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Martingale optimal transport in the Skorokhod space ☆

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Abstract

The dual representation of the martingale optimal transport problem in the Skorokhod space of multi dimensional *cádlág* processes is proved. The dual is a minimization problem with constraints involving stochastic integrals and is similar to the Kantorovich dual of the standard optimal transport problem. The constraints are required to hold for every path in the Skorokhod space. This problem has the financial interpretation as the robust hedging of path dependent European options.

MSC: 91G10; 60G44

Keywords: Model-free Hedging; Martingale Optimal Transport; Skorokhod Space

1. Introduction

Model independent approach to financial markets provides hedges without referring to a particular probabilistic structure. It is also shown to be closely connected to the classical Monge–Kantorovich optimal transportation problem. In this paper, we prove this connection for quite general financial markets that offer multi risky assets with *cádlág* (right continuous with left hand limits) trajectories. This generality is strongly motivated by the fact that investors use several assets in their portfolios and the observed stock price processes contain jump components [3,4]. The main result is a Kantorovich type duality for the super-replication cost

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of an exotic option G, which is simply a nonlinear function of the whole stock trajectory. It is well documented that this duality is central to understanding the financial markets. In particular, several other important results including the fundamental theorem of asset pricing follow from it.

As it is standard in these problems, following [17] we assume that a linear set of options $\mathcal{H}$ is available for static investment with a known price $\mathcal{L}(h)$ for $h \in \mathcal{H}$. In addition to this static investment, the investor can dynamically use stocks in her portfolio. Let an admissible predictable process γ represent this dynamic position in the stock whose price process is denoted by $\mathbb{S}$ with values in the positive orthant $\mathbb{R}^d_+$. An investment strategy (h, γ) super-replicates an exotic option if its final value at maturity T dominates G in all possible cases, i.e.,

$$h(\mathbb{S}) + \int_0^T \gamma_u(\mathbb{S})\, d\mathbb{S}_u \geq G(\mathbb{S}), \quad \forall\, \mathbb{S} \in \mathbb{D}, \tag{1.1}$$

where $\mathbb{D}$ is the set of all stock process $\mathbb{S}$ that are *cádlág*, $\mathbb{S}_0 = (1, \ldots, 1)$ and continuous at maturity T. Technical issues related to the stochastic integral and admissible strategies are discussed in Section 2, Definition 2.5. The minimal super-replicating cost is then given by

$$V(G) := \inf\{\mathcal{L}(h) : \text{there exists an admissible predictable process } \gamma \text{ so that } (h, \gamma) \text{ super-replicates } G\}.$$

As usual, the dual elements are martingale measures $\mathbb{Q}$ that are consistent with the given option data. Namely, let $\mathbb{M}_{\mathcal{L}}$ be the set of all measures on $\mathbb{D}$ so that the canonical process $\mathbb{S}$ is a martingale with the canonical filtration $\mathbb{F}$ and

$$\mathbb{E}_{\mathbb{Q}}[h] \leq \mathcal{L}(h), \quad h \in \mathcal{H}.$$

We then have the following duality result,

$$V(G) = \sup_{\mathbb{Q} \in \mathbb{M}_{\mathcal{L}}} \mathbb{E}_{\mathbb{Q}}[G]. \tag{1.2}$$

The above result is proved in Theorem 2.9 for G that is uniformly continuous in the Skorokhod topology and satisfies a certain growth condition. In Theorem 2.9 we assume that

$$|G(\mathbb{S})| \leq C(1 + |\mathbb{S}_T|) \tag{1.3}$$

for some constant $C > 0$. Then, we relax this condition to (5.1) in the last section.

In this paper, we study two classes of pairs $(\mathcal{H}, \mathcal{L})$. Namely, one and many marginal cases. In the first one, this pair is defined through a given probability measure μ. Then, we take $\mathcal{H}$ to be the set of all functions of the type $g(\mathbb{S}_T)$ with $g \in \mathbb{L}^1(\mathbb{R}^d_+, \mu)$ and set $\mathcal{L}(g) = \int g d\mu$. The only assumption on μ is that $\int x d\mu(x) = \mathbb{S}_0 = (1, \ldots, 1)$.

In the initial sections and in Theorem 2.9, we prove the duality for the single marginal case. Then, in Section 5, we both relax the growth assumption on G and consider the multi-marginal problem. Namely, we fix a partition $0 < T_1 < T_2 < \cdots < T_N = T$ and a probability measures $\mu_1 \preceq \mu_2 \preceq \cdots \preceq \mu_N$ on R^d_+, where $\preceq$ denotes convex order on probability measures, i.e.,

$$\mu \preceq \nu \Leftrightarrow \int \Phi d\mu \leq \int \Phi d\nu, \quad \forall\, \Phi \text{ convex, integrable.}$$

We extend the duality result (1.2) for the case where $\mathcal{H}$ is the set of all functions of the type $\sum_{i=1}^N g_i(\mathbb{S}_{T_i})$ with $g_i \in \mathbb{L}^1(\mathbb{R}^d_+, \mu_i)$. For this extension however, we need to assume that $\int |x|^p d\mu_N(x) < \infty$ for some $p > 1$. In particular, we assume that a power option is an element in the space of static positions $\mathcal{H}$.

Our approach, as in [13,14], relies on a discretization procedure. We then use a classical min–max theorem for the discrete approximation and a classical constrained duality result of Follmer and Kramkov [15]. The technical steps are to prove that the approximations on both side of the dual formula converge. The multi-dimensionality and the discontinuous behavior of the stock process introduce several technical difficulties. In particular, we introduce appropriate portfolio constraints in the approximate discrete markets. This new feature of the discretization is essential and enables us to control the error terms due to the multi-dimensionality and the possible discontinuities of the stock process.

Another technical difficulty originates from the fact the set of martingale measures $\mathbb{M}_{\mathcal{L}}$ is not compact. Therefore, passage to the limit in the dual side requires probabilistic constructions. In particular, we prove that the dual problem as seen as a function of the probability measure μ (with fixed G) has some continuity properties. This is proved in Section 4, Theorem 4.1.

For the multi-marginal case, it is not clear how to prove Theorem 4.1 via a probabilistic construction. Instead, we use an additional idea in the discretization procedure. This idea is based on a penalization technique and requires that the linear space $\mathcal{H}$ contains a power option.

The structure studied in this paper is similar to that of [17] and also of [7–13,16,19–23, 25]. We refer the reader to the excellent survey of Hobson [18] and to our previous papers [13,14] and to the references therein. A related issue is the fundamental theorem of asset pricing (FTAP) in these markets. This problem in the robust setting in discrete time is studied in [2,6,14]. [2] proves FTAP in the model independent framework with a general $\mathcal{H}$ containing a power option. Following the structure introduced in [29,30], [6] considers a discrete time market in which a set of probability measures $\mathcal{P}$ is assumed. The super replication is defined by demanding (1.1) not for every path $\mathbb{S}$ but $\mathbb{P}$ almost surely for every $\mathbb{P} \in \mathcal{P}$ (i.e., $\mathcal{P}$-quasi-surely). FTAP and duality (under the assumption of no-arbitrage) is proved for a finite dimensional $\mathcal{H}$ but possibly with no power option. The notions of no-arbitrage considered in [2,6] are different. In our earlier work [14] we prove model-independent duality for a discrete time market with proportional costs. FTAP follows as a consequence of the duality. However, the form of FTAP depends on the particular notion of no-arbitrage. A discussion of different notions is also provided in [14]. In continuous time, the desirable extension to the general quasi-sure setting remains open with the exception of [16] in which a certain class $\mathcal{P}$ is considered.

The paper is organized as follows. The main results are formulated in the next section. In Section 3, Theorem 2.9 is proved. In Section 4, we prove a continuity result for the dependence of the dual problem on the measure μ. The final section, is devoted to extensions.

Notation. We close this introduction with a list of some of the notations used in this paper.

- $\mathbb{R}_+ := (0, \infty)$ is the set of all positive real numbers.
- $\mathbb{N} := \{1, 2, \ldots\}$ is the set of positive integers.
- $\mathbb{D}$ is the set of all $\mathbb{R}^d_+$ valued *cádlág* processes $\mathbb{S}$ that are continuous at $t = T$ and also satisfy $\mathbb{S}_0 = (1, \ldots, 1)$; Section 2.
- The similar set $\mathbb{D}([0, T]; \mathbb{R}^d)$ for $\mathbb{R}^d$ valued processes is defined in Section 2.
- $\mathbb{S}$ is the canonical process and $\mathcal{F}$ is the canonical filtration on $\mathbb{D}$; Section 2.
- $\|\mathbb{S}\| = \sup\{|\mathbb{S}_t| : t \in [0, T]\}$.
- $\mathcal{H}$ is the set of statically tradable options. In this paper, it is the set of all functions of the form $h(\mathbb{S}) = g(\mathbb{S}_T)$, where $g \in \mathbb{L}^1(\mathbb{R}^d_+, \mu)$ for some probability measure μ; see Section 2.1.
- d is the Skorokhod metric on $\mathbb{D}$, see Section 2.4.
- For a positive integer n and $\mathbb{S} \in \mathbb{D}$, stopping times $\tau_k = \tau_k^{(n)}(\mathbb{S})$'s and the random integer $M = M^{(n)}(\mathbb{S})$ are defined in Section 3.1.

- For a positive integer n and $\mathbb{S} \in \mathbb{D}$, random times $\hat{\tau}_k = \hat{\tau}_k^{(n)}(\mathbb{S})$'s are defined in Section 3.3 as a function of the stopping times τ_k's.
- Maps $\hat{\Pi} : \mathbb{D} \to \hat{\mathbb{D}}$ and $\check{\Pi}$, $\Pi : \mathbb{D} \to \mathbb{D}$ are constructed in Section 3.3.

When possible we followed the convention that the notation $\hat{}$ is reserved for objects on the countable space $\hat{\mathbb{D}}$, such as $\hat{\mathbb{S}}$ is a generic point in $\hat{\mathbb{D}}$ and $\hat{\tau}_k$'s are its jump times.

2. Preliminaries and main results

The financial market consists of a savings account which is normalized to unity $B_t \equiv 1$ by discounting and of d risky assets with price process $\mathbb{S}_t \in \mathbb{R}^d_+$, $t \in [0, T]$, where $T < \infty$ is the maturity date. Without loss of generality we set the initial stock values to one, i.e., $\mathbb{S}_0 = (1, \dots, 1)$. We assume that each component of the price process is right continuous with left hand limits (i.e., a *cádlág* process) which is also continuous at maturity $t = T$. $\mathbb{D}$ denotes the set of all *cádlág* functions

$$\mathbb{S} = (\mathbb{S}^{(1)}, \dots, \mathbb{S}^{(d)}) : [0, T] \to \mathbb{R}^d_+,$$

that are continuous at $t = T$ and also satisfy $\mathbb{S}_0 = (1, \dots, 1)$. Then, any element of $\mathbb{D}$ can be a possible path for the stock price process. This is the only assumption that we make on our financial market.

We set $\mathbb{D}([0, T]; \mathbb{R}^d)$ be the set of all *cádlág* processes that take values in $\mathbb{R}^d$ (rather than $\mathbb{R}^d_+$ as in the case of $\mathbb{D}$) that start from $\mathbb{S}_0 = (1, \dots, 1)$ and are continuous at T.

Consider a European path dependent option with the payoff $X = G(\mathbb{S})$ where

$$G : \mathbb{D}([0, T]; \mathbb{R}^d) \to \mathbb{R}.$$

Although only the values of G on $\mathbb{D}$ are needed to define the problem, technically we require G to be defined on the larger space $\mathbb{D}([0, T]; \mathbb{R}^d)$. However, in almost all cases extension of a function defined on $\mathbb{D}$ to $\mathbb{D}([0, T]; \mathbb{R}^d)$ is straightforward; see Remark 2.8.

In probability theory, most processes are required to be either progressively measurable or predictable with respect to a filtration. In the context of this paper, the natural filtration is canonical filtration generated by the canonical process. Then, we have the following equivalent definition of progressive measurability.

Definition 2.1. We say that a process $\gamma : [0, T] \times \mathbb{D} \to \mathbb{R}^d$ is *progressively measurable* if for any $\mathbb{S}, \tilde{\mathbb{S}} \in \mathbb{D}$ and $t \in [0, T]$,

$$\mathbb{S}_u = \tilde{\mathbb{S}}_u, \quad \forall u \in [0, t] \Rightarrow \gamma_t(\mathbb{S}) = \gamma_t(\tilde{\mathbb{S}}). \quad \square \tag{2.1}$$

It is well known that if γ is left continuous and progressive measurable, then it is predictable with respect to the canonical filtration. Hence, in the sequel we check the predictability of any left continuous process by verifying (2.1).

2.1. Tradable options

$\mathcal{H}$ represents the set of all options available for trading. Although in this paper we use a specific class, in general it is assumed to be a linear subset of real-valued functions on $\mathbb{D}$. It is always assumed that

$$h_{cash}, h_1, \dots, h_d \in \mathcal{H}, \quad \text{where } h_{cash} \equiv 1, \qquad h_k(\mathbb{S}) := \mathbb{S}^{(k)}_T, \quad \forall\, k = 1, \dots, d.$$

The price of these options are given through an operator

$$\mathcal{L} : \mathcal{H} \to \mathbb{R}.$$

The essential assumptions on $\mathcal{L}$ are the convexity, an appropriate continuity and

$$\mathcal{L}(h_{cash}) = 1, \qquad \mathcal{L}(h_k) = \mathbb{S}_0^{(k)} = 1, \quad \forall\, k = 1, \ldots, d.$$

The last condition implies that the dual elements are martingale measures. So it might be interesting to relax it so as to allow for local martingale measures.

Example 2.2. In this example, we discuss the two examples $(\mathcal{H}, \mathcal{L})$ studied in this paper.

1. Let μ be a probability measure and let

$$\mathcal{H} = \{\, h(\mathbb{S}) = g(\mathbb{S}_T) : g \in \mathbb{L}^1(\mathbb{R}_+^d, \mu)\}. \tag{2.2}$$

The pricing operator given through the *probability measure* μ by,

$$\mathcal{L}(g) = \int_{\mathbb{R}_+^d} g \, d\mu. \tag{2.3}$$

We assume that μ satisfies

$$\int h_k d\mu = \int x_k d\mu(x) = \mathbb{S}_0^{(k)} = 1, \quad \forall\, k = 1, \ldots, d. \tag{2.4}$$

The above linear pricing rule is equivalent to assume that the distribution of $\mathbb{S}_T$ is known and equal to μ.

2. Consider a partition $0 < T_1 < T_2 < \cdots < T_N = T$ and probability measures $\mu_1 \preceq \mu_2 \preceq \cdots \preceq \mu_N$ on R_+^d. Assume that $\int |x|^p d\mu_N(x) < \infty$ for some $p > 1$. We also assume that μ_N is satisfying (2.4). Set

$$\mathcal{H} = \left\{ h(\mathbb{S}) = \sum_{i=1}^{N} g_i(\mathbb{S}_{T_i}) : g_i \in \mathbb{L}^1(\mathbb{R}_+^d, \mu_i) \right\}. \tag{2.5}$$

In this case, the linear pricing rule is equivalent to assume that for any k the distribution of $\mathbb{S}_{T_k}$ is equal to μ_k. We also assume the following analogue of (2.4),

$$\int x_k d\mu_i(x) = \mathbb{S}_0^{(k)} = 1, \quad \forall\, k = 1, \ldots, d, \; i = 1, \ldots, N. \quad \square \tag{2.6}$$

In this paper, to simplify the presentation we mainly consider the first case. Namely, we assume that $(\mathcal{H}, \mathcal{L})$ satisfy (2.2), (2.3), (2.4). In particular this case does not require the existence of a power option as an element in the space of all static positions. In Section 5, we assume the existence of a power option and extend the duality to the multi-marginal case (2.5).

Remark 2.3. Again let the tradable options to be of the form $h(\mathbb{S}) = g(\mathbb{S}_T)$. However, now assume that g is a bounded and continuous function of $\mathbb{R}_+^d$. Then, a careful analysis our proof shows that the duality result Theorem 2.9 holds for this problem with the same dual problem. Hence, the super-replication cost with this class of tradable options is the equal to the one with the larger class $g \in \mathbb{L}^1(\mathbb{R}_+^d, \mu)$. See Remark 3.8 and also Remark 2.10. $\square$

Next we discuss the importance of the power option.

Remark 2.4. The following example highlights the role of the power option assumed in (2.4) and it is communicated to us by Marcel Nutz.

Suppose $d = 1$. Let $h^*(\mathbb{S}) = \chi_{[0.5,\infty)}(\mathbb{S}_T)$ and $\mathcal{H}$ be the three dimensional space spanned by h^*, h_1, h_{cash}. Further let $\mathcal{L}$ be a linear functional on $\mathcal{H}$ with $\mathcal{L}(h_{cash}) = 1$ and $\mathcal{L}(h^*) = 0$. For an exotic option G, $V(G)$ is the super-replication cost. Let $\tilde{\mathcal{H}}$ be the extended market that also includes the power option with $\tilde{V}(G)$ as the corresponding super-replication cost.

In both markets, the investor can buy the digital option h^* with zero cost. Clearly, this implies some kind or arbitrage since $h^* \geq 0$ and is not identically equal to zero. However, for the market $\mathcal{H}$ this arbitrage while agreeing with the notion introduced in [6], does not agree with the one given in [2]. On the other hand, in $\tilde{\mathcal{H}}$ there is arbitrage in both senses and the super-replication cost $\tilde{V}(G) = -\infty$.

In the smaller market $\mathcal{H}$ it follows directly that $V(0) = 0$. But we claim that there is no martingale measure that is consistent with $\mathcal{L}$. Indeed, if there were a martingale measure $\mathbb{Q}$ satisfying

$$\mathbb{E}_{\mathbb{Q}}[h^*] \leq \mathcal{L}(h^*) = 0,$$

then the support of the distribution μ of $\mathbb{S}_T$ under $\mathbb{Q}$ must be a subset of $[0, 0.5]$. On the other hand, since $\mathbb{Q}$ is a martingale measure, $\int x d\mu(x) = \mathbb{S}_0 = 1$. Hence, the set $\mathbb{M}_{\mathcal{L}}$ is empty. This means that the duality (1.2) does not hold in $\mathcal{H}$ while it holds in the market $\tilde{\mathcal{H}}$ that contains the power option. (Note that by convention the supremum over an empty set is defined to be minus infinity.)

Although the duality does not hold in $\mathcal{H}$ with the dual set $\mathbb{M}_{\mathcal{L}}$, in this example it would hold if one relaxes the dual set of measures to include the local martingale measures as well. □

2.2. Martingale measures

Set $\Omega := \mathbb{D}$ and let $\mathcal{F}$ be the σ–algebra which is generated by the cylindrical sets. Let $\mathbb{S} = (\mathbb{S}_t)_{0\leq t\leq T}$ be the canonical process given by $\mathbb{S}_t(\omega) := \omega_t$, for all $\omega \in \Omega$.

A probability measure $\mathbb{Q}$ on the space $(\Omega, \mathcal{F})$ is a *martingale measure*, if the canonical process $(\mathbb{S}_t)_{t=0}^T$ is a martingale with respect to $\mathbb{Q}$ and $\mathbb{S}_0 = (1, \dots, 1)$, $\mathbb{Q}$-a.s.

For a probability measure μ on $\mathbb{R}_+^d$, let $\mathbb{M}_\mu$ be the set of all martingale measures $\mathbb{Q}$ such that the probability distribution of $\mathbb{S}_T$ under $\mathbb{Q}$ is equal to μ. Observe that condition $\int x_k d\mu(x) = 1$ in (2.4) is equivalent to $\mathbb{M}_\mu \neq \emptyset$.

2.3. Admissible portfolios

Next, we describe the continuous time trading in the underlying asset $\mathbb{S}$. We essentially adopt the path-wise approach which was already used in [13]. However, the present setup is more delicate than the one in [13]. Indeed, due to the possible discontinuities of the integrator $\mathbb{S}$, we require that the trading strategies are of bounded variation and left continuous. Indeed, observe that for any left continuous function $\gamma : [0, T] \to \mathbb{R}^d$ of bounded variation and a *cádlág* function $\mathbb{S} \in \mathbb{D}$, we may use integration by parts (see Section 1.7 in [26]) to define

$$\int_0^t \gamma_u d\mathbb{S}_u := \gamma_t \cdot \mathbb{S}_t - \gamma_0 \cdot \mathbb{S}_0 - \int_0^t \mathbb{S}_u \cdot d\gamma_u,$$

where for $a, b \in \mathbb{R}^d$, $a \cdot b$ is the usual scalar product. Furthermore, the last term in the above right hand side is the Lebesgue–Stieltjes integral and not the standard Riemann–Stieltjes integral which was used in [13].

In particular, when γ is also progressively measurable (cf., (2.1)) then for any martingale measure $\mathbb{Q} \in \mathbb{M}_\mu$, the stochastic integral $\int \gamma_u \mathbb{S}_u$ is well-defined and both the pathwise constructed integral and the stochastic integral agree $\mathbb{Q}$ almost surely. In the sequel, we use this equality repeatedly.

These considerations lead us to the following definition.

Definition 2.5. A *semi-static portfolio* is a pair $\phi := (g, \gamma)$, where $g \in \mathbb{L}^1(\mathbb{R}^d_+, \mu)$ and $\gamma : [0, T] \times \mathbb{D} \to \mathbb{R}^d$ is left continuous, progressively measurable and bounded variation where $\gamma_t(\mathbb{S})$ denotes the number of shares in the portfolio ϕ at time t, before a transfer is made at this time.

A semi-static portfolio is *admissible*, if for every $\mathbb{Q} \in \mathbb{M}_\mu$ the stochastic integral $\int \gamma_u d\mathbb{S}_u$ is a $\mathbb{Q}$ super-martingale.

An admissible semi-static portfolio is called *super-replicating*, if

$$g(\mathbb{S}_T) + \int_0^t \gamma_u(\mathbb{S}) d\mathbb{S}_u \geq G(\mathbb{S}), \quad \forall \mathbb{S} \in \mathbb{D}.$$

The (minimal) *super-hedging cost* of G is defined by,

$$V(G) := \inf \left\{ \int g d\mu : \exists \gamma \text{ such that } \phi := (g, \gamma) \text{ is super-replicating} \right\}. \quad \square$$

Remark 2.6. The condition of admissibility depends on the measure μ. Hence the set of admissible controls and the super-replication cost also have this dependence. One may remove this dependence by considering continuous and bounded g's instead of $\mathbb{L}^1(\mathbb{R}^d_+, \mu)$ functions. And for admissibility, instead of requiring that the stochastic integral $\int \gamma_u d\mathbb{S}_u$ is a $\mathbb{Q}$ super-martingale for every $\mathbb{Q} \in \mathbb{M}_\mu$, one may impose the condition that this integral is uniformly bounded from below in $\mathbb{S}$. A careful analysis of the proof of Theorem 2.9 reveals that the duality (under the hypothesis of Theorem 2.9) holds with this smaller class of admissible portfolios and hence the super-replication cost is not changed. See Remarks 3.8 and 2.10.

In the case when μ satisfies

$$\int |x|^p d\mu(x) < \infty \tag{2.7}$$

with an exponent $p > 1$, if there exists $C > 0$ satisfying

$$\int_0^t \gamma_u(\mathbb{S})\, d\mathbb{S}_u \geq -C \left(1 + \sup_{0 \leq u \leq t} |\mathbb{S}_u|^p \right), \quad \forall\, t \in [0, T],\ \mathbb{S} \in \mathbb{D}, \tag{2.8}$$

then the stochastic integral is a $\mathbb{Q}$ super-martingale for each $\mathbb{Q} \in \mathbb{M}_\mu$ due to Doob's inequality and (2.7).

In the sequel, we check the admissibility of γ by verifying either the above condition with $p > 1$ when (2.7) holds or again the above inequality but with $p = 0$ when we only have (2.4). $\square$

2.4. *Martingale optimal transport on the space* $\mathbb{D}$

We continue by stating the duality result. Since our approach relies on discretization, one requires the regularity of the exotic option. One may then relax this regularity through analytical methods as we have done in [14]. Since the emphasis of this paper is the possible discontinuity of the stock process and multi-dimensionality, we do not seek the most general condition on G. We first prove the duality when G is satisfying (1.3) and uniformly continuous in the Skorokhod topology. We then relax this condition in Section 5. To state the condition on G, recall the Skorokhod metric on $D([0,T];\mathbb{R}^d)$,

$$d(\omega,\tilde{\omega}) := \inf_{\lambda\in\Lambda[0,T]} \sup_{t\in[0,T]} \left(|\omega(t)-\tilde{\omega}(\lambda(t))| + |\lambda(t)-t|\right),$$

where $\Lambda[0,T]$ is the set of all strictly increasing onto functions $\lambda : [0,T] \to [0,T]$.

Assumption 2.7. We assume that the exotic option

$$G : \mathbb{D}([0,T];\mathbb{R}^d) \to \mathbb{R},$$

is satisfying (1.3) and uniformly continuous, i.e., there exists a continuous bounded function (modulus of continuity) $m_G : \mathbb{R}_+ \to \mathbb{R}_+$ so that

$$|G(\omega) - G(\tilde{\omega})| \le m_G\left(d(\omega,\tilde{\omega})\right), \quad \forall \omega, \tilde{\omega} \in \mathbb{D}([0,T];\mathbb{R}^d). \quad \square$$

Examples (for $d = 1$) of payoffs which satisfy Assumption 2.7 include lookback put options with fixed strike

$$G(\mathbb{S}) = (K - \min_{0\le t\le T} \mathbb{S}_t)^+$$

and lookback call option with floating strike

$$G(\mathbb{S}) = (\mathbb{S}_T - \min_{0\le t\le T} \mathbb{S}_t)^+.$$

In the last section, we relax the above assumption to Assumption 5.1. This extension allows for more options, in particular of Asian type.

Remark 2.8. For technical reasons, we assume that G defined not only on $\mathbb{D}$ but in the larger space $\mathbb{D}([0,T];\mathbb{R}^d)$. However, suppose that G is given only on its natural domain $\mathbb{D}$ rather than the whole space $\mathbb{D}([0,T];\mathbb{R}^d)$. Assume that G is uniformly continuous on $\mathbb{D}$. Then, one can extend G to the larger space still satisfying the above assumption and the main duality result is independent of the particular extension chosen.

Indeed, a direct closure argument extends G to a uniformly continuous function $\check{G}$ defined on $\mathbb{D}([0,T];[0,\infty)^d)$. Then, we define $\tilde{G}(\tilde{\mathbb{S}}) := \check{G}(\tilde{\mathbb{S}}')$ for every $\tilde{\mathbb{S}} \in \mathbb{D}([0,T];\mathbb{R}^d)$, *where* $\tilde{\mathbb{S}}_t'^{(i)} := |\tilde{\mathbb{S}}_t^{(i)}|$, $i = 1,\ldots,d$ and $t \in [0,T]$. $\square$

The following result is an extension of Theorem 2.7 in [13] to the case of multi-dimensional stock price process with possible jumps. Its proof is completed in the subsequent sections. Relaxations of Assumption 2.7 are provided in Section 5.

Theorem 2.9. *We assume that $(\mathcal{H},\mathcal{L})$ is as in (2.2), (2.3) and the probability measure μ satisfies (2.4). Then for any exotic option satisfying Assumption 2.7, we have the dual representation for the minimal super-replication cost defined in Definition 2.5,*

$$V(G) = \sup_{\mathbb{Q}\in\mathbb{M}_\mu} \mathbb{E}_{\mathbb{Q}}\left[G(\mathbb{S})\right],$$

where $\mathbb{E}_{\mathbb{Q}}$ denotes the expectation with respect to the probability measure $\mathbb{Q}$.

Proof. Let $\mathbb{Q} \in \mathbb{M}_\mu$. Then, for any admissible strategy γ, the path-wise integral $\int \gamma_u d\mathbb{S}_u$ agrees with stochastic integral $\mathbb{Q}$-almost surely and in view of Definition 2.5 this integral is a $\mathbb{Q}$ super-martingale. Now suppose that (g, γ) be an admissible super-replicating semi-static portfolio. Then,

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \gamma_u(\mathbb{S})\, d\mathbb{S}_u\right] \le 0, \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}}[g(\mathbb{S}_T)] = \int g d\mu.$$

We take the expected value with respect to $\mathbb{Q}$ in the super-replication inequality and use the above observations to arrive at,

$$V(G) \ge \sup_{\mathbb{Q}\in\mathbb{M}_\mu} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})].$$

The opposite inequality is proved in Corollary 3.7,

$$V(G) \le \liminf_{n\to\infty} V^{(n)}(G) \le \sup_{\mathbb{Q}\in\mathbb{M}_\mu} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})]. \tag{2.9}$$

We continue with a standard step that allows us to consider only bounded and non-negative claims.

Reduction to bounded non negative claims. Let $C > 0$ be the constant in the assumption (1.3) and set

$$\hat{G}(\mathbb{S}) := G(\mathbb{S}) + C\left[1 + \sum_{i=1}^{d} \mathbb{S}_T^{(i)}\right].$$

Then,

$$V(\hat{G}) = V(G) + (d+1)C,$$

and also

$$\sup_{\mathbb{Q}\in\mathbb{M}_\mu} \mathbb{E}_{\mathbb{Q}}\left[\hat{G}(\mathbb{S})\right] = \sup_{\mathbb{Q}\in\mathbb{M}_\mu} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})] + (d+1)C.$$

Since by (1.3) $\hat{G} \ge 0$, we may assume without loss of generality that the claim G is non negative and satisfies Assumption 2.7.

Next, for any constant $K \ge 0$ set $G_K := G \wedge C(dK+1)$ where C is again as in (1.3). Then, G_K is a bounded, non negative function. Then, in view of (1.3),

$$G = G_K + (G - C(dK+1))^+ \le G_K + C\sum_{i=1}^{d} (\mathbb{S}_T^{(i)} - K)^+.$$

Consequently, we have the following inequalities,

$$\begin{aligned} V(G_K) \le V(G) &\le V(G_K) + V\left(C\sum_{i=1}^{d} (\mathbb{S}_T^{(i)} - K)^+\right) \\ &= V(G_K) + C\sum_{i=1}^{d} \int (x_i - K)^+ d\mu(x). \end{aligned}$$

Also (2.4) implies that

$$\lim_{K\to\infty}\sum_{i=1}^{d}\int (x_i-K)^+ d\mu(x)=0.$$

Therefore, we conclude that $V(G)=\lim_{k\to\infty} V(G_K)$ and so if the inequality (2.9) holds for G_K, then it holds for G.

We conclude that without loss of generality, we may assume that G is a bounded non negative function satisfying Assumption 2.7. □

Remark 2.10. In the above proof, the lower bound for $V(G)$ follows from a classical direct argument. For this argument the minimal conditions for (g,γ) are the ones assumed in Definition 2.5. Namely, the integrability of g with respect to μ and the super-martingality of the stochastic integral. Therefore, any smaller class of semi-static portfolios would also satisfy the lower bound trivially. □

3. Proof of (2.9) for bounded non negative G

In this section and the next, we assume that G is bounded non negative and satisfies Assumption 2.7 and that the pair $(\mathcal{H},\mathcal{L})$ is as (2.2), (2.3).

3.1. Discretization of $\mathbb{R}_+^d$ and stopping times

In this subsection, we construct a sequence of stopping times that will be central to our discretization procedure.

For $n\in\mathbb{N}$ and $x\in\mathbb{R}_+^d$ define an open set by,

$$O(x,n):=\left\{y\in\mathbb{R}_+^d : |y-x|<\sqrt{d}\,2^{-n}\right\}.$$

For $\mathbb{S}\in\mathbb{D}$, set $\tau_0=0$ and define $\tau_{k+1}=\tau_{k+1}^{(n)}(\mathbb{S})$ by,

$$\tau_{k+1}:=T\wedge\left(\tau_k+\sqrt{d}\,2^{-n}\right)\wedge\inf\left\{t>\tau_k : \mathbb{S}_t\notin O(\mathbb{S}_{\tau_k},n)\right\},\qquad k=0,1,\ldots,$$

where we set $\tau_{k+1}=T\wedge(\tau_k+\sqrt{d}\,2^{-n})$ if the above set is empty. To ease the notation we suppress the dependence on n and $\mathbb{S}$ when this dependence is clear. Set

$$M=\mathbb{M}^{(n)}(\mathbb{S}):=\min\{\,k\in\mathbb{N}:\tau_k=T\,\}.$$

Since $\mathbb{S}$ is *cádlág* and $\mathbb{S}\in\mathbb{R}_+^d$, $M<\infty$. It is also clear that

$$0=\tau_0<\tau_1<\cdots<\tau_M=T$$

are stopping times with respect to the filtration which is generated by $\mathbb{S}$. Moreover, for $k=0,1,\ldots,M-1$,

$$|\tau_{k+1}-\tau_k|,\ \left|\mathbb{S}_t-\mathbb{S}_{\tau_k}\right|\le\sqrt{d}\,2^{-n},\qquad\forall t\in[\tau_k,\tau_{k+1}).\tag{3.1}$$

Also, by continuity of $\mathbb{S}$ at T, the above holds in the closed interval $[\tau_{M-1},T]$.

3.2. Approximation

In this subsection, we introduce a sequence of super-replication problems defined on a countable probability space. In the later sections, we show that this sequence approximates the original problem. Since the probability space is countable, robust (or equivalently point wise) and the probabilistic super-replications agree with a properly chosen probability measure. This allows us to use classical techniques to analyze the approximating problem.

We fix $n \in \mathbb{N}$ and define a sequence of probability spaces $\hat{\mathbb{D}} = \hat{\mathbb{D}}^{(n)}[0, T]$. Set

$$A^{(n)} := \left\{ 2^{-n} m : m = (m_1, \ldots, m_d) \in \mathbb{N}^d \right\},$$
$$B^{(n)} := \left\{ k\sqrt{d}\, 2^{-n} : k \in \mathbb{N} \right\} \cup \left\{ \sqrt{d}\, 2^{-n}/k : k \in \mathbb{N} \right\}.$$

Definition 3.1. A process $\hat{\mathbb{S}} \in \mathbb{D}$ belongs to $\hat{\mathbb{D}}$, if there exists a nonnegative integer M and a partition $0 = t_0 < t_1 = \sqrt{d} 2^{-n} < \cdots < t_M < T$ such that

$$\hat{\mathbb{S}}_t = \sum_{k=0}^{M-1} \hat{\mathbb{S}}_{t_k} \chi_{[t_k, t_{k+1})}(t) + \hat{\mathbb{S}}_{t_M} \chi_{[t_M, T]}(t)$$

where $\hat{\mathbb{S}}_0 = (1, \ldots, 1)$, $\hat{\mathbb{S}}_T = \hat{\mathbb{S}}_{t_M} \in A^{(n)}$ and

$$\hat{\mathbb{S}}_{t_k} \in A^{(n+k)}, \quad \forall\, k = 1, \ldots, M-1,\ t_k - t_{k-1} \in B^{(n+k)},\ \forall\, k = 2, \ldots, M. \quad \Box$$

Since the set $\hat{\mathbb{D}}$ is countable, there exists a probability measure $\mathbb{P} = \mathbb{P}^{(n)}$ on $\mathbb{D}$ with support contained in $\hat{\mathbb{D}}$, which gives positive weight to every element of $\hat{\mathbb{D}}$.

Let the probability structure $\Omega := \mathbb{D}$, the canonical map $\mathbb{S}$ and the filtration $\mathcal{F}$ be as in Section 2.4. Introduce a new filtration $\hat{\mathcal{F}} = (\hat{\mathcal{F}}_t)_{t\in[0,T]}$ by completing $\mathcal{F}$ by the null sets of $\mathbb{P}$. Note that all of this structure depends on n but this dependence is suppressed in our notation. Under the measure $\mathbb{P}$, the canonical map $\mathbb{S}$ has finitely many jumps. Let $M = M(\mathbb{S})$ be number of jumps and

$$0 < \hat{\tau}_1 < \cdots < \hat{\tau}_M < T$$

be the jump times of $\mathbb{S}$. We set $\hat{\tau}_0 = 0$, $\hat{\tau}_{M+1} = T$. We recall that the canonical process $\mathbb{S}$ is continuous at T.

A trading strategy on the filtered probability space $(\Omega, \{\hat{\mathcal{F}}_t\}_{t=0}^T, \mathbb{P})$ is simply a predictable stochastic process $\hat{\gamma}$ with respect to the filtration $\hat{\mathcal{F}}$. Next, consider a *constrained* financial market, in which the trading strategy satisfies the bound

$$\hat{\gamma} : [0, T] \times \mathbb{D} \to [-n\ ,\ n].$$

The statically tradable options are bounded real valued functions of $A^{(n)}$.

We also define a probability measure $\hat{\mu}$ on $A^{(n)}$ by,

$$\hat{\mu}(\{m2^{-n}\}) := \mu\left(\left\{x \in \mathbb{R}^d_+ : \pi^{(n)}(x) = m2^{-n}\right\}\right), \quad m \in \mathbb{N}^d,$$

where μ is the probability measure defining the operator $\mathcal{L}$ in Section 2.1 and

$$\pi^{(n)} : \mathbb{R}^d_+ \to A^{(n)} := \left\{2^{-n}k : k = (k_1, \ldots, k_d) \in \mathbb{N}^d\right\} \tag{3.2}$$

is given by

$$\pi^{(n)}(x)_i := 2^{-n}\ \lceil 2^n x_i \rceil, \quad i = 1, \ldots, d,$$

and for $a \in \mathbb{R}_+$, $\lceil a \rceil \in \mathbb{N}$ is smallest integer greater or equal to a.

We summarize this in the following by defining the probabilistic super-replication problem on the set $\hat{\mathbb{D}}$.

Definition 3.2. A (probabilistic) *semi-static portfolio* is a pair $(\hat{g}, \hat{\gamma})$ such that $\hat{g} : A^{(n)} \to \mathbb{R}$ is a bounded function, $\hat{\gamma} : [0, T] \times \mathbb{D} \to [-n, n]$ is predictable and the stochastic integral $\int \hat{\gamma}_u d\hat{\mathbb{S}}_u$ exists.

A semi-static portfolio is *admissible* if there exists $C > 0$ such that

$$\int_0^t \hat{\gamma}_u d\hat{\mathbb{S}}_u \geq -C, \quad \mathbb{P}\text{-a.s.},\ t \in [0, T].$$

A semi-static portfolio is $\mathbb{P}$-*super-replicating*, if

$$\hat{g}(\mathbb{S}_T) + \int_0^T \hat{\gamma}_u d\mathbb{S}_u \geq G(\mathbb{S}), \quad \mathbb{P}\text{-a.s.} \tag{3.3}$$

The (minimal) super-hedging cost of G is defined by,

$$V^{(n)}(G) := \inf \left\{ \int \hat{g} d\hat{\mu} : \exists \gamma \text{ such that } \hat{\phi} := (\hat{g}, \hat{\gamma}) \text{ is admissible and super-replicating} \right\}. \quad \square$$

We note that (3.3) is equivalent to having the same inequality for every $\hat{\mathbb{S}} \in \hat{\mathbb{D}}$.

Remark 3.3. The bound n that we place on the γ is somehow arbitrary. Indeed, any bound that converges to infinity with n and goes to zero when multiplied by 2^{-n} would suffice. This flexibility might be useful in possible future extensions. $\square$

3.3. Lifting

An important step in our approach is to "lift" a given probabilistic semi-static portfolio $\hat{\phi} = (\hat{h}, \hat{\gamma})$ to an admissible portfolio ϕ for the original financial market.

We start the construction of this lift by defining an approximation of the stopping times $\tau_k = \tau_k^{(n)}(\mathbb{S})$ defined in Section 3.1. Recall also the random integer $M = \mathbb{M}^{(n)}(\mathbb{S})$ defined in Section 3.1 and the set $B^{(i)}$ defined in Definition 3.1. Set

$$\hat{\tau}_0 := 0, \qquad \hat{\tau}_1 = \sqrt{d}\, 2^{-n}, \qquad \hat{\tau}_{M+1} := T.$$

For $k = 2, \ldots, M$ recursively define,

$$\hat{\tau}_k := \hat{\tau}_{k-1} + (1 - \sqrt{d}\, 2^{-n}/T) \sup \left\{ \Delta t > 0 \mid \Delta t \in B^{(n+k)} \text{ and } \Delta t < \tau_{k-1} - \tau_{k-2} \right\}.$$

We note that due to the definition of $B^{(i)}$ the above set is always non-empty. We collect some properties of these random times in the following lemma.

Lemma 3.4. *Random times $\hat{\tau}_k$'s satisfy,*

$$0 = \hat{\tau}_0 < \sqrt{d}\, 2^{-n} = \hat{\tau}_1 < \cdots < \hat{\tau}_M < \hat{\tau}_{M+1} = T,$$

and

$$|\hat{\tau}_k - \tau_k| \le \sqrt{d}\, 2^{-n+1}, \quad \forall\, k = 0, \ldots, M.$$

Proof. The above definitions yield,

$$\begin{aligned}
\hat{\tau}_M &= \hat{\tau}_1 + \sum_{k=2}^{M} [\hat{\tau}_k - \hat{\tau}_{k-1}] \\
&< \sqrt{d}\, 2^{-n} + (1 - \sqrt{d}\, 2^{-n}/T) \sum_{k=2}^{M} [\tau_{k-1} - \tau_{k-2}] \\
&= \sqrt{d}\, 2^{-n} + (1 - \sqrt{d}\, 2^{-n}/T)[\tau_{M-1} - \tau_0] \\
&< \sqrt{d}\, 2^{-n} + (1 - \sqrt{d}\, 2^{-n}/T)T = T.
\end{aligned}$$

This proves that

$$0 = \hat{\tau}_0 < \sqrt{d}\, 2^{-n} = \hat{\tau}_1 < \cdots < \hat{\tau}_M < \hat{\tau}_{M+1} = T.$$

Moreover, for any $k = 2, \ldots, M$,

$$\begin{aligned}
\hat{\tau}_k &= \hat{\tau}_1 + \sum_{j=2}^{k} [\hat{\tau}_j - \hat{\tau}_{j-1}] \\
&< \sqrt{d}\, 2^{-n} + (1 - \sqrt{d}\, 2^{-n}/T) \sum_{j=2}^{k} [\tau_{j-1} - \tau_{j-2}] \\
&= \sqrt{d}\, 2^{-n} + (1 - \sqrt{d}\, 2^{-n}/T)[\tau_{k-1} - \tau_0] = \tau_{k-1} + \sqrt{d}\, 2^{-n}(1 - \tau_{k-1}/T) \\
&< \tau_{k-1} + \sqrt{d}\, 2^{-n}.
\end{aligned}$$

The definition of $\hat{\tau}_k$ and the set $B^{(i)}$, imply that for any $j = 2, \ldots, M$,

$$\hat{\tau}_j - \hat{\tau}_{j-1} \ge \tau_{j-1} - \tau_{j-2} - \sqrt{d}\, 2^{-(n+j)}.$$

We use this to estimate $\hat{\tau}_k$ with $k = 2, \ldots, M$, from below as follows.

$$\begin{aligned}
\hat{\tau}_k &= \hat{\tau}_1 + \sum_{j=2}^{k} [\hat{\tau}_j - \hat{\tau}_{j-1}] \\
&\ge \sqrt{d}\, 2^{-n} + (1 - \sqrt{d}\, 2^{-n}/T) \sum_{j=2}^{k} [\tau_{j-1} - \tau_{j-2} - \sqrt{d}\, 2^{-(n+j)}] \\
&\ge \sqrt{d}\, 2^{-n} + (1 - \sqrt{d}\, 2^{-n}/T)[\tau_{k-1} - \tau_0] - \sqrt{d}\, 2^{-n} \\
&= \tau_{k-1} - \sqrt{d}\, 2^{-n} \tau_{k-1}/T \\
&> \tau_{k-1} - \sqrt{d}\, 2^{-n}.
\end{aligned}$$

Since $\hat{\tau}_{M+1} = \tau_M = T$, $\hat{\tau}_1 = \sqrt{d}\, 2^{-n}$, $\tau_0 = 0$, this proves that

$$|\hat{\tau}_k - \tau_{k-1}| \le \sqrt{d}\, 2^{-n}, \quad \forall\, k = 1, \ldots, M+1.$$

Also, by construction $|\tau_{k+1} - \tau_k| \le \sqrt{d}\; 2^{-n}$ for all $k = 0, \dots, M-1$. These inequalities complete the proof of the lemma. □

We now define a map $\hat{\Pi} = \hat{\Pi}^{(n)} : \mathbb{D} \to \hat{\mathbb{D}}$ by,

$$\hat{\Pi}_t(\mathbb{S}) := \sum_{k=0}^{M-1} \pi^{(n+k)}(\mathbb{S}_{\tau_k})\; \chi_{[\hat{\tau}_k, \hat{\tau}_{k+1})}(t) + \pi^{(n)}(\mathbb{S}_{\tau_M})\; \chi_{[\hat{\tau}_M, T]}(t), \tag{3.4}$$

where $\pi^{(n)}$ is defined in (3.2).

It is clear by the definition of $\pi^{(n)}$, $\hat{\tau}_k$'s and Definition 3.1, that $\hat{\Pi}(\mathbb{S}) \in \hat{\mathbb{D}}$ for every $\mathbb{S} \in \mathbb{D}$. We also note that $\mathbb{S}_{\tau_M} = S_T$ and that $\mathbb{S}$ is continuous at T. For comparison, we also define

$$\check{\Pi}_t(\mathbb{S}) := \sum_{k=0}^{M-2} \pi^{(n+k)}(\mathbb{S}_{\tau_k})\; \chi_{[\tau_k, \tau_{k+1})}(t) + \pi^{(n)}(\mathbb{S}_{\tau_{M-1}})\; \chi_{[\tau_{M-1}, T]}(t),$$

$$\Pi_t(\mathbb{S}) := \sum_{k=0}^{M-2} \mathbb{S}_{\tau_k}\; \chi_{[\tau_k, \tau_{k+1})}(t) + \mathbb{S}_{\tau_{M-1}}\; \chi_{[\tau_{M-1}, T]}(t).$$

Lemma 3.5. *Let d be the Skorokhod metric. Then, for every $\mathbb{S} \in \mathbb{D}$,*

$$d(\mathbb{S}, \Pi(\mathbb{S})),\; d(\Pi(\mathbb{S}), \check{\Pi}(\mathbb{S})) \le \sqrt{d}\; 2^{-n}, \qquad d(\check{\Pi}(\mathbb{S}), \hat{\Pi}(\mathbb{S})) \le 3\sqrt{d}\; 2^{-n}.$$

Suppose G satisfies Assumption 2.7. Then,

$$\left| G(\mathbb{S}) - G(\hat{\Pi}(\mathbb{S})) \right| \le 3 m_G (3\sqrt{d}\; 2^{-n}).$$

Proof. In view of (3.1), we have,

$$\begin{aligned} d(\mathbb{S}, \Pi(\mathbb{S})) &\le \|\mathbb{S} - \Pi(\mathbb{S})\|_\infty \\ &= \max_{k=0,\dots,M-1} \sup\{|\mathbb{S}_t - \mathbb{S}_{\tau_k}| : t \in [\tau_k, \tau_{k+1})\,\} \vee |\mathbb{S}_T - \mathbb{S}_{\tau_{M-1}}| \\ &\le \sqrt{d} 2^{-n}. \end{aligned}$$

Next we estimate directly that

$$d(\Pi(\mathbb{S}), \check{\Pi}(\mathbb{S})) \le \|\Pi(\mathbb{S}) - \check{\Pi}(\mathbb{S})\|_\infty \le \sup_{x \in \mathbb{R}^d_+,\; k \ge 0} |\pi^{(n+k)}(x) - x| \le \sqrt{d}\; 2^{-n}.$$

Define $\Lambda : [0, T] \to [0, T]$ by $\Lambda(0) = 0$, $\Lambda(\hat{\tau}_k) = \tau_k$ for $k = 1, \dots, M-1$,

$$\Lambda(\hat{\tau}_M) = [\tau_{M-1} + T]/2, \qquad \Lambda(\hat{\tau}_{M+1}) = \Lambda(T) = \tau_M = T,$$

and to be piecewise linear at other points. Then, it is clear that Λ is an increasing function and

$$\check{\Pi}_{\Lambda(t)}(\mathbb{S}) = \hat{\Pi}_t(\mathbb{S}), \quad \forall\, t \in [0, \hat{\tau}_{M-1}).$$

Moreover, for $t \in [\hat{\tau}_{M-1}, T]$,

$$\check{\Pi}_{\Lambda(t)}(\mathbb{S}) = \pi^{(n)}(\mathbb{S}_{\tau_{M-1}}).$$

Hence, by (3.1) and the continuity of $\mathbb{S}$ at T,

$$\sup_{t \in [0,T]} \left\{ \left| \check{\Pi}_{\Lambda(t)}(\mathbb{S}) - \hat{\Pi}_t(\mathbb{S}) \right| \right\} = \sup_{t \in [\hat{\tau}_{M-1}, T]} \left\{ \left| \check{\Pi}_{\Lambda(t)}(\mathbb{S}) - \hat{\Pi}_t(\mathbb{S}) \right| \right\} \le \sqrt{d}\; 2^{-n}.$$

We now use the above estimate together with Lemma 3.4 and the above Λ in the definition of the Skorokhod metric. The result is

$$d(\check{\Pi}(\mathbb{S}), \hat{\Pi}(\mathbb{S})) \leq \sup_{t\in[0,T]} \{|\check{\Pi}_{\Lambda(t)}(\mathbb{S}) - \hat{\Pi}_t(\mathbb{S})| + |\Lambda(t) - t|\}$$
$$= \sqrt{d}\, 2^{-n} + \max_{k=1,\dots,M-1} \{|\hat{\tau}_{k+1} - \tau_k|\} \leq \sqrt{d}\, 2^{-n} + \sqrt{d}\, 2^{-n+1}.$$

Suppose G satisfies Assumption 2.7. We now use the above estimates to obtain

$$\begin{aligned}|G(\mathbb{S}) - G(\hat{\Pi}(\mathbb{S}))| &\leq |G(\mathbb{S}) - G(\Pi(\mathbb{S}))| + |G(\Pi(\mathbb{S})) - G(\check{\Pi}(\mathbb{S}))| \\ &\quad + |G(\check{\Pi}(\mathbb{S})) - G(\hat{\Pi}(\mathbb{S}))| \\ &\leq 2m_G(\sqrt{d}2^{-n}) + m_G(3\sqrt{d}\, 2^{-n}) \\ &\leq 3m_G(3\sqrt{d}\, 2^{-n}). \quad \square\end{aligned}$$

We are ready to define the lift. Let $\hat{\phi} = (\hat{g}, \hat{\gamma})$ be a semi-static portfolio in the sense of Definition 3.2. Define a portfolio $\phi := \Psi(\hat{\phi}) := (g, \gamma)$ for the original problem by

$$g(x) := \hat{g}\left(\pi^{(n)}(x)\right), \quad x \in \mathbb{R}^d_+,$$
$$\gamma_t(\mathbb{S}) := \sum_{k=0}^{M-1} \hat{\gamma}_{\hat{\tau}_{k+1}(\mathbb{S})}(\hat{\Pi}(\mathbb{S}))\, \chi_{(\tau_k(\mathbb{S}),\tau_{k+1}(\mathbb{S})]}(t), \quad t \in [0, T]. \tag{3.5}$$

Observe that by definition $\gamma_0(\mathbb{S}) = 0$.

The following lemma provides the important properties of the above mapping.

Lemma 3.6. *For a semi-static portfolio $\hat{\phi} = (\hat{g}, \hat{\gamma})$ in the sense of Definition 3.2 and let $\phi = (g, \gamma)$ be defined as in (3.5). Then, ϕ is admissible in sense defined in Definition 2.5 and has the following properties,*

$$\int_{\mathbb{R}^d_+} g d\mu = \int_{A^{(n)}} \hat{g} d\hat{\mu},$$
$$\left| \int_0^T \hat{\gamma}_u(\hat{\Pi}(\mathbb{S})) d\hat{\Pi}_u(\mathbb{S}) - \int_0^T \gamma_u(\mathbb{S}) d\mathbb{S}_u \right| \leq \sqrt{d}\, n 2^{-n+1}, \quad \forall\, \mathbb{S} \in \mathbb{D}.$$

Proof. Using the definition of $\hat{\mu}$ and g, we directly calculate that

$$\begin{aligned}\int_{\mathbb{R}^d_+} g d\mu &= \sum_{m\in\mathbb{N}^d} \hat{g}\left(m2^{-n}\right) \mu\left(\{x : \pi^{(n)}(x) = m2^{-n}\}\right) \\ &= \sum_{m\in\mathbb{N}^d} \hat{g}\left(m2^{-n}\right) \hat{\mu}\left(\{m2^{-n}\}\right) = \int_{A^{(n)}} \hat{g} d\hat{\mu}.\end{aligned}$$

Since $\hat{\phi}$ is bounded by definition, the admissibility of ϕ would follow if γ is progressively measurable. We show this by verifying (2.1). Towards this goal, let $\mathbb{S}, \tilde{\mathbb{S}} \in \mathbb{D}$ and $t \in [0, T]$ be such that $\mathbb{S}_u = \tilde{\mathbb{S}}_u$ for all $u \leq t$. We have to show that $\gamma_t(\mathbb{S}) = \gamma_t(\tilde{\mathbb{S}})$.

Since $\gamma_0(\mathbb{S}) = \gamma_0(\tilde{\mathbb{S}}) = 0$, we may assume that $t > 0$. Let $0 \leq k_t(\mathbb{S})$ be the integer such that $t \in (\tau_{k_t}(\mathbb{S}), \tau_{k_t+1}(\mathbb{S})]$. Since by hypothesis $\mathbb{S}$ and $\tilde{\mathbb{S}}$ agree on $[0, t]$, their jump times up to time t also agree. In particular, $k_t(\mathbb{S}) = k_t(\tilde{\mathbb{S}}) =: k_t$ and

$$\tau_i(\mathbb{S}) = \tau_i(\tilde{\mathbb{S}}) < t \quad \text{and} \quad \mathbb{S}_{\tau_i(\mathbb{S})} = \tilde{\mathbb{S}}_{\tau_i(\tilde{\mathbb{S}})}, \quad \forall\, i = 1, \dots, k_t.$$

Since for any $k \geq 0$, $\hat{\tau}_{k+1}$ is defined directly by $\tau_1, \ldots, \tau_k$, we also conclude that

$$\hat{\tau}_i(\mathbb{S}) = \hat{\tau}_i(\tilde{\mathbb{S}}), \quad \forall\, i = 0, 1, \ldots, k_t + 1.$$

Set $\theta := \hat{\tau}_{k_t+1}(\mathbb{S}) = \hat{\tau}_{k_t+1}(\tilde{\mathbb{S}})$ so that

$$\gamma_t(\mathbb{S}) = \hat{\gamma}_\theta(\hat{\Pi}(\mathbb{S})) \quad \text{and} \quad \gamma_t(\tilde{\mathbb{S}}) = \hat{\gamma}_\theta(\hat{\Pi}(\tilde{\mathbb{S}})).$$

Since, $\hat{\gamma}$ is predictable, to prove $\gamma_t(\mathbb{S}) = \gamma_t(\tilde{\mathbb{S}})$ it suffices to show that

$$\hat{\Pi}_u(\mathbb{S}) = \hat{\Pi}_u(\tilde{\mathbb{S}}), \quad \forall\, u < \theta.$$

By the definition of $\hat{\Pi}$, for any $u < \theta$ there exists an integer $k \leq k_t$ (same for both $\mathbb{S}$ and $\tilde{\mathbb{S}}$) so that

$$\hat{\Pi}_u(\mathbb{S}) = \pi(\mathbb{S}_{\tau_k}) \quad \text{and} \quad \hat{\Pi}_u(\tilde{\mathbb{S}}) = \pi(\tilde{\mathbb{S}}_{\tau_k}).$$

Now recall that $\mathbb{S}$ and $\tilde{\mathbb{S}}$ agree on $[0, t]$ and $\tau_k \leq \tau_{k_t} < t$. Hence, $\mathbb{S}_{\tau_k} = \tilde{\mathbb{S}}_{\tau_k}$ and consequently $\hat{\Pi}_u(\mathbb{S}) = \hat{\Pi}_u(\tilde{\mathbb{S}})$. This proves that γ is progressively measurable.

We continue by estimating the difference of the two integrals. In view of the definitions, we have the following representations for the stochastic integrals,

$$\int_0^T \gamma_u(\mathbb{S}) d\mathbb{S}_u = \sum_{k=1}^M \hat{\gamma}_{\hat{\tau}_k(\mathbb{S})}(\hat{\Pi}(\mathbb{S})) \left(\mathbb{S}_{\tau_k(\mathbb{S})} - \mathbb{S}_{\tau_{k-1}(\mathbb{S})}\right)$$

and

$$\int_0^T \hat{\gamma}_u(\hat{\Pi}(\mathbb{S})) d\hat{\Pi}_u(\mathbb{S}) = \sum_{k=1}^M \hat{\gamma}_{\hat{\tau}_k(\mathbb{S})}(\hat{\Pi}(\mathbb{S})) \left(\pi^{(n+k)}(\mathbb{S}_{\tau_k(\mathbb{S})}) - \pi^{(n+k-1)}(\mathbb{S}_{\tau_{k-1}(\mathbb{S})})\right).$$

Set

$$\mathcal{I} := \int_0^T \hat{\gamma}_u(\hat{\Pi}(\mathbb{S})) d\hat{\Pi}_u(\mathbb{S}) - \int_0^T \gamma_u(\mathbb{S}) d\mathbb{S}_u.$$

Since the portfolio $\hat{\gamma}$ is bounded by n, we have the following estimate,

$$|\mathcal{I}| \leq 2\|\hat{\gamma}\|_\infty \sum_{k=1}^M \left|\pi^{(n+k)}(\mathbb{S}_{\tau_k(\mathbb{S})}) - \mathbb{S}_{\tau_k(\mathbb{S})}\right| \leq 2n \sum_{k=1}^M \sqrt{d}\, 2^{-(n+k)} \leq \sqrt{d}\, n 2^{-n+1}.$$

In view of the above results and the construction, g is bounded and therefore, $g \in \mathbb{L}^1(\mathbb{R}_+^d; \mu)$. Moreover, γ is shown to be progressively measurable and for $t \in [\tau_k, \tau_{k+1})$

$$\begin{aligned}\int_0^t \gamma_u(\mathbb{S}) d\mathbb{S}_u &\geq \int_0^{\hat{\tau}_k} \hat{\gamma}_u(\hat{\Pi}(\mathbb{S})) d\hat{\Pi}_u(\mathbb{S}) - 2\|\hat{\gamma}\|_\infty \sum_{k=1}^M \left|\pi^{(n+k)}(\mathbb{S}_{\tau_k(\mathbb{S})}) - \mathbb{S}_{\tau_k(\mathbb{S})}\right| - \frac{n}{2^n} \\ &\geq -C - \sqrt{d}\, n 2^{-n+1} - \frac{n}{2^n},\end{aligned}$$

where the last inequality follows from the fact that $\hat{\gamma}$ is admissible in the sense of Definition 3.2. Hence, the stochastic integral is bounded from below and consequently is a $\mathbb{Q}$ super-martingale for every $\mathbb{Q} \in \mathbb{M}_\mu$. These arguments imply that the lifted portfolio (g, γ) is admissible. $\square$

The above lifting result provides an immediate connection between $V(G)$ and $V^{(n)}(G)$.

Corollary 3.7. *Under the hypothesis of Theorem 2.9, the minimal super-replication costs satisfy*

$$V(G) \le V^{(n)}(G) + \sqrt{d}\, n 2^{-n+1} + 3m_G(3\sqrt{d}\, 2^{-n}).$$

In particular,

$$V(G) \le \liminf_{n\to\infty} V^{(n)}(G).$$

Proof. Let $\hat{\phi}$ and ϕ be as in Lemma 3.6. Further assume that $\hat{\phi}$ is super-replicating G on $\hat{\mathbb{D}}$. Let $\mathbb{S} \in \mathbb{D}$. Then, $\hat{\Pi}(\mathbb{S}) \in \hat{\mathbb{D}}$ and

$$\hat{g}(\hat{\Pi}_T(\mathbb{S})) + \int_0^T \hat{\gamma}_t(\hat{\Pi}(\mathbb{S}))d\hat{\Pi}(\mathbb{S})_t \ge G(\hat{\Pi}(\mathbb{S})).$$

By definition of g and $\hat{\Pi}$,

$$g(\mathbb{S}_T) = \hat{g}(\pi^{(n)}(\mathbb{S}_T)) = \hat{g}(\hat{\Pi}_T(\mathbb{S})).$$

Then, in view of Lemma 3.6,

$$\begin{aligned} g(\mathbb{S}_T) + \int_0^T \gamma_t(\mathbb{S})d\mathbb{S}_t &\ge \hat{g}(\hat{\Pi}_T(\mathbb{S})) + \int_0^T \hat{\gamma}_t(\hat{\Pi}(\mathbb{S}))d\hat{\Pi}(\mathbb{S})_t - \sqrt{d}\, n2^{-n+1} \\ &\ge G(\hat{\Pi}(\mathbb{S})) - \sqrt{d}\, n2^{-n+1} \\ &\ge \check{G}(\mathbb{S}) := G(\mathbb{S}) - \sqrt{d}\, n2^{-n+1} - 3m_G(3\sqrt{d}\, 2^{-n}). \end{aligned}$$

Hence, ϕ super-replicates $\check{G}$. This implies that $\int g d\mu \ge V(\check{G})$. Since by construction $\int g d\mu = \int \hat{g} d\hat{\mu}$, and since above inequality holds for every super-replicating $\hat{\phi}$, we conclude that $V(\check{G}) \le V^{(n)}(G)$. It is also clear that

$$\begin{aligned} V(G) &= V(\check{G}) + \sqrt{d}\, n2^{-n+1} + 3m_G(3\sqrt{d}\, 2^{-n}) \\ &\le V^{(n)}(G) + \sqrt{d}\, n2^{-n+1} + 3m_G(3\sqrt{d}\, 2^{-n}). \quad \square \end{aligned}$$

Remark 3.8. Observe that since $\hat{g}$ is bounded, so is the lifted static hedge g. Hence in Definition 2.5, one may use the class

$$\tilde{\mathcal{H}} := \{h(\mathbb{S}) = g(\mathbb{S}_T) : g \in \mathbb{L}^\infty(\mathbb{R}^d_+; \mu)\ \}.$$

Moreover, it is not difficult to construct g so that it agrees with $\hat{g}$ on $A^{(n)}$ and is continuous. This construction would enable us to consider the even smaller class $\check{\mathcal{H}}$ with bounded and continuous g's.

Moreover, in Definition 3.2 the stochastic integral $\hat{\gamma}_u d\mathbb{S}_u$ is assumed to be bounded from below by a constant C. In view of the above Lemma, also the lifted portfolio satisfies that the path wise integral $\int \gamma_u \mathbb{S}_u$ is also bounded from below, possibly with a slightly larger constant. This shows that in Definition 2.5 it would be sufficient to consider γ's so that the integrals are bounded from below, instead of assuming that their stochastic equivalents are $\mathbb{Q}$ super-martingales for every $\mathbb{Q} \in \mathbb{M}_\mu$.

The above corollary is the only place in the proof of the upper bound (under the hypothesis of Theorem 2.9) where the exact definition of admissibility is important. Hence the above discussions and Remark 2.10 show that for G satisfying the hypothesis of Theorem 2.9, the super-replication cost of G would be same if one considers the described smaller class of admissible strategies (g, γ). $\square$

3.4. Analysis of $V^{(n)}(G)$

In view of the previous corollary, to complete the proof of (2.9), we need to show the following inequality,

$$\limsup_{n\to\infty} V^{(n)}(G) \le \sup_{\mathbb{Q}\in\mathbb{M}_\mu} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})].$$

This is done in two steps. We first use a standard min–max theorem and the constrained duality result of [15] to get a dual representation for $V^{(n)}(G)$ (in fact we obtain an upper bound). We then analyze this dual by probabilistic techniques.

We start with a definition.

Definition 3.9. Let $\mathcal{P}$ be the set of all probability measures $\mathbb{Q}$ which are supported on $\hat{\mathbb{D}} = \mathbb{D}^{(n)}[0,T]$. For $c > 0$ let $\mathcal{M}(n,c) \subset \mathcal{P}$ be the set of all probability measures that has the following properties,

$$\sum_{m\in\mathbb{N}^d} \left|\mathbb{Q}\left(\hat{\mathbb{S}}_T = m2^{-n}\right) - \hat{\mu}\left(\{m2^{-n}\}\right)\right| \le \frac{c}{n}$$

and

$$\mathbb{E}_{\mathbb{Q}}\left[\sum_{k=1}^{M+1}\left|\mathbb{E}_{\mathbb{Q}}(\hat{\mathbb{S}}_{\hat{\tau}_k}|\hat{\mathcal{F}}_{\hat{\tau}_k -}) - \hat{\mathbb{S}}_{\hat{\tau}_{k-1}}\right|\right] \le \frac{c}{n}, \tag{3.6}$$

where as defined before, $\hat{\tau}_1(\hat{\mathbb{S}}) < \cdots < \hat{\tau}_M(\hat{\mathbb{S}})$ are the jump times of the piecewise constant process $\hat{\mathbb{S}} \in \hat{\mathbb{D}}$ and $\hat{\tau}_0 = 0,\ \hat{\tau}_{M+1} = T$.

We refer the reader to p. 105 in [26] for the definition of the σ-algebra $\hat{\mathcal{F}}_{\hat{\tau}_j -}$. Indeed, for any stopping time $\tau \in [0,T]$, $\hat{\mathcal{F}}_{\tau-}$ is defined to be the smallest σ-algebra that contains $\hat{\mathcal{F}}_0$ and all sets of the from $A \cap \{\tau > t\}$ for all $t \in (0,T]$ and $A \in \hat{\mathcal{F}}_t$. Clearly, $\hat{\mathcal{F}}\tau- \subset \hat{\mathcal{F}}_\tau$ and τ is $\hat{\mathcal{F}}\tau-$ measurable. Moreover, if X is a predictable process, then X_τ is $\hat{\mathcal{F}}\tau$-measurable (Theorem 8, p. 106 [26]).

The following lemma is proved by using the results of [15] on hedging under constraints, and applying a classical min–max theorem.

Lemma 3.10. *Suppose that* $0 \le G \le c$ *for some constant* $c > 0$. *Then,*

$$V^{(n)}(G) \le \left[\sup_{\mathbb{Q}\in\mathcal{M}(c,n)} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})]\right]^+,$$

where we set the right hand side to zero if $\mathcal{M}(c,n)$ *is empty.*

Proof. We proceed in several steps.

Step 1. In view of its definition, for any bounded function $\hat{g}$ on $A^{(n)}$, we have

$$V^{(n)}(G) \le \mathcal{V}^{(n)}(G \ominus \hat{g}) + \int g\, d\hat{\mu},$$

where $G \ominus \hat{g}(\mathbb{S}) := G(\mathbb{S}) - \hat{g}(\mathbb{S}_T)$ and for any bounded measurable real valued function ξ on $\mathbb{D}$,

$$\mathcal{V}^{(n)}(\xi) = \inf\left\{z \in \mathbb{R} : \exists\gamma \text{ such that} |\gamma| \le n,\ z + \int_0^T \gamma_u d\mathbb{S}_u \ge \xi,\ \mathbb{P}\text{-a.s.}\right\}$$

to be the "classical" super-hedging price of the European claim ξ under the constraint that absolute value of the number of the stocks in the portfolio is bounded by n. Furthermore, (as usual) we require that there exists $M > 0$ such that $\int_0^t \gamma_u d\mathbb{S}_u \geq -M$, for every $t \in [0, T]$.

Step 2. Under any measure $\mathbb{Q} \in \mathcal{P}$ the canonical process $\mathbb{S}$ on $\mathbb{D}$ is piecewise constant with jump times $0 < \hat{\tau}_1 < \cdots < \hat{\tau}_M < T$. So it is clear that the canonical process is a $\mathbb{Q}$ semi-martingale. Moreover, it has the following decomposition, $\mathbb{S} = \mathbb{M}^{\mathbb{Q}} - A^{\mathbb{Q}}$ where

$$A_t^{\mathbb{Q}} = \sum_{k=1}^{M} \chi_{[\hat{\tau}_k, \hat{\tau}_{k+1})}(t) \sum_{j=1}^{k} \left[\mathbb{S}_{\hat{\tau}_{j-1}} - \mathbb{E}_{\mathbb{Q}}(\mathbb{S}_{\hat{\tau}_j} | \hat{\mathcal{F}}_{\hat{\tau}_j -}) \right], \quad \forall\, t \in [0, T) \tag{3.7}$$

$$A_T^{\mathbb{Q}} := \lim_{t \uparrow T} A_t^{\mathbb{Q}},$$

a predictable process of bounded variation and $\mathbb{M}_t^{\mathbb{Q}} = A_t^{\mathbb{Q}} + \mathbb{S}_t$, $t \in [0, T]$, is a $\mathbb{Q}$ martingale. Then, from Example 2.3 and Proposition 4.1 in [15] it follows that

$$\mathcal{V}^{(n)}(\xi) = \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[\xi - n \sum_{k=1}^{M} \left| \mathbb{S}_{\hat{\tau}_{k-1}} - \mathbb{E}_Q(\mathbb{S}_{\hat{\tau}_j} | \hat{\mathcal{F}}_{\hat{\tau}_j -}) \right| \right].$$

Step 3. Set

$$\mathcal{Z} := \{\hat{g} : A^{(n)} \to \mathbb{R} : \|\hat{g}\|_\infty \leq n\}.$$

In view of the previous steps,

$$V^{(n)}(G) \leq \inf_{\hat{g} \in \mathcal{Z}} \sup_{\mathbb{Q} \in \mathcal{P}} \mathcal{G}(\hat{g}, \mathbb{Q}),$$

where $\mathcal{G} : \mathcal{Z} \times \mathcal{P} \to \mathbb{R}$ is given by

$$\mathcal{G}(\hat{g}, \mathbb{Q}) := \mathbb{E}_{\mathbb{Q}} \left[G - n \sum_{k=1}^{M} \left| \mathbb{E}_{\mathbb{Q}}(\mathbb{S}_{\hat{\tau}_k} | \hat{\mathcal{F}}_{\hat{\tau}_k -}) - \mathbb{S}_{\hat{\tau}_{k-1}} \right| \right] + \int \hat{g}\, d\hat{\mu} - \mathbb{E}_{\mathbb{Q}} \hat{g}(\mathbb{S}_T).$$

Step 4. This step is to interchange the order of the infimum and supremum by applying a standard min–max theorem. Indeed, consider the vector space $\mathbb{R}^{A^{(n)}}$ of all functions $\hat{g} : A^{(n)} \to \mathbb{R}$ equipped with the topology of point-wise convergence. Clearly, this space is locally convex. Also, since $A^{(n)}$ is countable, $\mathcal{Z}$ is a compact subset of $\mathbb{R}^{A^{(n)}}$. The set $\mathcal{P}_N$ can be naturally considered as a convex subspace of the vector space $\mathbb{R}_+^{\hat{\mathbb{D}}}$. In order to apply a min–max theorem, we also need to show continuity and concavity.

$\mathcal{G}$ is affine in the first variable, and by the bounded convergence theorem, it is continuous in this variable. We claim that $\mathcal{G}$ is concave in the second variable. To this purpose, it is sufficient to show that for any $k \geq 1$ the map

$$\mathbb{Q} \to \mathbb{E}_{\mathbb{Q}} |\mathbb{E}_{\mathbb{Q}}(\mathbb{S}_{\hat{\tau}_k} | \hat{\mathcal{F}}_{\hat{\tau}_k -}) - \mathbb{S}_{\hat{\tau}_{k-1}}|$$

is convex. Set $X = \mathbb{S}_{\hat{\tau}_k} - \mathbb{S}_{\hat{\tau}_{k-1}}$, $\hat{\mathcal{F}} := \hat{\mathcal{F}}_{\hat{\tau}_k -}$ and $Y = \mathbb{E}_{\mathbb{Q}}(X | \hat{\mathcal{F}})$. For probability measures $\mathbb{Q}_1, \mathbb{Q}_2$ and $\lambda \in (0, 1)$, set $Y_i = \mathbb{E}_{\mathbb{Q}_i}(X | \mathcal{F})$ and $\mathbb{Q} = \lambda \mathbb{Q}_1 + (1 - \lambda)\mathbb{Q}_2$. Then,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} |Y| &= \mathbb{E}_{\mathbb{Q}}(Y \chi_{\{Y > 0\}}) - \mathbb{E}_{\mathbb{Q}}(Y \chi_{\{Y < 0\}}) \\ &= \mathbb{E}_{\mathbb{Q}}(X \chi_{\{Y > 0\}}) - \mathbb{E}_{\mathbb{Q}}(X \chi_{\{Y < 0\}}) \\ &= \lambda \left(\mathbb{E}_{\mathbb{Q}_1}(X \chi_{\{Y > 0\}}) - \mathbb{E}_{\mathbb{Q}_1}(X \chi_{\{Y < 0\}}) \right) \end{aligned}$$

$$+ (1-\lambda)\left(\mathbb{E}_{\mathbb{Q}_2}(X\chi_{\{Y>0\}}) - \mathbb{E}_{\mathbb{Q}_2}(X\chi_{\{Y<0\}})\right)$$
$$= \lambda\left(\mathbb{E}_{\mathbb{Q}_1}(Y_1\chi_{\{Y>0\}}) - \mathbb{E}_{\mathbb{Q}_1}(Y_1\chi_{\{Y<0\}})\right)$$
$$+ (1-\lambda)\left(\mathbb{E}_{\mathbb{Q}_2}(Y_2\chi_{\{Y>0\}}) - \mathbb{E}_{\mathbb{Q}_2}(Y_2\chi_{\{Y<0\}})\right)$$
$$\leq \lambda\mathbb{E}_{\mathbb{Q}_1}|Y_1| + (1-\lambda)\mathbb{E}_{\mathbb{Q}_2}|Y_2|.$$

This yields the concavity of $\mathcal{G}$ in the $\mathbb{Q}$-variable.

Step 5. Next, we apply the min–max theorem, Theorem 45.8 in [31] to $\mathcal{G}$. The result is,

$$\inf_{\hat{g}\in\mathcal{Z}}\sup_{\mathbb{Q}\in\mathcal{P}}\mathcal{G}(\hat{g},\mathbb{Q}) = \sup_{\mathbb{Q}\in\mathcal{P}}\inf_{\hat{g}\in\mathcal{Z}}\mathcal{G}(\hat{g},\mathbb{Q}).$$

Together with Step 3, we conclude that

$$V^{(n)}(G) \leq \sup_{\mathbb{Q}\in\mathcal{P}}\inf_{h\in\mathcal{Z}}\mathcal{G}(\hat{g},\mathbb{Q}).$$

Finally, for any measure $\mathbb{Q}\in\mathcal{P}$, define $h^{\mathbb{Q}}\in\mathcal{Z}$ by

$$h^{\mathbb{Q}}\left(m2^{-n}\right) = c\; sign\left[\mathbb{Q}\left(\{\mathbb{S}_T = m2^{-n}\}\right) - \hat{\mu}\left(\{\mathbb{S}_T = m2^{-n}\}\right)\right], \quad m\in\mathbb{N}^d.$$

Then, by choosing $h^{\mathbb{Q}}$ in the min–max formula, we arrive at,

$$V^{(n)}(G) \leq \sup_{\mathbb{Q}\in\mathcal{P}}\mathcal{G}(h^{\mathbb{Q}},\mathbb{Q}).$$

Moreover,

$$\int h^{\mathbb{Q}}d\hat{\mu} - \mathbb{E}_{\mathbb{Q}}h^{\mathbb{Q}}(\mathbb{S}_T) = -n\sum_{m\in\mathbb{N}^d}\left|\mathbb{Q}\left(\hat{\mathbb{S}}_T = m2^{-n}\right) - \hat{\mu}\left(\{m2^{-n}\}\right)\right|.$$

Hence if $\mathbb{Q}$ does not belong to the set $\mathcal{M}(c,n)$, then

$$\mathcal{G}(h^{\mathbb{Q}},\mathbb{Q}) \leq \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})] - c.$$

By hypothesis, $0\leq G\leq c$ and therefore, $\mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})]\leq c$ and $V^{(n)}(G)\geq 0$. Hence, we may restrict the maximization to $\mathbb{Q}\in\mathcal{M}(c,n)$. Moreover, if $\mathcal{M}(c,n)$ is empty, then we can conclude that $V^{(n)}(G)\leq 0$. □

3.5. *Proof of (2.9) completed.*

In order to complete the proof of Theorem 2.9 it remains to establish the following result.

Lemma 3.11. *Suppose that* $0\leq G\leq c$ *and satisfies the Assumption 2.7. Then*

$$\limsup_{n\to\infty}\left[\sup_{\mathbb{Q}\in\mathcal{M}(c,n)}\mathbb{E}_{\mathbb{Q}}\left[G(\mathbb{S})\right]\right]^{+} \leq \sup_{\mathbb{Q}\in\mathbb{M}_\mu}\mathbb{E}_{\mathbb{Q}}\left[G(\mathbb{S})\right]. \tag{3.8}$$

Proof. Without loss of generality (by passing to a subsequence), we may assume that the sequence on the left hand side of (3.8) is convergent. Moreover, we may assume that for sufficiently large n the set $\mathcal{M}(c,n)$ is not empty, otherwise (3.8) is trivially satisfied.

Step 1. Choose $\mathbb{Q}_n\in\mathcal{M}(c,n)$ such that

$$\left[\sup_{\mathbb{Q}\in\mathcal{M}(c,n)}\mathbb{E}_{\mathbb{Q}}\left[G(\mathbb{S})\right]\right]^{+} \leq 2^{-n} + \mathbb{E}_{\mathbb{Q}_n}\left[G(\mathbb{S})\right].$$

Hence,

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[G(\mathbb{S})] = \lim\sup_{n\to\infty} \left[\sup_{\mathbb{Q}\in\mathcal{M}(c,n)} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})] \right]^+ .$$

Recall the decomposition given in the second step of the proof of Lemma 3.10. Set $\mathbb{M}^n := \mathbb{M}^{\mathbb{Q}_n}$, $A^n := A^{\mathbb{Q}_n}$. Since G is uniformly continuous in the Skorokhod metric,

$$\left|G(\mathbb{S}) - G(\mathbb{M}^n(\mathbb{S}))\right| \le m_G(n^{-1/2}), \quad \text{whenever } \sup_{t\in[0,T]} A^n_t(\mathbb{S}) \le n^{-1/2}.$$

Therefore, since $|G(\mathbb{S}) - G(\mathbb{M}^n(\mathbb{S}))| \le c$,

$$\left|\mathbb{E}_{\mathbb{Q}_n}\left[G(\mathbb{S}) - G(\mathbb{M}^n(\mathbb{S}))\right]\right| \le m_G(n^{-1/2}) + c\,\mathbb{Q}_n\left(\sup_{t\in[0,T]} A^n_t \ge n^{-1/2}\right).$$

We now use the representation (3.7) of A^n together with the Markov inequality. The result is,

$$\mathbb{Q}_n(\sup_{t\in[0,T]} A^n_t \ge n^{-1/2}) \le n^{1/2}\mathbb{E}_{\mathbb{Q}_n}\sum_{k=1}^{M}\left|\mathbb{E}_{\mathbb{Q}_n}(\mathbb{S}_{\hat{\tau}_k} \mid \mathcal{F}_{\hat{\tau}_k -}) - \mathbb{S}_{\hat{\tau}_{k-1}}\right| \le cn^{-1/2},$$

where the last inequality follows from the fact that $\mathbb{Q}_n \in \mathcal{M}(c, n)$ and (3.6). Therefore, we have concluded that

$$\lim\sup_{n\to\infty} \left[\sup_{\mathbb{Q}\in\mathcal{M}(c,n)} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})] \right]^+ = \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}\left[G(\mathbb{M}^n(\mathbb{S}))\right].$$

Step 2. As in Section 2.2, let $\tilde{\Omega} := \mathbb{D}([0,T];\mathbb{R}^d)$, $\tilde{\mathbb{F}}$ be the filtration generated by the canonical process $\tilde{\mathbb{S}}$. For a probability measure $\tilde{\mu}$ on $\mathbb{R}^d$, let $\tilde{\mathbb{M}}_{\tilde{\mu}}$ be a set of measures $\tilde{\mathbb{Q}}$ on $\mathbb{D}([0,T];\mathbb{R}^d)$ such that the canonical process is a martingale that starts at $\tilde{\mathbb{S}}_0 = (1,\ldots,1)$ and the distribution of $\tilde{\mathbb{S}}_T$ under $\tilde{\mathbb{Q}}$ is equal to $\tilde{\mu}$. Note that when the support of $\tilde{\mu}$ is on $\mathbb{R}^d_+$, then the support of any measure $\tilde{\mathbb{Q}} \in \tilde{\mathbb{M}}_{\tilde{\mu}}$ is included in $\mathbb{D}$. Hence, in that case $\tilde{\mathbb{M}}_{\tilde{\mu}}$ is the same as $\mathbb{M}_{\tilde{\mu}}$ defined earlier.

We set

$$v(\tilde{\mu}) := \sup_{\tilde{\mathbb{Q}}\in\tilde{\mathbb{M}}_{\tilde{\mu}}} \mathbb{E}_{\tilde{\mathbb{Q}}}[G(\tilde{\mathbb{S}})]. \tag{3.9}$$

Let $\tilde{\mathbb{Q}}_n$ be the measure on $\mathbb{D}([0,T];\mathbb{R}^d)$ induced by $\mathbb{M}^n$ under $\mathbb{Q}_n$, i.e., for any Borel subset $C \subset \mathbb{D}([0,T];\mathbb{R}^d)$,

$$\tilde{\mathbb{Q}}_n(C) := \mathbb{Q}_n\left(\left\{\mathbb{S}\in\mathbb{D} : \mathbb{M}^n(\mathbb{S}) \in C\right\}\right).$$

Further, let ν_n be the distribution of $\mathbb{M}^n_T$ under the measure $\mathbb{Q}_n$. Since $\mathbb{M}^n$ is a martingale, it is clear that $\tilde{\mathbb{Q}}_n \in \tilde{\mathbb{M}}_{\nu_n}$. Then, the previous step implies that

$$\lim\sup_{n\to\infty} \left[\sup_{\mathbb{Q}\in\mathcal{M}(c,n)} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})] \right]^+ \le \lim_{n\to\infty} v(\nu_n).$$

Step 3. Since $\mathbb{Q}_n \in \mathcal{M}(c,n)$, (3.6) implies that

$$\mathbb{E}_{\mathbb{Q}_n}\left|\mathbb{S}_T - \mathbb{M}^n_T(\mathbb{S})\right| = \mathbb{E}_{\mathbb{Q}_n}\left|A^n_T\right| \le \frac{c}{n}.$$

Let μ_n be the distribution of $\mathbb{S}_T$ under $\mathbb{Q}_n$. Then, by the definition of $\mathcal{M}(c,n)$, μ_n converges weakly to μ. Then, above inequalities imply that ν_n also converges weakly to μ.

Since each component $\mathbb{S}_t^{(k)} > 0$, for all $t \in [0,T]$ and $k = 1,\dots,d$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_n}[((\mathbb{M}^n)_T^{(k)}(\mathbb{S}))^-] &= \mathbb{E}_{\mathbb{Q}_n}[(-(\mathbb{M}^n)_T^{(k)}(\mathbb{S}))\ \chi_{\{(\mathbb{M}^n)_T^{(k)}(\mathbb{S})\le 0\}}] \\
&\le \mathbb{E}_{\mathbb{Q}_n}[(\mathbb{S}_T^{(k)} - (\mathbb{M}^n)_T^{(k)}(\mathbb{S}))\ \chi_{\{(\mathbb{M}^n)_T^{(k)}(\mathbb{S})\le 0\}}] \\
&\le \mathbb{E}_{\mathbb{Q}_n}\left|\mathbb{S}_T - \mathbb{M}_T^n(\mathbb{S})\right| \\
&= \mathbb{E}_{\mathbb{Q}_n}\left|A_T^n\right| \le \frac{c}{n}.
\end{aligned}$$

Hence, for each $k = 1,\dots,d$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} (x_k)^- d\nu_n(x) = 0.$$

Hence we are in a position to use the continuity result, Theorem 4.1 proved in the next section. This implies that

$$\lim_{n\to\infty} v(\nu_n) = v(\mu).$$

Since μ is supported on $\mathbb{R}_+^d$, as remarked before, $\tilde{\mathbb{M}}_\mu = \mathbb{M}_\mu$. We now combine all the steps of this proof to arrive at,

$$\begin{aligned}
\lim\sup_{n\to\infty} \left[\sup_{\mathbb{Q}\in\mathcal{M}(c,n)} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})]\right]^+ &= \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}\left[G(\mathbb{M}^n(\mathbb{S}))\right] \\
&\le \lim_{n\to\infty} v(\nu_n) = v(\mu) \\
&= \sup_{\mathbb{Q}\in\mathbb{M}_\mu} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})]. \quad \square
\end{aligned}$$

4. Continuity of the dual with respect to μ

In this section, we prove a continuity result for a martingale optimal transport problem on the space $\mathbb{D}$. Recall the functional $v(\tilde{\mu})$ defined in (3.9) and the set of martingale measures $\tilde{\mathbb{M}}_\nu$ again defined in (3.9).

Theorem 4.1. *Suppose G is bounded and satisfies the Assumption 2.7. Let ν_n be a sequence of probability measures on $\mathbb{R}^d$. Assume that ν_n converges weakly to a probability measure μ supported on $\mathbb{R}_+^d$. Further assume that for each component $k = 1,\dots,d$,*

$$\lim_{n\to\infty} \int x^{(k)} d\nu_n(x) = \int x^{(k)} d\mu(x) < \infty, \quad \textit{and} \quad \lim_{n\to\infty} \int (x^{(k)})^- d\nu_n(x) = 0.$$

Then,

$$\lim_{n\to\infty} v(\nu_n) = v(\mu).$$

Proof. To ease the notation, we take $d = 1$. First we prove that

$$\lim\sup_{n\to\infty} v(\nu_n) \le v(\mu). \tag{4.1}$$

In fact this the inequality that we used in the proof of Lemma 3.11. For each $n \in \mathbb{N}$ choose $\tilde{Q}_n \in \tilde{\mathbb{M}}_{\nu_n}$ such that

$$v(\nu_n) \le 2^{-n} + \mathbb{E}_{\mathbb{Q}_n}[G(\tilde{\mathbb{S}})].$$

Step 1. In the first step, we construct a martingale measure in $\mathbb{M}_\mu$ that is "close" to $\tilde{Q}_n$. This construction uses the Prokhorov's metric which we now recall. For any two probability measures ν, ρ on $\mathbb{R}$, the Prokhorov distance $\hat{d}(\nu, \rho)$ is defined to be the smallest $\delta > 0$ so that

$$\nu(C) \le \rho(C_\delta) + \delta, \quad \text{and} \quad \rho(C) \le \nu(C_\delta) + \delta,$$

for every Borel subset $C \subset \mathbb{R}$, where

$$C_\delta := \bigcup_{x \in C} (x - \delta, x + \delta).$$

It is well known that convergence in the Prokhorov metric is equivalent to weak convergence, (for more details on Prokhorov's metric we refer the reader to [27], Chapter 3, Section 7).

We now follow Theorem 4 on p. 358 in [27] and Theorem 1 in [28] to construct a random variable $\Lambda^{(n)}$ as follows. First construct a probability space $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{\mathbb{P}}_n)$ and a martingale $\mathbb{M}^{(n)}$ and a random variable $\xi^{(n)}$ uniformly distributed on $[0, T]$ such that:

a. $\xi^{(n)}$ and $\mathbb{M}^{(n)}$ are independent;
b. distribution of $\mathbb{M}^{(n)}$ under $\mathbb{P}_n$ is equal to the measure $\tilde{\mathbb{Q}}_n$ on $\mathbb{D}([0, T]; \mathbb{R})$. In particular,

$$\mathbb{E}_{\tilde{\mathbb{P}}_n}[G(\mathbb{M}^{(n)})] = \mathbb{E}_{\mathbb{Q}_n}[G(\tilde{\mathbb{S}})].$$

We may choose the filtration $\tilde{\mathcal{F}}$ to be the smallest right-continuous filtration that is generated by the processes $\mathbb{M}^{(n)}$ and $\xi_t^{(n)} := \xi^{(n)} \wedge t$. Recall that $\xi^{(n)}$ is uniformly distributed on $[0, T]$ and is independent of $\mathbb{M}^{(n)}$.

Moreover, in view of [27,28] there exists a measurable function $\psi^{(n)} : \mathbb{R}^2 \to \mathbb{R}$ such that the distribution of

$$\Lambda^{(n)} := \psi^{(n)}(\mathbb{M}_T^{(n)}, \xi^{(n)}),$$

on $\mathbb{R}$ is equal to μ and

$$\tilde{\mathbb{P}}_n\left(\left|\Lambda^{(n)} - \mathbb{M}_T^{(n)}\right| > \hat{d}(\nu_n, \mu)\right) < \hat{d}(\nu_n, \mu). \tag{4.2}$$

In particular, $\Lambda^{(n)} - \mathbb{M}_T^{(n)}$ converges to zero in probability.

We set

$$\mathbb{N}_t^{(n)} := \mathbb{E}_{\tilde{\mathbb{P}}_n}[\Lambda^{(n)} \mid \tilde{\mathcal{F}}_t], \quad t \in [0, T].$$

Then, clearly $\mathbb{N}_T^{(n)} = \Lambda^{(n)}$ and hence has the distribution μ. Moreover, the right-continuity of the filtration $\tilde{\mathcal{F}}$ implies that $\mathbb{N}^{(n)}$ has a *cádlág* modification (for details see [24] Chapter 3). However, $\mathbb{N}_0^{(n)}$ is not necessarily a constant as $\tilde{\mathcal{F}}_0$ may not be trivial.[1]

So we continue by modifying $\mathbb{N}^{(n)}$ to overcome this difficulty. Since $\mathbb{N}^{(n)}$ is right continuous than there exists $\delta > 0$ such that for any $t \le \delta$

$$\tilde{\mathbb{P}}_n[|\mathbb{N}_t^{(n)} - \mathbb{N}_0^{(n)}| > \hat{d}(\nu_n, \mu)/2] < \hat{d}(\nu_n, \mu)/2.$$

[1] Authors are grateful to Professor X. Tan of Paris, Dauphine for pointing out this.

We now define the *cádlág* martingale $\{\hat{\mathbb{N}}^{(n)}_t\}_{t=0}^T$ by $\hat{\mathbb{N}}^{(n)}_t = \int x d\mu(x) = 1$ for $t < \delta/2$, $\hat{\mathbb{N}}^{(n)}_t = \mathbb{N}^{(n)}_{2t-\delta}$ for $\delta/2 \le t < \delta$, and $\hat{\mathbb{N}}^{(n)}_t = \mathbb{N}^{(n)}_t$ for $t \ge \delta$. Let $\hat{\mathcal{F}}$ be the completion of the filtration generated by $\hat{\mathbb{N}}^{(n)}$. Then, one can directly verify that $\hat{\mathbb{N}}^{(n)}$ is a $\hat{\mathcal{F}}$ martingale. Therefore, the measure on $\mathbb{D}$ induced by $\hat{\mathbb{N}}^{(n)}$ under $\tilde{\mathbb{P}}_n$ is an element in $\mathbb{M}_\mu$. In particular,

$$\mathbb{E}_{\tilde{\mathbb{P}}_n}[G(\hat{\mathbb{N}}^{(n)})] \le v(\mu).$$

In view of Assumption 2.7, for any $\epsilon > 0$

$$|G(\hat{\mathbb{N}}^{(n)}) - G(\mathbb{N}^{(n)})| \le m_G(\hat{d}(\nu_n, \mu) + 2\epsilon), \quad \text{on the set } \mathcal{A}_{n,\epsilon},$$

where

$$\mathcal{A}_{n,\epsilon} := \left\{ \sup_{0 \le t \le T} \left| \hat{\mathbb{N}}^{(n)}_t - \mathbb{N}^{(n)}_t \right| > \hat{d}(\nu_n, \mu) + 2\epsilon \right\}.$$

Thus from the choice of delta we get

$$\begin{aligned} |\mathbb{E}_{\tilde{\mathbb{P}}_n}[G(\hat{\mathbb{N}}^{(n)})] - \mathbb{E}_{\tilde{\mathbb{P}}_n}[G(\mathbb{N}^{(n)})]| &\le m_G(d(\nu_n, \mu) + \epsilon) + \|G\|_\infty \, \tilde{\mathbb{P}}_n\left(\mathcal{A}_{n,\epsilon}\right) \\ &\le m_G(\hat{d}(\nu_n, \mu) + \epsilon) + \|G\|_\infty \left(\hat{d}(\nu_n, \mu) + \tilde{\mathbb{P}}_n \left(\left| \mathbb{N}^{(n)}_0 - \int x d\mu(x) \right| > 2\epsilon \right) \right). \end{aligned} \tag{4.3}$$

Step 2.

In view of Assumption 2.7,

$$G(\mathbb{M}^{(n)}) - G(\mathbb{N}^{(n)}) \le m_G(\epsilon), \quad \text{on the set } \mathcal{A}^{(n)}_\epsilon,$$

where

$$\mathcal{A}^{(n)}_\epsilon := \left\{ \sup_{0 \le t \le T} \left| \mathbb{M}^{(n)}_t - \mathbb{N}^{(n)}_t \right| > \epsilon \right\}.$$

Hence,

$$\mathbb{E}_{\tilde{\mathbb{P}}_n}[G(\mathbb{M}^{(n)})] - \mathbb{E}_{\tilde{\mathbb{P}}_n}[G(\mathbb{N}^{(n)})] \le m_G(\epsilon) + \|G\|_\infty \, \tilde{\mathbb{P}}_n\left(\mathcal{A}^{(n)}_\epsilon\right).$$

Step 3. Observe that $\lim_{n\to\infty} \mathbb{M}^{(n)}_0 = \int x d\mu(x)$ and so for sufficiently large n $\{|\mathbb{N}^{(n)}_0 - \int x d\mu(x)| > 2\epsilon\} \subseteq \mathcal{A}^{(n)}_\epsilon$. Thus in view of Steps 1–2, (4.1) would follow if

$$\lim_{n\to\infty} \tilde{\mathbb{P}}_n\left(\mathcal{A}^{(n)}_\epsilon\right) = 0,$$

for each $\epsilon > 0$. Towards this goal, we first observe that both $\mathbb{M}^{(n)}$ and $\mathbb{N}^{(n)}$ are $(\tilde{\mathbb{P}}_n, \tilde{\mathcal{F}})$ martingales. Hence, by Doob's maximal inequality,

$$\tilde{\mathbb{P}}_n\left(\mathcal{A}^{(n)}_\epsilon\right) \le \frac{1}{\epsilon} \mathbb{E}_{\tilde{\mathbb{P}}_n} \left| \mathbb{M}^{(n)}_T - \mathbb{N}^{(n)}_T \right|.$$

Recall that by construction $\mathbb{M}^{(n)}_T$ has distribution ν_n and $\mathbb{N}^{(n)}_T$ has distribution μ. Also by hypothesis, in the limit as n tends to infinity first moments of ν_n are equal to those of μ. Hence,

$$\begin{aligned} \limsup_{n\to\infty} \mathbb{E}_{\tilde{\mathbb{P}}_n} |\mathbb{N}^{(n)}_T - \mathbb{M}^{(n)}_T| &= \limsup_{n\to\infty} \left[2\mathbb{E}_{\tilde{\mathbb{P}}_n} (\mathbb{N}^{(n)}_T - \mathbb{M}^{(n)}_T)^+ - \mathbb{E}_{\tilde{\mathbb{P}}_n} (\mathbb{N}^{(n)}_T - \mathbb{M}^{(n)}_T) \right] \\ &= 2 \limsup_{n\to\infty} \mathbb{E}_{\tilde{\mathbb{P}}_n} (\mathbb{N}^{(n)}_T - \mathbb{M}^{(n)}_T)^+. \end{aligned}$$

Step 4. In view of (4.2), $\mathbb{N}_T^{(n)} - \mathbb{M}_T^{(n)} = \Lambda^{(n)} - \mathbb{M}_T^{(n)}$ converges to zero in probability. Hence the previous step gives us the final reduction of (4.1). Namely, to prove (4.1) it suffices to show the uniform integrability of the sequence of random variables $\mathbb{M}_T^{(n)} - \mathbb{N}_T^{(n)}$.

We first briefly recall that $\mathbb{M}_T^{(n)}$ has the distribution ν_n, $\mathbb{N}_T^{(n)}$ has the distribution μ, μ is supported on the positive real line $\mathbb{R}_+$ and by hypothesis

$$\lim_{n\to\infty} \int_{\mathbb{R}} (x)^- d\nu_n(x) = -\lim_{n\to\infty} \int_{-\infty}^{0} x d\nu_n(x) = 0.$$

For brevity, set

$$X_n := \mathbb{N}_T^{(n)}, \quad Y_n := \mathbb{M}_T^{(n)},$$

and denote by $\mathbb{E}_n$ the expectation under the measure $\tilde{\mathbb{P}}_n$. We directly estimate that

$$\begin{aligned}\mathbb{E}_n \left[\chi_{\{X_n - Y_n > c\}} (X_n - Y_n)^+\right] &= \mathbb{E}_n \left[\chi_{\{X_n - Y_n > c\}}\ \chi_{\{X_n > -Y_n\}} (X_n - Y_n)^+\right] \\ &\quad + \mathbb{E}_n \left[\chi_{\{X_n - Y_n > c\}}\ \chi_{\{X_n < -Y_n\}} (X_n - Y_n)^+\right] \\ &\le 2\mathbb{E}_n \left[\chi_{\{2X_n > c\}}\ X_n\right] + 2\mathbb{E}_n \left[\chi_{\{2Y_n < -c\}}\ |Y_n|\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{c\uparrow\infty} \sup_{n\in\mathbb{N}} \mathbb{E}_n \left[\chi_{\{X_n - Y_n > c\}} (X_n - Y_n)^+\right] &\le 2 \lim_{c\uparrow\infty} \int_{c/2}^{\infty} x d\mu(x) \\ &\quad + 2 \lim_{c\uparrow\infty} \sup_{n\in\mathbb{N}} \int_{-\infty}^{-c/2} |x| d\nu_n(x) = 0.\end{aligned}$$

This proves the uniform integrability of the sequence $\mathbb{N}_T^{(n)} - \mathbb{M}_T^{(n)}$. Hence, (4.1) follows.

The opposite inequality is proved similarly by replacing the roles of ν_n and μ. □

5. Extensions

This section discusses the relaxations of the Assumption 2.7. Furthermore, in this section we consider the multi-marginal case. Thus let $0 < T_1 < T_2 < \cdots < T_N = T$ and $\mu_1 \preceq \mu_2 \preceq \cdots \preceq \mu_N$ be probability measures on $\mathbb{R}_+^d$ satisfying (2.6) We also assume that μ_N satisfies (2.7) for some $p > 1$. The space of static positions is given by (2.5).

In this section we enrich the set of trading strategies, in order to deal with possible jumps at the times $T_1, \ldots, T_{N-1}$, ($T_N = T$ is a continuity point). A trading strategy $\gamma = \{\gamma_t\}_{t=0}^T$ is an admissible trading strategy if it has the decomposition $\gamma = \gamma^{(1)} + \sum_{i=1}^{N-1} \beta_i \chi_{\{T_i\}}(t)$ where $\gamma^{(1)}$ is a portfolio trading strategy satisfies the same assumptions as in Definition 2.5 and β_i is $\mathcal{F}_{T_i -}$ measurable and bounded. The value of such trading strategy is given by

$$\int_0^t \gamma_u d\mathbb{S}_u = \int_0^t \gamma_u^{(1)} d\mathbb{S}_u + \sum_{i=1}^{N-1} (\mathbb{S}_{T_i} - \mathbb{S}_{T_i -}) \beta_i \chi_{\{T_i \le t\}}.$$

Thus an admissible semi-static portfolio is a vector $(g_1, \ldots, g_N, \gamma)$ where for any i, γ is of the above form and $g_i \in \mathbb{L}^1(\mathbb{R}_+^d, \mu_i)$. An admissible semi-static portfolio is called super-replicating, if

$$\sum_{i=1}^{N} g_i(\mathbb{S}_{T_i}) + \int_0^t \gamma_u(\mathbb{S}) d\mathbb{S}_u \ge G(\mathbb{S}), \quad \forall \mathbb{S} \in \mathbb{D}.$$

The minimal super-hedging cost of G is defined by,

$$V(G) := \inf\left\{\sum_{i=1}^{N}\int g_i d\mu_i : \exists\gamma \text{ such that } \phi := (g_1,\dots,g_N,\gamma) \text{ is super-replicating}\right\}.$$

Assumption 5.1. We modify the Skorokhod metric and define

$$\check{d}(\mathbb{S},\tilde{\mathbb{S}}) = d(\mathbb{S},\tilde{\mathbb{S}}) + \left|\int_0^T \mathbb{S}_u du - \int_0^T \tilde{\mathbb{S}}_u du\right|.$$

It is clear that

$$d(\mathbb{S},\tilde{\mathbb{S}}) \le \check{d}(\mathbb{S},\tilde{\mathbb{S}}) \le (1+T)\|\mathbb{S}-\tilde{\mathbb{S}}\|.$$

We assume that there is a modulus continuity of: i,.e., a continuous function $m_G : [0,\infty) \to [0,\infty)$ with $m_G(0) = 0$ that satisfies

$$\left|G(\mathbb{S}) - G(\tilde{\mathbb{S}})\right| \le m_G(\check{d}(\mathbb{S},\tilde{\mathbb{S}})), \quad \forall\, \mathbb{S},\tilde{\mathbb{S}} \in \mathbb{D}([0,T];\mathbb{R}^d).$$

Furthermore, we still assume that G satisfies the following growth condition instead of (1.3),

$$|G(\mathbb{S})| \le C\left(1+\|\mathbb{S}\|\right), \tag{5.1}$$

for some constant C. □

Clearly Assumption 5.1 is more general than Assumption 2.7. In particular Assumption 5.1 allows to include Asian call/put options with fixed and floating strikes

$$G(\mathbb{S}) = \left(\frac{1}{T}\int_0^T \mathbb{S}_t dt - K\right)^+, \qquad G(\mathbb{S}) = \left(\mathbb{S}_T - \frac{1}{T}\int_0^T \mathbb{S}_t dt\right)^+,$$
$$G(\mathbb{S}) = \left(K - \frac{1}{T}\int_0^T \mathbb{S}_t dt\right)^+, \qquad G(\mathbb{S}) = \left(\frac{1}{T}\int_0^T \mathbb{S}_t dt - \mathbb{S}_T\right)^+,$$

and lookback call (respectively put) options with fixed (respectively floating) strike,

$$G(\mathbb{S}) = \left(\max_{0\le t\le T} \mathbb{S}_t - K\right)^+, \qquad G(\mathbb{S}) = \max_{0\le t\le T} \mathbb{S}_t - \mathbb{S}_T.$$

Denote by $\mathbb{M}_{\mu_1,\dots,\mu_N}$ the set of all martingale measures $\mathbb{Q}$ on $(\Omega,\mathcal{F})$ such that for any $k \le N$ the probability distribution of $\mathbb{S}_{T_k}$ under $\mathbb{Q}$ is equal to μ_k. Observe that from the relations $\mu_1 \preceq \mu_2 \preceq \cdots \preceq \mu_N$, (2.6) and the fact that μ_N is satisfying (2.4) it follows that $\mathbb{M}_{\mu_1,\dots,\mu_N} \neq \emptyset$.

The aim of this section is to prove the following result.

Theorem 5.2. *Suppose that G satisfies Assumption 5.1. Further assume (2.6) and that μ_N satisfies (2.7) for some $p > 1$. Then,*

$$V(G) = \sup_{\mathbb{Q}\in\mathbb{M}_{\mu_1,\dots,\mu_N}} \mathbb{E}_{\mathbb{Q}}\left[G(\mathbb{S})\right].$$

5.1. Preparation towards the proof of Theorem 5.2

Towards the proof of the above theorem, we need several auxiliary lemmas and modifications or previous constructions.

Set $\tau_0^{(1)} =$ and define the sequence of stopping times $\tau_k^{(i)},\ i = 1, \dots, N,\ k \in \mathbb{N}$ by

$$\tau_1^{(1)} := \sqrt{d}\, 2^{-n} \wedge \inf\{t > 0 : \mathbb{S}_t \notin O(\mathbb{S}_0, n)\},$$

and for $k = 1, \dots,$

$$\tau_{k+1}^{(1)} := T_1 \wedge \left(\tau_k^{(1)} + \left[\sqrt{d}\, 2^{-n} \wedge \Delta\tau_k^{(1)}\right]\right) \wedge \inf\left\{t > \tau_k^{(1)} : \mathbb{S}_t \notin O(\mathbb{S}_{\tau_k^{(1)}}, n)\right\},$$

where, $\Delta\tau_k^{(1)} = \tau_k^{(1)} - \tau_{k-1}^{(1)}$. Set M_1 to be the smallest integer such that $\tau_{M_1}^{(1)} = T_1$. Assume that we have defined $\tau_k^{(i)}$, $i < j$, $k \in \mathbb{N}$ and M_i is the smallest integer such that $\tau_{M_i}^{(i)} = T_i$. Then, we define

$$\tau_1^{(j)} := T_{j-1} + \sqrt{d}\, 2^{-n} \wedge \inf\left\{t > T_{j-1} : \mathbb{S}_t \notin O(\mathbb{S}_{T_{j-1}}, n)\right\},$$

and for $k = 1, \dots,$

$$\tau_{k+1}^{(j)} := T_{j-1} \wedge \left(\tau_k^{(j)} + \left[\sqrt{d}\, 2^{-n} \wedge \Delta\tau_k^{(j)}\right]\right) \wedge \inf\left\{t > \tau_k^{(j)} : \mathbb{S}_t \notin O(\mathbb{S}_{\tau_k^{(j)}}, n)\right\}.$$

We fix $n \in \mathbb{N}$ and define a sequence of probability spaces $\hat{\mathbb{D}} = \hat{\mathbb{D}}^{(n)}[0, T]$. A process $\hat{\mathbb{S}} \in \mathbb{D}$ belongs to $\hat{\mathbb{D}}$, if there exists nonnegative integers $M_1, \dots, M_N$ and a partition

$$\begin{aligned} 0 &= t_0^{(1)} < t_1^{(1)} = \sqrt{d}2^{-n} < \cdots < t_{M_1}^{(1)} \\ &= T_1 = t_0^{(2)} < t_1^{(2)} = T_1 + \sqrt{d}2^{-n} < \cdots < t_{M_2}^{(2)} \\ &= T_2 = t_0^{(3)} < \cdots < t_{M_{N-1}}^{(N-1)} \\ &= T_{N-1} = t_0^{(N)} < t_1^{(2)} = T_{N-1} + \sqrt{d}2^{-n} < \cdots < t_{M_N}^{(N)} < T, \end{aligned}$$

so that

$$\hat{\mathbb{S}}_t = \sum_{i=1}^{N} \sum_{k=0}^{M_i - 1} \hat{\mathbb{S}}_{t_k^{(i)}} \chi_{[t_k^{(i)}, t_{k+1}^{(i)})}(t) + \hat{\mathbb{S}}_{t_{M_N}^{(N)}} \chi_{[t_{M_N}^{(N)}, T]}(t)$$

where $\hat{\mathbb{S}}_0 = (1, \dots, 1)$, and for any $i \le N$ and $1 \le k < M_i$,

$$\hat{\mathbb{S}}_{T_i} \in A^{(n)}, \qquad \hat{\mathbb{S}}_{t_k^{(i)}} \in A^{(n+k)}, \qquad t_{k+1}^{(i)} - t_k^{(i)} \in B^{(n+k+1)}.$$

Once again, the set $\hat{\mathbb{D}}$ is countable, thus there exists a probability measure $\mathbb{P} = \mathbb{P}^{(n)}$ on $\mathbb{D}$ with support contained in $\hat{\mathbb{D}}$, which gives positive weight to every element of $\hat{\mathbb{D}}$.

The hedging problem on the countable space is given as follows.

Definition 5.3. A (probabilistic) *semi-static portfolio* is a pair $(\hat{g}_1, \dots, \hat{g}_N, \hat{\gamma})$ such that for any i, $\hat{g}_i : A^{(n)} \to \mathbb{R}$ is a bounded function and $\hat{\gamma} : [0, T] \times \mathbb{D} \to [-n, n]$ is admissible trading strategy (in the same sense as in Definition 3.2).

A semi-static portfolio is $\mathbb{P}$-*super-replicating*, if

$$\sum_{i=1}^{N} \hat{g}_i(\mathbb{S}_{T_i}) + \int_0^T \hat{\gamma}_u d\mathbb{S}_u \ge G(\mathbb{S}), \quad \mathbb{P}\text{-a.s.}$$

The (minimal) super-hedging cost of G is defined by,

$$V^{(n)}(G) := \inf\left\{\sum_{i=1}^{N}\int \hat{g}_i d\hat{\mu}_i : \exists\gamma \text{ such that } \hat{\phi} := (\hat{g}_1,\ldots,\hat{g}_N,\hat{\gamma}) \right.$$

$$\left. \text{is admissible and super-replicating}\right\},$$

where $\hat{\mu}_1,\ldots,\hat{\mu}_N$ are probability measures on $A^{(n)}$ given by,

$$\hat{\mu}_i(\{m2^{-n}\}) := \mu_i\left(\left\{x\in\mathbb{R}^d_+ : \pi^{(n)}(x) = m2^{-n}\right\}\right), \quad m\in\mathbb{N}^d. \quad \square$$

Next, we define the lifting. Set $T_0 = 0$. For any $i = 1,\ldots,N$ introduce the stopping times

$$\hat{\tau}^{(i)}_0 := T_{i-1}, \qquad \hat{\tau}^{(i)}_1 = \sqrt{d}\,2^{-n}.$$

For $k = 2,\ldots,M_i - 1$ recursively define,

$$\hat{\tau}^{(i)}_k := \hat{\tau}^{(i)}_{k-1} + (1-\sqrt{d}\,2^{-n}/T_i)\sup\left\{\Delta t > 0 \mid \Delta t\in B^{(n+k)} \text{ and } \Delta t < \tau^{(i)}_{k-1} - \tau^{(i)}_{k-2}\right\}.$$

Also set $\hat{\tau}^{(i)}_{M_i} = T_i$.

Define,

$$\hat{\Pi}_t(\mathbb{S}) := \sum_{i=1}^{N}\sum_{k=0}^{M_i-1}\pi^{(n+k)}(\mathbb{S}_{\tau^{(i)}_k})\,\chi_{[\hat{\tau}^{(i)}_k,\hat{\tau}^{(i)}_{k+1})}(t) + \pi^{(n)}(\mathbb{S}_T)\chi_T(t), \tag{5.2}$$

$$\check{\Pi}_t(\mathbb{S}) := \sum_{i=1}^{N}\sum_{k=0}^{M_i-1}\pi^{(n+k)}(\mathbb{S}_{\tau^{(i)}_k})\,\chi_{[\tau^{(i)}_k,\tau^{(i)}_{k+1})}(t) + \pi^{(n)}(\mathbb{S}_T)\chi_T(t),$$

$$\Pi_t(\mathbb{S}) := \sum_{i=1}^{N}\sum_{k=0}^{M_i-1}\mathbb{S}_{\tau^{(i)}_k}\,\chi_{[\tau^{(i)}_k,\tau^{(i)}_{k+1})}(t) + \mathbb{S}_T\chi_T(t).$$

Similarly to Lemma 3.5 we get that

$$d(\mathbb{S},\Pi(\mathbb{S})),\ d(\Pi(\mathbb{S}),\check{\Pi}(\mathbb{S})) \le \sqrt{d}\,2^{-n}, \qquad d(\check{\Pi}(\mathbb{S}),\hat{\Pi}(\mathbb{S})) \le 3N\sqrt{d}\,2^{-n}. \tag{5.3}$$

The first two inequalities are proved in the same way as in Lemma 3.5. The third inequality done in a similar way as in Lemma 3.5 by modifying the map $\Lambda : [0,T]\to[0,T]$ as follows. Define $\Lambda(\hat{\tau}^{(i)}_k) = \tau^{(i)}_k$ for $i = 1,\ldots,N$, $k = 0,\ldots,M_i-1$, and to be piecewise linear at other points.

Now we estimate $|\int_0^T \mathbb{S}_u du - \int_0^T \hat{\Pi}_u(\mathbb{S})du|$. Fix $i < N$. Clearly,

$$\begin{aligned}\left|\int_{T_{i-1}}^{T_i}\mathbb{S}_u du - \int_{T_{i-1}}^{T_i}\hat{\Pi}_u(\mathbb{S})du\right| &\le 2\sqrt{d}2^{-n}\Delta T_i\|\mathbb{S}\| + \left|\int_{T_{i-1}}^{T_i}\check{\Pi}(\mathbb{S})_u du - \int_{T_{i-1}}^{T_i}\hat{\Pi}_u(\mathbb{S})du\right| \\ &\le 2\sqrt{d}2^{-n}\Delta T_i\|\mathbb{S}\| + \|\mathbb{S}\|\left[(T-\tau^{(i)}_{M_i-1}) + (T-\hat{\tau}^{(i)}_{M_i-1})\right] \\ &\quad + \sum_{k=0}^{M_i-2}\left|\pi^{(n+k)}(\mathbb{S}_{\tau^{(i)}_k})\right|\left|\Delta\tau^{(i)}_{k+1} - \Delta\hat{\tau}^{(i)}_{k+1}\right|.\end{aligned}$$

Observe that for any $k = 2, \dots, M_i$,

$$\Delta\hat{\tau}_k^{(i)} \le (1 - \sqrt{d}\, 2^{-n}/T)\Delta\hat{\tau}_{k-1}^{(i)}, \qquad \Delta\hat{\tau}_1^{(i)} = \sqrt{d}\, 2^{-n}$$

and

$$\left|\pi^{(n+k)}(\mathbb{S}_{\tau_k^{(i)}})\right| \le \|\mathbb{S}\| + \sqrt{d}\, 2^{-n},$$
$$T_i - \tau_{M-1}^{(i)} = \Delta\tau_M^{(i)} \le \Delta\tau_1^{(i)}\sqrt{d}\, 2^{-n},$$
$$T_i - \hat{\tau}_{M_i-1}^{(i)} \le \Delta\tau_M^{(i)} + \sqrt{d}2^{-n}/T_i \le \sqrt{d}\, 2^{-n}(1 + 1/T_i).$$

Hence,

$$\begin{aligned}
&\left|\int_{T_{i-1}}^{T_i} \mathbb{S}_u du - \int_{T_{i-1}}^{T_i} \hat{\Pi}_u(\mathbb{S}) du\right| \le \hat{c}_1 2^{-n}\|\mathbb{S}\| + \sum_{k=0}^{M_i-2} \left|\pi^{(n+k)}(\mathbb{S}_{\tau_k^{(i)}})\right| \left|\Delta\tau_{k+1}^{(i)} - \Delta\hat{\tau}_{k+1}^{(i)}\right| \\
&\quad \le [\|\mathbb{S}\| + \sqrt{d}\, 2^{-n}] \left|\Delta\tau_1^{(i)} - \sqrt{d}\, 2^{-n}\right| + \hat{c}_1 2^{-n}\|\mathbb{S}\| \\
&\qquad + \left[\|\mathbb{S}\| + \sqrt{d}\, 2^{-n}\right] \sum_{k=1}^{M_i-2} \left|\Delta\tau_{k+1}^{(i)} - (1 - \sqrt{d}\, 2^{-n}/T_i)\Delta\tau_k^{(i)}\right| \\
&\quad \le \hat{c}_2 2^{-n}\|\mathbb{S}\| + \|\mathbb{S}\| \sum_{k=1}^{M_i-2} \left|\Delta\tau_{k+1}^{(i)} - \Delta\tau_k^{(i)}\right| + \|\mathbb{S}\|(\sqrt{d}\, 2^{-n}/T_i) \sum_{k=1}^{M_i-2} \Delta\tau_k^{(i)} \\
&\quad \le \hat{c}_2 2^{-n}\|\mathbb{S}\| + \|\mathbb{S}\|[\Delta\tau_M^{(i)} - \Delta\tau_1^{(i)}] + \|\mathbb{S}\|\sqrt{d}\, 2^{-n} \\
&\quad \le \hat{c}_3 2^{-n}\|\mathbb{S}\|,
\end{aligned}$$

where $\hat{c}_1, \hat{c}_2, \hat{c}_3$ are appropriate constants (independent of n and $\mathbb{S}$). Hence,

$$\left|\int_0^T \mathbb{S}_u du - \int_0^T \hat{\Pi}_u(\mathbb{S}) du\right| \le c_1\|\mathbb{S}\|2^{-n} \tag{5.4}$$

for some constant c_1.

Finally, let $\hat{\phi} = (\hat{g}_1, \dots, \hat{g}_N, \hat{\gamma})$ be a semi-static portfolio in the sense of Definition 3.2. Define a portfolio $\phi := \Psi(\hat{\phi}) := (g_1, \dots, g_N, \gamma)$ for the original problem by

$$g_i(x) := \hat{g}_i\left(\pi^{(n)}(x)\right), \quad i = 1, \dots, N\ x \in \mathbb{R}_+^d,$$
$$\gamma_t(\mathbb{S}) := \sum_{i=1}^{N}\sum_{k=0}^{M_i-1} \hat{\gamma}_{\hat{\tau}_{k+1}^{(i)}(\mathbb{S})}(\hat{\Pi}(\mathbb{S}))\, \chi_{(\tau_k^{(i)}(\mathbb{S}), \tau_{k+1}^{(i)}(\mathbb{S})]}(t) + \sum_{i=1}^{N-1}(\hat{\gamma}_{T_i} - \hat{\gamma}_{\hat{\tau}_{M_i-1}^{(i)}})\chi_{\{T_i\}}(t).$$

As in Lemma 3.6, we have that for any i,

$$\int_{\mathbb{R}_+^d} g_i d\mu_i = \int_{A^{(n)}} \hat{g}_i d\hat{\mu}_i. \tag{5.5}$$

Furthermore,

$$\int_{T_{i-1}}^{T_i} \gamma_u(\mathbb{S}) d\mathbb{S}_u = \sum_{k=1}^{M_i-1} \hat{\gamma}_{\hat{\tau}_k^{(i)}(\mathbb{S})}(\hat{\Pi}(\mathbb{S}))\left(\mathbb{S}_{\tau_k^{(i)}(\mathbb{S})} - \mathbb{S}_{\tau_{k-1}^{(i)}(\mathbb{S})}\right) + (\hat{\gamma}_{T_i} - \hat{\gamma}_{\hat{\tau}_{M_i-1}^{(i)}})(\mathbb{S}_{T_i} - \mathbb{S}_{T_i-})$$

and

$$\int_{T_{i-1}}^{T_i} \hat{\gamma}_u(\hat{\Pi}(\mathbb{S}))d\hat{\Pi}_u(\mathbb{S}) = \sum_{k=1}^{M_i-1} \hat{\gamma}_{\hat{\tau}_k^{(i)}(\mathbb{S})}(\hat{\Pi}(\mathbb{S}))\left(\pi^{(n+k)}(\mathbb{S}_{\tau_k^{(i)}(\mathbb{S})}) - \pi^{(n+k-1)}(\mathbb{S}_{\tau_{k-1}^{(i)}(\mathbb{S})})\right)$$
$$+ (\hat{\gamma}_{T_i} - \hat{\gamma}_{\hat{\tau}^{(i)}_{M_i-1}})(\pi^{(n)}(\mathbb{S}_{T_i}) - \pi^{(n+M_i-1)}(\mathbb{S}_{T_i-})).$$

Again, by using the fact that the portfolio $\hat{\gamma}$ is bounded by n, we obtain the following estimate,

$$\left|\int_0^T \gamma_u(\mathbb{S})d\mathbb{S}_u - \int_0^T \hat{\gamma}_u(\hat{\Pi}(\mathbb{S}))d\hat{\Pi}_u(\mathbb{S})\right| \le \sum_{i=1}^N \left|\int_{T_{i-1}}^{T_i} \gamma_u(\mathbb{S})d\mathbb{S}_u - \int_{T_{i-1}}^{T_i} \hat{\gamma}_u(\hat{\Pi}(\mathbb{S}))d\hat{\Pi}_u(\mathbb{S})\right|$$
$$\le 2\|\hat{\gamma}\|_\infty \left(2N\sqrt{d}2^{-n} + \sum_{i=1}^N \sum_{k=1}^{M_i} \left|\pi^{(n+k)}(\mathbb{S}_{\tau_k^{(i)}}) - \mathbb{S}_{\tau_k^{(i)}}\right|\right)$$
$$\le 4N\sqrt{d}n2^{-n} + 2n\sum_{i=1}^N \sum_{k=1}^{M_i} \sqrt{d}2^{-n-k}$$
$$\le 6N\sqrt{d}n2^{-n}. \tag{5.6}$$

By applying similar arguments as in Lemma 3.6 we observe that γ is progressively measurable and $\int_0^t \gamma_u(\mathbb{S})d\mathbb{S}_u$ is uniformly bounded from below. The following lemma ends our preparations towards the proof of Theorem 5.2.

Lemma 5.4. i. *Let $p > 1$ given by (2.7). Then,*

$$V(\|\mathbb{S}\|^p) < \infty.$$

ii. *Let $\epsilon > 0$. Define the stopping times $\tau_0^{(\epsilon)} = 0$ and for $j > 0$*

$$\tau_j^{(\epsilon)} = T \wedge \min\{t > \tau_{j-1}^{(\epsilon)} : t \in \{T_1, \ldots, T_{N-1}\} \text{ or } |\Pi_t(\mathbb{S}) - \Pi_{\tau_{j-1}^{(\epsilon)}}(\mathbb{S})| \ge \epsilon\}.$$

Set $M^{(\epsilon)} = \min\{k : \tau_k^{(\epsilon)} = T\}$. Consider the random variable

$$X_\epsilon = \sqrt{\sum_{i=1}^{M^{(\epsilon)}} |\Pi_{\tau_i^{(\epsilon)}}(\mathbb{S}) - \Pi_{\tau_{i-1}^{(\epsilon)}}(\mathbb{S})|^2}.$$

Then

$$V(X_\epsilon) < 3dV(\|\mathbb{S}\|^p).$$

Proof. i. Fix $n \in \mathbb{N}$. Let τ_k and n be as in Section 3.1. We define a portfolio (g, γ) as follows. Set $\gamma_0 = 0$. For $k = 0, 1, \ldots, n-1$ and $t \in (\tau_k, \tau_{k+1}]$, let

$$\gamma_t(\mathbb{S}) = -\frac{p^2}{(p-1)}\left(\max_{0\le i\le k}(\mathbb{S}_{\tau_i}^{(1)})^{p-1}, \ldots, \max_{0\le i\le k}(\mathbb{S}_{\tau_i}^{(d)})^{p-1}\right),$$

and

$$g(x) = \left(\frac{p}{p-1}\right)^p \sum_{i=1}^d x_i^p - \frac{pd}{p-1}, \quad x \in \mathbb{R}_+^d.$$

We use Proposition 2.1 in [1] to conclude that for any $k = 0, 1, \dots, n-1$ and $t \in (\tau_k, \tau_{k+1}]$,

$$g(\mathbb{S}_t) + \int_0^t \gamma_u d\mathbb{S}_u \geq \max(|\mathbb{S}_t|^p, \max_{0\leq i\leq k} |\mathbb{S}_{\tau_i}|^p).$$

Therefore, $\phi^{(n)} := (g, \gamma)$ is admissible. Also at $t = T$,

$$g(\mathbb{S}_T) + \int_0^T \gamma_u d\mathbb{S}_u \geq \max_{0\leq i\leq n} |\mathbb{S}_{\tau_i}|^p.$$

In view of the definitions of τ_k's, for sufficiently large n,

$$\max_{0\leq i\leq n} |\mathbb{S}_{\tau_i}|^p \geq \left(\|\mathbb{S}\| - \sqrt{d}2^{-n}\right)^p \geq \frac{\|\mathbb{S}\|^p}{2^p} - 1.$$

Combining all the above, we arrive at

$$V(\|\mathbb{S}\|^p) \leq 2^p \left(1 + \int g d\mu_N\right) < \infty.$$

ii. Define the trading strategy $\gamma_t = \sum_{i=1}^{M(\epsilon)} \gamma_i \chi_{(\tau_{i-1}^{(\epsilon)}, \tau_i^{(\epsilon)}]}(t)$ where $\gamma_i = (\gamma_i^{(1)}, \dots, \gamma_i^{(d)})$ is given by

$$\gamma_i^{(k)} = \left(\frac{-\Pi_{\tau_{i-1}^{(\epsilon)}}(\mathbb{S}^{(k)})}{\sqrt{\sum_{j=1}^{i-1} |\Pi_{\tau_j^{(\epsilon)}}(\mathbb{S}^{(k)}) - \Pi_{\tau_{j-1}^{(\epsilon)}}(\mathbb{S}^{(k)})|^2 + \max_{0\leq j\leq i-1} \Pi^2_{\tau_j^{(\epsilon)}}(\mathbb{S}^{(k)})}} \right).$$

From Theorem 1.2 in [5] it follows that for any i,

$$\int_0^{\tau_i^{(\epsilon)}} \gamma_u d\mathbb{S}_u + 3d \max_{0\leq j\leq i} \Pi_{\tau_j^{(\epsilon)}}(\mathbb{S}) \geq \sqrt{\sum_{j=1}^{i} |\Pi_{\tau_j^{(\epsilon)}}(\mathbb{S}) - \Pi_{\tau_{j-1}^{(\epsilon)}}(\mathbb{S})|^2}.$$

This together with the fact that $|\gamma| \leq \sqrt{d}$ yields that γ is admissible trading strategy, and $V(X_\epsilon - 3d\|S\|) \leq 0$. Thus from the linearity of the market and the fact that $\|\mathbb{S}\| \geq \|\mathbb{S}_0\| = \sqrt{d}$ we get

$$V(X_\epsilon) \leq 3dV(\|S\|) \leq 3dV(\|S\|^p)$$

and the result follows. □

5.2. *Proof of Theorem 5.2*

Proof. We start with the proof of the inequality

$$V(G) \geq \sup_{\mathbb{Q}\in\mathbb{M}_{\mu_1,\dots,\mu_N}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})]. \tag{5.7}$$

Let $\mathbb{Q} \in \mathbb{M}_{\mu_1,\dots,\mu_N}$. Consider a trading strategy of the form $\gamma = \gamma^{(1)} + \sum_{i=1}^{N-1} \beta_i \chi_{\{T_i\}}(t)$ where $\gamma^{(1)}$ is a portfolio trading strategy satisfies the same assumptions as in Definition 2.5 and β_i is $\mathcal{F}_{T_i-}$ measurable and bounded. Clearly, $\mathbb{E}_{\mathbb{Q}}(\mathbb{S}_{T_i}|\mathcal{F}_{T_i-}) = \mathbb{S}_{T_i-}$, and so

$$\mathbb{E}_{\mathbb{Q}}\left[\sum_{i=1}^{N-1} \beta_i(\mathbb{S}_{T_i} - \mathbb{S}_{T_i-})\right] = 0.$$

Thus,

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \gamma_u(\mathbb{S})\, d\mathbb{S}_u\right] = \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \gamma_u^{(1)}(\mathbb{S})\, d\mathbb{S}_u\right] \le 0.$$

Now suppose that $(g_1, \dots, g_N, \gamma)$ is an admissible super-replicating semi-static portfolio. Then,

$$\sum_{i=1}^N \int g_i d\mu_i = \mathbb{E}_{\mathbb{Q}}\left[\sum_{i=1}^N g_i(\mathbb{S}_{T_i})\right] \ge \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})],$$

and we conclude (5.7).

Next, prove the inequality

$$V(G) \le \sup_{\mathbb{Q}\in\mathbb{M}_{\mu_1,\dots,\mu_N}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})]. \tag{5.8}$$

The proof will be done in four steps.

Step 1: In this step we show that if (5.8) holds for a bounded non negative G, then it holds for general function satisfying Assumption 5.1. A similar reduction is already done in the proof of Theorem 2.9. However, that proof uses the growth assumption (1.3) while we now assume a weaker condition (5.1). The proof below is essentially the same as the one given in our earlier papers [13,14].

First, assume that G is a claim satisfying Assumption 5.1 that is also bounded from below. Thus there exists $M > 0$ such that $G \ge -M$. For $K > 0$ large, set

$$G_K := G \wedge c(K+1) + M.$$

Then, G_K is bounded non negative, and so (5.8) applies to G_K yielding,

$$V(G_K) \le \sup_{\mathbb{Q}\in\mathbb{M}_{\mu_1,\dots,\mu_N}} \mathbb{E}_{\mathbb{Q}}[G_K(\mathbb{S})] \le \sup_{\mathbb{Q}\in\mathbb{M}_{\mu_1,\dots,\mu_N}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})] + M.$$

Moreover, by the upper bound on G, the set $\{G(\mathbb{S}) \ge c(K+1)\}$ is included in the set $\{\|\mathbb{S}\| \ge K\}$. Hence,

$$\begin{aligned} G(\mathbb{S}) &\le G_K(\mathbb{S}) + c\,(\|\mathbb{S}\|+1)\,\chi_{\{\|\mathbb{S}\|\ge K\}}(\mathbb{S}) - M \\ &\le G_K(\mathbb{S}) + c\frac{(\|\mathbb{S}\|+1)^p}{K^{p-1}} - M \\ &\le G_K(\mathbb{S}) + \frac{c2^p}{K^{p-1}}\,\|\mathbb{S}\|^p - M. \end{aligned}$$

By the linearity of the market, this inequality implies that

$$V(G) \le V(G_K) + \frac{c2^p}{K^{p-1}} V(\|\mathbb{S}\|^p) - M.$$

Thus, for any $K > 0$,

$$V(G) \le \sup_{\mathbb{Q}\in\mathbb{M}_{\mu_1,\dots,\mu_N}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})] + \frac{c2^p}{K^{p-1}} V(\|\mathbb{S}\|^p).$$

We let K tend to infinity and apply Lemma 5.4 to conclude duality holds for all G satisfying the Assumption 5.1 and bounded from below.

Now suppose that G is a general function satisfying Assumption 5.1. For $K > 0$ large, set

$$\check{G}_K := G \vee (-c[K+1]).$$

Then, $\check{G}_K$ is bounded from below and duality holds. Again, the linear upper bound implies that $\check{G}_K(\mathbb{S}) \le G(\mathbb{S}) + \check{e}_K(\mathbb{S})$, where the error function is

$$\check{e}_K(\mathbb{S}) := c\,(\|\mathbb{S}\| + 1)\,\chi_{\{\|\mathbb{S}\| \ge K\}}(\mathbb{S}) \le \frac{c2^p}{K^{p-1}}\,\|\mathbb{S}\|^p.$$

Since $G \le \check{G}_K$ and duality holds for $\check{G}_K$,

$$\begin{aligned} V(G) \le V(\check{G}_K) &= \sup_{\mathbb{Q}\in\mathbb{M}_{\mu_1,\ldots,\mu_N}} \mathbb{E}_{\mathbb{Q}}[\check{G}_K] \le \sup_{\mathbb{Q}\in\mathbb{M}_{\mu_1,\ldots,\mu_N}} \mathbb{E}_{\mathbb{Q}}[G + \check{e}_K] \\ &\le \sup_{\mathbb{Q}\in\mathbb{M}_{\mu_1,\ldots,\mu_N}} \mathbb{E}_{\mathbb{Q}}[G] + \sup_{\mathbb{Q}\in\mathbb{M}_{\mu_1,\ldots,\mu_N}} \mathbb{E}_{\mathbb{Q}}[\check{e}_K]. \end{aligned}$$

Moreover, using the Doob's inequality for the $\mathbb{Q} \in \mathbb{M}_{\mu_1,\ldots,\mu_N}$ martingale $\mathbb{S}$, we obtain,

$$\begin{aligned} \sup_{\mathbb{Q}\in\mathbb{M}_{\mu_1,\ldots,\mu_N}} \mathbb{E}_{\mathbb{Q}}\left[\check{e}_K(\mathbb{S})\right] &\le \frac{c2^p}{K^{p-1}} \sup_{\mathbb{Q}\in\mathbb{M}_{\mu_1,\ldots,\mu_N}} \mathbb{E}_{\mathbb{Q}}\left(\|\mathbb{S}\|^p\right) \\ &\le C_p \frac{c2^p}{K^{p-1}} \sup_{\mathbb{Q}\in\mathbb{M}_{\mu_1,\ldots,\mu_N}} \mathbb{E}_{\mathbb{Q}}\left(|\mathbb{S}_T|^p\right) \\ &= C_p \frac{c2^p}{K^{p-1}} \int |x|^p d\mu_N(x), \end{aligned}$$

where C_p is the constant in the Doob's inequality. Once again, we let K tend to infinity to arrive at (5.8).

Step 2: From know on, we assume that $0 \le G \le c$ for some $c > 0$. Fix $\epsilon > 0$ and $n \in \mathbb{N}$. On the space $\hat{\mathbb{D}}$ define the stopping times $\hat{\tau}_0^{(\epsilon)} = 0$ and for $j > 0$

$$\hat{\tau}_j^{(\epsilon)} = T \wedge \min\{t > \hat{\tau}_{j-1}^{(\epsilon)} : t \in \{T_1, \ldots, T_{N-1}\} \text{ or } |\hat{\mathbb{S}}_t - \hat{\mathbb{S}}_{\hat{\tau}_{j-1}^{(\epsilon)}}| \ge \epsilon\}.$$

Set $\hat{M}^{(\epsilon)} = \min\{k : \hat{\tau}_k^{(\epsilon)} = T\}$. Introduce the random variable

$$\hat{X}_\epsilon := F(\hat{\mathbb{S}}) := \sqrt{\sum_{i=1}^{\hat{M}^{(\epsilon)}} |\hat{\mathbb{S}}_{\hat{\tau}_i^{(\epsilon)}} - \hat{\mathbb{S}}_{\tau_{i-1}^{(\epsilon)}}|^2}$$

and consider the bounded claim

$$Y = G(\hat{\mathbb{S}}) - \left(\frac{c}{\epsilon} \wedge \epsilon \hat{X}_\epsilon\right).$$

Define the set $\mathcal{M}(n, c)$ of all probability measures which satisfy

$$\sum_{m\in\mathbb{N}^d} \left|\mathbb{Q}\left(\hat{\mathbb{S}}_{T_i} = m2^{-n}\right) - \hat{\mu}_i\left(\{m2^{-n}\}\right)\right| \le \frac{c}{n}, \quad i = 1, \ldots, N \tag{5.9}$$

and (3.6). Also let $\mathcal{M}(n, c, \epsilon) \subset \mathcal{M}(n, c)$ be the set of all probability measures $\mathbb{Q}$ which in addition satisfy $\mathbb{E}_{\mathbb{Q}}\left[\frac{c}{\epsilon} \wedge \epsilon \hat{X}_\epsilon\right] \le c$. From the Markov inequality it follows that for any $\mathbb{Q} \in \mathcal{M}(n, c, \epsilon)$

$$\mathbb{Q}\left(\hat{X}_\epsilon \ge \frac{c}{\epsilon^2}\right) \le \epsilon. \tag{5.10}$$

Using similar arguments as in Lemma 3.10 it follows that

$$V^{(n)}(Y) \leq \left[\sup_{\mathbb{Q}\in\mathcal{M}(c,n)} \mathbb{E}_{\mathbb{Q}} Y\right]^+ = \left[\sup_{\mathbb{Q}\in\mathcal{M}(c,n,\epsilon)} \mathbb{E}_{\mathbb{Q}} Y\right]^+, \tag{5.11}$$

where the last equality follows from the fact that $G \leq c$.

Next, from the linearity of the market and Lemma 5.4 we have

$$V(G) \leq V\left(G - \frac{c}{\epsilon} \wedge \epsilon X_\epsilon\right) + \epsilon V(X_\epsilon) \leq V\left(G - \frac{c}{\epsilon} \wedge \epsilon X_\epsilon\right) + c_2\epsilon \tag{5.12}$$

for some constant c_2.

Finally, we estimate the term $V\left(G - \frac{c}{\epsilon} \wedge \epsilon X_\epsilon\right) - V^{(n)}(Y)$, from above.

From Assumption 5.1, (5.3)–(5.4) and the fact $0 \leq G \leq c$ we obtain that for n sufficiently large,

$$|G(\mathbb{S}) - G(\hat{\Pi}(\mathbb{S}))| \leq \epsilon + c\chi_{\|S\|\geq\epsilon^{-1}} \leq \epsilon + c\epsilon^{p-1}\|S\|^p.$$

Observe that $X_\epsilon = F(\hat{\Pi}(\mathbb{S}))$. Thus from (5.6), Lemma 5.4 and the linearity of the market we get

$$V\left(G - \frac{c}{\epsilon} \wedge \epsilon X_\epsilon\right) - V^{(n)}(Y) \leq 6N\sqrt{d}n2^{-n} + \epsilon + \epsilon^{p-1}V(\|S\|^p) \leq c_3\epsilon^{p-1} \tag{5.13}$$

for some constant c_3. From (5.11)–(5.13) it follows that for n sufficiently large,

$$V(G) \leq c_4\epsilon^{p-1} + \left[\sup_{\mathbb{Q}\in\mathcal{M}(c,n,\epsilon)} \mathbb{E}_{\mathbb{Q}}[G(\hat{\mathbb{S}})]\right]^+ \tag{5.14}$$

for some constant c_4.

Step 3: In order to complete the proof of the theorem it remains to establish that

$$\limsup_{n\to\infty} \left[\sup_{\mathbb{Q}\in\mathcal{M}(c,n,\epsilon)} \mathbb{E}_{\mathbb{Q}}[G(\hat{\mathbb{S}})]\right]^+ \leq \sup_{\mathbb{Q}\in\mathbb{M}_{\mu_1,\dots,\mu_N}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})] + m(\epsilon) \tag{5.15}$$

where $m : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function with $m(0) = 0$. Then by letting $\epsilon \downarrow 0$ we obtain the duality.

Clearly, we can assume that for n sufficiently large the set $\mathcal{M}(c, n, \epsilon) \neq \emptyset$ is not empty, otherwise the left hand side of (5.15) $= 0$ and the statement is trivial.

We start with a modification of the process $\hat{\mathbb{S}}$. Namely, we will modify the stochastic process $\hat{S}$, such that the new process will have a finitely many (uniformly bounded) jumps. This modification will allow us to obtain tightness.

Thus fix $n \in \mathbb{N}$ (sufficiently large). There exists a probability measure $\mathbb{Q}_n \in \mathcal{M}(c, n, \epsilon)$ such that

$$\mathbb{E}_{\mathbb{Q}_n}[G(\hat{\mathbb{S}})] > \left[\sup_{\mathbb{Q}\in\mathcal{M}(c,n,\epsilon)} \mathbb{E}_{\mathbb{Q}}[G(\hat{\mathbb{S}})]\right]^+ - 1/n. \tag{5.16}$$

Define the process

$$\tilde{\mathbb{S}}_t = \sum_{i=1}^N \hat{\mathbb{S}}_{T_{i-1}+\alpha_i(t-T_{i-1})}\chi_{[T_{i-1},T_i-\epsilon)}(t) + \mathbb{S}_{T_i}\chi_{[T_i-\epsilon,T_i]}(t)$$

where $\alpha_i = \frac{T_i - T_{i-1}}{T_i - T_{i-1} - \epsilon}$, $i = 1, \ldots, N$. Observe that the Skorokhod distance between $\tilde{\mathbb{S}}$ and $\hat{\mathbb{S}}$ satisfies $d(\tilde{\mathbb{S}}, \hat{\mathbb{S}}) \le \epsilon$ and

$$\left| \int_0^T \hat{\mathbb{S}}_u du - \int_0^T \tilde{\mathbb{S}}_u du \right| \le 2N\epsilon \|S\|.$$

Thus $\check{d}(\tilde{\mathbb{S}}, \hat{\mathbb{S}}) \le (2N+1)\epsilon\|S\|$. This together with Assumption 5.1 and the fact that $0 \le G \le c$ yields

$$|G(\tilde{\mathbb{S}}) - G(\hat{\mathbb{S}})| \le m_G((4N+2)d\sqrt{\epsilon}) + c\chi_{\|\hat{\mathbb{S}}\| \ge 2d\epsilon^{-1/2}}. \tag{5.17}$$

Similarly to Lemma 3.10 we have the decomposition $\hat{\mathbb{S}} = \mathbb{M}^{\mathbb{Q}_n} - A^{\mathbb{Q}_n}$. Denote $\mathbb{M}^{\mathbb{Q}_n} = (M^{(1)}, \ldots, M^{(d)})$ and $A^{\mathbb{Q}_n} = (A^{(1)}, \ldots, A^{(d)})$. Observe that (since $\mathbb{Q}_n \in \mathcal{M}(c, n, \epsilon)$) for any i, $\mathbb{E}_{\mathbb{Q}_n} M_T^{(i)} = 1$ and $\mathbb{E}_{\mathbb{Q}_n} \|A^{(i)}\| \le \frac{c}{n}$. Thus from the Doob inequality and the Markov inequality we obtain

$$\begin{aligned} \mathbb{Q}_n(\|\hat{\mathbb{S}}\| \ge 2d\epsilon^{-1/2}) &\le \sum_{i=1}^d [\mathbb{Q}_n(\|M^{(i)}\| \ge \epsilon^{-1/2}) + \mathbb{Q}_n(\|A^{(i)}\| \ge \epsilon^{-1/2})] \\ &\le d\sqrt{\epsilon}(1 + c/n). \end{aligned} \tag{5.18}$$

This together with (5.17) gives

$$|\mathbb{E}_{\mathbb{Q}_n}[G(\tilde{\mathbb{S}})] - \mathbb{E}_{\mathbb{Q}_n}[G(\hat{\mathbb{S}})]| \le m_G(c_5\sqrt{\epsilon}) + c_5\sqrt{\epsilon} \tag{5.19}$$

for some constant c_5.

Next, set $\Theta = \lceil N + c^2/\epsilon^6 \rceil$ and $\delta = \frac{\epsilon}{4\Theta^2}$. Define $\tilde{\tau}_0 = 0$ and for $1 \le j \le \Theta$ define

$$\tilde{\tau}_j = (T - \delta) \wedge \min\{t > \tilde{\tau}_{j-1} : t \in \{T_1, \ldots, T_{N-1}\} \text{ or } |\tilde{\mathbb{S}}_t - \tilde{\mathbb{S}}_{\tilde{\tau}_{j-1}}| \ge \epsilon\}.$$

For $j > \Theta$ we set $\tilde{\tau}_j = (T - \delta) \wedge \min\{T_i : T_i > \tilde{\tau}_{j-1}\}$. Observe that $\tilde{\tau}_{N+\Theta} = T - \delta$.

Let $\sigma_0 = 0$ and for $k > 0$ let $\sigma_k = \tilde{\tau}_k + \delta k$ if $\tilde{\tau}_k \notin \{T_1, \ldots, T_{N-1}, T - \delta\}$ and $\sigma_k = \tilde{\tau}_k$ otherwise. Define the process

$$\check{\mathbb{S}}_t = \sum_{i=0}^{\Theta+N-1} \tilde{\mathbb{S}}_{\tilde{\tau}_i} \chi_{[\sigma_i, \sigma_{i+1})}(t) + \hat{\mathbb{S}}_T \chi_{[T-\delta, T]}(t).$$

Recall the inequality (5.10). Observe that on the event $\{\hat{X}_\epsilon \ge \frac{c}{\epsilon^2}\}$ we have

$$\min\{k : \tilde{\tau}_k = T - \delta\} \le \Theta.$$

Thus

$$\mathbb{Q}_n(\tilde{\sigma}_\Theta = T - \delta) \ge 1 - \epsilon \tag{5.20}$$

and so,

$$d(\check{\mathbb{S}}, \tilde{\mathbb{S}}) \le 2\epsilon + \max_{1 \le i \le M+\Theta} [\sigma_i - \tau_i] \le 3\epsilon.$$

Furthermore, similarly to (5.4) we get

$$\left| \int_0^T \check{\mathbb{S}}_t dt - \int_0^T \tilde{\mathbb{S}}_t dt \right| \le 2\epsilon T + 2\epsilon\|\hat{\mathbb{S}}\| \le c_6\epsilon\|\hat{\mathbb{S}}\|$$

for some constant c_6. This observation together with (5.10) yields that

$$\left|\mathbb{E}_{\mathbb{Q}_n}[G(\tilde{\mathbb{S}})]-\mathbb{E}_{\mathbb{Q}_n}[G(\check{\mathbb{S}})]\right| \leq c\epsilon + c\mathbb{Q}_n(\|\hat{\mathbb{S}}\| \geq 2d\epsilon^{-1/2}) + m_G(3\epsilon+2dc_6\sqrt{\epsilon})$$
$$\leq c\epsilon + cd\sqrt{\epsilon}(1+c/n) + m_G(3\epsilon+2dc_6\sqrt{\epsilon}). \tag{5.21}$$

From (5.16), (5.19) and (5.21) it follows that in order to establish (5.15) it sufficient to show

$$\limsup_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[G(\check{\mathbb{S}})] \leq \sup_{\mathbb{Q}\in\mathbb{M}_{\mu_1,\ldots,\mu_N}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})] + m_G(\epsilon) + c\epsilon. \tag{5.22}$$

Step 4: Finally, we establish (5.22) by using weak convergence on the Skorokhod space $\mathbb{D}$. Without loss of generality (by passing to a subsequence) we assume that the limit in the left hand side of (5.22) is exists.

In this step we denote the process $\tilde{\mathbb{S}}, \check{\mathbb{S}}$ and the stopping times $\tilde{\tau}_k, \sigma_k$, which constructed for $n \in \mathbb{N}$ by $\tilde{\mathbb{S}}^{(n)}, \check{\mathbb{S}}^{(n)}$ and $\tilde{\tau}_n^{(k)}, \sigma_k^{(n)}$, respectively.

Introduce the martingale

$$\tilde{\mathbb{M}}_t^{(n)} = \sum_{i=1}^{N} \mathbb{M}^{\mathbb{Q}_n}_{T_{i-1}+\alpha_i(t-T_{i-1})}\chi_{[T_{i-1},T_i-\epsilon)}(t) + \mathbb{M}^{\mathbb{Q}_n}_{T_i}\chi_{[T_i-\epsilon,T_i]}(t).$$

For $k = 0, 1, \ldots, N+\Theta$ let

$$X_k^{(n)} = \check{\mathbb{S}}^{(n)}_{\sigma_k^{(n)}} = \tilde{\mathbb{S}}^{(n)}_{\tilde{\tau}_k^{(n)}}, \qquad Y_k^{(n)} = \tilde{\mathbb{M}}^{(n)}_{\tilde{\tau}_k^{(n)}}, \qquad Z_k^{(n)} = \tilde{\mathbb{M}}^{(n)}_{\tilde{\tau}_k^{(n)}-} \quad \text{and} \quad W_k^{(n)} = \tilde{\mathbb{S}}^{(n)}_{\tilde{\tau}_k^{(n)}-}.$$

From (5.9) it follows that we have a weak convergence $\check{\mathbb{S}}_T^{(n)} \Rightarrow \mu_N$. In addition, from the fact that $\check{\mathbb{S}}_T^{(n)} \geq 0$ and

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[\check{\mathbb{S}}_T^{(n)}] = \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[\mathbb{M}_T^{\mathbb{Q}_n}] = (1,\ldots,1) = \int x\,d\mu_N(x)$$

it follows that the sequence $\{\check{\mathbb{S}}_T^{(n)}\}_{n=1}^{\infty}$ is uniformly integrable. In addition, the equality $\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}\|A^{\mathbb{Q}_n}\| = 0$ yields that $\{\mathbb{M}_T^{\mathbb{Q}_n}\}_{n=1}^{\infty}$ is uniformly integrable, and since $M^{\mathbb{Q}_n}$ is a martingale we can replace T by any stopping time. Thus, we conclude that the sequence

$$\left(X_0^{(n)},\ldots,X_{N+\Theta}^{(n)},Y_0^{(n)},\ldots,Y_{N+\Theta}^{(n)},Z_0^{(n)},\ldots,Z_{N+\Theta}^{(n)},\sigma_0^{(n)},\ldots,\sigma_{N+\Theta}^{(n)}\right), \quad n \in \mathbb{N}$$

is uniformly integrable, and in particular its tight on the space $\mathbb{R}^{4N+4\Theta+4}$. Thus there is a subsequence (which we still denote by n) which converge weakly. From the Skorokhod representation theorem it follows that we can redefine the above sequence on a new probability space such that it converge a.s. Denote the limit by

$$\left(X_0,\ldots,X_{N+\Theta},Y_0,\ldots,Y_{N+\Theta},Z_0,\ldots,Z_{N+\Theta},\sigma_0,\ldots,\sigma_{N+\Theta}\right)$$

and introduce the *cádlág* processes

$$U_t = \sum_{i=0}^{N+\Theta-1} X_i\chi_{[\sigma_i,\sigma_{i+1})}(t) + X_{N+\Theta}\chi_{[T-\delta,T]}(t).$$

Observe that for any k, n $\sigma_k^{(n)} - \sigma_{k-1}^{(n)} > \delta$ provided that $\sigma_{k-1}^{(n)} < T-\delta$. Thus we get the same property for the limit, namely $\sigma_k - \sigma_{k-1} > \delta$ provided that $\sigma_{k-1} < T-\delta$. We conclude that

$\check{\mathbb{S}}^{(n)} \to U$ a.s with respect to the Skorokhod topology on the space $\mathbb{D}$. Thus $G(\check{\mathbb{S}}^{(n)}) \to G(U)$ a.s, and so from the bounded convergence theorem it follows that

$$\mathbb{E}[G(U)] = \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[G(\check{\mathbb{S}})]. \tag{5.23}$$

Let us notice that U is not a martingale, and so we modify U.

Let $\mathcal{G}_t^{(i)}$ be the right continuous filtration which is given by

$$\mathcal{G}_t^{(i)} = \bigcap_{u>t} \sigma\{Y_0, \dots, Y_i, \sigma_0, \dots, \sigma_i, u \wedge \sigma_{i+1}\}.$$

Introduce the *cádlág* process

$$\tilde{U}_t = \sum_{i=0}^{N+\Theta-1} \mathbb{E}(Z_{i+1}|\mathcal{G}_t^{(i)})\chi_{[\sigma_i,\sigma_{i+1})}(t) + X_{N+\Theta}\chi_{[T-\delta,T]}(t).$$

From the fact that $\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}\|A^{\mathbb{Q}_n}\| = 0$ it follows that

$$X_k = Y_k, \qquad Z_k = W_k, \quad k = 0, 1, \dots, N+\Theta, \tag{5.24}$$
$$\text{and} \quad |W_k - X_{k-1}| \le \epsilon, \quad k = 0, 1, \dots, \Theta.$$

Next, observe that for a given n we have

$$\mathbb{E}_{\mathbb{Q}_n}(Z_{k+1}^{(n)}|\sigma_1^{(n)}, \dots, \sigma_k^{(n)}, Y_1^{(n)}, \dots, Y_k^{(n)}) = Y_k^{(n)}$$

and

$$\mathbb{E}_{\mathbb{Q}_n}(Y_{k+1}^{(n)}|\sigma_1^{(n)}, \dots, \sigma_{k+1}^{(n)}, Z_1^{(n)}, \dots, Z_{k+1}^{(n)}, Y^{(n)}1, \dots, Y_k^{(n)}) = Z_{k+1}^{(n)}.$$

This together with uniform integrability yields

$$\mathbb{E}(Z_{k+1}|\sigma_1, \dots, \sigma_k, Y_1, \dots, Y_k) = Y_k \tag{5.25}$$

and

$$\mathbb{E}(Y_{k+1}|\sigma_1, \dots, \sigma_{k+1}, Z_1, \dots, Z_{k+1}, Y_1, \dots, Y_k) = Z_{k+1}. \tag{5.26}$$

From (5.25)–(5.26) and the chain rule for conditional expectation it follows that $\tilde{U}$ is a martingale. From (5.24)–(5.25) we have $\tilde{U}_{\sigma_k} = Y_k = X_k = U_{\sigma_k}$. Observe that if $\sigma_k = T_i$ for some i, then for sufficiently large n we have $\sigma_i^{(n)} = T$. Thus

$$U_{T_i} = \tilde{U}_{T_i} = \lim_{n\to\infty} \hat{\mathbb{S}}_{T_i}^{(n)}.$$

This together with (5.9) gives that for any i the distribution of $\tilde{U}_{T_i}$ is equals to μ_i, we conclude that the law of $\tilde{U}$ is an element in $\mathbb{M}_{\mu_1,\dots,\mu_N}$.

Finally, we estimate $\mathbb{E}[G(U)] - \mathbb{E}[G(\tilde{U})]$. Let $k < \Theta$. On the event $t \in [\sigma_k, \sigma_{k+1})$ (which is $\mathcal{G}_t^{(i)}$ measurable) we apply (5.24)–(5.25) to obtain

$$|\tilde{U}_t - U_t| = |\mathbb{E}(Z_{k+1} - Y_k|Y_1, \dots, Y_k, \sigma_1, \dots, \sigma_k)| \le \epsilon.$$

Thus on the event $\sigma_\Theta = T - \delta$ we get $\|U - \tilde{U}\| \le \epsilon$. We conclude that

$$|\mathbb{E}G(U) - \mathbb{E}G(\tilde{U})| \le c\mathbb{P}(\sigma_\Theta < T - \delta) + m_G(\epsilon) \le c\epsilon + m_G(\epsilon) \tag{5.27}$$

where the last inequality follows from (5.20). By combining (5.23) with (5.27) we obtain (5.22), and complete the proof. □

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Further reading

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