|A\|_{L^6(\Lambda^1\mathbb{R}^3)} \le C\|A\|_{\dot{H}^1_*(\Lambda^1\mathbb{R}^3)}.$$

In view of standard results about the Hodge decomposition, given any 1-form $\tilde{A}$ such that $\tilde{A} \in A_{\epsilon,ex} + \dot{H}^1(\mathbb{R}^3;\Lambda^1\mathbb{R}^3)$, we can write $\tilde{B} := \tilde{A} - A_{\epsilon,ex} \in \dot{H}^1$ in the form

$$\tilde{B} = B + d\phi, \qquad \text{where } B \in \dot{H}^1_* \text{ and } d\phi \in \dot{H}^1(\mathbb{R}^3;\Lambda^1\mathbb{R}^3).$$

Thus given any pair $(\tilde{u},\tilde{A}) \in H^1(\Omega;\mathbb{C}) \times [A_{ex} + \dot{H}^1(\Lambda^1\mathbb{R}^3)]$, there exists an equivalent pair $(u,A) = (\tilde{u}e^{-i\phi}, \tilde{A} - d\phi)$ in $H^1(\Omega;\mathbb{C}) \times [A_{\epsilon,ex} + \dot{H}^1_*]$, so that in restricting $\mathcal{F}_\epsilon$ to $H^1(\Omega;\mathbb{C}) \times [A_{\epsilon,ex} + \dot{H}^1_*]$, we do not sacrifice any generality.

2.2. Hodge decompositions. We will need several Hodge decompositions. First, on a bounded open domain Ω with C^1 boundary, we have, for every integer $k \in \{0,\dots,n\}$ the decompositions

$$L^2(\Lambda^k\Omega) = dH^1(\Lambda^{k-1}\Omega) \oplus d^*H^1_N(\Lambda^{k+1}\Omega) \oplus \mathcal{H}_N(\Lambda^k\Omega) \tag{2.3}$$

and

$$L^2(\Lambda^k\Omega) = dH^1_\top(\Lambda^{k-1}\Omega) \oplus d^*H^1(\Lambda^{k+1}\Omega) \oplus \mathcal{H}_\top(\Lambda^k\Omega). \tag{2.4}$$

These are known from work of Morrey (see also [20], Thm. 5.7). The first of these, for example, means that every $\omega \in L^2(\Lambda^k\Omega)$ can be written in the form $\omega = d\alpha + d^*\beta + \gamma$, where $\alpha \in H^1(\Lambda^{k-1}\Omega)$, $\beta \in d^*H^1_N(\Lambda^{k+1}\Omega)$, and $\gamma \in \mathcal{H}^k_N$, and moreover $d\alpha$, $d^*\beta$, and γ are mutually orthogonal in L^2.

We will sometimes use the notation

$$\ker d = H^1(\Omega) \oplus \mathcal{H}_N(\Lambda^1\Omega), \qquad (\ker d)^\perp = d^*H^1_N(\Lambda^2\Omega). \tag{2.5}$$

This is justified by the following considerations. First, we claim that for $v \in L^2(\Lambda^1\Omega)$,

$$dv = 0 \text{ as a distribution on } \Omega \iff v \in dH^1(\Omega) \oplus \mathcal{H}_N(\Lambda^1\Omega). \tag{2.6}$$

Indeed, to prove that $v \in dH^1(\Omega) \oplus \mathcal{H}_N(\Lambda^1\Omega)$, it suffices by (2.3) to verify that $v \perp d^*H^1_N(\Lambda^2\Omega)$. Fix any $\beta \in H^1_N(\Lambda^2\Omega)$, and let $\chi_\epsilon \in C^\infty_c(\Omega)$ be a sequence of functions such that $\chi_\epsilon = 1$ in $\{x \in \Omega : \text{dist}(x,\partial\Omega) > \epsilon\}$, $\|\nabla\chi_\epsilon\|_\infty \le C\epsilon$. Then the assumption that $dv = 0$ in Ω implies that

$$0 = \int_\Omega v \cdot d^*(\chi_\epsilon\beta) = \int_\Omega \chi_\epsilon v \cdot d^*\beta + \int_\Omega v \cdot \star \cdot (d\chi_\epsilon \wedge \star\beta)$$

for every ϵ. Thus

$$\int_\Omega v \cdot d^*\beta = \lim_{\epsilon\to 0}\int_\Omega \chi_\epsilon v \cdot d^*\beta = -\lim_{\epsilon\to 0}\int_\Omega v \cdot \star(d\chi_\epsilon \wedge \star\beta) = 0,$$

where the last equality follows from the fact that $\beta_N = 0$. This proves one implication in (2.6), and the other is obvious.

Similarly, the Hodge decomposition implies that if $\omega \in L^2(\Lambda^k\Omega)$, then

$$\omega \in d^* H^1_N(\Lambda^{k+1}\Omega) \quad \Longleftrightarrow \quad \int_\Omega \omega\cdot\phi = 0 \quad \forall\, \phi \in H^1(\Lambda^k\Omega) \text{ with } d\phi = 0. \tag{2.7}$$

We also define

$$P := L^2(\Lambda^1\Omega) \text{ orthogonal projection onto } d^* W^{1,2}_N(\Lambda^2\Omega) = (\ker d)^\perp. \tag{2.8}$$

Given $A \in \dot H^1(\Lambda^1\mathbb{R}^3)$ for example, we will generally abuse notation and write PA_1 instead of $P(A_1|_\Omega)$. We remark that

$$\|PB\|^2_{L^2(\Omega)} = \inf\{\|B+\gamma\|^2_{L^2(\Omega)} \ : \gamma \in H^1(\Omega), d\gamma = 0\}. \tag{2.9}$$

For applications to Bose-Einstein condensates we will need a Hodge decomposition in the weighted Hilbert space $L^2_\rho(\Lambda^k\Omega)$. In particular, in the notation from the introduction (compare (1.29), (1.31)), we may decompose[2] $\phi \in L^2_\rho(\Lambda^1\Omega)$ as

$$\phi = \omega + \frac{d^*\beta}{\rho}, \qquad \omega = P_\rho\phi \in (\ker d)_\rho, \qquad \frac{d^*\beta}{\rho} \in L^2_\rho(\Lambda^1\Omega).$$

For refined results assuming ρ and ϕ sufficiently smooth, see [27].

2.3. Duality. We will frequently use the following basic result, whose proof can be found for example in [16].

Lemma 8. *Assume that H is a Hilbert space, and that $I : H \to (-\infty,\infty]$ is a convex function and that $I(x) < \infty$ for some $x \in H$.*

Let $G(x) := I(x) + \frac{1}{2}\|x\|^2_H$.

Let I^* denote the Legendre-Fenchel transform of I, so that

$$I^*(\xi) := \sup_{x\in H}\left((\xi, x)_H - I(x)\right).$$

Then if we define $G^\dagger(x) := I^*(-x) + \frac{1}{2}\|x\|^2_H$, the following hold:

(1) *There exists a unique $x_0 \in H$ such that $G(x_0) = \min_H G(\cdot)$.*
(2) *The same $x_0 \in H$ is the unique minimizer of $G^\dagger$ in H.*
(3) *$G(x) + G^\dagger(y) \geq 0$, and $G(x) + G(y) = 0$ if and only if $(x, y) = (x_0, x_0)$.*

[2] Notice that our notation is inconsistent, with P_ρ = projection onto $(\ker d)_\rho$ for Bose-Einstein, and $P :=$ projection onto $(\ker d)^\perp$ for superconductivity. These conventions are convenient however, and we do not think they can lead to any confusion.

3. Vortex Density in 3d Superconductors

3.1. A dual variational problem. We start with the proof of Theorem 2, in which we identify a variational problem dual to that of miminizing $\mathcal{F}$, which describes the limiting density of vortex lines in a superconducting material subjected to an applied magnetic field. We then use this dual problem to prove Theorem 3, giving a necessary and sufficient condition for the limiting vorticity to vanish.

In the next section we present several different and, actually, simpler derivations of (an equivalent but different-looking characterization for) the critical field. The approach presented here, although a little more complicated, has the advantage of yielding the dual problem of the statement of Theorem 2, which clearly generalizes, in an interesting way, the obstacle problem identified in the 2d literature, see [32].

Proof of Theorem 2. Step 0. Let us write $\xi = A|_\Omega - v$ and $\zeta = A - A_{ex}$, so that in terms of the ξ, ζ variables,

$$\mathcal{F}(v, A) = \frac{1}{2}\int_\Omega |\xi|^2 + |d(\zeta - \xi + A_{ex})| + \frac{1}{2}\int_{\mathbb{R}^3} |d\zeta|^2 =: F(\xi, \zeta).$$

Also, let $H := L^2(\Lambda^1\Omega) \times \dot{H}^1_*$. Note that H is a Hilbert space with the norm

$$\|(\xi, \zeta)\|^2_H := \|\xi\|^2_{L^2(\Omega)} + \|d\zeta\|^2_{L^2(\mathbb{R}^3)}$$

and the corresponding inner product. We next define

$$I(\xi, \zeta) := \frac{1}{2}\int_\Omega |d(\zeta - \xi + A_{ex})|$$

so that

$$F(\xi, \zeta) = \frac{1}{2}\|(\xi, \zeta)\|^2_H + I(\xi, \zeta).$$

As usual, I is understood to equal $+\infty$ if $d(\zeta - \xi + A_{ex})$ is not a Radon measure. Let I^* denote the Legendre-Fenchel transform of I, so that

$$I^*(\xi, \zeta) = \sup_{(\xi^*, \zeta)\in H} \big\{((\xi, \zeta), (\xi^*, \zeta^*))_H - I(\xi^*, \zeta^*)\big\}.$$

Let us further write

$$F^\dagger(\xi, \zeta) = \frac{1}{2}\|(\xi, \zeta)\|^2_H + I^*(\xi, \zeta).$$

Step 1. As remarked in Lemma 8 above, (ξ_0, ζ_0) minimizes F if and only if $(-\xi_0, -\zeta_0)$ minimizes $F^\dagger$. To compute I^*, note that for $(\xi, \zeta) \in H$,

$$\begin{aligned} I^*(\xi, \zeta) &= \sup_{(\xi^*, \zeta^*)\in H} \left\{((\xi, \zeta), ((\xi^* - A_{ex}) + A_{ex}, \zeta^*))_H - \frac{1}{2}\int_\Omega |d(\zeta^* - (\xi^* - A_{ex}))|\right\} \\ &= (\xi, A_{ex})_{L^2(\Omega)} + \sup_{(\xi^*, \zeta^*)\in H} \left\{((\xi, \zeta), (\xi^*, \zeta^*))_H - \frac{1}{2}\int_\Omega |d(\zeta^* - \xi^*)|\right\}. \end{aligned} \tag{3.1}$$

It is clear the supremum on the right-hand side equals zero if (ξ, ζ) satisfies

$$\int_{\mathbb{R}^3} \mathbf{1}_\Omega \xi \cdot \xi^* + d\zeta \cdot d\zeta^* \le \frac{1}{2}\int_\Omega |d(\zeta^* - \xi^*)| \qquad \text{for all } (\xi^*, \zeta^*) \in H, \tag{3.2}$$

and if this condition fails to hold, then (by homogeneity) the sup in (3.1) is infinite. Thus

$$I^*(\xi,\zeta) = \begin{cases} (\xi, A_{ex})_{L^2(\Omega)} & \text{if (3.2) holds} \\ +\infty & \text{if not.} \end{cases}$$

It follows that

$$F^\dagger(\xi,\zeta) = \begin{cases} \frac{1}{2}\|(\xi + A_{ex},\zeta)\|_H^2 - \frac{1}{2}\|A_{ex}\|_{L^2(\Omega)}^2 & \text{if (3.2) holds} \\ +\infty & \text{if not.} \end{cases}$$

Step 2. We want to rewrite $F^\dagger$ in a more useful form. To this end, we first claim that $(\xi,\zeta) \in H$ satisfies (3.2) if and only if

$$\int_{\mathbb{R}^3} d\zeta \cdot d\zeta^* \quad \le \quad \frac{1}{2}\int_\Omega |d\zeta^*| \quad \text{for all } \zeta^* \in \dot{H}^1(\Lambda^1\mathbb{R}^3) \tag{3.3}$$

and

$$\zeta \in H^2_{loc} \cap H^1_*, \quad \text{and} \quad d^*d\zeta + \mathbf{1}_\Omega \xi = 0. \tag{3.4}$$

Step 2a. First assume that (3.2) holds. Note that since $(\xi,\zeta) \in H$.

$$((\xi,\zeta),(\xi^*,\zeta^*))_H \le \frac{1}{2}\int_\Omega |d(\zeta^* - \xi^*)| \qquad \text{for all } (\xi,\zeta) \in L^2(\Lambda^1\Omega) \times \dot{H}^1(\Lambda^1\mathbb{R}^3).$$

This follows from (3.2), since we can write $(\xi^*,\zeta^*) \in L^2(\Omega) \times \dot{H}^1(\mathbb{R}^3)$ as $(\xi^*,\zeta') + (0,\zeta'')$ with $(\xi^*,\zeta') \in H$ and $\zeta'' \perp \dot{H}^1_*$, so that $d\zeta'' \equiv 0$.

Now we immediately obtain (3.3) by taking (ξ^*,ζ^*) of the form $(0,\zeta^*)$ in the above inequality. Similarly, by choosing (ξ^*,ζ^*) of the form $\pm(\zeta^*|_\Omega,\zeta^*)$ we find that

$$\int_{\mathbb{R}^3} (d\zeta \cdot d\zeta^* + \mathbf{1}_\Omega \xi \cdot \zeta^*) = 0 \qquad \text{for all } \zeta^* \in \dot{H}^1(\Lambda^1\mathbb{R}^3). \tag{3.5}$$

Since $d^*\zeta = 0$ for all $\zeta \in \dot{H}^1_*$, we see from (3.5) that $-\Delta\zeta + \mathbf{1}_\Omega\xi = 0$ as distributions, and hence from elliptic regularity that $\zeta \in H^2_{loc}(\mathbb{R}^3)$ and that $d^*d\zeta + \mathbf{1}_\Omega\xi = 0$ a.e. in $\mathbb{R}^3$, so that (3.4) holds.

Step 2b. Conversely, suppose that (3.3), (3.4) hold. Clearly (3.4) implies (3.5), so for $(X^*,\zeta^*) \in \dot{H}^1(\Lambda^1\mathbb{R}^3) \times \dot{H}^1(\Lambda^1\mathbb{R}^3)$,

$$\int_\Omega \xi \cdot X^* + \int_{\mathbb{R}^3} d\zeta \cdot d\zeta^* \overset{(3.5)}{=} \int_{\mathbb{R}^3} d\zeta \cdot d(\zeta^* - X^*) \overset{(3.3)}{\le} \frac{1}{2}\int_\Omega |d(\zeta^* - X^*)|.$$

Thus (3.2) follows whenever ξ^* is the restriction to Ω of some $X^* \in \dot{H}^1(\Lambda^1\mathbb{R}^3)$.

We next deduce from this that (3.2) holds whenever $\xi^* \in L^2(\Lambda^1\Omega)$. We may assume that $d\xi^*$ is a measure, as otherwise the right-hand side of (3.2) is infinite and there is nothing to prove. Then, given (ξ^*,ζ^*), it suffices to find $(X^*_\epsilon,\zeta^*_\epsilon) \in \dot{H}^1(\Lambda^1\mathbb{R}^3) \times \dot{H}^1(\Lambda^1(\mathbb{R}^3)$ such that

$$\begin{aligned} &X^*_\epsilon|_\Omega \rightharpoonup \xi^* \text{ weakly in } L^2(\Lambda^1\Omega), \\ &d\zeta^*_\epsilon \rightharpoonup d\zeta^* \text{ weakly in } L^2(\Lambda^1\mathbb{R}^3), \text{ and} \\ &\int_\Omega |d(\zeta^*_\epsilon - X^*_\epsilon)| \rightharpoonup \int_\Omega |d(\zeta^* - \xi^*)|. \end{aligned} \tag{3.6}$$

To do this, we start by fixing, for all ϵ sufficiently small, a C^1 diffeomorphism $\Psi_\epsilon : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\Psi_\epsilon(\{x \in \mathbb{R}^3 : \operatorname{dist}(x, \Omega) < \epsilon\}) \subset \Omega, \quad \Psi^\epsilon(x) = x \quad \text{if } \operatorname{dist}(x, \partial\Omega) > \sqrt{\epsilon} \tag{3.7}$$

and

$$\|D\Psi_\epsilon - I\|_\infty \le C\sqrt{\epsilon}, \qquad \|D\Psi_\epsilon^{-1} - I\|_\infty \le C\sqrt{\epsilon}. \tag{3.8}$$

For example we may take Ψ_ϵ in $\{x \in \mathbb{R}^3 : \operatorname{dist}(x, \partial\Omega) \le \sqrt{\epsilon}\}$ to have the form $\Psi_\epsilon(s) = x - f_\epsilon(d(x))\bar{\nu}(x)$, where $\bar{\nu}(x)$ is the outer unit normal to $\partial\Omega$ at the point of $\partial\Omega$ closest to x, and $d(x)$ is the signed distance (positive outside Ω, negative in Ω) to $\partial\Omega$, and f_ϵ is a nonnegative function with compact support in $(-\sqrt{\epsilon}, \sqrt{\epsilon})$ such that $|f_\epsilon'| \le C\sqrt{\epsilon}$ and $f_\epsilon(\epsilon) > \epsilon$.

Next, let $\bar{\xi}^*$ denote some extension of ξ to an element of $L^2(\Lambda^1\mathbb{R}^3)$, and let ψ_ϵ be a smooth nonnegative radially symmetric mollifier with support in $B(0, \epsilon/2)$ and such that $\int \phi_\epsilon = 1$.

Then we define

$$X_\epsilon^* := \psi_\epsilon * (\Psi_\epsilon^\# \bar{\xi}^*), \quad \text{and} \quad \zeta_\epsilon^* := \psi_\epsilon * (\Psi_\epsilon^\# \zeta^*).$$

Then the verification of (3.6) follows by a reasonably straightforward, classical argument. (See for example the proof of Lemma 15, at the end of Sect. 4.1, where similar computations are carried out in detail in a somewhat more complicated setting.)

Step 3. In view of (3.4), we can eliminate ξ from the expression for $F^\dagger$ to find that

$$F^\dagger(\xi, \zeta) = \begin{cases} \frac{1}{2}\|(-d^*d\zeta + A_{ex}, \zeta)\|_H^2 - \frac{1}{2}\|A_{ex}\|_{L^2(\Omega)}^2 & \text{if (3.3), (3.4) hold} \\ +\infty & \text{if not.} \end{cases}$$

We now rewrite everything in terms of $A = \zeta + A_{ex}$, $v = A|_\Omega - \xi = (\zeta + A_{ex})|_\Omega - \xi$, and $B = d(A - A_{ex}) = d\zeta$.

First, the constraints (3.3), (3.4) are equivalent to the conditions appearing under part **1** of the statement of the theorem, that is,

$$B \in \mathcal{C}, \quad \text{and} \quad d^*B + \mathbf{1}_\Omega(A - v) = 0, \tag{3.9}$$

where the constraint set $\mathcal{C}$ is defined in (1.9).

Second, it follows from Lemma 8 that

$$\begin{aligned} &(v_0, A_0) \text{ minimizes } \mathcal{F} \text{ in } L^2(\Omega) \times [A_{ex} + \mathcal{H}_0] \\ &\iff (A_0 - v_0, A_0 - A_{ex}) \text{ minimizes } F \text{ in } H \\ &\iff (v_0 - A_0, A_{ex} - A_0) \text{ minimizes } F^\dagger \text{ in } H, \\ &\iff (v_0, A_0) \text{ minimizes } \mathcal{F}^\dagger \text{ in } L^2(\Omega) \times [A_{ex} + \mathcal{H}_0], \end{aligned}$$

where $\mathcal{F}^\dagger(v, A) := F^\dagger(v - A, A_{ex} - A) + \frac{1}{2}\|A_{ex}\|_{L^2(\Omega)}^2$.

Thus

$$\mathcal{F}^\dagger(v, A) = \begin{cases} \frac{1}{2}\|(v - A + A_{ex}, A_{ex} - A)\|_H^2 & \text{if (3.9) holds, and} \\ +\infty & \text{if not.} \end{cases} \tag{3.10}$$

Rewriting $\mathcal{F}^\dagger$ in terms of $B := d(A - A_{ex})$, it follows that (v_0, A_0) minimizes $\mathcal{F}^\dagger$ if and only if conditions **1** and **2** from the statement of the theorem hold. Remark finally that (1.12) follows from (1.11) and the relation

$$\frac{1}{2}\int_\Omega |dv_0| + \int_\Omega (v_0 - A_0)\cdot v_0 = 0 \tag{3.11}$$

which in turn follows by stationarity of $\mathcal{F}(v_0, A_0)$ with respect to variations $v_t = e^t \cdot v_0$ around $t = 0$. □

3.2. First characterization of the critical applied magnetic field. We next want to prove Theorem 3, which gives a necessary and sufficient condition for the vorticity of a minimizer of $\mathcal{F}$ to be nonzero. Recall that this criterion involves the minimizer of an energy $\mathcal{E}_0$ in a space $\mathcal{C}'$, defined in (1.15). We first establish some facts about $\mathcal{C}'$. Given a function v defined on Ω, we use the notation $\mathbf{1}_\Omega v$ to denote its extension to the function, defined on $\mathbb{R}^3$, that vanishes away from Ω.

Lemma 9. *Assume that $B \in H^1(\Lambda^2\mathbb{R}^3) \cap d\dot{H}^1(\Lambda^1\mathbb{R}^3)$. If $B \in \mathcal{C}'$, then* $\mathrm{supp}(d^*B) \subset \bar{\Omega}$, *and $(d^*B)|_\Omega \in d^*H^1_N(\Lambda^2\Omega) = (\ker d)^\perp$. Conversely, given any $\phi \in d^*H^1_N(\Lambda^2\Omega)$, there exists $B_\phi \in \mathcal{C}'$ such that $d^*B_\phi = \mathbf{1}_\Omega\phi$ Finally,*

$$\mathcal{C}' \subset \bigcap_{1<p\le 2} \dot{W}^{1,p} \subset \bigcap_{\frac{3}{2}<q\le 6} L^q. \tag{3.12}$$

Remark 13. The proof will show that $B_\phi = d(-\Delta)^{-1}(\mathbf{1}_\Omega\phi)$, where $(-\Delta)^{-1}$ denotes convolution with the fundamental solution for the Laplacian on $\mathbb{R}^3$, with $(-\Delta)^{-1}(\mathbf{1}_\Omega\phi) \in \cap_{1<p\le 2}\dot{W}^{2,p} \subset \cap_{r>3} L^r$.

Proof. Step 1. We first claim that if $B \in \mathcal{C}'$, then

$$\int_{\mathbb{R}^3} d^*B\cdot v = 0 \qquad \text{for all } v \in L^2(\Lambda^1\mathbb{R}^3) \text{ such that } dv = 0 \text{ in } \Omega. \tag{3.13}$$

(Recall that by definition of $\mathcal{C}'$, this identity holds for $v \in H^1(\Lambda^1\mathbb{R}^3)$ such that $dv = 0$ in Ω.) To see this, define a diffeomorphism $\Psi_\epsilon : \mathbb{R}^3 \to \mathbb{R}^3$ as in (3.7), (3.8), and let ψ_ϵ denote a symmetric approximate identity supported in $B(0, \epsilon/2)$. Given $v \in L^2(\Lambda^1\mathbb{R}^3)$ such that $dv = 0$ in Ω, define $v_\epsilon := \psi_\epsilon * (\Psi_\epsilon^\# v) \in H^1(\Lambda^1\mathbb{R}^3)$. Clearly $v_\epsilon \to v$ in $L^2(\mathbb{R}^3)$, as $\epsilon \to 0$, and we also claim that $dv_\epsilon = 0$ in Ω. To see this, note that for any $\phi \in C^1_c(\Lambda^2\Omega)$,

$$\int_\Omega dv_\epsilon\cdot\phi = \int_\Omega v_\epsilon\cdot d^*\phi = \int_{\mathbb{R}^3} \Psi_\epsilon^\# v\cdot(\psi_\epsilon * d^*\phi) = \int_{\mathbb{R}^3} \Psi_\epsilon^\# v \wedge d\star(\psi_\epsilon * \phi).$$

Since $\Psi_\epsilon^\# v \wedge d\star(\psi_\epsilon * \phi) = \Psi_\epsilon^\#[v \wedge (\Psi_\epsilon^{-1})^\#(d\star(\psi_\epsilon * \phi))]$, it follows that

$$\int_\Omega dv_\epsilon\cdot\phi = \int_{\Psi_\epsilon(\mathbb{R}^3)} v \wedge d(\Psi_\epsilon^{-1})^\#(\star(\psi_\epsilon * \phi)). = \int_{\mathbb{R}^3} v\cdot d^*\star\phi_\epsilon \tag{3.14}$$

for $\phi_\epsilon := (\Psi_\epsilon^{-1})^\#(\star(\psi_\epsilon * \phi))$. The definitions of Ψ_ϵ and ψ_ϵ imply that ϕ_ϵ has compact support in Ω. Thus $\int_\Omega dv_\epsilon \cdot \phi = \int_\Omega dv \cdot \phi_\epsilon = 0$ for every $\psi \in C_c^1(\Lambda^2\Omega)$, and it follows that $dv_\epsilon = 0$ in Ω. Then if $B \in \mathcal{C}'$,

$$\int_{\mathbb{R}^3} d^*B \cdot v = \lim_{\epsilon\to 0} \int_{\mathbb{R}^3} B \cdot dv_\epsilon = 0.$$

Step 2. Now for $B \in \mathcal{C}'$, if $\chi \in C_c^\infty(\mathbb{R}^3 \backslash \Omega)$, then $\chi d^*B \in L^2(\Lambda^1\mathbb{R}^3)$ and $d(\chi d^*B) = 0$ in Ω, so $\int_{\mathbb{R}^3} \chi |d^*B|^2 = 0$ by (3.13). Hence $\operatorname{supp}(d^*B) \subset \bar{\Omega}$. Then (3.13) implies that for any $v \in L^2(\Lambda^1\Omega)$ such that $dv = 0$,

$$0 \;=\; \int_{\mathbb{R}^3} d^*B \cdot (\mathbf{1}_\Omega v) = \int_\Omega d^*B \cdot v.$$

Thus $(d^*B)|_\Omega \in (\ker d)^\perp = d^*H_N^1(\Lambda^2\Omega)$.

Step 3. Now, given $\phi \in d^*H_N^1(\Lambda^2\omega)$, let $\psi = (-\Delta)^{-1}(\mathbf{1}_\Omega\phi)$, and let $B_\phi := d\psi$.

Then the fact that $\phi \in d^*H_N^1$ implies that $d^*(\mathbf{1}_\Omega\phi) = 0$ on $\mathbb{R}^3$. It follows that $d^*\psi = 0$, and hence that $d^*B_\phi = d^*d\psi = (d^*d + dd^*)\psi = -\Delta\psi = \mathbf{1}_\Omega\phi$. In particular $\operatorname{supp}(d^*\psi) \subset \bar{\Omega}$.

Finally, to see that $B_\phi \in \mathcal{C}'$, observe that $\mathbf{1}_\Omega\phi \in \cap_{1\le p\le 2} L^p(\Lambda^1\mathbb{R}^3)$, so elliptic regularity and embedding theorems imply $B_\psi \in \cap_{1<p\le 2} W^{1,p} \subset \cap_{3/2<q<6} L^q$, $B_\phi \in H^1(\Lambda^2\mathbb{R}^3) \cap d\dot{H}^1(\Lambda^1\mathbb{R}^3)$, and in addition (3.12) holds. It is clear that $\operatorname{supp}(d^*B_\phi) = \operatorname{supp}(\mathbf{1}_\Omega\phi) \subset \bar{\Omega}$, so $B_\phi \in \mathcal{C}'$. □

We need one more easy fact about $\mathcal{C}'$.

Lemma 10. *If $B_1, B_2 \in \mathcal{C}'$, then there exists $\psi_1 \in \cap_{1<p\le 2} \dot{W}^{2,p}(\Lambda^1\mathbb{R}^3)$ such that $d\psi_1 = B_1$ and*

$$\int_{\mathbb{R}^3} B_1 \cdot B_2 \;=\; \int_\Omega \psi_1 \cdot d^*B_2.$$

Proof. Let $\psi_1 = (-\Delta)^{-1} d^*B$, so that in view of Remark 13,

$$\int_{\mathbb{R}^3} B_1 \cdot B_2 \;=\; \int_{\mathbb{R}^3} d\psi_1 \cdot B_2 \;=\; \int_{\mathbb{R}^3} \psi_1 \cdot d^*B_2 \;=\; \int_\Omega \psi_1 \cdot d^*B_2$$

where the integration by parts is easily justified in view of the decay properties recorded in (3.12) and Remark 13. □

Now we give the

Proof of Theorem 3. Step 1. We first assume that $\|B_*\|_* \le \frac{1}{2}$. Then, recalling (1.11), and recalling that $dB_0 = 0$ since $B_0 = d(A_0 - A_{ex})$, we must show that

$$0 = dv_0 = dd^*B_0 + dA_0 = dd^*B_0 + B_0 + H_{ex} \quad \text{in } \Omega.$$

Since B_0 and B_* minimize $\mathcal{E}_0$ in $\mathcal{C}$ and $\mathcal{C}'$ respectively, and since $\mathcal{C} \subset \mathcal{C}'$, it is clear that $B_* = B_0$ if and only if $B_* \in \mathcal{C}$, which holds if and only if $\|B_*\|_* \le \frac{1}{2}$. So it suffices to check that

$$dd^*B_* + B_* + H_{ex} = 0 \quad \text{in } \Omega. \tag{3.15}$$

To do this, we take first variations of $\mathcal{E}_0$ in $\mathcal{C}'$ to find that B_* satisfies

$$\int_{\mathbb{R}^3} B_* \cdot B + \int_\Omega (d^* B_* + A_{ex}) \cdot d^* B = 0 \qquad \text{for all } B \in \mathcal{C}'. \tag{3.16}$$

By Lemma 10, we may rewrite this as

$$\int_\Omega (\psi + d^* B_* + A_{ex}) \cdot d^* B = 0 \qquad \text{for all } B \in \mathcal{C}',$$

where $\psi = (-\Delta)^{-1} d^* B_*$, so that $d\psi = B_*$. Then we conclude from Lemma 9 that $(\psi + d^* B_* + A_{ex})|_\Omega \in ((\ker d)^\perp)^\perp = \ker d$, and hence that $d(\psi + d^* B_* + A_{ex}) = 0$ in Ω, which is (3.15).

Step 2. Now we assume that $dv_0 = 0$ in Ω. We will show that in this case, $\mathcal{E}_0(B_0) = \mathcal{E}_0(B_*)$. Since B_* is the unique minimizer of $\mathcal{E}_0$ in $\mathcal{C}'$ and $B_0 \in \mathcal{C} \subset \mathcal{C}'$, this implies that $B_0 = B_*$, and hence that $B_* \in \mathcal{C}$ or equivalemtly, that $\|B_*\|_* \le \frac{1}{2}$.

First note that

$$\int_\Omega v_0 \cdot d^* B = 0 \qquad \text{for any } B \in \mathcal{C}', \tag{3.17}$$

since $d^* B \in d^* H^1_N(\Lambda^2 \Omega) = (\ker d)^\perp$ by Lemma 9. Applying this to $B = B_0$ and recalling that $v_0 = d^* B_0 + A_0 = d^* B_0 + (A_0 - A_{ex}) + A_{ex}$ in Ω, we obtain

$$\begin{aligned} 0 &= \int_\Omega (d^* B_0 + (A_0 - A_{ex}) + A_{ex}) \cdot d^* B_0 \\ &= \int_{\mathbb{R}^3} (d^* B_0 + (A_0 - A_{ex}) + A_{ex}) \cdot d^* B_0 \\ &= \int_{\mathbb{R}^3} |d^* B_0|^2 + |B_0|^2 + A_{ex} \cdot d^* B_0. \end{aligned}$$

(The integration by parts is easily justified using (3.12).) Using this to rewrite the definition of $\mathcal{E}_0$ yields

$$\mathcal{E}_0(B_0) = \frac{1}{2} \int_\Omega A_{ex} \cdot d^* B_0 + |A_{ex}|^2. \tag{3.18}$$

Step 3. Next, taking B_* as a test function in (3.16), we obtain

$$\int_{\mathbb{R}^3} |B_*|^2 + \mathbf{1}_\Omega (|d^* B_*|^2 + A_{ex} \cdot d^* B) = 0.$$

It follows that

$$\mathcal{E}_0(B_*) = \frac{1}{2} \int_\Omega (d^* B_* \cdot A_{ex} + |A_{ex}|^2). \tag{3.19}$$

From (3.16) we also have

$$0 = \int_{\mathbb{R}^3} B_* \cdot B_0 + \mathbf{1}_\Omega (d^* B_* + A_{ex}) \cdot d^* B_0 = 0. \tag{3.20}$$

On the other hand, again using (3.17), we compute

$$0 = \int_\Omega v_0 \cdot d^* B_* = \int_\Omega (d^* B_0 + (A_0 - A_{ex}) + A_{ex}) \cdot d^* B_*$$
$$= \int_{\mathbb{R}^3} (d^* B_0 + (A_0 - A_{ex}) + \mathbf{1}_\Omega A_{ex}) \cdot d^* B_*$$
$$= \int_{\mathbb{R}^3} d^* B_0 \cdot d^* B_* + B_0 \cdot B_* + \mathbf{1}_\Omega A_{ex} \cdot d^* B_*,$$

recalling that $d(A_0 - A_{ex}) = B_0$. And by comparing this and (3.20), we find that

$$\int_\Omega A_{ex} \cdot d^* B_* = \int_\Omega A_{ex} \cdot d^* B_0.$$

This, together with (3.18) and (3.19), shows that $\mathcal{E}_0(B_*) = \mathcal{E}_0(B_0)$, completing the proof. $\square$

3.3. An alternate characterization of the critical applied field. Our next result gives a different characterization of the critical field.

Theorem 11. *Let* (v_0, A_0) *minimize* $\mathcal{F}$ *in* $L^2(\Omega; \Lambda^1\mathbb{R}^3) \times [A_{ex} + \dot{H}^1_*]$.
Further, let

$$\mathcal{E}_1(A) = \frac{1}{2} \int_{\mathbb{R}^3} \mathbf{1}_\Omega |PA|^2 + |dA - H_{ex}|^2\, dx, \tag{3.21}$$

where P *is defined in* (2.8)*, and let* A_1 *minimize* $\mathcal{E}_1$ *in* $A_{ex} + H^1(\mathbb{R}^3; \Lambda^1\mathbb{R}^3)$. *Let* $\alpha_1 \in H^1(\Omega; \Lambda^2\mathbb{R}^3)$ *be such that*

$$d^*\alpha_1 = PA_1,\ d\alpha_1 = 0 \quad in\ \Omega, \qquad \alpha_{1,N} = 0 \quad on\ \partial\Omega. \tag{3.22}$$

(Such an α_1 *exists by definition of* P*.) Note that* A_1 *and hence* α_1 *depend on* A_{ex}*.*
Then $dv_0 = 0$ *if and only if*

$$\|\alpha_1\|_{**} := \sup_{|dv|(\Omega)\le 1} \int_\Omega dv \cdot \alpha_1 \le 1/2. \tag{3.23}$$

Moreover, if $dv_0 = 0$ *then* $A_0 = A_1$*.*

3.3.1. Theorem 11 via a splitting of $\mathcal{F}$*.* We will give three proofs of this theorem. We first present the most direct proof, which does not use convex duality at all.

First proof of Theorem 11. Recall from (2.6) that for $v \in L^2(\Lambda^1\Omega)$, $dv = 0$ in Ω if and only if $v \in dH^1(\Omega) \oplus \mathcal{H}_N(\Lambda^1\Omega) = \ker d$, see (2.6). Define

$$\tilde{\mathcal{F}}(v, A) := \inf\{\mathcal{F}(v + \gamma, A) : \gamma \in \ker d\}$$
$$\overset{(2.9)}{=} \frac{1}{2}\left[\int_\Omega |dv| + |P(v - A)|^2 dx \ + \int_{\mathbb{R}^3} |dA - H_{ex}|^2\, dx\right].$$

(Note that the definition (3.21) of $\mathcal{E}_1$ can be rewritten $\mathcal{E}_1(A) = \tilde{\mathcal{F}}(0, A)$.) It is clear that

$$(v_0, A_0) \text{ minimizes } \tilde{\mathcal{F}} \quad \Longleftrightarrow \quad (v_0 + \gamma, A_0) \text{ minimizes } \mathcal{F} \text{ for some } \gamma \in \ker d.$$

Since we are interested here in dv_0, we may consider $\tilde{\mathcal{F}}$ instead of $\mathcal{F}$. We rewrite

$$\tilde{\mathcal{F}}(v, A) = \mathcal{E}_1(A) + \frac{1}{2}\int_\Omega |dv| + |Pv|^2 - 2Pv \cdot PA\, dx.$$

Since A_1 minimizes $\mathcal{E}_1$,

$$\int_{\mathbb{R}^3} \mathbf{1}_\Omega PA_1 \cdot PB + (dA_1 - H_{ex}) \cdot dB\; dx = 0 \tag{3.24}$$

for all $B \in \dot{H}^1(\mathbb{R}^3; \Lambda^1\mathbb{R}^3)$, so that

$$\mathcal{E}_1(A_1 + B) = \mathcal{E}_1(A_1) + \frac{1}{2}\int_{\mathbb{R}^3} \mathbf{1}_\Omega |PB|^2 + |dB|^2\; dx$$

for B as above. Given any A, let us write $A = A_1 + B$. Then

$$\begin{aligned}\tilde{\mathcal{F}}(v, A_1 + B) = \mathcal{E}_1(A_1) + \frac{1}{2}\int_\Omega |PB|^2\, dx\; + \int_{\mathbb{R}^3} |dB|^2\; dx \\ + \frac{1}{2}\int_\Omega |dv| + |Pv|^2 - 2Pv \cdot (PA_1 + PB)\, dx \\ = \frac{1}{2}\int_\Omega |P(B - v)|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |dB|^2 + \int_\Omega \frac{1}{2}|dv| - Pv \cdot PA_1.\end{aligned}$$

For α_1 as in the statement of the theorem,

$$\int_\Omega Pv \cdot PA_1\; = \int_\Omega Pv \cdot d^*\alpha_1\; dx\; = \int_\Omega dv \cdot \alpha_1\; dx, \tag{3.25}$$

where the boundary terms arising from integration by parts have vanished due to the fact that $\alpha_{1,N} = 0$. Thus

$$\tilde{\mathcal{F}}(v, A) = \mathcal{E}_1(A_1) + \frac{1}{2}\int_{\mathbb{R}^3} |dB|^2 + \mathbf{1}_\Omega |P(v - B)|^2 + \int_\Omega (\frac{1}{2}|dv| - dv \cdot \alpha_1). \tag{3.26}$$

If condition (3.23) holds, then $\int_\Omega(\frac{1}{2}|dv| - dv \cdot \alpha_1) \geq 0$ for all $v \in L^2(\Omega)$, and thus $\tilde{\mathcal{F}}(v, A) \geq \mathcal{E}_1(A_1)$ for all (v, A). Moreover, if $(v_0, A_0) = (v_0, A_1 + B_0)$ attains this minimum, then $\frac{1}{2}\int_{\mathbb{R}^3} |dB_0|^2 + \mathbf{1}_\Omega |P(v_0 - B_0)|^2 = 0$, and this implies that $dv_0 = 0$.

And if (3.23) fails, then there exists some v_1 such that $\int_\Omega -dv_1 \cdot \alpha_1 + \frac{1}{2}|dv_1| < 0$, and then it is clear that $\tilde{\mathcal{F}}(\lambda v_1, A_1) < \mathcal{E}_1(A_1) = \tilde{\mathcal{F}}(0, A_1)$ for all sufficiently small $\lambda > 0$. Thus $\tilde{\mathcal{F}}(v_0, A_0) < \mathcal{E}_1(A_1)$ for any minimizing (v_0, A_0), and then (3.26) implies that $dv_0 \neq 0$.

Finally, if $dv_0 = 0$ then it is clear from (3.26) that $\tilde{\mathcal{F}}(0, A_1) = \mathcal{E}_1(A_1) = \min \mathcal{F}$, and and hence that $A_0 = A_1$. □

3.3.2. Theorem 11 via partial convex duality. We next prove Theorem 11 by a duality computation that differs slightly from the one used in the proof of Theorem 2. The result of this computation is summarized in the following

Lemma 12. *Let*

$$N := \left\{ \zeta \in L^2(\Omega) : (\zeta, \xi)_{L^2(\Omega)} \le \frac{1}{2} \int |d\xi| \quad \text{for all } \xi \in L^2(\Omega) \right\}, \tag{3.27}$$

and define

$$\mathcal{F}^\ddagger(A) := \frac{1}{2} \int_{\mathbb{R}^3} (|d(A - A_{ex})|^2 + \mathbf{1}_\Omega |A|^2)\, dx - \frac{1}{2} \mathrm{dist}^2_{L^2(\Omega)}(A, N). \tag{3.28}$$

Then (v_0, A_0) *minimizes* $\mathcal{F}$ *in* $L^2(\Omega; \Lambda^1 \mathbb{R}^3) \times [A_{ex} + \dot{H}^1_*]$ *if and only if*

1. A_0 *minimizes* $\mathcal{F}^\ddagger$ *in* $[A_{ex} + \dot{H}^1_*]$, and
2. $A_0|_\Omega - v_0 \in N$, *and* $\|A_0 - v_0\|_{L^2(\Omega)} = \mathrm{dist}_{L^2(\Omega)}(A_0, N)$.

It is clear from the definition that $N \subset (\ker d)^\perp = \mathrm{Image}(P)$, and it follows that

$$\begin{aligned} \mathcal{F}^\ddagger(A) &= \frac{1}{2} \int_{\mathbb{R}^3} (|d(A - A_{ex})|^2 + \mathbf{1}_\Omega |PA|^2)\, dx - \frac{1}{2} \mathrm{dist}^2_{L^2(\Omega)}(PA, N) \\ &= \mathcal{E}_1(A) - \frac{1}{2} \mathrm{dist}^2_{L^2(\Omega)}(PA, N). \end{aligned} \tag{3.29}$$

Proof. We will compute the convex dual of $\mathcal{F}$ with respect to the "v" variable only, treating A as a parameter. Thus, let $\xi = A|_\Omega - v$, and write $\tilde{F}(\xi; A) = \mathcal{F}(v, A)$, so that

$$\tilde{F}(\xi; A) = \frac{1}{2} \int_\Omega |\xi|^2 + |d(A - \xi)| + \frac{1}{2} \int_{\mathbb{R}^3} |d(A - A_{ex})|^2.$$

Let

$$\tilde{I}(\xi; A) := \frac{1}{2} \int_\Omega |d(A - \xi)| + c_A, \qquad c_A := \frac{1}{2} \int_{\mathbb{R}^3} |d(A - A_{ex})|^2.$$

Then $\tilde{F}(\xi; A) = \tilde{I}(\xi; A) + \frac{1}{2}\|\xi\|_2^2$.

Next, let

$$\tilde{I}^*(\xi^*; A) := \sup_{\xi \in L^2} \left\{ (\xi^*, \xi)_{L^2(\Omega)} - \tilde{I}(\xi; A) \right\}.$$

A short computation like that in the proof of Theorem 2 shows that

$$\tilde{I}^*(\xi^*; A) = \begin{cases} (\xi^*, A) - c_A & \text{if } \xi^* \in N \\ +\infty & \text{if not} \end{cases} \tag{3.30}$$

for N as defined in (3.27). Now let

$$\begin{aligned} \tilde{F}^\ddagger(\xi^*, A) &:= \tilde{I}^*(-\xi^*; A) + \frac{1}{2}\|\xi\|^2 \\ &= \begin{cases} (-\xi^*, A) + \frac{1}{2}\|\xi\|^2 - c_A & \text{if } \xi^* \in N \\ +\infty & \text{if not} \end{cases} \\ &= \begin{cases} \frac{1}{2}\|\xi^* - A\|^2 - \frac{1}{2} \int_{\mathbb{R}^3} (|d(A - A_{ex})|^2 + \mathbf{1}_\Omega |A|^2)\, dx & \text{if } \xi^* \in N \\ +\infty & \text{if not.} \end{cases} \end{aligned}$$

Then it is clear that

$$-\inf_{\xi^*} \tilde{F}^{\ddagger}(\xi^*; A) = \mathcal{F}^{\ddagger}(A)$$

as defined above, and that the infimum is attained by a unique ξ, the closest point to A in the (closed convex) set N. Recall from Lemma 8 that ξ minimizes $\tilde{F}(\cdot; A)$ if and only if it minimizes $\tilde{F}^{\ddagger}(\cdot; A)$, and moreover that $\min_\xi \tilde{F}(\cdot; A) = -\min_\xi \tilde{F}^{\ddagger}(\cdot; A)$. Thus

$$\min_{A,v} \mathcal{F}(v; A) = \min_A \min_v \mathcal{F}(v; A) = \min_A (-\min_\xi \tilde{F}(\xi, A)) = \min_A \mathcal{F}^{\ddagger}(A),$$

and (v_0, A_0) minimizes $\mathcal{F}$ if and only if A_0 minimizes $\mathcal{F}^{\ddagger}$ and $v_0 = A_0|_\Omega - \xi_0$, where ξ_0 is the closest point in N to A. □

Now we use Lemma 12 to give a

Second proof of Theorem 11. Fix (v_0, A_0) minimizing $\mathcal{F}$, and A_1 minimizing $\mathcal{E}_0$. By Lemma 12, A_0 minimizes $\mathcal{F}^{\ddagger}$.

We write $\xi_0 = A_0|_\Omega - v_0$ as above, so that ξ_0 is the closest point to A_0 in N. From the definition (2.8) of P we know that $dPA_0 = dA_0$, so that

$$dv_0 = 0 \iff d(PA_0 - \xi_0) = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Also, $PA_0 - \xi_0 \in (\ker d)^\perp = d^* W^{1,2}_N(\Lambda^2\Omega)$, since the definitions of N and P imply that $N \subset (\ker d)^\perp$ and $\text{Image}(P) = (\ker d)^\perp$. Then (2.6) implies that $d(PA_0 - \xi_0) = 0$ in $\mathcal{D}'$ if and only if $PA_0 - \xi_0 \in d^* W^{1,2}_N(\Lambda^2\Omega) \cap \big(dH^1(\Omega) \oplus \mathcal{H}(\Lambda^1\Omega)\big) = \{0\}$. In other words, $dv_0 = 0$ if and only if $PA_0 = \xi_0$. But since ξ_0 is the closest point in N to A_0, and hence to PA_0, we conclude that

$$dv_0 = 0 \quad \iff \quad PA_0 \in N. \tag{3.31}$$

Next, note that $A \mapsto \mathcal{F}^{\ddagger}(A_{ex} + A)$ is strictly convex in H^1_*, so that the minimizers A_0 of $\mathcal{F}^{\ddagger}$ and A_1 of $\mathcal{E}_1$ are unique. Also, (3.29) implies that if $PA \in N$ and $A' \in H^1_*$, then

$$\mathcal{F}^{\ddagger}(A + A') \le \mathcal{E}_1(A + A') \le \mathcal{F}^{\ddagger}(A) + C\|A'\|^2_{L^2(\Omega)} \le \mathcal{F}^{\ddagger}(A) + C\|A'\|^2_{H_*}.$$

Thus any critical point A of $\mathcal{E}_1$ such that $PA \in N$ must also be a critical point of $\mathcal{F}^{\ddagger}$, and conversely.

$$PA_0 \in N \text{ if and only if } PA_1 \in N. \tag{3.32}$$

It follows along the same lines that if $dv_0 = 0$ then $A_0 = A_1$. Finally, recalling the definitions (3.22) of α_1 and (3.27) of N, and integrating by parts as in (3.25), we conclude that

$$PA_1 \in N \quad \iff \quad \|\alpha_1\|_\star \le \frac{1}{2}.$$

By combining this with (3.31) and (3.32), we conclude this proof of Theorem 11. □

3.3.3. Equivalence of Theorem 11 and Theorem 3. In Theorem 3 and Theorem 11, we have derived two necessarily equivalent but rather different-looking necessary and sufficient conditions for the vorticity dv_0 of a minimizing pair (v_0, A_0) to vanish. In this section we elucidate the connection between the auxiliary functions B_*, defined in Theorem 3, and α_1, defined in (3.23).

Proposition 13. *If $dv_0 = 0$ in Ω, then $PA_1 = d^*\alpha_1 = (d^*B_*)|_\Omega$, and $\|\alpha_1\|_{**} = \|B_*\|_*$.*

This can be seen as a third proof of Theorem 11.

Proof. If $dv_0 = 0$, then $Pv_0 = 0$, and we have seen that $A_0 = A_1$ and $B_0 = B_*$. As a result,

$$(d^*B_*)|_\Omega = (d^*B_0)|_\Omega = P(d^*B_0) = P(A_0 - v_0) = PA_0 = PA_1 = d^*\alpha_1$$

by Theorem 3, Lemma 9, and (1.11). Thus for every $v \in H^1(\Lambda^2\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} dv \cdot B_* = \int_{\mathbb{R}^3} v \cdot d^*B = \int_\Omega v \cdot d^*B = \int_\Omega v \cdot d^*\alpha_1 = \int_\Omega dv \cdot \alpha_1.$$

Now the conclusion follows from the definitions of the norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$, see (1.10) and (3.23). □

4. Vortex Density in 3d Bose-Einstein Condensates

In this section we use the results of [7] to prove convergence as $\epsilon \to 0$ of Gross-Pitaevskii functional $\mathcal{G}_\epsilon$, defined in (1.25) to the limiting $\mathcal{G}$, defined in (1.28). We also establish some results describing minimizers of $\mathcal{G}$.

4.1. Γ-convergence. Our first theorem makes precise the sense in which $\mathcal{G}$ is a limiting functional associated to the sequence of functionals $(\mathcal{G}_\epsilon)_{\epsilon\in(0,1]}$. The statement of the result uses some notation that is introduced in Sect. 1.2.

Theorem 14. *Assume that $\Phi_\epsilon = |\log\epsilon|\Phi$ for $\Phi \in L^4_{loc}(\Lambda^1\mathbb{R}^3)$ and that $|\Phi(x)|^2 \le C(a(x)+1)$ for all $x \in \mathbb{R}^3$.*

(i) **Compactness**. *Assume that $(u_\epsilon)_{\epsilon\in(0,1]} \subset H^1_{a,m}$ and that there exists some $C > 0$ such that*

$$\mathcal{G}_\epsilon(u_\epsilon) \le C|\log\epsilon|^2 \qquad \text{for all } \epsilon \in (0, 1/2]. \tag{4.1}$$

Then there exists $j \in L^{4/3}(\Lambda^1\mathbb{R}^3)$, supported in $\bar{\Omega}$, such that if we define $v = \frac{1}{\rho} j|_\Omega$ and pass to a subsequence if necessary, we have

$$\frac{ju_\epsilon}{|\log\epsilon|} \rightharpoonup j = \rho v \text{ weakly in } L^{4/3}(\mathbb{R}^3), \quad \text{and} \quad v \in L^2_\rho(\Lambda^1\Omega) \text{ with } \int_\Omega \rho|dv| < \infty. \tag{4.2}$$

(ii) **Lower bound inequality**. *There exists a sequence of numbers (κ_ϵ) such that if we assume the above hypotheses and (4.2), then*

$$\liminf_{\epsilon\to 0} |\log\epsilon|^{-2}\left(\mathcal{G}_\epsilon(u_e) - \kappa_\epsilon\right) \ge \mathcal{G}(v). \tag{4.3}$$

(iii) **Upper bound inequality.** *Given any* $v \in L^2_\rho(\Lambda^1\Omega)$ *such that* dv *is a measure on* Ω *with* $\int_\Omega \rho|dv| < \infty$*, there exists a sequence* $u_\epsilon \in H^1_{a,m}$ *such that* (4.2) *holds and* $\lim_{\epsilon\to 0} |\log\epsilon|^{-2} (\mathcal{G}_\epsilon(u_e) - \kappa_\epsilon) = \mathcal{G}(v)$.

Remark 14. Actually, we have $\kappa_\epsilon = \mathcal{G}_\epsilon(f_\epsilon)$, where the real-valued function f_ϵ minimizes $\mathcal{G}_\epsilon$ in $H^1_a(\mathbb{R}^3;\mathbb{R})$.

The theorem states that the functionals $|\log\epsilon|^{-2}(\mathcal{G}_\epsilon(\,\cdot\,) - \kappa_\epsilon)$ converge to $\mathcal{G}$ in the sense of Γ-convergence, with respect to the convergence (4.2). As remarked in the Introduction, Proposition 5 is a direct corollary of Theorem 14 and basic properties of Γ-convergence.

Remark 15. In fact we prove a more general result than Theorem 14, since we also allow higher rotations $\Phi_\epsilon = \sqrt{g_\epsilon}\Phi$, with $|\log\epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$. In fact we show that for such Φ_ϵ, if $\mathcal{G}_\epsilon(u_\epsilon) \le C g_\epsilon$, then after passing to a subsequence, $\frac{ju_\epsilon}{\sqrt{g_\epsilon}} \rightharpoonup j$ weakly in $L^{4/3}(\Lambda^1\mathbb{R}^3)$, with $j = \mathbf{1}_\Omega \rho v$ for some $v \in L^2_\rho(\Lambda^1\Omega)$, and

$$g_\epsilon^{-1}\left(\mathcal{G}_\epsilon(\cdot) - \kappa_\epsilon\right) \xrightarrow{\Gamma} \tilde{\mathcal{G}}(\cdot)\,, \qquad \text{where } \tilde{\mathcal{G}}(v) = \int_\Omega \rho\left(\frac{v^2}{2} - \Phi\cdot v\right). \tag{4.4}$$

Proposition 6 and Remark 8 in the Introduction follow as immediate corollaries.

The proofs rely at certain points on Theorem 2 in [7].

Proof of Theorem 14 and Remark 15. Let $\mathcal{G}_\epsilon(u_\epsilon) \le C g_\epsilon$, for $|\log\epsilon|^2 \le g_\epsilon \ll \epsilon^{-2}$, and let $\Phi_\epsilon = \sqrt{g_\epsilon}\Phi$.

Step 1. First we control the potentially negative term in $\mathcal{G}_\epsilon(u_\epsilon)$. To do this, recall our assumption that $|\Phi|^2 \le C(a+1)$. Since $|ju| \le |u|\,|du|$, it follows that

$$|\Phi_\epsilon\cdot ju_\epsilon| \;\le\; \frac{1}{2}g_\epsilon|\Phi|^2|u|^2 + \frac{1}{2}|du|^2 \;\le\; Cg_\epsilon(a+1)|u|^2 + \frac{1}{2}|du|^2.$$

But (1.24) implies that $a(x) = w(x) - \rho(x) + \lambda \le w(x) + \lambda$, so it follows that

$$|\Phi_\epsilon\cdot ju_\epsilon| \le Cg_\epsilon(w+\lambda+1)|u_\epsilon|^2 + \frac{1}{2}|du_\epsilon|^2.$$

Integrating this over $\mathbb{R}^3$ and recalling that $\|u_\epsilon\|_2^2 = m$, we obtain

$$\int_{\mathbb{R}^3} |\Phi_\epsilon\cdot ju_\epsilon| \le C\mathcal{G}_\epsilon(u_\epsilon) + Cg_\epsilon(\lambda+1)\int_{\mathbb{R}^3}|u_\epsilon|^2 \le Cg_\epsilon.$$

It follows from this that

$$\int_{\mathbb{R}^3} \frac{1}{2}|du_\epsilon|^2 + \frac{1}{4\epsilon^2}(\rho - |u_\epsilon|^2)^2 + \frac{w}{2\epsilon^2}|u_\epsilon|^2 = \mathcal{G}_\epsilon(u_\epsilon) + \int_{\mathbb{R}^3}\Phi_\epsilon\cdot ju_\epsilon \le Cg_\epsilon. \tag{4.5}$$

In particular $|u_\epsilon|^2 \to \rho$ in $L^2(\mathbb{R}^3)$.

Step 2. Next,

$$\|u_\epsilon\|^4_{L^4(\mathbb{R}^3)} = \|\,|u_\epsilon|^2\|^2_{L^2} \le C(\|\,|u_\epsilon|^2 - \rho\|^2_{L^2} + \|\rho\|^2_{L^2}) \le C + \epsilon^2\mathcal{G}_\epsilon(u_\epsilon) \le C,$$

and Step 1 implies that $\|du_\epsilon\|_{L^2} \le C\sqrt{g_\epsilon}$. Since $\|ju_\epsilon\|_{L^{4/3}} \le \|u_\epsilon\|_{L^4}\|du_\epsilon\|_{L^2}$, we conclude that $\{\frac{1}{\sqrt{g_\epsilon}} ju_\epsilon\}$ is uniformly bounded in $L^{4/3}(\mathbb{R}^3)$, and it follows that there exists some $j \in L^{4/3}(\mathbb{R}^3)$ such that

$$\frac{ju_\epsilon}{\sqrt{g_\epsilon}} \rightharpoonup j \text{ weakly in } L^{4/3} \text{ along some subsequence.} \tag{4.6}$$

Step 3. Now let f_ϵ denote the minimizer in $H^1_a(\mathbb{R}^3;\mathbb{R})$ of $\mathcal{G}_\epsilon(\cdot)$, where $H^1_a(\mathbb{R}^3;\mathbb{R})$ is defined by analogy with $H^1_a(\mathbb{R}^3;\mathbb{C})$, see (1.23).

Note that when f is real-valued, $jf = 0$, so the forcing term $\Phi \cdot jf$ vanishes on $H^1_a(\mathbb{R}^3;\mathbb{R})$. It is standard that f_ϵ does not vanish, and we will assume that $f_\epsilon > 0$. Then for any $u \in H^1_{a,m}$ we may define $U := u/f_\epsilon$, and it is known that

$$\mathcal{G}_\epsilon(u) = \mathcal{G}_\epsilon(f_\epsilon U) = \mathcal{G}_\epsilon(f_\epsilon) + H_\epsilon(U) + \int_{\mathbb{R}^3} f_\epsilon^2 \Phi_\epsilon \cdot jU \ , \tag{4.7}$$

where $H_\epsilon(U) = H_\epsilon(U;\mathbb{R}^3)$, and for a measurable subset $A \subset \mathbb{R}^3$ we write

$$H_\epsilon(U;A) := \int_A \frac{f_\epsilon^2}{2}|dU|^2 + \frac{f_\epsilon^4}{4\epsilon^2}(|U|^2-1)^2\,dx. \tag{4.8}$$

See Lemma 3.1 [27] for a proof in exactly the situation we consider here,[3] following ideas that originated in [25] and have been used extensively in the literature on Bose-Einstein condensates. It is also known that $f_\epsilon^2 \to \rho$ uniformly in $\mathbb{R}^3$, see [27], Lem. B.1.

Step 4. Now let Ω' denote a subset of Ω such that $\Omega' \subset\subset \Omega$, so that $\rho \ge 2c'$ in Ω' for some c', and hence $f_\epsilon^2 \ge c'$ for all sufficiently small ϵ. Let $U_\epsilon = u_\epsilon/f_\epsilon$, and note that Step 1 implies that $H_\epsilon(U_\epsilon) \le Cg_\epsilon$. Thus the functional

$$\tilde{H}_\epsilon(U_\epsilon) = \int_{\Omega'} \frac{1}{2}|dU_\epsilon|^2 + \frac{1}{\epsilon^2}W(u) \ \le \ C'g_\epsilon, \qquad \text{where } W(U) = \frac{c'}{4}(|U|^2-1)^2,$$

verifies in Ω' hypothesis (H_q) of Theorem 2 in [7], for $q = 4$. Hence Theorem 2 of [7] (or arguments such as those in Steps 1 and 2 above) imply that there exists some v' in $L^2(\Lambda^1\Omega')$ such that, after passing to a further subsequence if necessary, $\frac{jU_\epsilon}{\sqrt{g_\epsilon}} \rightharpoonup v'$ weakly in $L^{4/3}(\Lambda^1\Omega')$. Since $u_\epsilon = f_\epsilon U_\epsilon$, one easily checks that $ju_\epsilon = f_\epsilon^2\, jU_\epsilon$, and then it follows from (4.6) and the uniform convergence $f_\epsilon^2 \to \rho$ that $j = \rho v'$ in Ω', and hence that $v' = j/\rho =: v$ in Ω' and is independent of Ω'. It also follows that the chosen subsequence is independent of Ω'. Let moreover

$$\mu_\epsilon := \frac{1}{g_\epsilon}\left(\frac{1}{2}|dU_\epsilon|^2 + \frac{1}{\epsilon^2}W(u)\right)dx,$$

be the energy density of $\tilde{H}_\epsilon$, and notice also that $\int_{\Omega'}\mu_\epsilon$ is uniformly bounded, so that after passing to a subsequence, we may assume that there exist a measure μ_0 Ω' such that $\mu_\epsilon \rightharpoonup \mu_0$ weakly as measures in Ω'. It then follows from Theorem 2 and Remark

[3] That is, the Gross-Pitaevskii functional on $\mathbb{R}^3$ with a rather general trapping potential a and forcing term Φ_ϵ, considered in the function space $H^1_{a,m}$. The Gross-Pitaevskii integrand is written in a slightly different way in [27], but this is purely a cosmetic difference.

4 in [7], that, in the case $g_\epsilon \le C|\log\epsilon|^2$, $\mu_0 \ge \frac{1}{2}(|v|^2\,dx + |dv|)$, in the sense that $|dv|$ is a Radon measure, and

$$\mu_0(U) \ge \frac{1}{2}\int_U (|v|^2\,dx + |dv|)$$

for every open $U \subset \Omega'$, while for $|\log\epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$ we have $\mu_0 \ge \frac{1}{2}|v|^2\,dx$. In either case we deduce (using basic facts about weak convergence of measures) that

$$\liminf_{\epsilon\to 0} \frac{1}{g_\epsilon} H_\epsilon(U_\epsilon;\Omega') = \liminf_{\epsilon\to 0} \int_{\Omega'} f_\epsilon^2 \mu_\epsilon \ge \int_{\Omega'} \rho\mu_0\,, \tag{4.9}$$

which yields

$$\liminf_{\epsilon\to 0} \frac{1}{g_\epsilon} H_\epsilon(U_\epsilon;\Omega') \ge \frac{1}{2}\int_{\Omega'} \rho(|v|^2\,dx + |dv|) \qquad \text{if } g_\epsilon \le C|\log\epsilon|^2, \tag{4.10}$$

and

$$\liminf_{\epsilon\to 0} \frac{1}{g_\epsilon} H_\epsilon(U_\epsilon;\Omega') \ge \frac{1}{2}\int_{\Omega'} \rho|v|^2\,dx \qquad \text{if } |\log\epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}. \tag{4.11}$$

Since this holds for all $\Omega' \subset \Omega$, it follows in particular that $v \in L^2_\rho(\Lambda^1\Omega)$ and (in case $g_\epsilon \le C|\log\epsilon|^2$) that dv is a measure on all of Ω with $\int_\Omega \rho|dv| < \infty$, nearly completing the proof of (4.2). (We still need however to prove that j is supported in $\bar{\Omega}$.)

Step 5. We next claim that

$$\frac{1}{g_\epsilon}\int_{\mathbb{R}^3} f_\epsilon^2 \Phi_\epsilon \cdot jU_\epsilon \to \int_{\mathbb{R}^3} \rho\,\Phi\cdot v. \tag{4.12}$$

To prove (4.12), since $\Phi_\epsilon = \sqrt{g_\epsilon}\Phi$ and $\frac{1}{\sqrt{g_\epsilon}} ju_\epsilon = \frac{1}{\sqrt{g_\epsilon}} f_\epsilon^2\; jU_\epsilon \rightharpoonup j = \rho v$ weakly in $L^{4/3}(\Omega)$, it is clear that

$$\frac{1}{g_\epsilon}\int_\Omega f_\epsilon^2\Phi_\epsilon\cdot jU_\epsilon \to \int_\Omega \rho\,\Phi\cdot v,$$

and we only need to show that

$$\int_{\mathbb{R}^3\setminus\Omega} \Phi\cdot\frac{ju_\epsilon}{\sqrt{g_\epsilon}} \to 0.$$

Since $\rho = 0$ outside Ω, in this set we have

$$\begin{aligned}|j(u_\epsilon)| \le |u_\epsilon|\,|du_\epsilon| &\le \frac{1}{4\epsilon}|u_\epsilon|^4 + \frac{3\epsilon^{1/3}}{4}|du_\epsilon|^{4/3}\\ &= \frac{1}{4\epsilon}(|u_\epsilon|^2-\rho)^2 + \frac{3\epsilon^{1/3}}{4}|du_\epsilon|^{4/3},\end{aligned}$$

whence, for any compact $K \subset \mathbb{R}^3$, we see from (4.5) that

$$\frac{1}{\sqrt{g_\epsilon}}\int_{K\setminus\Omega} |ju_\epsilon| \le C\left(\epsilon\sqrt{g_\epsilon} + (\epsilon\sqrt{g_\epsilon})^{1/3}\right).$$

Thus $\frac{ju_\epsilon}{\sqrt{g_\epsilon}} \to 0$ in $L^1(\Lambda^1(K\backslash\Omega))$ for any compact K. This implies that $j = 0$ outside Ω, so that the identity $j = \rho v$ holds in all of $\mathbb{R}^3$, finally completing the proof of (4.2), and it also implies that

$$\frac{1}{g_\epsilon}\int_{K\backslash\Omega} \Phi_\epsilon \cdot ju_\epsilon = \int_{K\backslash\Omega} \Phi \cdot \frac{ju_\epsilon}{\sqrt{g_\epsilon}} \to 0$$

for K compact. Next, due to (1.22), (1.24) and the assumption that $|\Phi|^2 \le C(a+1)$, we can find a compact K such that $|\Phi|^2 \le Cw$ outside of K, so that (arguing as in Step 1)

$$|\Phi \cdot ju_\epsilon| \le \frac{\epsilon}{2}|du|^2 + \frac{w}{2\epsilon}|u|^2 \quad \text{outside of } K.$$

It follows from this and Step 1 that

$$\frac{1}{g_\epsilon}\int_{\mathbb{R}^3\backslash K} |\Phi_\epsilon \cdot ju_\epsilon| \ \le \ C\epsilon.$$

By combining these inequalities, we obtain the claim (4.12).

Step 6. We now complete the proof of the lower bound inequality. Note that, by combining (4.12) with (4.10) and recalling (4.7), we find that, in case $g_\epsilon \le C|\log\epsilon|^2$,

$$\liminf_{\epsilon\to 0} \frac{1}{|\log\epsilon|^2}(\mathcal{G}_\epsilon(u_\epsilon) - \mathcal{G}_\epsilon(f_\epsilon)) \ge \frac{1}{2}\int_{\Omega'} \rho(|v|^2 + |dv|) - \int_\Omega \rho\Phi\cdot v$$

for any open Ω' compactly contained in Ω. Taking the supremum over all such Ω', we obtain (4.3) with $\kappa_\epsilon = \mathcal{G}_\epsilon(f_\epsilon)$. Analogously, in case $|\log\epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$, using (4.11) in place of (4.10) we obtain the lower bound part in (4.4),

$$\liminf_{\epsilon\to 0} \frac{1}{g_\epsilon}(\mathcal{G}_\epsilon(u_\epsilon) - \mathcal{G}_\epsilon(f_\epsilon)) \ge \int_\Omega \rho\left(\frac{|v|^2}{2} - \Phi\cdot v\right).$$

Step 7. Let us prove the upper bound inequality in case $g_\epsilon \le C|\log\epsilon|^2$. The proof in the case $|\log\epsilon|^2 \ll g_\epsilon \ll \epsilon^{-2}$ follows the same lines and hence is omitted. We will use the following lemma.

Lemma 15. *Suppose that $v \in L^2_\rho(\Lambda^1\Omega)$ and that dv is a locally finite measure with $\int_\Omega \rho|dv| < \infty$. Then for every $\delta > 0$, there exists $v_\delta \in C^\infty_c(\Lambda^1\mathbb{R}^3)$ such that*

$$\int_\Omega \rho|v_\delta - v|^2 \ < \ \delta, \qquad \int_\Omega \rho|dv_\delta| \ \le \ \int_\Omega \rho|dv| + \delta. \tag{4.13}$$

The proof is given at the end of this section. Now we use the lemma to complete the proof of the theorem.

Fix v and v_δ as in the statement of the lemma.

It is proved in [27], Lem. B.1, that the (positive) function f_ϵ appearing in the decomposition (4.7) satisfies $\|f_\epsilon\|_{L^\infty(\mathbb{R}^3)} \le c_0$, for c_0 independent of $\epsilon \in (0,1]$, and moreover there exists $R > 0$ such that $\Omega \subset\subset B_R$, and $0 < f_\epsilon(x) \le Ce^{-R/\epsilon^{2/3}}$ whenever $|x| \ge R$.

Now for every $\delta > 0$, it follows from Theorem 2 and Remark 4 in [7] that there exists a sequence $U^\delta_\epsilon \in H^1(B_{R+1};\mathbb{C})$ such that $\frac{1}{|\log\epsilon|} jU^\delta_\epsilon \rightharpoonup v_\delta$ weakly in $L^{4/3}(B_{R+1})$, and

$$\frac{1}{|\log \epsilon|}\left(\frac{1}{2}|dU_\epsilon^\delta|^2 + \frac{1}{4\epsilon^2}(|U_\epsilon^\delta|^2 - 1)^2\right) \rightharpoonup \frac{1}{2}(|v_\delta|^2 + |dv_\delta|) \tag{4.14}$$

weakly as measures in B_{R+1}. Let $\epsilon' := c_0\epsilon$, and let $u_\epsilon^\delta := f_\epsilon \chi U_{\epsilon'}^\delta$, where $\chi \in C_c^\infty(B_{R+1})$ is a function such that $\chi \equiv 1$ on B_R and $|d\chi| \leq C$. Also, set $u_\epsilon^\delta := 0$ on $\mathbb{R}^3 \backslash B_{R+1}$. Then as in (4.7), and recalling that $\Phi_\epsilon = |\log \epsilon|\Phi$, we have

$$\frac{1}{|\log \epsilon|^2}(\mathcal{G}_\epsilon(u_\epsilon^\delta) - \mathcal{G}_\epsilon(f_\epsilon)) \;=\; \frac{1}{|\log \epsilon|^2} H_\epsilon(\chi U_{\epsilon'}^\delta) + \frac{1}{|\log \epsilon|^2}\int_{\mathbb{R}^3} \chi f_\epsilon^2 \Phi_\epsilon \cdot jU_{\epsilon'}^\delta.$$

The second term on the right-hand side converges to $\int_{\mathbb{R}^3} \chi\rho\Phi \cdot v_\delta = \int_{\mathbb{R}^3} \rho\Phi \cdot v_\delta$ as $\epsilon \to 0$. The proof of this statement is like that of (4.12), but easier. We break the other term into two pieces. The first is

$$\begin{aligned}\frac{1}{|\log \epsilon|^2} H_\epsilon(\chi U_{\epsilon'}^\delta; B_R) &= \frac{1}{|\log \epsilon|^2}\int_{B_R} f_\epsilon^2 \left(\frac{1}{2}|dU_{\epsilon'}^\delta|^2 + \frac{f_\epsilon^2}{4\epsilon^2}(|U_{\epsilon'}^\delta|^2 - 1)^2\right)\\ &\leq \frac{(1+o(1))}{|\log \epsilon'|^2}\int_{B_R} f_\epsilon^2\left(\frac{1}{2}|dU_{\epsilon'}^\delta|^2 + \frac{1}{4\epsilon'^2}(|U_{\epsilon'}^\delta|^2 - 1)^2\right).\end{aligned}$$

Then, since $\mathrm{supp}(\rho) \subset \bar{\Omega} \subset\subset B_R$, it follows from (4.14) and the uniform convergence $f_\epsilon^2 \to \rho$ that

$$\limsup_{\epsilon\to 0} \frac{1}{|\log \epsilon|^2} H_\epsilon(\chi U_{\epsilon'}^\delta; B_R) \leq \int \rho(|v_\delta|^2 + |dv_\delta|).$$

And from properties of χ and exponential smallness of f_ϵ outside of B_R, it easily follows that $H_\epsilon(\chi U_{\epsilon'}^\delta; \mathbb{R}^3\backslash B_R) \to 0$ as $\epsilon \to 0$. By combining the above inequalities, we find that

$$\limsup_{\epsilon\to 0} \frac{1}{|\log \epsilon|^2}(\mathcal{G}_\epsilon(u_\epsilon^\delta) - \mathcal{G}_\epsilon(f_\epsilon)) \;\leq \mathcal{G}(v_\delta) \leq \mathcal{G}(v) + C\delta.$$

Note also that

$$\frac{ju_\epsilon^\delta}{|\log \epsilon|} = (1+o(1))(f_\epsilon\,\chi)^2 \frac{jU_{\epsilon'}^\delta}{|\log \epsilon'|} \to \rho v_\delta \quad \text{weakly in } L^{4/3}(\mathbb{R}^3).$$

Conclusion (iii) now follows by setting $u_\epsilon := u_\epsilon^{\delta(\epsilon)}$ for $\delta(\epsilon)$ converging to 0 sufficiently slowly. □

We conclude this section with the proof of the approximation lemma used above.

Proof of Lemma 15. We introduce some auxiliary functions. First, for $r \in (0, 1]$, let

$$\Omega_r := \{x \in \mathbb{R}^3 : \mathrm{dist}(x, \Omega) < r\}, \qquad \Omega_{-r} := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > r\}.$$

Next, for sufficiently small $\sigma > 0$, let $\Psi_\sigma : \Omega_\sigma \to \Omega$ be the $W^{1,\infty}$ diffeomorphism given by

$$\Psi_\sigma(x) := \begin{cases} x - \sqrt{\sigma}(d(x) + \sqrt{\sigma} - \sigma)\bar{\nu}(x) & \text{if } x \in \Omega_\sigma \backslash \Omega_{\sigma-\sqrt{\sigma}} \\ x & \text{if } x \in \Omega_{\sigma-\sqrt{\sigma}} \end{cases},$$

where $\bar{\nu}(x)$ is the outer unit normal to $\partial\Omega$ at the point of $\partial\Omega$ closest to x, and $d(x)$ is the signed distance (positive outside Ω, negative in Ω) to $\partial\Omega$. Note that

$$\|D\Psi_\sigma - I\|_\infty \le C\sqrt{\sigma}, \qquad \|D\Psi_\sigma^{-1} - I\|_\infty \le C\sqrt{\sigma}, \tag{4.15}$$

where I denotes the identity matrix. In addition, $\rho(\Psi_\sigma(x)) \ge \rho(x)$ for all x, whenever σ is sufficiently small, since $D\rho(y) = -c(y)\nu(y)$ for $y \in \partial\Omega$, with $c(y) \ge c > 0$ for all y, by (1.27).

Next, let $\chi_\sigma \in C_c^\infty(\mathbb{R}^3)$ be a nonnegative function such that $\chi_\sigma = 1$ in $\Omega_{\sigma/4}$ and χ_σ has compact support in $\Omega_{3\sigma/4}$. Finally, for $\tau \in (0, 1]$ let η_τ be a smooth nonnegative even mollifier with support in $B(0, \tau)$ with $\int \eta_\tau = 1$.

We define $v_\delta := \eta_\tau * (\chi_\sigma \cdot \Psi_\sigma^\# v)$, where τ, σ will be fixed below. Here $\Psi_\sigma^\# v$ denotes the pullback of v by Ψ_σ, which is a one-form on Ω_σ, and the product $\chi_\sigma \cdot \Psi_\sigma^\# v$ is understood to equal zero on $\mathbb{R}^3\backslash\Omega_\sigma$.

Note that $\chi_\sigma \cdot \Psi_\sigma^\# v$ is integrable on $\mathbb{R}^3$, so that the convolution in the definition of v_δ makes sense. Indeed, (4.15) implies that $|\Psi_\sigma^\# v(x)| \le (1 + C\sqrt{\sigma})|v(\Psi_\sigma(x))|$ for all x, so that by a change of variables,

$$\int_{\mathbb{R}^3} |\chi_\sigma \cdot \Psi_\sigma^\# v| \le \int_{\Omega_{3\sigma/4}} |\Psi_\sigma^\# v| \le C \int_{\Psi_\sigma(\Omega_{3\sigma/4})} |v| \ < \ \infty.$$

The final estimate follows from $v \in L^2_\rho$, as well as the fact that ρ is bounded away from 0 in $\Psi_\sigma(\Omega_{3\sigma/4})$, since this set is compactly contained in Ω.

We will take $\tau < \sigma/4$, so that $v_\delta = \eta_\tau * \Psi_\sigma^\# v$ in Ω. Then

$$\int_\Omega \rho|v_\delta - v|^2 \le 2\int_\Omega \rho|\eta_\tau * \Psi_\sigma^\# v - \Psi_\sigma^\# v|^2 + 2\int_\Omega \rho|\Psi_\sigma^\# v - v|^2.$$

The definition of Ψ_σ implies that $\Psi_\sigma^\# v - v = 0$ if $\mathrm{dist}(x, \mathbb{R}^3\backslash\Omega) > \sigma$, and $|\Psi_\sigma^\# v - v| \le C|v|$, so that the second term on the right-hand side tends to 0 as $\sigma \to 0$, by the dominated convergence theorem, and can be made less than $\delta/2$ by choosing σ appropriately. Then we can clearly make the first term on the right-hand side less than $\delta/2$ by taking τ smaller if necessary.

To estimate $\int \rho|dv_\delta|$, we consider the action of dv_δ on some $\phi \in C_c^\infty(\Lambda^2\Omega)$. Exactly as in (3.14), we can rewrite

$$\int_\Omega \rho\,\phi \cdot dv_\delta = \int_{\Psi_\sigma(\Omega)} v \cdot d^* \star (\Psi_\sigma^{-1})^\#(\eta_\tau * (\rho \star \phi)).$$

Since $d^* \star (\Psi_\sigma^{-1})^\#(\eta_\tau * (\rho \star \phi))$ is smooth, and thus continuous,

$$\int_\Omega \rho\,\phi \cdot dv_\delta \ = \ \int_{\Psi_\sigma(\Omega)} dv \cdot \star(\Psi_\sigma^{-1})^\#(\eta_\tau * (\rho \star \phi)),$$

where the right-hand side indicates the integral of the continuous function $\star(\Psi_\sigma^{-1})^\#(\cdots)$ with respect to the measure dv. It follows that

$$\int_\Omega \rho\,\phi \cdot dv_\delta \le \sup_{x\in\Psi_\sigma(\Omega)} \left|\frac{1}{\rho}(\Psi_\sigma^{-1})^\#(\eta_\tau * (\rho \star \phi))(x)\right| \int_\Omega \rho|dv|.$$

And for $x \in \Psi_\sigma(\Omega)$, since $| \star \phi(x)| = |\phi(x)|$ for all x,

$$\left| \frac{1}{\rho} (\Psi_\sigma^{-1})^\# (\eta_\tau * (\rho \star \phi))(x) \right| \overset{(4.15)}{\leq} \frac{(1 + C\sqrt{\sigma})}{\rho(x)} (\eta_\tau * |\rho \star \phi|)(\Psi_\sigma^{-1}(x))$$
$$\leq \|\phi\|_\infty \frac{(1 + C\sqrt{\sigma})}{\rho(x)} (\eta_\tau * \rho)(\Psi_\sigma^{-1}(x))$$
$$\leq \|\phi\|_\infty \frac{(1 + C\sqrt{\sigma})}{\rho(x)} \left(\sup_{B_\tau(\Psi_\sigma^{-1}(x))} \rho \right). \tag{4.16}$$

And writing $y := \Psi_\sigma^{-1}(x)$, if σ is small enough then $D\rho \cdot \nu \leq -c < 0$ in the set where Ψ_σ is not the identity, so it follows from the mean value theorem and the definition of Ψ_σ that $\rho(\Psi_\sigma(y)) \geq \rho(y) + c\sqrt{\sigma}(d(y) + \sqrt{\sigma} - \sigma)^+$, where $(\cdots)^+$ denotes the positive part. Thus

$$\frac{1}{\rho(\Psi_\sigma(y))} \left(\sup_{B_\tau(y)} \rho \right) \leq \frac{\rho(y) + C\tau}{\rho(y) + c\sqrt{\sigma}(d(y) + \sqrt{\sigma} - \sigma)^+}.$$

We insist that $\sigma < 1/16$ (in addition to other smallness conditions), so that in $\Omega \setminus \Omega_{-\sigma}$, we have the inequality $d(y) + \sqrt{\sigma} - \sigma \geq \sqrt{\sigma} - 2\sigma \geq \frac{1}{2}\sqrt{\sigma}$. In this set, then,

$$\frac{\rho(y) + C\tau}{\rho(y) + c\sqrt{\sigma}(d(y) + \sqrt{\sigma} - \sigma)^+} \leq \frac{\rho(y) + C\tau}{\rho(y) + (c/2)\sigma} \leq 1 \quad \text{for } \tau \text{ sufficiently small.}$$

By taking τ small enough, we can make $C\tau/\rho(y)$ as small as we like in the set $\Omega_{-\sigma}$, where $\rho \geq c\sigma$. Thus by taking τ still smaller, if necessary, we can guarantee that the right-hand side of (4.16) is bounded by $(1 + \delta)\|\phi\|_\infty$. Inserting this into the above estimates, we conclude that

$$\int_\Omega \rho\phi \cdot dv_\delta \leq \|\phi\|_\infty (1 + \delta) \int_\Omega \rho |dv|$$

for all $\phi \in C_c^\infty(\Lambda^2\Omega)$, and hence that v_δ satisfies (4.13). □

4.2. A dual problem and critical forcing. In this section we give the proof of Theorem 7. We will use notation introduced in Sect. 1.2.

Proof of Theorem 7. Step 1. We first formulate a dual problem. Let

$$I_0(w) := -\int_\Omega \rho w \cdot \Phi + \frac{1}{2}\int_\Omega \rho |dw| \;\; = \;\; -(w, \Phi)_\rho + \frac{1}{2}\int_\Omega \rho |dw|$$

so that $\mathcal{G}(v) = I_0(v) + \frac{1}{2}\|v\|_\rho^2$. Then

$$I_0^*(v) := \sup_{w \in L^2_\rho} \big((v, w)_\rho - I_0(w)\big)$$
$$= \sup_{w \in L^2_\rho} \left((v + \Phi, w)_\rho - \frac{1}{2}\int_\Omega \rho |dw| \right).$$

It is clear that if $(v + \Phi, w)_\rho - \frac{1}{2}\int_\Omega \rho|dw| > 0$ for any w, then the supremum on the right-hand side above is unbounded, so we conclude that

$$I_0^*(v) = \begin{cases} 0 & \text{if } v + \Phi \in N \quad \text{(defined below)} \\ +\infty & \text{if not,} \end{cases}$$

for

$$N := \left\{ \xi \in L^2_\rho(\Lambda^1\Omega) : (\xi, w)_\rho \leq \frac{1}{2}\int_\Omega \rho|dw| \ \text{ for all } w \in L^2_\rho(\Lambda^1\Omega) \right\}. \tag{4.17}$$

Then it follows from basic facts about duality, see Lemma 8, that the unique minimizer v_0 of $\mathcal{G}$ in L^2_ρ is also the unique minimizer of

$$\mathcal{G}^\dagger(v) := I_0^*(-v) + \frac{1}{2}\|v\|^2_\rho = \begin{cases} \frac{1}{2}\|v\|^2_\rho & \text{if } \Phi - v \in N \\ +\infty & \text{if not.} \end{cases} \tag{4.18}$$

Step 2. We next rewrite the dual problem. It is immediate from the definition (4.17) of N and (1.30) that $N \subset (\ker d)^\perp_\rho$. Hence, writing P_ρ for L^2_ρ-orthogonal projection onto $(\ker d)_\rho$, it follows that

$$\Phi - v \in N \qquad \text{if and only if} \qquad P_\rho\Phi = P_\rho v \quad \text{and} \quad P^\perp_\rho\Phi - P^\perp_\rho v \in N,$$

In particular,

$$\frac{1}{2}\|v\|^2_\rho = \frac{1}{2}\|P_\rho\Phi\|^2_\rho + \frac{1}{2}\|P^\perp_\rho v\|^2 \quad \text{if } \Phi - v \in N$$

so that minimizing the L^2_ρ norm of v, subject to the constraint $\Phi - v \in N$, is equivalent to minimizing the L^2_ρ norm of $P^\perp_\rho v$, subject to the constraint $P^\perp_\rho\Phi - P^\perp_\rho v \in N$.

Now recall from the description (1.31) of $(\ker d)^\perp_\rho = \text{Image}(P^\perp_\rho)$ that every element of $(\ker d)^\perp_\rho$ can be written in the form $\frac{d^*\beta}{\rho}$ for some $d^*\beta \in H^1_N(\Lambda^2\Omega)$. In particular, if we write $P^\perp_\rho\Phi = \frac{d^*\beta_\Phi}{\rho}$ and $P^\perp_\rho v_0 = \frac{d^*\beta_0}{\rho}$, then

$$v_0 = P_\rho\Phi + \frac{d^*\beta_0}{\rho} \quad \text{where } \beta_0 \text{ minimizes}$$
$$\beta \mapsto \frac{1}{2}\|\frac{d^*\beta}{\rho}\|^2_\rho \text{ in } \left\{ \beta \in H^1_N(\Lambda^2\Omega) \ : \ \frac{d^*(\beta_\Phi - \beta)}{\rho} \in N \right\}. \tag{4.19}$$

As usual, we understand $\|\frac{d^*\beta}{\rho}\|_\rho$ to equal $+\infty$ if $\frac{d^*\beta}{\rho}$ does not belong to $L^2_\rho(\Lambda^1\Omega)$.

We now rewrite the constraint by noting that for any smooth $w \in C^\infty(\Lambda^1\bar{\Omega})$ and for $\beta \in H^1_N(\Lambda^2\Omega)$ such that $\frac{d^*\beta}{\rho} \in L^2_\rho(\Lambda^1\Omega)$,

$$(\frac{d^*(\beta_\Phi - \beta)}{\rho}, w)_\rho \ = \ \int_\Omega d^*(\beta_\Phi - \beta)\cdot w \ = \ \int_\Omega (\beta_\Phi - \beta)\cdot dw.$$

Here the boundary terms vanish due to the fact that $(\beta_\Phi)_N = \beta_N = 0$. Then the definition (4.17) of N and facts about density of smooth functions established in Lemma 15 imply that

$$\frac{d^*(\beta_\Phi - \beta)}{\rho} \in N \iff (\frac{d^*(\beta_\Phi - \beta)}{\rho}, w) \le \frac{1}{2}\int_\Omega \rho|dw| \quad \text{for all } w \in C^\infty(\Lambda^1\bar{\Omega})$$
$$\iff \int_\Omega (\beta_\Phi - \beta)\cdot dw \le \frac{1}{2}\int_\Omega \rho|dw| \quad \text{for all } w \in C^\infty(\Lambda^1\bar{\Omega})$$
$$\iff \|\beta_\Phi - \beta\|_{\rho*} \le \frac{1}{2} \tag{4.20}$$

where we recall the definition

$$\|\gamma\|_{\rho*} := \sup\{\int_\Omega \gamma\cdot dw \ : \ w \in C^\infty(\Lambda^1\bar{\Omega}),\ \int_\Omega \rho|dw| \le 1\}.$$

Now by combining (4.19) and (4.20), we obtain the characterization of v_0 appearing in Theorem 3, see (1.33), (1.34), (1.35).

Observe further that, by stationarity of (1.28) with respect to variations $t \mapsto e^t v_0$ around $t = 0$ we obtain

$$\frac{1}{2}\int_\Omega \rho|dv_0| + \int_\Omega \rho v_0\cdot(v_0 - \Phi) = 0. \tag{4.21}$$

Recalling that $\Phi = P_\rho\Phi + \frac{d^*\beta_\Phi}{\rho}$, we have $\rho(v_0 - \Phi) = d^*\beta_0 - d^*\beta_\Phi$. Inserting in (4.21) yields (1.36) after integration by parts.

Step 3. It remains to check that $dv_0 = 0$ if and only if $\|\beta_\Phi\|_{\rho*} \le \frac{1}{2}$.

The global minimizer of the functional $\beta \mapsto \frac{1}{2}\|\frac{d^*\beta}{\rho}\|^2_\rho$ in $H^1_N(\Lambda^2\Omega)$ is attained by $\beta = 0$, and this satisfies the constraint (4.20) if and only if $\|\beta_\Phi\|_{\rho*} \le \frac{1}{2}$.

Thus if $\|\beta_\Phi\|_{\rho*} \le \frac{1}{2}$, then $v_0 = P_\rho\Phi \in (\ker d)_\rho$, and in this case clearly $dv_0 = 0$.

On the other hand, if $\|\beta_\Phi\|_{\rho*} > \frac{1}{2}$, then $v_0 - P_\rho\Phi = \frac{d^*\beta_0}{\rho}$ is a (nonzero) element of $(\ker d)^\perp_\rho$, and hence in this case $0 \ne d(v_0 - P_\rho\Phi) = dv_0$. □

5. Further Remarks

5.1. Symmetry reduction. In the presence of rotational symmetry, the functionals we study in this paper reduce to simpler 2-dimensional models. We discuss this first for the functional $\mathcal{G}$, defined in (1.28), arising in case of Bose-Einstein condensates.

Lemma 16. *Consider the functional* $\mathcal{G}(v) = \int_\Omega \rho\left(\frac{|v|^2}{2} - v\cdot\Phi + \frac{1}{2}|dv|\right)$, *and assume that there exist some* $\tilde{\Omega} \subset [0,\infty)\times\mathbb{R}$ *and some* $\tilde{\rho} : \tilde{\Omega} \to (0,\infty)$ *such that*

$$\Omega = \{(r\cos\alpha, r\sin\alpha, z) : (r,z) \in \tilde{\Omega}, \alpha \in \mathbb{R}\},$$
$$\rho(r\cos\alpha, r\sin\alpha, z) = \tilde{\rho}(r,z) \quad \forall\alpha \in \mathbb{R}.$$

Assume moreover that there exists some $\phi : \tilde{\Omega} \to \mathbb{R}$ *such that*

$$\Phi(r\cos\alpha, r\sin\alpha, z) = \phi(r,z)d\theta \qquad \textit{for all } \alpha.$$

Then the unique minimizer v_0 of $\mathcal{G}$ is given in cylindrical coordinates by $v_0 = w_0(r,z)d\theta$, where w_0 minimizes the functional

$$\mathcal{G}^{red}(w) := \frac{1}{2}\int_{\tilde{\Omega}} \tilde{\rho}\left(|\nabla w| + \frac{(w-\phi)^2}{r}\right) dr\, dz \tag{5.1}$$

in the space of functions $w : \tilde{\Omega} \to \mathbb{R}$ such that $\int_{\tilde{\Omega}} \frac{\tilde{\rho}}{r} w^2\, dr\, dz < \infty$.

We set $\mathcal{G}^{red}(w) = +\infty$ if dw is not a Radon measure in $\tilde{\Omega}$ or if $r\tilde{\rho}$ is not $|dw|$-integrable.

As noted in the Introduction, $\mathcal{G}^{red}$ is exactly a (weighted) version of a functional that has been studied in the context of image denoising, see for example [29,10].

Proof. 1. Let $R_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ denote rotation by an angle α around the x_3 axis. Equivalently, in cylindrical coordinates, R_α is the map $(r,\theta,z) \mapsto (r,\theta+\alpha,z)$. Then our assumptions imply that

$$R_\alpha(\Omega) = \Omega, \qquad \rho \circ R_\alpha = \rho, \qquad R_\alpha^\# \Phi = \Phi$$

for all α. It easily follows that $\mathcal{G}(R_\alpha^\# v) = \mathcal{G}(v)$ for all v and α. By uniqueness of the minimizer v_0 of $\mathcal{G}$, which follows from strict convexity, we conclude that

$$v_0 \in \tilde{L}^2_\rho(\Lambda^1\Omega) := \left\{ v \in L^2_\rho(\Lambda^1\Omega) \ : \ R_\alpha^\# v = v \text{ for all } \alpha \right\}.$$

It is then immediate that v_0 minimizes $\mathcal{G}$ in $\tilde{L}^2_\rho(\Lambda^1\Omega)$.

2. Any $v \in \tilde{L}^2_\rho(\Lambda^1\Omega)$ can be written in polar coordinates as

$$v = v^\theta(r,z)d\theta + v^r(r,z)dr + v^z(r,z)dz. \tag{5.2}$$

We claim that for any such v,

$$\mathcal{G}(v) \geq \mathcal{G}(v^\theta d\theta) = 2\pi\mathcal{G}^{red}(v^\theta) - C(\Phi). \tag{5.3}$$

Clearly, this together with Step 1 implies the conclusion of the lemma. To prove (5.3), note that if v is smooth and has the form (5.2), then

$$|dv| = |\partial_r v^\theta dr \wedge d\theta + \partial_z v^\theta dz \wedge d\theta + (\partial_r v^z - \partial_z v^r) dr \wedge dz| \geq |d(v^\theta d\theta)|.$$

In the general case, the same conclusion follows from the density of smooth functions in $v \in L^2_\rho(\Lambda^1\Omega)$ in the sense of Lemma 15. It is also clear that if $v \in \tilde{L}^2_\rho(\Lambda^1\Omega)$ and $\Phi = \phi(r,z)d\theta$, then

$$|v-\Phi|^2 = \frac{(v^\theta-\phi)^2}{r^2} + (v^r)^2 + (v^z)^2 \geq \frac{(v^\theta-\phi)^2}{r^2} = |v^\theta d\theta - \Phi|^2$$

pointwise, so that

$$\int_\Omega \rho(\frac{|v|^2}{2} - v\cdot\Phi) = \int_\Omega \frac{\rho}{2}\left(|v-\Phi|^2 - |\Phi|^2\right) \geq \int_\Omega \rho(\frac{|v^\theta d\theta|^2}{2} - (v^\theta d\theta)\cdot\Phi).$$

Combining these estimates, we conclude that $\mathcal{G}(v) \geq \mathcal{G}(v^\theta d\theta)$ for all $v \in \tilde{L}^2_\rho(\Lambda^1\Omega)$.

Finally, the identity

$$\mathcal{G}(v^\theta d\theta) = 2\pi\, \mathcal{G}^{red}(v^\theta) - C(\Phi), \qquad C(\Phi) := \int_\Omega \frac{\rho}{2}|\Phi|^2$$

is clear if v^θ is smooth, and in the general case can be verified either by a density argument similar to the one given above, or by directly relating the definitions of the total variation measures associated with $d(v^\theta d\theta)$ in Ω and ∇v^θ in $\tilde{\Omega}$, respectively. □

Remark 16. As in Theorem 7, one can use duality to rewrite the problem of minimizing $\mathcal{G}^{red}$ as a constrained variational problem. For example, one can verify that v_0 minimizes $\mathcal{G}^{red}$ if and only if it minimizes the functional

$$w \mapsto \int_{\tilde{\Omega}} \frac{\tilde{\rho}}{r} w^2 \, dr\, dz \tag{5.4}$$

subject to the constraint

$$\int_{\tilde{\Omega}} \frac{\tilde{\rho}}{r}(\phi - w)\zeta \; dr\, dz \le \frac{1}{2}\int_{\tilde{\Omega}} \tilde{\rho}|\nabla\zeta| \qquad \text{for all } \zeta \in C^\infty(\tilde{\Omega}), \tag{5.5}$$

analogous to (4.18). One could also reformulate this as a problem of minimizing a weighted Dirichlet energy of a 1-form on $\tilde{\Omega}$ with a nonlocal constraint like that of (1.34), but in this setting this seems to us less natural, since the formulation in terms of functions rather than 1-forms seems simpler.

Remark 17. For velocity field represented by the 1-form $v = w(r,z)d\theta$, the associated vorticity 2-form is $dv = \partial_r w\, dr \wedge d\theta + \partial_z w\, dz \wedge d\theta$. The vorticity vector field, that is, the vector field dual to dv, is then $\frac{1}{r}(\partial_r w\, \hat{e}_z - \partial_z w\, \hat{e}_r)$, where $\hat{e}_z$ and $\hat{e}_r$ denote unit vectors in the (upward) vertical and (outward) radial directions respectively. It is natural to interpret integral curves of this vector field as "vortex curves". Since the vorticity vector field has no $\hat{e}_\theta$ component and is always tangent to level surfaces of w, we conclude that, formally, vortex curves have the form "$\theta =$ constant, $w =$ constant" (at least for regular values of w).

Thus in the reduced 2d model, we interpret level sets of a minimizer w_0, or more precisely sets of the form $\partial\{(r,z) : w_0(r,z) > t\}$, as representing vortex curves.

For similar reasons, one should think of the "vorticity measure" as being given by $\nabla^\perp w_0$, rather than ∇w_0.

Similarly, we have

Lemma 17. *Consider the functional*

$$\mathcal{F}(v, A) = \frac{1}{2}\int_\Omega |dv| + |v - A|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |d(A - A_{ex})|^2$$

and assume that there exist some $\tilde{\Omega} \subset [0,\infty) \times \mathbb{R}$ *and* $\phi : \tilde{\Omega} \to \mathbb{R}$ *such that*

$$\Omega = \{(r\cos\alpha, r\sin\alpha, z) : (r,z) \in \tilde{\Omega}, \alpha \in \mathbb{R}\},$$
$$A_{ex}(r\cos\alpha, r\sin\alpha, z) = \phi(r,z)d\theta \qquad \textit{for all } \alpha.$$

Then the unique minimizer (v_0, A_0) *of* $\mathcal{F}$ *is given in cylindrical coordinates by* $(v_0, A_0) = (w_0(r,z)d\theta, b_0(r,z)d\theta)$, *where* (w_0, b_0) *minimizes the functional*

$$\mathcal{F}^{red}(w, b) := \frac{1}{2}\int_{\tilde{\Omega}} |\nabla w| + \frac{(w-b)^2}{r}\, dr\, dz \;+\; \frac{1}{2}\int_{\tilde{\mathbb{R}}^2} \frac{|\nabla(b-\phi)|^2}{r}\, dr\, dz \quad (5.6)$$

in the space of $(w, b) : \tilde{\mathbb{R}}^2 \to \mathbb{R}^2$ *for which* $\mathcal{F}^{red}(w, b)$ *is well-defined and finite, where* $\tilde{\mathbb{R}}^2 := \{(r, z) : r > 0\}$.

We omit the proof, which is extremely similar to that of Lemma 16.

5.2. Contact curves and vortex curves. It is interesting to ask whether one can define a useful analog of the "contact set" (as normally defined for classical obstacle problems) for the variational problems with nonlocal constraints formulated in Theorems 2 and 7. We address this question first for Bose-Einstein condensates in the presence of rotational symmetry, as discussed immediately above. Thus, we assume that $w_0 : \tilde{\Omega} \to \mathbb{R}$ minimizes the functional (5.4) subject to the constraint (5.5). An approximation argument starting from (5.5) shows that if E is a set of locally finite perimeter in $\tilde{\Omega}$, then

$$\int \frac{\tilde{\rho}}{r}(\phi - w_0)\mathbf{1}_E\, dr\, dz \le \frac{1}{2}\int \tilde{\rho}|\nabla \mathbf{1}_E|. \quad (5.7)$$

We say that ∂E is a *contact curve* if equality holds in the above (where ∂E should be understood as the 1-dimensional set that carries $|\nabla \mathbf{1}_E|$).

Lemma 18. *For a.e.* t, $\partial\{w_0 > t\}$ *is a contact curve.*

As argued in Remark 17, it is natural to interpret $\partial\{w_0 > t\}$ as a "vortex curve", so the lemma states, heuristically, that every vortex curve for w_0 is also a contact curve.

Proof. By using rotational symmetry to reduce (1.36) to the (r, z) variables, or by using the fact that $0 = \frac{d}{dt}\mathcal{G}^{red}(e^t w_0)\big|_{t=0}$, we find that

$$\frac{1}{2}\int \tilde{\rho}|\nabla w_0| + \int \frac{\tilde{\rho}}{r}(w_0 - \phi)w_0\, dr\, dz = 0.$$

Using the coarea formula, we rewrite this as

$$\int_{-\infty}^{\infty}\left(\frac{1}{2}\int \tilde{\rho}|\nabla \mathbf{1}_{\{w_0>t\}}| \;+ \int \frac{\tilde{\rho}}{r}(w_0 - \phi)\mathbf{1}_{\{w_0>t\}}\, dr\, dz\right) dt = 0. \quad (5.8)$$

It follows from (5.7) that

$$\frac{1}{2}\int \tilde{\rho}|\nabla \mathbf{1}_{\{w_0>t\}}| \;+ \int \frac{\tilde{\rho}}{r}(w_0 - \phi)\mathbf{1}_{\{w_0>t\}}\, dr\, dz \ge 0$$

for every t, and then (5.8) implies that in fact equality holds for a.e. t. □

It is almost certainly not true that every contact curve for the minimizer w_0 is also a vortex curve, in the generality that we consider here, due to the possibility of degenerate (nonlocal) obstacles, as in the classical obstacle problem. One might hope, however, that the vortex curves and contact curves coincide under reasonable physical assumptions (for example, $\Phi = r^2 d\theta$, corresponding to rotation of a condensate around the z axis, probably also with some conditions on ρ.)

The situation is more complicated for Bose-Einstein condensates in a general domain $\Omega \subset \mathbb{R}^3$ without rotational symmetry, since in this case the analogs of vortex curves and contact curves may not in fact be curves and do not in general admit a very easy concrete characterization. Abstractly, they may be described as follows: if we write $\mathcal{Z}$ to denote the closure (in the sense of distributions) of

$$\{d\alpha : \alpha \in L^2(\Lambda^1\Omega), \int_\Omega \rho|d\alpha| \le 1\},$$

then one can think of the set extr $\mathcal{Z}$ of extreme points of (the convex set) $\mathcal{Z}$ as analogous to the objects — distributional boundaries of sets of finite weighted perimeter — used above to describe vortex and contact curves. Indeed, by arguments exactly like those of Remark 3 of [34], general convexity considerations and a bit of functional analysis imply that extr $\mathcal{Z}$ is a nonempty Borel subset of a metric space, and for any T in the vector space generated by $\mathcal{Z}$ (that is, the space $\cup_{\lambda>0}\lambda\mathcal{Z}$), there is a measure μ_T on extr $\mathcal{Z}$ such that

$$T = \int_{\mathrm{extr}\,\mathcal{Z}} \omega \, d\mu_T(\omega), \tag{5.9}$$

and in addition

$$\int_\Omega \rho \, d|T| \;=\; \int_{\mathrm{extr}\,\mathcal{Z}} \left(\int_\Omega \rho \, d|\omega| \right) \, d\mu_T(\omega). \tag{5.10}$$

We remark that in the closely related situation of divergence-free vector fields on $\mathbb{R}^n$, a concrete characterization of elements of the analog of extr $\mathcal{Z}$ as "elementary solenoids" is established in [34].

With this notation, an analog of Lemma 18 is

Lemma 19. *Let β_0 be the minimizer of the constrained variational problem (1.33), (1.34), so that $v_0 = P_\rho\Phi + \frac{d^*\beta_0}{\rho}$ is the minimizer of $\mathcal{G}(\cdot)$. Then*

$$\int_\Omega (\beta_\Phi - \beta_0)\cdot d\omega \;\le\; \frac{1}{2}\int_\Omega \rho d|\omega|, \tag{5.11}$$

for every $\omega \in \mathcal{Z}$. We say that $\omega \in$ extr$Z$ is a "generalized contact curve" if the above holds with equality.

Furthermore, let μ_{dv_0} denote a measure on extr $\mathcal{Z}$ *satisfying (5.9), (5.10) (with T replaced by dv_0). Then μ_{dv_0} a.e. ω is a generalized contact curve.*

The proof is exactly like that of Lemma 18, except that (5.9), (5.10) are substituted for the coarea formula. Then (5.11) follows immediately from the fact that β_0 satisfies (1.34), and the last assertion is a consequence of (1.36).

A version of Lemma 19 could be formulated for the functional $\mathcal{F}$ arising in the description of superconductivity and the associated constrained variational problem described in Theorem 2, using a measurable decomposition (5.9), (5.10) of the vorticity dv_0 to deduce from (1.12) a precise form of the assertion that every (generalized) vortex curve is a (generalized) contact curve.

It would presumably be rather easy to adapt results of [34] to the closely related situations considered here, to obtain concrete descriptions of extr $\mathcal{Z}$, or the corresponding objects relevant for superconductivity, although we are not sure that this would add much insight. It would also be interesting to know whether, if we consider the model case of uniform rotation about the z axis (for Bose-Einstein) or a constant applied magnetic field (for Ginzburg-Landau), the complexities sketched above do not in fact occur, and the vortex curves and contact curves for minimizers can in fact be identified with curves of finite length; this seems likely to us to be the case.

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