Utility Maximization in an Illiquid Market *

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Abstract

We consider a stochastic optimization problem of maximizing the expected utility from terminal wealth in an illiquid market. A discrete time model is constructed with few additional state variables. The dynamic programming approach is then developed and used for numerical studies. No-arbitrage conditions were also discussed.

Key words: Liquidity risk, limit order book, price impact, utility maximization, dynamic programming.
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1 Introduction

A classical Merton problem [9, 10] of a rational investor is to maximize her expected utility from final wealth

$$E\left[U\left(y+\int_0^T z_u dS_u\right)\right]$$

over all admissible trading strategies z with a given utility function U. The process z_t represents the shares of stocks held at time t, while S_t is the spot price. It is well known that the dynamic programming approach yields a nonlinear parabolic partial differential equation which admits explicit solutions for power, logarithmic or exponential utilities.

This paper studies the same problem but in an *illiquid* financial market. We adopt the modeling approach of Roch and Soner [11] developed for continuous time. In particular, price impact is included in our model. This impact, described in Section 2, is random and hence the liquidity risk is not simply deterministic. We start with an appropriate discretization of the model in Section 3. Then, the dynamic programming

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approach is developed. As well known in the literature, when not properly introduced, price impact may lead to arbitrage in the market. Especially in models with random impact, no-arbitrage is not immediately implied by the existence of a risk neutral measure for the price process with no-impact. We investigate this question and obtain conditions for no-arbitrage. In particular, we prove that the resulting value function is less than the Merton value when there is no initial price impact. This is the content of Theorem .1, below. However, when there exists initial liquidity premium in the observed stock price, then it is intuitively expected, that the investor may and does use this to achieve a value larger than the Merton one. This is clearly demonstrated in the numerical examples.

The paper is organized as follows. In Section 2, we briefly describe the liquidity risk model of Roch and Soner [11]. In Section 3, we explain the discrete version of that model. In Section 4, we define our stochastic optimization problem and show that the dynamic programming principle holds. Section 5 is dedicated to numerical results.

2 A liquidity risk model

In this section, we briefly introduce the model for liquidity risk proposed by Roch and Soner in [11]. Many existing price impact models can be seen as a particular or limiting case of it. Their starting point is the seminal paper of Kyle [8], where three important dimensions of illiquidity: depth, resilience and tightness, are identified. Depth is the size of the order flow required to change prices by one monetary unit, resilience is the degree to which prices recover from the price impact and tightness is the bid-ask spread.

Asset prices are frequently obtained through a limit order book (LOB). There are two types of trades: limit orders, i.e., orders to buy or sell a given amount of shares at a specific price and market orders which are executed against the limit orders. Limit orders are kept in the LOB until a market order comes in that matches to one of the existing limit orders. They provide liquidity by filling the LOB whereas the market orders deplete it. In [11] a trader is considered who only makes market orders. His trades have an impact on the prices.

We are given a trading horizon T and a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P})$ satisfying the usual conditions. The filtration $\mathscr{F}_t$ represents the information the trader has at time t. At every time $t \leq T$ he observes the LOB and knows the average price to pay for a transaction of size z via a market order. We represent the LOB at time t by the function $\rho_t^+(s)$, which denotes the density of the number of shares offered at some ask price s and by $\rho_t^-(s)$, which denotes the density of the number of shares offered at some bid price s.

Now suppose that a trader wants to buy (resp. sell) $z > 0$ number of shares at time t through a market order. We denote the ask price by a_t and the bid price by b_t. The total amount paid (resp. received) is then

$$\int_{a_t}^{s_z} s\rho_t^+(s)ds, \qquad (\text{resp. } \int_{s_z}^{b_t} s\rho_t^-(s)ds),$$

where s_z solves the equation

$$\int_{a_t}^{s_z} \rho_t^+(s)ds = z, \qquad (\text{resp. } \int_{s_z}^{b_t} \rho_t^-(s)ds = z).$$

A trade occurs when a market order is placed. Hence, the limit orders in the LOB are executed with the cheapest to the most expensive until the total number of shares ordered is reached. The optimal strategies typically consist of small trades, such that the price paid is never too far away from the best bid and ask price. A consequence is that one may assume that the LOB has a constant density outside the bid-ask spread at time t, i.e.

$$\rho_t^+(s) = \frac{1}{2m_t}\chi_{\{s\geq a_t\}} \qquad (\text{resp. } \rho_t^-(s) = \frac{1}{2m_t}\chi_{\{s\leq b_t\}}).$$

Here, m_t is an adapted càdlàg process and $\frac{1}{2m_t}$ is exactly the depth of the LOB. Under this assumption, when buying z number of shares we obtain $s_z = a_t + 2m_t z$ and the total amount paid is $a_t z + m_t z^2$. After buying z number of shares the best bid price b_t does not change, whereas the best ask price a_t moves to $a_t + 2m_t z$. So for prices between a_t and $a_t + 2m_t z$ the density of the LOB is zero. Elsewhere it remains unchanged.

We denote the equilibrium price process by S_t. It may not be observable and is the theoretical value of the stock. Furthermore, it is only observable in the long run and when the trader stops trading. In the model, also the mid-quote price process S_t^* is used which depends on the portfolio activity of the trader. In the long run and when the trader stops trading S_t^* converges to S_t. Also the bid and ask price processes, a_t and b_t converge to S_t^* when the impact on the trades vanishes and in the long run a_t and b_t converge to S_t. As either a_t or b_t may converge to S_t^* faster then the other process, S_t^* is typically not the average of a_t and b_t. Note that the mid-quote price process is important, as it is affected by the depth, whereas the bid-ask spread (tightness) is associated to the proportional cost. This will allow us to consider a reduced model, where the proportional transaction costs vanish.

We consider a simple trading strategy $z_t = z_0 + \sum_{k=0}^n \xi_k \chi_{\{\tau_k \leq t\}}$, where

$$0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_n \leq T$$

are stopping times and ξ_k is $\mathscr{F}_{\tau_k}$-measurable. If at time τ_k, we buy $\xi_k > 0$ number of shares, the ask price increases by $\Delta a_{\tau_k} = 2m_{\tau_k}\xi_k$. The increment of $S_{\tau_k}^*$ is $\Delta S_{\tau_k}^* = 2m_{\tau_k}\xi_k$. Using this simple strategy Roch and Soner postulate in [11] the dynamics of the difference of the mid-quote price and the equilibrium price process $l_t := S_t^* - S_t$

$$dl_t = -\kappa_t l_t dt + 2m_t dz_t,$$

where κ_t is the resilience. Let x_t be the position in the money market account at time t. We denote the liquidation value of the portfolio (after ignoring the bid-ask spread) by $Y_t = x_t + z_t(S_t^* - m_t z_t)$. Then, according to [11] its dynamics after ignoring the bid-ask spread is

$$dY_t = z_t(dS_t - \kappa_t l_t dt) - z_t^2 dm_t.$$

The case with infinite resilience corresponds to the model proposed by Çetin, Jarrow and Protter in [2]. Later it was studied in [3, 7]. Similar models are also used in the optimal execution problems. We refer the reader to [6, 12] and to the references therein.

3 Model assumptions in discrete time

We consider the discrete version of the liquidity risk model introduced in [11]. The financial market consists of a risky asset and a risk-free asset. The risk-free asset is taken to be a numeraire and for simplicity we set the interest rate $r = 0$. We are given a finite trading horizon T and divide the trading period $[0,T]$ into $N \in \mathbb{N}$ equal intervals of length $h = \frac{T}{N}$, such that the agents trade at the times $t_n = nh$, for $n = \{0,1,\ldots,N\}$. Furthermore, we have a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where the sample space Ω is given by

$$\Omega = \{(\xi_1,\ldots,\xi_N):\ \xi_k \in \{+1,-1\},\ \forall 1 \le k \le N\}.$$

For the σ-algebra we take $\mathscr{F} = 2^{\Omega}$. The random variables $\{\xi_k\}_{k=1}^N$ are i.i.d and satisfy

$$\mathbb{P}[\xi_k = \pm 1] = \frac{1}{2}, \quad \forall 1 \le k \le N.$$

We denote the equilibrium stock price process by $S = \{S_n\}_{n=1}^N$ meaning $S_n := S_{t_n}$ and choose the binomial model for it. Thus, we have

$$S_{n+1} = S_n(1 + \mu h + \sigma\sqrt{h}\xi_{n+1}),$$

for all $n \in \{0,\ldots,N-1\}$ with $S_0 = s > 0$. We take the filtration $\mathbb{F} = \{\mathscr{F}_n\}_{n=1}^N$ to be the σ-algebra generated by S, i.e. $\mathscr{F}_n = \sigma(S_1,\ldots,S_n)$ for $n \in \{1,\ldots,N\}$ and $\mathscr{F}_0 = \{\emptyset, \Omega\}$. By $z = \{z_n\}_{n=1}^N$ we denote the portfolio process which is adapted to the filtration $\mathbb{F}$. This choice makes the dynamic programming principle easier to state. Furthermore, we have the mid-quote price process which we denote by $S^* = \{S_n^*\}_{n=1}^N$.

Following the continuous time model proposed in [11] we postulate that S^* converges to S geometrically in the absence of trading. Hence,

$$S_{n+1}^* - S_n^* = -\kappa h(S_n^* - S_n) + (S_{n+1} - S_n) + 2m_{n+1}(z_{n+1} - z_n), \tag{3.1}$$

where $\kappa > 0$ is the resilience parameter, which is assumed to be a constant for simplicity, and $1/2m_n$ is the depth. For $n \in \{0,\ldots,N\}$, set

$$l_n := S_n^* - S_n,$$

so that in view of (3.1) it solves the equation

$$l_{n+1} = (1 - \kappa h)l_n + 2m_{n+1}(z_{n+1} - z_n).$$

Let $Y = \{Y_n\}_{n=1}^N$ be the liquidation value after ignoring the bid-ask spread,

$$Y_n := x_n + (S_n^* - m_n z_n)z_n,$$

where x_n is the position in the money market. Then, Y solves

$$Y_{n+1} - Y_n = x_{n+1} - x_n + (S_{n+1}^* - m_{n+1}z_{n+1})z_{n+1} - (S_n^* - m_n z_n)z_n,$$

where $x_{n+1} - x_n$ is the money needed to purchase $z_{n+1} - z_n$ number of shares at the time step $n+1$. The average price paid for $z_{n+1} - z_n$ number of shares is

$$\begin{aligned} S_n^* + m_{n+1}(z_{n+1} - z_n) &= S_{n+1}^* - m_{n+1}(z_{n+1} - z_n) \\ &= S_{n+1} + (1-\kappa h)l_n + m_{n+1}(z_{n+1} - z_n). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} Y_{n+1} - Y_n = &-S_{n+1}z_{n+1} + S_{n+1}z_n - l_n(1-\kappa h)z_{n+1} + l_n(1-\kappa h)z_n \\ &- m_{n+1}(z_{n+1} - z_n)^2 + l_{n+1}z_{n+1} + S_{n+1}z_{n+1} - m_{n+1}z_{n+1}^2 - l_n z_n \\ &- S_n z_n + m_n z_n^2 \\ = &(S_{n+1} - S_n)z_n + (l_{n+1} - l_n)z_{n+1} + \kappa h l_n z_{n+1} - \kappa h l_n z_n \\ &- m_{n+1}(z_{n+1} - z_n)^2 - m_{n+1}z_{n+1}^2 + m_n z_n^2 \\ = &(S_{n+1} - S_n)z_n - \kappa h l_n z_n - (m_{n+1} - m_n)z_n^2. \end{aligned}$$

We sum the above to arrive at

$$Y_r = Y_0 + \sum_{n=0}^{r-1} \Big[(S_{n+1} - S_n)z_n - \kappa h l_n z_n - (m_{n+1} - m_n)z_n^2 \Big].$$

The difference between the value $Y_0 + \sum_{n=0}^{r-1}(S_{n+1} - S_n)z_n$ that one obtains in the frictionless market and the liquidation value above is given by,

$$L_r := \sum_{n=0}^{r-1} \kappa h l_n z_n + (m_{n+1} - m_n)z_n^2.$$

Hence, L_r is the liquidity cost of the LOB. The following result is similar to Theorem .2 in [11] and is central to our no-arbitrage analysis.

Lemma .1 *Let $\eta_n = l_n - 2m_n z_n$ and assume that $\eta_0 = 0$. If m_n is a non-negative constant and $\kappa < 2/h$, then $L_r \geq 0$ for all $r \geq 1$. In general, if*

$$\phi_n := \frac{\Lambda_n^2}{m_n}, \qquad \textit{with} \qquad \Lambda_n = (1-\kappa h)^n$$

is a supermartingale, then $E[L_r] \geq 0$. Furthermore, L_r has the representation

$$L_r = \frac{1}{4}\Big(\frac{\eta_r^2}{m_r} - \frac{\eta_0^2}{m_0}\Big) - \frac{1}{4}\sum_{n=0}^{r-1} l_n^2 \Lambda_n^{-2}(\phi_{n+1} - \phi_n). \tag{3.2}$$

Proof. We first consider the case $m_n = M > 0$. The definition of η implies that

$$z_n = \frac{l_n - \eta_n}{2M}$$

and η solves

$$\eta_n - \eta_{n+1} = l_n - l_{n+1} + 2M(z_{n+1} - z_n) = \kappa h l_n.$$

This leads to

$$L_r = \sum_{n=0}^{r-1} \frac{(\eta_n - \eta_{n+1})(l_n - \eta_n)}{2M} = \sum_{n=0}^{r-1} \frac{\eta_{n+1}^2 - \eta_n^2 - (\eta_n - \eta_{n+1})^2 + 2l_n(\eta_n - \eta_{n+1})}{4M}$$
$$= \sum_{n=0}^{r-1} \frac{(\eta_{n+1}^2 - \eta_n^2 + (2\kappa h - (\kappa h)^2)l_n^2}{4M} = \frac{1}{4M}(\eta_r^2 - \eta_0^2 + \sum_{n=0}^{r-1}(2\kappa h - (\kappa h)^2)l_n^2 \geq 0,$$

provided that $0 < \kappa < \frac{2}{h}$. Next, we consider the general case. We use the notation $\Delta m_{n+1} = m_{n+1} - m_n$ and $\Delta\frac{1}{m_{n+1}} = \frac{1}{m_{n+1}} - \frac{1}{m_n}$. Recall the definition of η to compute,

$$\begin{aligned}
\kappa h l_n z_n &= \frac{1}{2m_n}(\eta_n - \eta_{n+1})(l_n - \eta_n) - 2z_n^2(\Delta m_{n+1}) \\
&= \frac{1}{4m_n}[\eta_{n+1}^2 - \eta_n^2 - (\eta_n - \eta_{n+1})^2 + 2l_n(\eta_n - \eta_{n+1})] - 2z_n^2(\Delta m_{n+1}) \\
&= \frac{1}{4m_n}\Big[\eta_{n+1}^2 - \eta_n^2 - (\kappa h l_n + 2z_n(\Delta m_{n+1}))^2 + 2l_n(\kappa h l_n + 2z_n(\Delta m_{n+1}))\Big] \\
&\quad - 2z_n^2(\Delta m_{n+1}) \\
&= \frac{\eta_{n+1}^2}{4m_{n+1}} - \frac{\eta_n^2}{4m_n} - \frac{\eta_n^2}{4}\Big(\Delta\frac{1}{m_{n+1}}\Big) - \frac{\eta_{n+1}^2 - \eta_n^2}{4}\Big(\Delta\frac{1}{m_{n+1}}\Big) - 2z_n^2(\Delta m_{n+1}) \\
&\quad + (2\kappa h - (\kappa h)^2)\frac{l_n^2}{4m_n} + (1 - \kappa h)\frac{z_n l_n}{m_n}(\Delta m_{n+1}) - \frac{z_n^2}{m_n}(\Delta m_{n+1})^2 \\
&= \frac{\eta_{n+1}^2}{4m_{n+1}} - \frac{\eta_n^2}{4m_n} + (2\kappa h - (\kappa h)^2)\frac{l_n^2}{4m_n} - \frac{l_n^2}{4}\Big(\Delta\frac{1}{m_{n+1}}\Big) \\
&\quad + \Big(m_n z_n l_n - m_n^2 z_n^2 - \frac{\eta_{n+1}^2 - \eta_n^2}{4}\Big)\Big(\Delta\frac{1}{m_{n+1}}\Big) - 2z_n^2(\Delta m_{n+1}) && (3.3) \\
&\quad + (1 - \kappa h)\frac{z_n l_n}{m_n}(\Delta m_{n+1}) - \frac{z_n^2}{m_n}(\Delta m_{n+1})^2. && (3.4)
\end{aligned}$$

Then, a tedious but otherwise direct computation shows that

$$\begin{aligned}
(3.3) + (3.4) &= \Big(m_n z_n l_n - m_n^2 z_n^2 - \frac{(\eta_{n+1} - \eta_n)^2 + 2(l_n - 2m_n z_n)(\eta_{n+1} - \eta_n)}{4}\Big)\Big(\Delta\frac{1}{m_{n+1}}\Big) \\
&\quad + (1 - \kappa h)\frac{z_n l_n}{m_n}(\Delta m_{n+1}) - \frac{z_n^2}{m_n}(\Delta m_{n+1})^2 - 2z_n^2(\Delta m_{n+1}) \\
&= \frac{1}{4}(2\kappa h - (\kappa h)^2)l_n^2\Big(\Delta\frac{1}{m_{n+1}}\Big) + \Big[m_n z_n l_n - m_n^2 z_n^2 - z_n^2(\Delta m_{n+1})^2 \\
&\quad + (1 - \kappa h)z_n l_n(\Delta m_{n+1}) - \kappa h l_n m_n z_n - 2m_n z_n^2(\Delta m_{n+1})\Big]\Big(\Delta\frac{1}{m_{n+1}}\Big) \\
&\quad + (1 - \kappa h)\frac{z_n l_n}{m_n}(\Delta m_{n+1}) - \frac{z_n^2}{m_n}(\Delta m_{n+1})^2 - 2z_n^2(\Delta m_{n+1}) \\
&= \frac{1}{4}(2\kappa h - (\kappa h)^2)l_n^2\Big(\Delta\frac{1}{m_{n+1}}\Big) - z_n^2(\Delta m_{n+1}).
\end{aligned}$$

Hence,

$$\kappa h l_n z_n = \frac{1}{4}\Big(\frac{\eta_{n+1}^2}{m_{n+1}} - \frac{\eta_n^2}{m_n}\Big) - \frac{l_n^2}{4}\Big(\Delta\frac{1}{m_{n+1}}\Big) + (2\kappa h - (\kappa h)^2)\frac{l_n^2}{4m_{n+1}} - z_n^2(\Delta m_{n+1}).$$

Summing up the liquidity costs leads to

$$\begin{aligned} L_r &= \sum_{n=0}^{r-1}\Big[\kappa h l_n z_n + (m_{n+1} - m_n) z_n^2\Big] \\ &= \frac{1}{4}\Big(\frac{\eta_r^2}{m_r} - \frac{\eta_0^2}{m_0}\Big) - \frac{1}{4}\sum_{n=0}^{r-1} l_n^2\Big[\Big(\frac{1}{m_{n+1}} - \frac{1}{m_n}\Big) + \Big((\kappa h)^2 - 2\kappa h\Big)\frac{1}{m_{n+1}}\Big]. \end{aligned} \tag{3.5}$$

Recall that $\phi_n = \Lambda_n^2/m_n$ with $\Lambda_n = \big(1-\kappa h\big)^n$. Hence,

$$\phi_{n+1} - \phi_n = \Lambda_n^2\Big(\frac{1}{m_{n+1}} - \frac{1}{m_n}\Big) + (\Lambda_{n+1}^2 - \Lambda_n^2)\frac{1}{m_{n+1}}$$

and

$$\Lambda_n^{-2}(\phi_{n+1} - \phi_n) = \Big(\frac{1}{m_{n+1}} - \frac{1}{m_n}\Big) + ((\kappa h)^2 - 2\kappa h)\frac{1}{m_{n+1}}.$$

So equation (3.5) can be written as (3.2). Thus, we see that L_r is non-negative in expectation provided that ϕ_n is a supermartingale. □

Remark .1 *Suppose $m_n = MS_n$, where M is a non-negative constant. For h sufficiently small ϕ_n is a supermartingale under $\mathbb{P}$ (resp. under the equivalent martingale measure $\mathbb{Q}$) only if*

$$\kappa > \frac{\sigma^2 - \mu}{2} \qquad (\textit{resp.}\ \kappa > \frac{\sigma^2}{2}).$$

Indeed, we directly compute the expected value under the measure $\mathbb{P}$ to obtain

$$E\Big[\frac{\phi_{n+1}}{\phi_n}\Big|\mathscr{F}_n\Big] = E\Big[(1-\kappa h)^2\frac{S_n}{S_{n+1}}\Big|\mathscr{F}_n\Big] \quad = (1-\kappa h)^2\left(\frac{1+\mu h}{1+2\mu h - \sigma^2 h + \mu^2 h^2}\right).$$

For h sufficiently small, ϕ_n is a supermartingale provided that

$$(1-2\kappa h)\Big(\frac{1+\mu h}{1+2\mu h - \sigma^2 h}\Big) < 1,$$

which leads to $\kappa > \frac{\sigma^2-\mu}{2}$. Similarly, under the measure $\mathbb{Q}$ we obtain $\kappa > \frac{\sigma^2}{2}$. □

In [11], it is shown that the above result implies that this model is arbitrage free. From the classical no-arbitrage theory we know that if there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the stock price process S is a martingale, then there are no arbitrage opportunities. Here in the liquidity risk setup, the additional supermartingale condition on the process ϕ_n must also hold under the equivalent measure $\mathbb{Q}$. Having an arbitrage free model allows us to consider the expected utility from final wealth problem. This will be discussed in the next section.

4 Expected utility from terminal wealth

In this section, we study the optimal stochastic problem of maximizing the expected utility from terminal wealth. From Remark .1 it follows that $m_n = MS_n$, with $M > 0$ is a good choice for the depth parameter. We first formulate our stochastic optimization problem. We consider a dynamic system

$$X := (S_n, l_n, Y_n, z_n) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R},$$

which is characterized by its state at any time step $n \in \{0, 1, \ldots, N\}$. It evolves in an uncertain environment and solves the equations

$$\begin{aligned} S_{n+1} &= S_n(1 + \mu h + \sigma\sqrt{h}\xi_{n+1}) \\ l_{n+1} &= (1 - \kappa h)l_n + 2MS_{n+1}(z_{n+1} - z_n) \\ Y_{n+1} &= Y_n + (S_{n+1} - S_n)z_n - \kappa h l_n z_n - M(S_{n+1} - S_n)z_n^2 \\ z_{n+1} &= z_n + (z_{n+1} - z_n), \end{aligned} \tag{4.1}$$

with initial states $S_k^{k,s} = s, l_k^{k,s,l,\Delta z} = l, Y_k^{k,s,l,y,z,\Delta z} = y$ and $z_k^{k,z,\Delta z} = z$ at the time step k. To fully represent our stochastic control problem the process z is needed as a state variable, whereas $\Delta z = \{\Delta z_n\}_{n=k+1}^{N-1}$ with $\Delta z_n = z_n - z_{n-1}$ is the control variable. We see in equation (4.1) that the process l is influenced by the control Δz. The set of admissible controls will be denoted by $\mathscr{A}$. Then the objective is to maximize over all admissible controls a cost functional $J(k, \tilde{x}, \Delta z)$ which is defined below.

For a given utility function U our objective functional has the form

$$J(k, \tilde{x}, \Delta z) = E[U(Y_N^{k,\tilde{x},\Delta z})],$$

where $\tilde{x} = (s, l, y, z)$ is the initial condition at the time step k. The value function is then defined to be the maximum value

$$v(k, \tilde{x}) = \sup_{\Delta z \in \tilde{\mathscr{A}}} J(k, \tilde{x}, \Delta z),$$

where $\tilde{\mathscr{A}} = \mathscr{A}_k(\tilde{x}) = \{\Delta z \text{ s.t. } Y_t^{k,\tilde{x},\Delta z} \geq 0 \; \forall t \in \{k, \ldots, N\} \text{ a.s.}\}$ is the class of admissible controls.

Let v^M be the corresponding Merton value function of the liquid market. The no-arbitrage consideration implies that the v^M should dominate v for initial values in which there is no liquidity premium, i.e., when $\eta = 0$ or equivalently when $l = 2Msz$. The following result proves this.

Theorem .1 *Suppose that ϕ_n is a $\mathbb{Q}$-supermartingale. Then, for all $\tilde{x} = (s, l, y, z)$ such that $l = 2Msz$, we have*

$$v(k, \tilde{x}) \leq v^M(k, y).$$

Proof. We use (3.2) to rewrite the dynamics of Y as follows,

$$\begin{aligned} Y_N &= Y_{N-1} + (S_N - S_{N-1})z_{N-1} + (L_N - L_{N-1}) \\ &= Y_{N-1} + (S_N - S_{N-1})z_{N-1} - \frac{1}{4M}\Big(\frac{\eta_N^2}{S_N} - \frac{\eta_{N-1}^2}{S_{N-1}}\Big) + \frac{1}{4} l_{N-1}^2 \Lambda_{N-1}^{-2}(\phi_N - \phi_{N-1}). \end{aligned}$$

By Doob's decomposition theorem, $\phi_N = \tilde{S}_N + A_N$, where $\tilde{S}$ is a $\mathbb{Q}$-martingale and A is a predictable, decreasing process with $A_0 = 0$. Hence, there is an adapted process $\tilde{\alpha}$ such that $\phi_N - \phi_{N-1} = \tilde{\alpha}_{N-1}(S_N - S_{N-1}) + A_N - A_{N-1}$. Set $\alpha_n = \frac{1}{4} l_n^2 \Lambda_n^{-2} \tilde{\alpha}_n$ to compute,

$$\frac{1}{4} l_{N-1}^2 \Lambda_{N-1}^{-2} (\phi_N - \phi_{N-1}) = \alpha_{N-1}(S_N - S_{N-1}) + \frac{1}{4} l_{N-1}^2 \Lambda_{N-1}^{-2} (A_N - A_{N-1}).$$

Then, we directly estimate that

$$\begin{aligned}
E[U(Y_N^{k,\tilde{x},\Delta z})] &= E\Big[E\Big[U\Big(Y_{N-1} + (S_N - S_{N-1})(z_{N-1} + \alpha_{N-1}) \\
&\quad - \frac{1}{4M}\Big(\frac{\eta_N^2}{S_N} - \frac{\eta_{N-1}^2}{S_{N-1}}\Big) + \frac{1}{4} l_{N-1}^2 \Lambda_{N-1}^{-2}(A_N - A_{N-1})\Big)\Big|\mathscr{F}_{N-1}\Big]\Big] \\
&\le E\Big[E\Big[U\Big(Y_{N-1} + (S_N - S_{N-1})(z_{N-1} + \alpha_{N-1}) \\
&\quad - \frac{1}{4M}\Big(\frac{\eta_N^2}{S_N} - \frac{\eta_{N-1}^2}{S_{N-1}}\Big)\Big)\Big|\mathscr{F}_{N-1}\Big]\Big] \\
&= \cdots \\
&\le E\Big[U\Big(y + \sum_{n=k}^{N-1} (S_{n+1} - S_n)(z_n + \alpha_n) + \frac{1}{4M}\frac{\eta_k^2}{S_k}\Big)\Big].
\end{aligned}$$

In the last step, we use the assumption that $\eta_k = l - 2Msz = 0$. Since the supremum over z is the same as the supremum over $z + \alpha$, we obtain the claimed inequality.

□

4.1 Reduction of one state variable

Our stochastic control problem has four state variables and one control variable. Clearly, less state variables would be desirable especially from the numerical point of view. The idea of this reduction is to use z itself and not its increments as the control process. However, the dynamics of l contains the increments. So as before we introduce the variable $\eta_n = l_n - 2MS_n z_n$. Then, its dynamics are given by

$$\eta_{n+1} - \eta_n = -\kappa h(\eta_n + 2MS_n z_n) - 2Mz_n(S_{n+1} - S_n). \tag{4.2}$$

Then, we rewrite the dynamics of Y in terms of η and obtain

$$Y_{n+1} - Y_n = (S_{n+1} - S_n)z_n - \kappa h \eta_n z_n - 2M\kappa h S_n z_n^2 - M(S_{n+1} - S_n)z_n^2. \tag{4.3}$$

The difference equation for S remains the same since the process l does not influence S. We now notice that the state equations (4.2), (4.3) and the equation of S for the variables (S, η, Y) depend only on themselves and z but not on the increment Δz. Hence, we can define a new control problem with three state variables (S, η, Y) and one control variable $z = \{z_n\}_{n=k+1}^{N-1}$. Let $x := (s, \eta, y)$ be the initial condition and define the corresponding value function by,

$$V(k,x) = \sup_{z \in \mathscr{A}} J(k,x,z), \tag{4.4}$$

where $\mathscr{A} = \mathscr{A}_k(x) = \{z \text{ s.t. } Y_t^{k,x,z} \geq 0 \ \forall\ t \in \{k,\dots,N\} \text{ a.s.}\}$. We have the following relation between the two value functions v and V

$$V(k,x) = V(k,s,\eta,y) = \sup_{\bar{z}\in\mathbb{R}} v(k,s,\eta + 2Ms\bar{z},y,\bar{z}) = \sup_{\bar{z}\in\mathbb{R}} v(k,\tilde{x}).$$

Since this fact is not used in this paper, we do not provide its and refer to [13].

In this reduced setup, we continue with a of the dynamic programming principle (DPP). We first introduce the following notation. We denote the controlled process by $X_n^{k,x,z} := (S_n^{k,s}, \eta_n^{k,s,\eta,z}, Y_n^{k,s,\eta,y,z})$ and the initial condition by $x := (s,\eta,y)$. Furthermore, we use the following abbreviations $u := \mu h + \sigma\sqrt{h}$ and $d := \mu h - \sigma\sqrt{h}$.

Theorem .2 *The value function $V(k,x)$ is continuous in $x \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ =: Q$ and for every $k \in \{0,1,\dots,N-1\}$ the dynamic programming principle holds ,*

$$V(k,x) = \sup_{z\in\mathscr{I}(x)} E[V(k+1,X_{k+1}^{k,x,z})],$$

where

$$\mathscr{I}(x) = \{z \in \mathbb{R} :\ y + (us - \kappa h\eta)z - (2Ms\kappa h + Mus)z^2 \geq 0 \textit{ and}$$
$$y + (ds - \kappa h\eta)z - (2Ms\kappa h + Mds)z^2 \geq 0\}.$$

Proof. In view of (4.3), the definition of $\mathscr{I}(x)$ implies that $Y_{k+1}^{k,x,z} \geq 0$ for every $z \in \mathscr{I}(x)$ with probability one (i.e, for two possible values). We first show that $\mathscr{I}(x)$ is a bounded set. Indeed, set

$$\begin{aligned} y_{up}(z) &:= y + (us - \kappa h\eta)z - (2Ms\kappa h + Mus)z^2, \text{if } \xi_N = 1 \\ y_{down}(z) &:= y + (ds - \kappa h\eta)z - (2Ms\kappa h + Mds)z^2, \text{if } \xi_N = -1. \end{aligned}$$

Then, the zeroes of $y_{up}(z)$ are

$$z_{1,2}^{up} = \frac{(us - \kappa h\eta) \mp \sqrt{(us - \kappa h\eta)^2 + 4(2Ms\kappa h + Mus)y}}{2(2Ms\kappa h + Mus)}$$

and those of $y_{down}(z)$ are

$$z_{1,2}^{down} = \frac{(ds - \kappa h\eta) \mp \sqrt{(ds - \kappa h\eta)^2 + 4(2M\kappa h + Mds)y}}{2(2M\kappa sh + Mds)}.$$

Furthermore, the coefficient of z^2 of the function $y_{up}(z)$ is always non-positive, whereas the coefficient of z^2 of the function $y_{down}(z)$ is non-negative, whenever $\kappa \leq -\frac{d}{2h}$ and non-positive otherwise. So if $z \in [z_1^{up}, z_2^{up}] =: I_1$, then $y_{up}(z) \geq 0$. Whenever $\kappa \leq -\frac{d}{2h}$ we have $y_{down}(z) \geq 0$ if $z \in (-\infty, z_1^{down}] \cup [z_2^{down}, \infty) =: I_2$, otherwise we have $y_{down}(z) \geq 0$ if $z \in [z_1^{down}, z_2^{down}] := \tilde{I}_2$. Thus, we see that $y_{up}(z), y_{down}(z) \geq 0$ if $z \in I_3 := I_1 \cap I_2$ or $z \in \tilde{I}_3 := I_1 \cap \tilde{I}_2$, Note that I_3 resp. $\tilde{I}_3$ is not empty as it contains $z = 0$.

We prove the statement by induction. So we observe that at the final step N, the value function is continuous as $V(N,x) = U(y)$. Also, the dynamic programming principle holds trivially at the time step N. Indeed, it is the definition of $V(N-1,x)$.

Next, we show that the value function $V(N-1,\cdot)$ is continuous in Q at the time step $N-1$. Let $x^* := (s^*,\eta^*,y^*)$ be the limit of an arbitrary sequence $x_k = (s_k,\eta_k,y_k)$. Then take an ε-optimal control $z^\varepsilon \in \mathrm{int}(\mathscr{I}(x^*))$, where $\varepsilon > 0$. Hence, there exists a K^* such that for all $k \geq K^*$ we have $z^\varepsilon \in \mathscr{I}(x_k)$. Furthermore, by continuity of U and $Y_N^{N-1,x,z}$ we have

$$U(Y_N^{N-1,x^*,z^\varepsilon}) = \lim_{k\to\infty} U(Y_N^{N-1,x_k,z^\varepsilon}).$$

Then, for any $\varepsilon > 0$ and by the dominated convergence theorem we see

$$\begin{aligned} V(N-1,x^*) &\leq E[U(Y_N^{N-1,x^*,z^\varepsilon})] + \varepsilon = \lim_{k\to\infty} E[U(Y_N^{N-1,x_k,z^\varepsilon})] + \varepsilon \\ &\leq \liminf_{k\to\infty} V(N-1,x_k) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it remains to show that

$$\limsup_{k\to\infty} V(N-1,x_k) \leq V(N-1,x^*). \tag{4.5}$$

Given $\varepsilon > 0$ there exists an ε-optimal $z_k^\varepsilon \in \mathscr{I}(x_k)$. For any admissible control z we have $z \in I_3$. Since I_3 is a bounded interval we have

$$\sup_{k,\varepsilon} |z_k^\varepsilon| < \infty.$$

Therefore, there exists a convergent subsequence $z_{k_n}^\varepsilon$ such that $\lim_{n\to\infty} z_{k_n}^\varepsilon = z^\varepsilon$. By the dominated convergence theorem and ε-optimality of z_k^ε,

$$\begin{aligned} V(N-1,x^*) &\geq E[U(Y_N^{N-1,x^*,z^\varepsilon})] = \lim_{n\to\infty} E[U(Y_N^{N-1,x_{k_n},z_{k_n}^\varepsilon})] \\ &\geq \limsup_{n\to\infty} V(N-1,x_{k_n}) - \varepsilon. \end{aligned}$$

Hence, (4.5) follows and the continuity of $V(N-1,\cdot)$ is proved. What remains to prove is the dynamic programming principle at the time step $N-1$, i.e., we need to show that

$$V(N-2,x) = \sup_{z_{N-2}\in\mathscr{I}(x)} E[V(N-1,X_{N-1}^{N-2,x,z_{N-2}})].$$

Set $z = (z_{N-2}, z_{N-1})$, and use the Markovian structure of the state variables to arrive at

$$E[U(Y_N^{N-2,x,z})] = E[E[U(Y_N^{N-1,X_{N-1}^{N-2,x,z_{N-1}},z_{N-1}})|\mathscr{F}_{N-1}]] \leq E[V(N-1,X_{N-1}^{N-2,x,z_{N-2}})].$$

Taking the supremum on both sides leads to

$$V(N-2,x) = \sup_{z\in\mathscr{A}} E[U(Y_N^{N-2,x,z})] \leq \sup_{z_{N-2}\in\mathscr{I}(x)} E[V(N-1,X_{N-1}^{N-2,x,z_{N-2}})].$$

The opposite inequality is proved using the continuity of $V(N-1,\cdot)$. Fix an initial condition $x_0 \in Q$ and set

$$(\alpha,\beta) := (z_{N-2}, z_{N-1}) \qquad \text{and} \qquad X^\alpha := X_{N-1}^{N-2,x_0,\alpha}.$$

Fix $\varepsilon > 0$, and choose $\alpha \in \mathscr{I}(x_0)$ such that

$$\sup_{\alpha \in \mathscr{I}(x_0)} E[V(N-1, X^\alpha)] \le E[V(N-1, X^\alpha)] + \varepsilon. \tag{4.6}$$

For every $x \in Q$, choose $\beta^\varepsilon(x) \in \mathscr{I}(x)$ such that

$$V(N-1, x) \le E[U(Y_N^{N-1,x,\beta^\varepsilon(x)})] + \varepsilon = J(N-1, x, \beta^\varepsilon(x)) + \varepsilon. \tag{4.7}$$

The family of open sets

$$\begin{aligned} \mathscr{O}_x^\varepsilon := \{y \in Q : \ &|J(N-1, y, \beta^\varepsilon(x)) - J(N-1, x, \beta^\varepsilon(x))| < \varepsilon \\ &\text{and } |V(N-1, y) - V(N-1, x)| < \varepsilon\} \end{aligned} \tag{4.8}$$

for $x \in Q$ is an open cover of Q. Since any open cover of a separable space contains a countable subcover, there exists a subsequence $x_n \in Q$ such that $\cup_n \mathscr{O}_{x_n}^\varepsilon = Q$. Set $C_1 := \mathscr{O}_{x_1}^\varepsilon$ and $C_{n+1} = \mathscr{O}_{x_{n+1}}^\varepsilon \setminus \cup_{k=1}^n \mathscr{O}_{x_k}^\varepsilon$. Then, for all $x \in Q$ there exists a $k \in \mathbb{N}$ such that $x \in C_k$. We denote that unique integer by $k(x)$. Note that $k(x)$ is by construction a measurable function. Since C_k's are disjoint, $Q = \cup_{k(x)} \mathscr{O}_{x_{k(x)}}^\varepsilon$. Also note that the map

$$(\alpha, \omega) \mapsto X^\alpha(\omega) =: X^\alpha$$

is measurable. Hence, so is $k(X^\alpha)$. Define $\hat{\beta} = \beta^\varepsilon(x_{k(X^\alpha)})$ and set $z^\varepsilon = (\alpha, \hat{\beta}) \in \mathscr{A}$. Since $\hat{\beta}$ is a composition of measurable functions, it also is measurable. We now directly calculate that

$$\begin{aligned} J(N-2, x_0, z^\varepsilon) &= E[U(Y_N^{N-2,x_0,z^\varepsilon})] = E[E[U(Y_N^{N-1,X^\alpha,\hat{\beta}})|\mathscr{F}_{N-1}]] \\ &= E[J(N-1, X^\alpha, \beta^\varepsilon(x_{k(X^\alpha)})] \\ &\overset{(4.8)}{\ge} E[J(N-1, x_{k(X^\alpha)}, \beta^\varepsilon(x_{k(X^\alpha)}))] - \varepsilon \\ &\overset{(4.7)}{\ge} E[V(N-1, x_{k(X^\alpha)})] - 2\varepsilon \\ &\overset{(4.8)}{\ge} E[V(N-1, X^\alpha)] - 3\varepsilon. \end{aligned}$$

Hence,

$$\begin{aligned} V(N-2, x_0) \ge J(N-2, x_0, z^\varepsilon) &\ge E[V(N-1, X^\alpha)] - 3\varepsilon \\ &\overset{(4.6)}{\ge} \sup_{\alpha \in \mathscr{I}(x_0)} E[V(N-1, X^\alpha)] - 4\varepsilon. \end{aligned}$$

The general case follows by induction. The induction hypothesis is that $V(N-k, \cdot)$ is continuous and that the dynamic programming principle holds at the time step $N-k-1$, i.e.

$$V(N-k-1, x) = \sup_{z_{N-k-1} \in \mathscr{I}(x)} E[V(N-k, X_{N-k}^{N-k-1,x,z_{N-k-1}}].$$

Then, the continuity of $V(N-k-1,\cdot)$ is proved exactly as the continuity of $V(N-1,\cdot)$ by using the induction hypothesis. The dynamic programming principle at the time step $N-k-1$, i.e.

$$V(N-k-2,x) = \sup_{z_{N-k-2}\in\mathscr{I}(x)} E[V(N-k-1, X_{N-k-1}^{N-k-2,x,z_{N-k-2}})].$$

is proved exactly as in the time step $N-1$. □

Proposition .1 *There exists an optimal feedback strategy z for the optimal control problem* (4.4).

Proof. Since $\mathscr{I}(x)$ is a bounded set, by Theorem .2 we have

$$V(k,x) = \max_{z\in\mathscr{I}(x)} E[V(k+1, X_{k+1}^{k,x,z})].$$

Set $z_k^*(x) = \arg\max_z\{E[V(k+1, X_{k+1}^{k,x,z})]\}$. We construct an optimal feedback strategy in the following way: We start at the time step k with x and choose $z_k^* \in z_k^*(x)$. Having the dynamics of the state variables we obtain a random variable X_{k+1}^*. At the time step $k+1$ we choose $z_{k+1}^* \in z_{k+1}^*(X_{k+1}^*)$ and continue with the iteration. The optimality of the feedback control follows from the DPP. □

5 Numerical results

In this section, we solve the optimization problem numerically. Hence, dimension reduction of the problem (4.4) is desirable. One may achieve this by assuming that the utility function is of power type,

$$U(y) = \begin{cases} \frac{y^{1-\gamma}}{1-\gamma}, & 0<\gamma, \\ \log(y), & \gamma = 1. \end{cases}$$

Indeed, the homothety of the utility function leads to

$$V(k,x) = V(k,s,\eta,y) = s^{1-\gamma}V(k,1,\eta/s,y/s) =: s^{1-\gamma}w(k,\psi,\tilde{y}). \tag{5.1}$$

Then, Theorem .2 and (5.1) provide a dynamic programming equation for w as well,

$$w(k,\psi,\tilde{y}) = \sup_{z\in\mathscr{I}(\psi,\tilde{y})} E\Big[(1+\mu h+\sigma\sqrt{h}\xi_{k+1})^{1-\gamma} w(k+1, \psi_{k+1}^{k,\psi,z}, \tilde{Y}_{k+1}^{k,\psi,\tilde{y},z})\Big]. \tag{5.2}$$

The state variables ψ and $\tilde{Y}$ solve the equations

$$\psi_{n+1} = \frac{(1-\kappa h)\psi_n - 2M\kappa h z_n - 2M z_n(\mu h + \sigma\sqrt{h}\xi_{n+1})}{1+\mu h+\sigma\sqrt{h}\xi_{n+1}} \tag{5.3}$$

$$\tilde{Y}_{n+1} = \frac{\tilde{Y}_n - \kappa h\psi_n z_n + (\mu h+\sigma\sqrt{h}\xi_{n+1})(1-Mz_n)z_n - 2M\kappa h z_n^2}{1+\mu h+\sigma\sqrt{h}\xi_{n+1}}. \tag{5.4}$$

We continue by describing the algorithm. We first discretize the continuous state space $(\psi, \tilde{Y})$ and the control space z with grid sizes $\Delta\psi, \Delta\tilde{Y}$ and Δz. The discretized state variables ψ and $\tilde{Y}$ lie in closed intervals $\mathscr{I}_\psi$ and $\mathscr{I}_{\tilde{Y}}$. The same holds for the control variable z, i.e., $z \in \mathscr{I}_z$. We start the algorithm at the final time step N, and compute $w(N, \psi, \tilde{Y}) = U(\tilde{Y})$, for all discrete values of $\tilde{Y} \in \mathscr{I}_{\tilde{Y}}$. For the recursion step, we use (5.2), (5.3) and (5.4). If the state variables $(\psi, \tilde{Y})$ at the next time step fall outside the interval $\mathscr{I}_\psi \times \mathscr{I}_{\tilde{Y}}$, we use an extrapolation to compute the value function w for these values.

We set $\gamma = 0.75, \sigma = 0.3, \mu = 0.04, M = 0.2, \kappa = 12, T = 2$ and $N = 20$. The value function w is plotted in Figure 5. Figure 5 is the plot of the difference $w - v^M$. This graph shows that Merton's value function v^M dominates w for values of ψ around the origin. This is more apparent if one considers Figure 5, where we fix ψ and plot $w - v^M$. However, there is also local arbitrage since the opposite inequality also holds for some values of ψ. The corresponding optimal strategy z^* is plotted in Figure 5. Note that the optimal strategy grows almost linearly in ψ, whereas in $\tilde{y}$ it changes less.

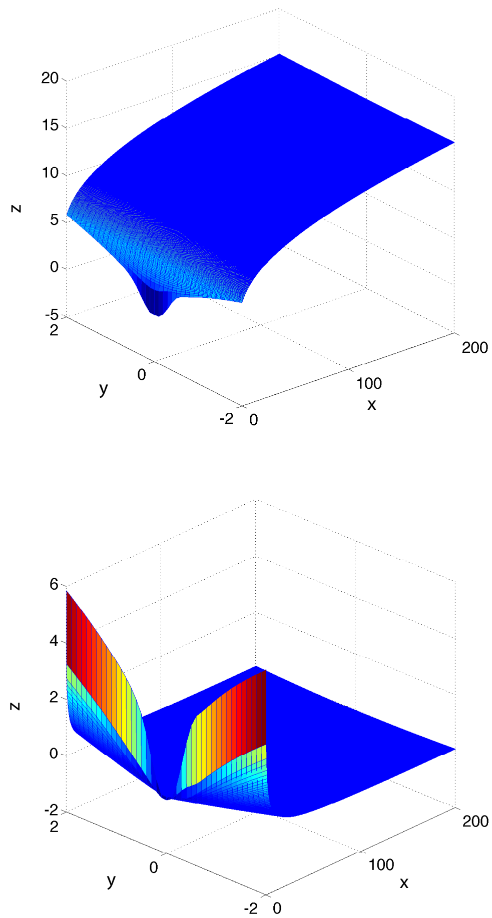

Figure 1: Numerical results for w and $w - v^M$

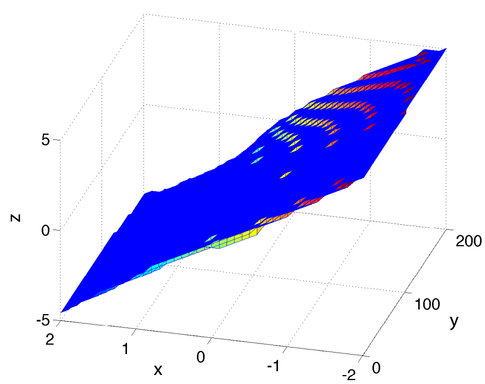

Figure 2: Numerical results for $w - v^{\mathrm{M}}$ for fixed ψ and the optimal strategy z^*

6 Concluding remarks

In this paper, we consider the expected utility from terminal wealth in a liquidity risk model proposed by Roch and Soner [11], in discrete time. We prove the dynamic programming principle and use it to compute the value function for some parameter values. We also show that the resulting discrete time model is arbitrage free for some parameter values. Indeed, for these parameter values, the value function with liquidity is bounded from above by Merton's value function, provided that the initial liquidity premium is zero. This is confirmed by numerical calculations which also show local arbitrage when there is initial price impact, or equivalently when the state variable ψ is initially not zero.

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