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# TRADING WITH SMALL PRICE IMPACT

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An investor trades a safe and several risky assets with linear price impact to maximize expected utility from terminal wealth. In the limit for small impact costs, we explicitly determine the optimal policy and welfare, in a general Markovian setting allowing for stochastic market, cost, and preference parameters. These results shed light on the general structure of the problem at hand, and also unveil close connections to optimal execution problems and to other market frictions such as proportional and fixed transaction costs.

KEY WORDS: price impact, portfolio choice, asymptotics, homogenization, viscosity solutions.

## 1. INTRODUCTION

Even in the most liquid financial markets, only small quantities can be traded quickly without adversely affecting market prices. For large investors, it is therefore crucial to balance the gains generated by trading against the corresponding price impact costs.

This problem has received a lot of attention in the optimal execution literature, which studies how to efficiently split up a single exogenously given order (cf., e.g., Bertsimas and Lo 1998; Almgren and Chriss 2001; Huberman and Stanzl 2005; Obizhaeva and Wang 2013, as well as many more recent studies). In contrast, less is known about dynamic portfolio choice with price impact, i.e., the problem of how to endogenously determine the optimal order flow from market dynamics and investors' preferences. Here, previous work has focused on price impact linear in the order size, in concrete models with specific market dynamics and preferences (Garleanu and Pedersen 2013a, 2013b; Almgren and Li 2011; Collin-Dufresne et al. 2012; Guasoni and Weber 2014, 2017); see Section 5.1 for a detailed discussion. In the present study, we also focus on linear price impact. However, we allow for arbitrary preferences, as well as for general Markovian dynamics of market prices and impact parameters. Despite this generality, we obtain explicit formulas for the optimal policy and welfare, *asymptotically* for small price impacts.

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These results shed new light on the general structure of the problem at hand, and also reveal deep connections to other market frictions. As in previous studies (Garleanu and Pedersen 2013a, 2013b; Almgren and Li 2011; Guasoni and Weber 2014, 2017), it turns out to be optimal to always trade from the current position $\theta_t^\Lambda$ toward the frictionless target $\theta_t^0$ at a finite rate $\dot{\theta}_t^\Lambda$. For a single risky asset,[1] traded with small linear price impact $\Lambda_t$, this asymptotically optimal trading rate is given explicitly by:

$$\dot{\theta}_t^\Lambda = \sqrt{\frac{(\sigma_t^S)^2}{2\Lambda_t R_t}} \left(\theta_t^0 - \theta_t^\Lambda\right). \tag{1.1}$$

Here, $\sigma_t^S$ is the risky asset's volatility and $R_t$ is the frictionless investor's "indirect risk-tolerance process," i.e., the risk tolerance of the frictionless value function. Thus, the current position $\theta_t^\Lambda$ is pushed back more aggressively to the frictionless target $\theta_t^0$ if i) the current deviation $\theta_t^\Lambda - \theta_t$ is large, ii) market volatility $\sigma_t^S$ is high, iii) trading costs $\Lambda_t$ are low, or iv) the investor's risk tolerance $R_t$ is low. For constant market, cost, and preference parameters, this reduces to the formulas obtained by Garleanu and Pedersen (2013); Almgren and Li (2011); Guasoni and Weber (2017). In the general setting considered here, these quantities are updated continuously with the current volatility, price impact, and (indirect) risk tolerance. Hence, the optimal policy is "myopic" in the sense that it trades toward the current frictionless maximizer (rather a projected future optimum) with a speed determined by current market and preference parameters.[2]

This observation is in analogy to results for small proportional (Martin 2012; Soner and Touzi 2013; Kallsen and Muhle-Karbe 2013, 2015; Kallsen and Li 2013) and fixed transaction costs (Korn 1998; Altarovici et al. 2015), where "myopic" policies are also optimal asymptotically. With these frictions, the risky fraction is *always* kept between two trading boundaries around the frictionless target position. In contrast, with price impact, it is no longer optimal to remain uniformly close. Instead, the optimal deviation follows a diffusion process with fluctuations driven by the frictionless optimizer and mean reversion induced by the control (1.1). Hence, the "fine" structure of the optimal policy crucially depends on the specific market friction under consideration. Yet, the "coarse" structure is the same in each case, in that the *average* squared deviation from the frictionless target is kept below some threshold, determined by the same inputs.[3] Indeed, with small linear price impact $\Lambda_t$, this threshold is given by:

$$\sqrt{2}\left(\frac{R_t\Lambda_t}{(\sigma_t^S)^2}\right)^{1/2}\left(\sigma_t^{\theta^0}\right)^2,$$

[1]The results readily extend to multiple risky assets, cf. Theorems 4.3 and 4.7. For ease of exposition, we focus on a single risky asset in this introduction.

[2]Hedging against the future evolution of the frictionless target is studied by Garleanu and Pedersen (2013).

[3]This is the (leading-order) stationary variance obtained when considering a small time interval around $t$, and then i) changing time to stretch it to the entire half-line, and ii) normalizing the deviation by the dynamic threshold accordingly. See Kallsen and Muhle-Karbe (2013, 2015); Kallsen and Li (2013) for more details.

where $\sigma_t^{\theta^0} = \sqrt{d\langle\theta^0\rangle_t/dt}$ is the volatility of the frictionless target strategy.[4] For small proportional transaction costs $\Lambda_t$, the analogous bound reads as follows:[5]

$$\frac{1}{\sqrt[3]{12}}\left(\frac{R_t\Lambda_t}{(\sigma_t^S)^2}\right)^{2/3}\left(\sigma_t^{\theta^0}\right)^{4/3}.$$

Similarly, for small fixed trading costs $\Lambda_t$, the corresponding threshold is given by:[6]

$$\frac{1}{\sqrt{3}}\left(\frac{R_t\Lambda_t}{(\sigma_t^S)^2}\right)^{1/2}\sigma_t^{\theta^0}.$$

Hence, there is a different universal constant in each case, and the powers to which the input parameters are raised also depend on the specific friction at hand. The inputs $R_t$, $\Lambda_t$, $\sigma_t^S$, and $\sigma_t^{\theta^0}$, however, are the same in each model. As a result, the corresponding comparative statics are universal: the frictionless target is tracked tightly, on average, if price risk is high relative to risk tolerance, if trading costs are low, or if the frictionless target strategy is relatively inactive and can therefore be implemented with few adjustments.

The optimal trading rate (1.1) also reveals a close connection to the optimal execution literature. Indeed, for small price impacts, (1.1) locally corresponds to the optimal execution strategy of Almgren and Chriss (2001) as well as Schied and Schöneborn (2009), with the order to be executed given by the deviation from the frictionless target.[7] Hence, dynamic portfolio choice with small price impacts can be interpreted as "optimally liquidating toward the frictionless target," where the latter as well as market, impact, and preference parameters all are updated continuously.

The performance of the optimal policy and in turn the welfare loss due to finite market depth can also be quantified. At the leading order, the certainty equivalent loss due to small price impact, i.e., the cash equivalent of trading without frictions, is given by:

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \sqrt{\frac{(\sigma_t^S)^2\Lambda_t}{2R_t}}\left(\sigma_t^{\theta^0}\right)^2 dt\right]. \tag{1.2}$$

As a result, price impact has a substantial welfare effect if i) market risk measured by the volatility $\sigma_t^S$ is high compared to the investor's risk tolerance $R_t$, ii) the trading costs $\Lambda_t$ are large, or iii) the frictionless target strategy is highly active with large volatility $\sigma_t^{\theta^0}$. As all of these quantities generally are time-dependent and random, they have to be averaged suitably, across both time and states. Here, averaging across states is carried out

[4] If $\theta_t^0 = \Delta(t, S_t)$ is a delta-hedge in a complete Markovian setting then this is the "Cash-Gamma," i.e., the second derivative of the option price with respect to the underlying, multiplied by the squared value of the latter.

[5] This bound is derived by noticing that the deviations from the frictionless target are approximately uniform in this case (Janeček and Shreve 2004; Rogers 2004; Goodman and Ostrov 2010; Kallsen and Muhle-Karbe 2013, 2015; Kallsen and Li 2013), so that the corresponding average squared deviation equals one-third of the halfwidth of the no-trade region determined in Martin (2012); Soner and Touzi (2013); Kallsen and Muhle-Karbe (2013, 2015); Kallsen and Li (2013).

[6] To see this, note that the approximate probability density of the deviation is a "hat function" in this case, so that the corresponding average squared deviation is given by one-sixth of the halfwidth of the no-trade region determined by Korn (1998); Altarovici et al. (2015).

[7] This correspondence remains true with several risky assets, where optimal liquidation has been studied by Schied et al. (2010); Schöneborn (2011).

with respect to the frictionless investor's "marginal pricing measure" $\mathbb{Q}$,[8] i.e., the effect of the small friction is priced like a marginal path-dependent option.

For frictionless models that can be solved in closed form, Representation (1.2) readily yields explicit formulas. In general, this expression allows to shed further light on the connections between price impact and other market frictions. Indeed, close analogues of Formula (1.2) for the certainty equivalent loss due to small price impact remain true for different trading costs. Only the universal constant and the powers of the inputs have to be changed, like for the average squared deviation from the frictionless target. For example, with small proportional transaction costs $\Lambda_t$, the analogue of (1.2) reads as follows (Soner and Touzi 2013; Kallsen and Muhle-Karbe 2013, 2015):

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \sqrt[3]{\frac{9(\sigma_t^S)^2\Lambda_t}{32R_t}}\left(\sigma_t^{\theta^0}\right)^{4/3}dt\right].$$

Hence, the monotonicity in the model inputs $\sigma_t^S$, $\Lambda_t$, $R_t$, and $\sigma^{\theta^0}$ remains unchanged, and the corresponding comparative statics are the same for each small friction.

For investors with constant absolute risk tolerance, i.e., with exponential utilities, our results readily allow to incorporate random endowments by a change of measure. This in turn allows us to obtain utility-indifference prices and hedging strategies. As volatilities are invariant under equivalent measure changes, it follows that the trading rate (1.1) is truly universal, in that it applies both to optimal investment and to hedging; only the frictionless inputs need to be changed accordingly. Formula (1.2) for the corresponding welfare loss in turn leads to utility-based derivative prices in the spirit of Hodges and Neuberger (1989) as well as Davis, Panas, and Zariphopoulou (1993).[9]

We use dynamic programming and matched asymptotics to prove the results discussed above. To outline this methodology, let $v^0$ be the frictionless value function of the initial data $\zeta$.[10] Also let $v^\lambda$ be its counterpart for small linear price impact $\Lambda_t = \lambda\Lambda(\cdot)$.[11] Due to the friction, $v^\lambda$ depends not only on $\zeta$ but also on the number $\vartheta$ of shares the investor currently holds. Then, the main technical objective is to understand the limit behavior of

$$\bar{u}^\lambda(\zeta,\vartheta) := \frac{v^0(\zeta) - v^\lambda(\zeta,\vartheta)}{\lambda^{1/2}} \geq 0, \quad \text{as } \lambda \downarrow 0.$$

The viscosity approach developed by Evans (1992) to problems in homogenization is suitable for this analysis. Indeed, it provides a technique to derive the equation satisfied by the relaxed semilimits $\bar{u}^*$ and $\bar{u}_*$ of $\bar{u}^\lambda$ as $\lambda \downarrow 0$. Then, by a comparison result, one concludes that these limits are equal to each other. In particular, this proves the local uniform convergence of $\bar{u}^\lambda$.

[8]That is, the dual martingale measure linked to the primal optimizer by the usual first-order condition. Expectations under this measure correspond to utility indifference prices for infinitesimally small claims (Davis 1997; Karatzas and Kou 1996; Kramkov and Sirbu 2007), whence the name "marginal pricing measure."

[9]For related asymptotics with small proportional costs, cf. Whalley and Wilmott (1997); Bichuch (2014); Kallsen and Muhle-Karbe (2013, 2015); Bouchard et al. (2013); Possamai and Royer (2014).

[10]As is well known, the frictionless value function depends on time $t$, the current values $s$ and $y$ of the risky assets and state variables, and the investor's wealth $x$. These are collected in $\zeta = (t, s, y, x)$.

[11]Here, $\lambda \sim 0$ is the small parameter for the asymptotic expansion, and $\Lambda(\cdot)$ is a given deterministic function of time, the current values of asset prices and state variables, and the investor's wealth.

In this approach, it is crucial that the limit functions depend only on the "original" variable $\zeta$. However, in our context, the relaxed semilimits $\bar{u}^*$ and $\bar{u}_*$ depend also on the $\vartheta$-variable and we need to identify this dependence separately. Indeed, we first show that $\bar{u}^*$ and $\bar{u}_*$ are sub- and supersolutions, respectively, of an Eikonal-type equation as studied in Kružkov (1975); Ishii (1987):

$$(D_\vartheta \bar{u})^2 = \mathfrak{n},$$

where $\mathfrak{n}$ is a smooth nonnegative function, quadratic in the $\vartheta$-variable. In general, there is no comparison principle for the above equation. However, using a transformation technique, we prove a comparison result for nonnegative solutions. This implies the existence of a smooth quadratic function $\varpi$ of the difference between the actual position $\vartheta$ and the frictionless optimal position $\theta^0(\zeta)$ such that the there is no $\vartheta$-dependence for the relaxed semilimits of

$$\bar{u}^\lambda(\zeta, \vartheta) - \varpi(\zeta, \vartheta).$$

We then proceed by analyzing these limits using the viscosity technique outlined above.

Similar asymptotic results have been recently obtained for utility maximization with proportional transaction costs in Soner and Touzi (2013), for several risky assets in Possamai et al. (2013), for random endowments in Bouchard et al. (2013), and for models with fixed transaction costs in Altarovici et al. (2015). In these models, the semilimits can be shown to be independent of the $\vartheta$-variable due to the gradient constraint in the dynamic programming equation, because a single trade from the actual position to the frictionless target is negligible at the leading order. In contrast, such bulk trades are impossible in our framework as they incur infinite price impact. This necessitates the novel analysis through the Eikonal equation.

The remainder of this paper is organized as follows. The model is set up in Section 2. Afterwards, we state the dynamic programming equations without and with frictions, before turning to the corrector equations governing their asymptotic relationship for small price impacts. For better readability, we first derive the corrector equations heuristically in a simple setting, and then state their general versions. The subsequent Section 4 contains our main results, an asymptotic expansion of the value function for small price impacts and a corresponding almost optimal trading policy. These results, their implications, and connections to the literature are discussed in Section 5, and proved in Section 6. Afterwards, in Section 7, we provide a set of sufficient conditions for our technical assumptions, which are standard for verification results (cf., e.g., Touzi 2013, theorem 4.1). Finally, in Section 8, we show how to verify the conditions of Section 7 in a concrete model.

**Notation**. Throughout, $\mathbb{M}^{d\times m}$ denotes the space of $d \times m$ matrices, and $\mathbb{S}^d$ the subspace of symmetric $d \times d$ matrices. For $k \geq 1$, $x \in \mathbb{R}^k$ and $r > 0$, we write $B_r(x)$ for the open ball of radius $r$ centered at $x$; $\bar{B}_r(x)$ and $\partial B_r(x)$ denote its closure and boundary, respectively.

For a smooth function $\varphi : (t, x_1, \ldots, x_k) \to \mathbb{R}$, we write $\partial_t \varphi$, $\partial_{x_i} \varphi$ for the corresponding partial derivatives. The second-order derivatives are denoted by $\partial_{x_i x_j} \varphi$ etc. We write $D\varphi$ and $D^2\varphi$ for the gradient vector and Hessian matrix of $\varphi$ with respect to the spatial components, respectively. For any subset $I \subset \{1, \cdots, k\}$, $D_{(x_i)_{i\in I}}$ and $D^2_{(x_i)_{i\in I}}$ refer to the gradient and Hessian with respect to $(x_i)_{i\in I}$.

$C^i$ denotes the $i$-times continuously differentiable functions, $C^i_b$ is the subspace with bounded derivatives, and $C^{1,2}$ refers to the functions once resp. twice continuously differentiable in the time resp. space variables.

Finally, for any locally bounded function $v$, the corresponding lower- and upper-semicontinuous envelopes are denoted by $v_*$, $v^*$.

## 2. MODEL

### 2.1. Unaffected Prices

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space supporting a $q$-dimensional Brownian motion $W$. Fix a finite time horizon $T > 0$, and let $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ be the augmented filtration generated by $W$.

We consider a financial market with $d + 1$ assets. The first one is safe, and its price is assumed to be normalized to one. The other $d$ assets are risky, with unaffected best quotes $S := (S^1, \dots, S^d)$ following

$$dS_r = \mu_S(r, S_r, Y_r)dr + \sigma_S(r, S_r, Y_r)dW_r, \qquad S_t = s, \tag{2.1}$$

for a state variable $Y$ taking values in an open subset $\mathcal{Y}$ of $\mathbb{R}^m$, with dynamics

$$dY_r = \mu_Y(r, Y_r)dr + \sigma_Y(r, Y_r)dW_r, \qquad Y_t = y. \tag{2.2}$$

The mappings $(\mu_S, \sigma_S) : [0, T] \times (0, \infty)^d \times \mathcal{Y} \longmapsto \mathbb{R}^d \times \mathbb{M}^{d\times q}$ and $(\mu_Y, \sigma_Y) : [0, T] \times \mathcal{Y} \longmapsto \mathbb{R}^m \times \mathbb{M}^{m\times q}$ are continuous and Lipschitz-continuous in $(s, y)$. Moreover, $\sigma_S$ belongs to $C^{1,2}$ and satisfies the following local ellipticity condition: for any compact subset $B \subset [0, T] \times (0, \infty)^d \times \mathcal{Y}$, there is a constant $\gamma_B > 0$ such that:

$$\left|\mathbf{x}^\top\sigma_S\right|^2 = \mathbf{x}^\top\sigma_S\sigma_S^\top\mathbf{x} \geq \gamma_B\,|\mathbf{x}|^2, \quad \text{for all } \mathbf{x}\in\mathbb{R}^d \text{ on } B. \tag{2.3}$$

As a result, for any initial data $(t, s, y) \in [0, T] \times (0, \infty)^d \times \mathcal{Y}$, there is a unique strong solution of the SDEs (2.1)-(2.2), that we denote by $(S^{t,s,y}, Y^{t,y})$.

REMARK 2.1. The condition $\sigma_S \in C^{1,2}$ allows to produce a smooth solution of the First Corrector Equation (3.13) in Lemma 4.1. This assumption could be weakened using a mollification argument as in Possamai et al. (2013).

### 2.2. Linear Price Impact

The unaffected best quotes $S$ from (2.1) represent the idealized prices at which minimal amounts can be traded slowly without adversely affecting market prices. In contrast, if $\Delta\theta$ shares are traded over a time interval $\Delta t$, then this order is filled at an average price per share of

$$S_t + \Lambda_t\frac{\Delta\theta}{\Delta t}$$

instead of $S_t$. This price impact is purely "transient," in that prices immediately return to their unaffected value after each trade is filled.[12] Moreover, impact is linear in the

[12] For models also taking into account persistent price impact, cf., e.g., Bertsimas and Lo (1998); Almgren and Chriss (2001); Huberman and Stanzl (2005); Gatheral (2010); Obizhaeva and Wang (2013); Alfonsi, Fruth, and Schied (2010); Roch and Soner (2013); Garleanu and Pedersen (2013), and the references therein.

trading rate $\Delta\theta/\Delta t$. This is described by the process $\Lambda_t = \lambda\Lambda(t, S_t, Y_t, X_t)$, where $\lambda > 0$ is a small parameter and $\Lambda(t, S_t, Y_t, X_t)$ is a $C^{1,2}$-function of time $t$, current prices $S_t$, the state variable $Y_t$, and the investor's current (paper) wealth $X_t$, taking values in the symmetric, positive definite $d \times d$ matrices.[13] For $\lambda = 0$, the usual frictionless model obtains, where arbitrary quantities $\Delta\theta$ can be traded over any time interval $\Delta t$ at the same price $S_t$, for a total execution price of $\Delta\theta S_t$. With a nontrivial $\lambda > 0$, trading prices become less favorable in that each order $\Delta\theta$ incurs an additional cost which is quadratic[14] in quantities traded, and inversely proportional to the trade's execution time:

$$\frac{\Delta\theta^\top}{\Delta t}\Lambda_t\frac{\Delta\theta}{\Delta t}\Delta t.$$

These considerations motivate the following continuous-time model.[15] For any absolutely continuous trading strategy

$$d\theta_r = \dot{\theta}_r dr\ , \quad \theta_t = \vartheta, \tag{2.4}$$

the corresponding (paper) wealth has dynamics

$$dX_r = \theta_r dS_r - \lambda\dot{\theta}_r^\top \Lambda(r, S_r, Y_r, X_r)\dot{\theta}_r dr\ , \qquad X_t = x. \tag{2.5}$$

To wit, the usual frictionless dynamics are adjusted for trading costs quadratic in the trading rate $\dot{\theta}$. For notational simplicity, we write

$$\zeta := (t, s, y, x) \in \mathfrak{D},$$

where

$$\mathfrak{D} := \mathfrak{D}_< \cup \partial_T\mathfrak{D}$$

with

$$\mathfrak{D}_< := [0, T) \times (0,\infty)^d \times \mathbb{R}^m \times \mathbb{R} \quad \text{and} \quad \partial_T\mathfrak{D} := \{T\} \times (0,\infty)^d \times \mathbb{R}^m \times \mathbb{R}.$$

With this notation, the set of controls $\Theta_0^\lambda$ consists of the $\mathbb{F}$-progressively measurable trading rates $\dot{\theta}$ for which the system (2.4-2.5) admits a unique strong solution $(\theta^{t,\vartheta}, X^{\zeta,\vartheta,\dot{\theta},\lambda})$ for all initial data $(\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d$.

[13]As pointed out by Garleanu and Pedersen (2013), symmetry of $\Lambda$ can be assumed without loss of generality because otherwise the symmetrized version $(\Lambda + \Lambda^\top)/2$ leads to the same trading costs. Positive definiteness means that each transaction has a positive cost. The wealth dependence of the price impact parameter permits the incorporation of feedback effects of the investor's actions on market liquidity. For example, price impact inversely proportional to the investor's current wealth corresponds to the representative investor model of Guasoni and Weber (2014, 2017), where impact is constant relative to the total market capitalization.

[14]Quadratic trading costs can also be motivated by a block-shaped limit order book (Obizhaeva and Wang 2013) or a microstructure model based on the inventory risk accumulated by market makers (Garleanu and Pedersen 2013). The empirical literature consistently finds convex trading costs (e.g., Engle, Ferstenberg, and Russell 2008; Lillo, Farmer, and Mantegna 2003). Some studies actually report quadratic costs (Breen, Hodrick, and Korajczyk 2002; Kyle and Obizhaeva 2011), whereas others point toward sublinear price impact with trading costs between linear and quadratic (e.g., Almgren et al. 2005; Tóth et al. 2011).

[15]Convergence of the respective optimizers is proved in a related model by Garleanu and Pedersen (2013).

REMARK 2.2. To ease notation and because the time-derivative plays a special role, for any smooth function $\varphi : \mathfrak{D} \to \mathbb{R}$, (resp. $\varphi : \mathfrak{D} \times \mathbb{R}^d \to \mathbb{R}$) we write $D\varphi$ (resp. $D_\zeta \varphi$) for the gradient of $\varphi$ with respect to its spatial components $(s, y, x)$. Derivatives with respect to time $t$ are denoted by $\partial_t \varphi$ throughout.

### 2.3. Preferences and Liquidation

In the above market with linear price impact, an investor trades to maximize expected utility from terminal wealth at some finite planning horizon $T > 0$. Her utility function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is nondecreasing, as well as smooth and strictly concave on the interior of its effective domain.

As the investment horizon is finite, liquidation at the terminal time $T$ has to be taken into account. For small proportional or fixed trading costs, a single bulk trade is negligible at the leading order, so that this issue disappears asymptotically. With price impact, however, liquidation becomes a nontrivial (and potentially costly) issue. Because we focus here on the dynamic trading before $T$, we separate the liquidation problem as follows. We suppose that the model parameters are simply frozen at time $T$ and the investor's terminal position $\theta_T$ is liquidated quickly toward the frictionless target $\theta_T^0 = \theta^0(T, S_T, Y_T, X_T^\theta)$ using the deterministic mean-variance optimal strategy from Schöneborn (2011), with constant risk-tolerance $R_T = -U'(X_T^\theta)/U''(X_T^\theta)$. This leads to risk-adjusted liquidation costs (Schöneborn 2011, equation (11)) of $\lambda^{1/2}\mathfrak{P}(T, S_T, Y_T, X_T^\theta, \theta_T)$, where (Schöneborn 2011, theorem 4.1):

$$\mathfrak{P}(\zeta, \vartheta) := (\vartheta - \theta^0(\zeta))^\top \frac{\Lambda^{1/2}(\Lambda^{-1/2}\sigma_S\sigma_S^\top\Lambda^{-1/2})^{1/2}\Lambda^{1/2}}{(2R)^{1/2}}(\zeta)(\vartheta - \theta^0(\zeta)).$$

As these liquidation costs are small for small price impacts ($\Lambda \sim 0$), we in turn define the investor's frictional value function as suggested by Taylor's theorem:

$$v^\lambda(\zeta, \vartheta) := \sup_{\dot\theta \in \dot\Theta^\lambda_{\zeta,\vartheta}} \mathbb{E}\left[U\big(X_T^{\zeta,\vartheta,\dot\theta,\lambda}\big) - U'\big(X_T^{\zeta,\vartheta,\dot\theta,\varepsilon}\big)\lambda^{1/2}\mathfrak{P}\big(T, S_T^\zeta, Y_T^\zeta, X_T^{\zeta,\vartheta,\dot\theta,\varepsilon}, \theta_T^{\zeta,\vartheta}\big)\right], \tag{2.6}$$

for initial data $(\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}$. Here, $\dot\theta$ runs through the set $\dot\Theta^\lambda_{\zeta,\vartheta}$ of *admissible* controls. These have to satisfy

$$U\big(X_T^{\zeta,\vartheta,\dot\theta,\varepsilon}\big) - U'\big(X_T^{\zeta,\vartheta,\dot\theta,\varepsilon}\big)\lambda^{1/2}\mathfrak{P}\big(T, S_T^\zeta, Y_T^\zeta, X_T^{\zeta,\vartheta,\dot\theta,\varepsilon}, \theta_T^{\zeta,\vartheta}\big) \in L^1. \tag{2.7}$$

Moreover, one needs to be able to approximate the corresponding wealth processes using simple strategies as in Biagini and Černý (2011). The first condition is evidently needed to make the terminal utility well defined. The second assumption is an economically meaningful class of strategies that is small enough to exclude doubling strategies,[16] but large enough to contain the optimizer under weak assumptions; see Biagini and Černý (2011) for more details. For utilities which are only finite on the positive half-line, the approximation property is replaced by requiring the wealth process to be positive on $[0, T]$.

[16]With superlinear frictions, doubling strategies need not be ruled out a priori to make the frictional problem well posed (Guasoni and Rásonyi 2014). However, even if doubling strategies are not scalable at will, their availability may still cause the value function to become discontinuous at the terminal time $T$, ruling out classical verification theorems as in Section 7. Therefore, we do not allow doubling strategies here.

REMARK 2.3. The liquidation penalty $\mathfrak{P}$ disappears in the following two important special cases:

1. For infinite-horizon problems as in Garleanu and Pedersen (2013); Guasoni and Weber (2014, 2017), liquidation is not an issue. Indeed, as the horizon grows, the cost of the terminal liquidation program remains the same, whereas the accumulated benefits from trading grow indefinitely.
2. Suppose that the initial allocation is close to the frictionless target. Then, for strategies that always trade quickly toward the latter, the deviation always remains small in expectation. Hence, the liquidation penalty is of higher order in this case, and can be neglected asymptotically.

For finite-horizon problems and arbitrary initial endowments, however, liquidation has to be taken into account explicitly, see Almgren and Li (2011).

REMARK 2.4. Instead of requiring liquidation to the frictionless optimizer in (2.6), one could also impose liquidation to a full cash position, or no liquidation penalty at all. Both of these alternatives are economically meaningful, but complicate the problem substantially. The reason is that unlike for proportional or fixed costs, one cannot set up or liquidate a given portfolio with a single block trade and trading costs negligible at the leading asymptotic order. Therefore, with no liquidation penalty, investors with a short horizon will only trade very little if their initial position is far from the frictionless target to save trading costs. In contrast, with a longer horizon, they will trade much more aggressively to reap the gains from an optimal position in the long run. Requiring full liquidation leads to similar inhomogeneities. Indeed, as the horizon nears, the investor's focus then gradually shifts from rebalancing to maintain an optimal risk-return tradeoff to a liquidation program. In contrast, liquidation toward the frictionless target leads to a "stationary" version of the (asymptotic) problem, where the effects of setting up and liquidating the portfolio are disregarded, to be dealt with as separate optimal execution problems.

## 3. DYNAMIC PROGRAMMING AND CORRECTOR EQUATIONS

In this section, we state the dynamic programming equations solved by the frictionless and frictional value functions, respectively. For small price impacts, their difference is described by the solution of the so-called "corrector equations." To provide some intuition, we first derive these heuristically for a single risky asset and state variable. Afterwards, we state the general multidimensional versions.

### 3.1. The Frictionless Case

Without price impact, the diffusions $(S^{t,s,y}, Y^{t,y})$ are still defined as the strong solutions of the SDEs (2.1-2.2) but, without trading costs, the wealth dynamics (2.5) reduce to

$$dX_r^{\zeta,\theta} = \theta_r dS_r, \quad X_t^{\zeta,\theta} = x.$$

Here, the—now no longer necessarily absolutely continuous—control $\theta$ denotes the numbers of risky shares held in the portfolio. The control set consists of the $\mathbb{F}$-progressively measurable processes taking values in $\mathbb{R}^d$ such that the above SDE admits a unique strong solution $X^{\zeta,\theta}$. As above, we restrict ourselves to the subset $\Theta^0_\zeta$ of *admissible* controls for which $U(X^{\zeta,\theta}_T) \in L^1$, and for which the corresponding wealth processes can be approximated by simple strategies as in Biagini and Černý (2011). The frictionless value function is then defined as follows:

$$v^0(\zeta) := \sup_{\theta \in \Theta^0_\zeta} \mathbb{E}\big[U\big(X^{\zeta,\theta}_T\big)\big]. \tag{3.1}$$

Standard arguments (compare, e.g., Fleming and Soner 2006) show that the frictionless value function $v^0$ solves the Dynamic Programming Equation (henceforth DPE) for the problem at hand:

PROPOSITION 3.1. *Assume that $v^0$ is locally bounded. Then it is a (discontinuous) viscosity solution of*

$$\begin{cases} \inf_{\vartheta \in \mathbb{R}^d} \{-\mathcal{L}^\vartheta v^0\} = 0, & \text{on } \mathfrak{D}_<, \\ v^0(T, \zeta) = U(x), & \text{on } \partial_T \mathfrak{D}, \end{cases} \tag{3.2}$$

*where, for $\psi \in C^{1,2}$ and $(\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d$:*

$$\mathcal{L}^\vartheta \psi(\zeta, \vartheta) := \left\{\partial_t \psi + \mu_\vartheta \cdot D_\zeta \psi + \frac{1}{2} \operatorname{Tr}\left[\sigma_\vartheta \sigma_\vartheta^\top D^2_\zeta \psi\right]\right\}(\zeta, \vartheta),$$

*with*

$$\mu_\vartheta(\zeta) := \begin{pmatrix} \mu_S \\ \mu_Y \\ \vartheta \cdot \mu_S \end{pmatrix}(\zeta) \quad \text{and} \quad \sigma_\vartheta(\zeta) := \begin{pmatrix} \sigma_S \\ \sigma_Y \\ \vartheta^\top \sigma_S \end{pmatrix}(\zeta).$$

REMARK 3.2. Suppose that $v^0$ is smooth with $\partial_{xx} v^0 < 0$. Then, as $\sigma_S$ satisfies the ellipticity condition (2.3), it follows that $v^0$ is a classical solution of

$$\mathcal{L}^{\theta^0} v^0(\zeta) = 0, \tag{3.3}$$

for all $\zeta \in \mathfrak{D}_<$ or, equivalently,

$$\left\{\partial_t v^0 + \mu_0 D v^0 + \frac{1}{2} \operatorname{Tr}\left[\bar\sigma_0 \bar\sigma_0^\top D^2_{(s,y)} v^0\right]\right\}(\zeta) = \left(\frac{1}{2}(\theta^0)^\top \sigma_S \sigma_S^\top \theta^0 \partial_{xx} v^0\right)(\zeta), \tag{3.4}$$

where the optimal investment strategy $\theta^0(\zeta)$ satisfies

$$-\,(\partial_{xx} v^0 \sigma_S \sigma_S^\top \theta^0)(\zeta) := \mu_S \partial_x v^0 + \sigma_S \bar\sigma_0^\top D_{(s,y)}(\partial_x v^0)(\zeta), \tag{3.5}$$

with

$$\bar\sigma_0 := \begin{pmatrix} \sigma_S \\ \sigma_Y \end{pmatrix}.$$

Indeed, given sufficient regularity of the coefficients of the SDEs, standard verification arguments (compare, e.g., Touzi 2013) show that the Markovian feedback policy

$$\theta^0_u := \theta^0\left(u, S^{t,s,y}_u, \hat X^{t,s,y,x,\theta^0}_u, Y^{t,y}_u\right), \quad u \in [t, T],$$

is optimal for (3.1) in this case. Note that—with an abuse of notation—we use the same symbol to denote both the feedback description of a strategy and its evolution as a stochastic process.

### 3.2. The Dynamic Programming Equation with Price Impact

Given that the frictionless value function $v^0$ is locally bounded, its frictional counterpart $v^\lambda$ is evidently locally bounded from above because any absolutely continuous control in $\dot{\Theta}^\lambda_{\zeta,\vartheta}$ can be reproduced by a control in $\Theta^0_\zeta$, the utility function $U$ is nondecreasing, and the penalty function $\mathfrak{P}$ is nonnegative. We assume in addition that $v^\lambda$ is also locally bounded from below, i.e., there exists at least one strategy that closes out any initial position with finite utility.[17]

Next, we turn to the corresponding DPE with linear price impact. Without state constraints, i.e., for utilities that are finite on the whole real line, the latter can be derived from the weak dynamic programming principle of Bouchard and Touzi (2011). It is expected that this remains true if wealth is required to remain positive for utilities finite only on $\mathbb{R}_+$, see Bouchard and Nutz (2012). Making this rigorous in the presence of frictions is more delicate, though, see Altarovici et al. (2015); Soner and Vukelja (2014) for some specific examples. Therefore, we simply state the DPE as an assumption in the general setting considered here:

ASSUMPTION 3.3. *The frictional value function $v^\lambda$ is locally bounded and a (discontinuous) viscosity solution of*

$$\begin{cases} -\mathcal{L}^\vartheta v^\lambda - \mathcal{H}^\lambda v^\lambda = 0, & \text{on } \mathfrak{D}_< \times \mathbb{R}^d, \\ v^\lambda = U - U'\lambda^{1/2}\mathfrak{P}, & \text{on } \partial_T\mathfrak{D} \times \mathbb{R}^d, \end{cases} \tag{3.6}$$

*where, for $\psi \in C^{1,2}$ and $(\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d$:*

$$\mathcal{H}^\lambda\psi(\zeta,\vartheta) := \sup_{\dot\vartheta \in \mathbb{R}^d} \left\{ \dot\vartheta \cdot D_\vartheta\psi - \lambda\dot\vartheta^\top\Lambda\dot\vartheta\partial_x\psi \right\}(\zeta,\vartheta), \tag{3.7}$$

*and the liquidation penalty $\mathfrak{P}$ is defined as in Section 2.3.*

REMARK 3.4. The PDE (3.6) generally has to be understood in terms of the semicontinuous envelopes $\mathcal{H}^{\lambda,*}$, $\mathcal{H}^\lambda_*$ of

$$\mathcal{H}^\lambda : (\zeta, q_x, q_\vartheta) \in \mathfrak{D} \times \mathbb{R} \times \mathbb{R}^d \longmapsto \sup_{\dot\vartheta \in \mathbb{R}^d} \left\{ \dot\vartheta \cdot q_\vartheta - \lambda\dot\vartheta^\top\Lambda(\zeta)\dot\vartheta q_x \right\}.$$

(We use the shorthand notation $\mathcal{H}^\lambda\psi(\zeta,\vartheta) := \mathcal{H}^\lambda(\zeta, \partial_x\psi(\zeta,\vartheta), D_\vartheta\psi(\zeta,\vartheta))$.)

However, we have $\mathcal{H}^{\lambda,*} = \mathcal{H}^\lambda_* = \mathcal{H}^\lambda$ on $\mathfrak{D} \times (0,\infty) \times \mathbb{R}^d$ so that this relaxation is superfluous for smooth test function $\psi$ satisfying $\partial_x\psi > 0$ on $\mathfrak{D} \times \mathbb{R}^d$. Moreover, in this case, positive-definiteness of $\Lambda$ gives that the first line in (3.6) can be rewritten as

$$-\left(\mathcal{L}^\vartheta\psi + \frac{(D_\vartheta\psi)^\top\Lambda^{-1}D_\vartheta\psi}{4\lambda\partial_x\psi}\right)(\zeta,\vartheta) = 0, \quad \text{for all } (\zeta,\vartheta) \in \mathfrak{D}_< \times \mathbb{R}^d, \tag{3.8}$$

[17] For any initial wealth, this is evidently possible with a single bulk trade for sufficiently small proportional or fixed costs. With linear price impact, only absolutely continuous trading strategies can be implemented. Therefore, one has to restrict to long-only portfolios for utilities defined on the positive half-line, and impose sufficient integrability on the asset dynamics even for utilities defined on the whole real line, see Section 8 for more details.

where we have used the pointwise optimizer in (3.7):

$$\dot{\vartheta}^{\lambda}(\zeta,\vartheta) := \frac{\Lambda^{-1} D_{\vartheta}\psi}{2\lambda\partial_x\psi}(\zeta,\vartheta). \tag{3.9}$$

3.3. Heuristic Expansion for a Single Risky Asset

Our goal is to show that, for all $(\zeta,\vartheta)\in\mathfrak{D}\times\mathbb{R}^d$, the frictional value function has the asymptotic expansion

$$v^{\lambda}(\zeta,\vartheta) = v^0(\zeta) - \lambda^{1/2}u(\zeta) - \lambda\varpi\circ\boldsymbol{\xi}_{\lambda}(\zeta,\vartheta) + o(\lambda^{1/2}). \tag{3.10}$$

Here, we write

$$\varpi\circ\boldsymbol{\xi}_{\lambda}(\zeta,\vartheta) := \varpi(\zeta,\boldsymbol{\xi}_{\lambda}(\zeta,\vartheta))$$

for $\varpi : (\zeta,\xi)\in\mathfrak{D}\times\mathbb{R}^d \longmapsto \varpi(\zeta,\xi)$, and the “fast” variable

$$\boldsymbol{\xi}_{\lambda}(\zeta,\vartheta) := \frac{\vartheta-\theta^0(\zeta)}{\lambda^{1/4}} \tag{3.11}$$

measures the deviation of the actual position from the frictionless target (3.5), rescaled to be of order one as $\lambda\to 0$.

REMARK 3.5. The asymptotic scalings for the value function and the optimal policy are motivated by the corresponding results of Guasoni and Weber (2017).

To motivate the corrector equations describing the asymptotics (cf. Section 3.4), let us first informally derive them for a single risky asset ($d=1$) and a single state variable ($m=1$).[18] Both processes are driven by a two-dimensional Brownian motion ($q=2$), with volatilities

$$\sigma_S := (\sigma_{S,1} \quad 0) \quad \text{and} \quad \sigma_Y := (\sigma_{Y,1} \quad \sigma_{Y,2}),$$

so that price and state shocks are correlated for $\sigma_{Y,1}\neq 0$. In this simple framework, the price impact matrix $\Lambda$ is simply a positive, smooth, scalar function on $\mathfrak{D}$. Suppose that $v^0$ and $v^{\lambda}$ are classical solutions of (3.2) and (3.6), respectively, satisfying $\partial_x v^0\wedge(-\partial_{xx}v^0)\wedge\partial_x v^{\lambda}>0$. Assume furthermore that the functions $\theta^0$, $u$, $\varpi$ and $\boldsymbol{\xi}_{\lambda}$ belong to $C^{1,2}$, and introduce the local quadratic variation of the frictionless optimizer:

$$c_{\theta^0}(\zeta) := \frac{d\langle\theta^0\rangle}{dt}(\zeta) = \left(\sigma_S\partial_s\theta^0 + \sigma_{SY}\partial_y\theta^0 + \sigma_S\theta^0\partial_x\theta^0\right)^2(\zeta) + \left(\sigma_Y\partial_y\theta^0\right)^2(\zeta) \geq 0. \tag{3.12}$$

Notice that $\theta^0$ refers to the evolution of the optimal frictionless strategy as a stochastic process, where the appropriate state variables are plugged into its feedback description.

*3.3.1.* **The corrector equations**.
Inserting the ansatz (3.10-3.11) into the frictional DPE (3.8) leads to

$$\begin{aligned}0 = &-\mathcal{L}^{\theta^0}v^0 - \lambda^{1/4}\boldsymbol{\xi}_{\lambda}\left(\mu_S\partial_x v^0 + \sigma_{S,1}^2\partial_{sx}v^0 + \sigma_S\sigma_{Y,1}\partial_{xy}v^0 + \sigma_{S,1}^2\theta^0\partial_{xx}v^0\right)\\ &-\lambda^{1/2}\left(-\mathcal{L}^{\theta^0}u + \frac{1}{2}\sigma_{S,1}^2\partial_{xx}v^0\boldsymbol{\xi}_{\lambda}^2 - \frac{1}{2}c_{\theta^0}\partial_{\xi\xi}\varpi + \frac{(\partial_{\xi}\varpi)^2}{4\Lambda\partial_x v^0}\right) + o(\lambda^{1/2}).\end{aligned}$$

[18]The corresponding calculations for several assets and state variables are analogous, but more tedious.

Here, the first line vanishes by the frictionless DPE (3.3) and the first-order condition (3.5) for the frictionless optimizer. Observe that the map $u$ in (3.10) is independent of $\vartheta$, hence $\mathcal{L}^{\theta^0}u$ is a function of $\zeta$ only as well. As a consequence, the remaining terms at the order $\lambda^{1/2}$ in the previous equation should not depend on $\vartheta$ either. Therefore, we first look for a function $a : \mathfrak{D} \to \mathbb{R}$ such that the pair $(\varpi, a)$ is solution, for fixed $(t, s, x, y) \in \mathfrak{D}_<$, of the *first corrector equation*

$$\frac{1}{2}\sigma_{S,1}^2\xi^2\partial_{xx}v^0 - \frac{1}{2}c_{\theta^0}\partial_{\xi\xi}\varpi + \frac{\Lambda^{-1}(\partial_\xi\varpi)^2}{4\partial_x v^0} + a = 0, \tag{3.13}$$

and then identify $u$ as the solution on $\mathfrak{D}_<$ of the *second corrector equation*

$$-\mathcal{L}^{\theta^0}u - a = 0. \tag{3.14}$$

Now, insert the ansatz (3.10) into the terminal condition (3.6) for the frictional value function $v^\lambda$ and use the terminal condition (3.2) for its frictionless counterpart $v^0$. This shows that the corresponding terminal condition for $u$ is given by

$$\lambda^{1/2}u + \lambda\varpi \circ \boldsymbol{\xi}_\lambda = U'\lambda^{1/2}\mathfrak{P}, \quad \text{on } \partial_T\mathfrak{D}. \tag{3.15}$$

Let $R := -\partial_x v^0/\partial_{xx}v^0$ denote the risk tolerance of the frictionless value function. As $R > 0$ because we assumed $-\partial_{xx}v^0 \wedge \partial_x v^0 > 0$, the First Corrector Equation (3.13) is readily rewritten as

$$\frac{\sigma_{S,1}^2}{2\Lambda R}\xi^2 + \frac{c_{\theta^0}}{2\Lambda\partial_x v^0}\partial_{\xi\xi}\varpi - \left(\frac{\partial_\xi\varpi}{2\Lambda\partial_x v^0}\right)^2 - \frac{a}{\Lambda\partial_x v^0} = 0.$$

Evidently, there should be no penalty for deviating when the actual position coincides with the frictionless target. Hence, we impose the additional constraint $\varpi(\cdot, 0) = 0$, obtaining the explicit solution $(\varpi, a)$ with

$$\varpi(\zeta, \xi) = k_2(\zeta)\xi^2,$$

as well as

$$k_2 = \pm(\Lambda\partial_x v^0)\sqrt{\sigma_{S,1}^2/(2\Lambda R)}, \qquad a = c_{\theta^0}k_2. \tag{3.16}$$

Via (3.9), (3.10), and (3.11), this identifies the optimal trading rate for small price impact ($\lambda \sim 0$) as

$$\dot{\theta}^\lambda(\zeta, \vartheta) \sim -\frac{\lambda^{3/4}\partial_\xi\varpi(\zeta, \boldsymbol{\xi}(\zeta, \vartheta))}{2\lambda\Lambda(\zeta)\partial_x v^0(\zeta)} = -\left(\pm\sqrt{\frac{\sigma_{S,1}(t,s,y)^2}{2\lambda\Lambda(\zeta)R(\zeta)}}(\vartheta - \theta^0(\zeta))\right).$$

As one should evidently always trade toward the frictionless position $\theta^0$ rather than away from it, the positive sign for $k_2$ is the correct one in (3.16). Hence, asymptotically for small $\lambda$, the optimal policy prescribes trading toward the target portfolio at rate $\sqrt{\sigma_{S,1}^2/(2\lambda\Lambda R)}$, in line with (1.1).

Observe furthermore that the explicit form of $k_2$ gives $\lambda\varpi \circ \boldsymbol{\xi}_\lambda = \lambda^{1/2}\varpi \circ \boldsymbol{\xi}_1 = U'\lambda^{1/2}\mathfrak{P}$ on $\partial_T\mathfrak{D}$, so that the terminal condition for $u$ in (3.15) reads as

$$u = 0, \quad \text{on } \partial_T\mathfrak{D}. \tag{3.17}$$

### 3.4. Corrector Equations in the General Multidimensional Case

Let us now state the general multidimensional counterparts of the Corrector Equations (3.13-3.14, 3.17). To this end, we first introduce the $d$-dimensional counterpart of the local quadratic variation $c_{\theta^0}$ defined in (3.12):

$$c_{\theta^0}(\zeta) := \frac{d\langle \theta^0 \rangle_t}{dt}(\zeta) = \left(D_\zeta \theta^0\right)^\top \sigma_{\theta^0}\sigma_{\theta^0}^\top D_\zeta \theta^0. \tag{3.18}$$

With this notation, the corrector equations in the general multivariate case read as follows:

DEFINITION 3.6. **(Corrector Equations)** For a given point $\zeta \in \mathfrak{D}$, the first corrector equation for the unknown pair $(a(\zeta), \varpi(\zeta, \cdot)) \in \mathbb{R} \times C^2(\mathbb{R})$ is

$$\left\{ \frac{1}{2}\left|\xi^\top \sigma_S\right|^2 \partial_{xx} v^0 - \frac{1}{2}\operatorname{Tr}\left[c_{\theta^0} D^2_{\xi\xi}\varpi(\cdot, \xi)\right] + \frac{(D_\xi \varpi)^\top \Lambda^{-1} D_\xi \varpi}{4\partial_x v^0}(\cdot, \xi) + a \right\}(\zeta) = 0, \tag{3.19}$$

together with the normalization $\varpi(\zeta, 0) = 0$.

The second corrector equation uses the constant term $a(\zeta)$ from the first corrector, and is a simple linear equation for the function $u : \mathfrak{D} \to \mathbb{R}$:

$$\begin{cases} -\mathcal{L}^{\theta^0} u = a, & \text{on } \mathfrak{D}_<, \\ u = 0, & \text{on } \partial_T \mathfrak{D}. \end{cases} \tag{3.20}$$

We say that the pair $(u, \varpi)$ is a solution of the corrector equations.

For a single risky asset ($d = 1$) and a single state variable ($m = 1$), one readily verifies that these definitions coincide with the equations derived heuristically in Section 3.3 above.

## 4. MAIN RESULTS

Our main results are an asymptotic expansion of the value function $v^\lambda$ for small price impact $\Lambda_t = \lambda\Lambda(\cdot) \sim 0$, and an "almost optimal" trading policy that achieves the optimal performance at the leading order. To formulate these results, set

$$\bar{u}^\lambda(\zeta, \vartheta) := \frac{v^0(\zeta) - v^\lambda(\zeta, \vartheta)}{\lambda^{1/2}} \geq 0. \tag{4.1}$$

Then, the leading-order behavior of this difference can be analyzed under our Standing Assumption 3.3 that the frictional value function is a viscosity solution of the corresponding DPE and the following abstract conditions:[19]

ASSUMPTION A.

*(A1) (Regularity of the frictionless problem) The frictionless value function $v^0$ and optimal investment strategy $\theta^0$ belong to $C^{1,2}$. Moreover, $\partial_x v^0 \wedge (-\partial_{xx} v^0) > 0$.*

[19] Convenient sufficient conditions for their validity are provided in Section 7, and verified in a specific setting in Section 8. As in related results for proportional and fixed costs (Soner and Touzi 2013; Altarovici et al. 2015), these "verification theorems" are based on the availability of classical smooth solutions.

*(A2) (Locally uniform bound) For any $(\zeta_o, \vartheta_o) \in \mathfrak{D} \times \mathbb{R}^d$, there exist $r_o, \lambda_o > 0$ such that*

$$\sup\left\{\bar{u}^\lambda(\zeta, \vartheta) : (\zeta, \vartheta) \in B_{r_o}(\zeta_o, \vartheta_o) \cap (\mathfrak{D} \times \mathbb{R}^d) \text{ and } \lambda \in (0, \lambda_o]\right\} < \infty.$$

*(A3) (Comparison) A viscosity solution $u$ of the Second Corrector Equation* (3.20) *exists. Moreover, there is a class of functions $\mathcal{C}$ which contains $u, \bar{u}_*(\cdot, \theta^0(\cdot))$ and $\bar{u}^*(\cdot, \theta^0(\cdot))$ such that $u_1 \geq u_2$ for all $u_1, u_2 \in \mathcal{C}$ with $u_1$ (resp. $u_2$) being a lower-semicontinuous (resp. upper-semicontinuous) viscosity supersolution (resp. subsolution) of the Second Corrector Equation* (3.20).[20] *Here, $\bar{u}^*, \bar{u}_*$ denote the following relaxed semilimits:*

$$\begin{aligned} \bar{u}^*(\zeta, \vartheta) &:= \limsup_{\lambda \to 0, (\zeta', \vartheta') \to (\zeta, \vartheta)} \bar{u}^\lambda(\zeta', \vartheta'), \\ \bar{u}_*(\zeta, \vartheta) &:= \liminf_{\lambda \to 0, (\zeta', \vartheta') \to (\zeta, \vartheta)} \bar{u}^\lambda(\zeta', \vartheta'), \end{aligned} \tag{4.2}$$

*for all $(\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d$, which are well-defined upper- resp. lower-semicontinuous functions under Assumption (A2).*

Assumptions (A1) and (A3) are technical and can be guaranteed by imposing sufficient regularity conditions on the coefficient functions of the model. The crucial assumption is (A2), which postulates that the leading-order correction of the value function due to small price impact $\lambda\Lambda$ is indeed of order $O(\lambda^{1/2})$ as $\lambda \to 0$. This condition needs to be verified with more specific arguments. See Sections 7 and 8 for a verification theorem that achieves this for sufficiently regular classical solutions of the dynamic programming equations.

LEMMA 4.1. *Suppose Assumption (A1) is satisfied. Then, the First Corrector Equation* (3.19) *is solved by the locally bounded function*

$$a(\zeta) = \operatorname{Tr}\left[c_{\theta^0} k_2\right](\zeta) \tag{4.3}$$

*and the map*

$$\varpi : \xi \longmapsto \xi^\top k_2(\zeta)\xi,$$

*where $c_{\theta^0} = d\langle\theta^0\rangle/dt$ is the local quadratic variation of the frictionless target strategy $\theta^0$, and the positive semidefinite function $k_2 \in C^{1,2}(\mathfrak{D}; \mathbb{S}^d)$ is defined as*

$$k_2(\zeta) = \frac{\partial_x v^0}{\sqrt{-2\partial_x v^0/\partial_{xx} v^0}} \left[\Lambda^{1/2}(\Lambda^{-1/2}\sigma_S\sigma_S^\top\Lambda^{-1/2})^{1/2}\Lambda^{1/2}\right](\zeta).$$

*If, in addition, Assumption (A2) holds, then, evaluated along the frictionless optimal strategy $\theta^0$, the semilimits $\bar{u}^*(\cdot, \theta^0(\cdot)), \bar{u}_*(\cdot, \theta^0(\cdot))$ are viscosity sub- and supersolutions, respectively, of the Second Corrector Equation* (3.20).

*Proof.* Under (A1), the first part of the assertion is readily verified by direct computation. For the second part, first notice that the relaxed semilimits are finite by Assumption

[20]In particular, $u$ is the unique viscosity solution of (3.20) in the class $\mathcal{C}$.

(A2) and are upper- resp. lower-semicontinuous by definition. Using Assumptions (A1) and (A2), we show in Propositions 6.3, 6.4, and 6.5 that $\zeta \in \mathfrak{D} \longmapsto \bar{u}^*(\zeta, \theta^0(\zeta))$ and $\zeta \in \mathfrak{D} \longmapsto \bar{u}_*(\zeta, \theta^0(\zeta))$ are viscosity sub- resp. supersolutions of the Second Corrector Equation (3.20) with $a$ defined as in (4.3). □

REMARK 4.2. For later use, observe that the function $\varpi$ satisfies, for all $\xi \in \mathbb{R}^d$:

$$\frac{(|\varpi| + |D_{(t,\zeta)}\varpi|)(\cdot, \xi)}{1 + |\xi|^2} + \frac{(|D_\xi \varpi| + |D_{(t,\zeta)}(D_\xi \varpi)|)(\cdot, \xi)}{1 + |\xi|} + |D^2_{\xi\xi}\varpi|(\cdot, \xi) \leq \varrho, \quad \text{on } \mathfrak{D}, \tag{4.4}$$

for some continuous function $\varrho : \mathfrak{D} \to \mathbb{R}$.

We now state our main result, which determines the leading-order coefficient of the value function, under the Assumption (A2) that the first nontrivial term in its expansion is of order $O(\lambda^{1/2})$:

THEOREM 4.3. **(Expansion of the Value Function)** *Suppose Assumptions 3.3 and A are satisfied. Then, for any initial data* $(\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d$*:*

$$\bar{u}^\lambda(\zeta, \vartheta) \longrightarrow u(\zeta) + \varpi\left(\zeta, \vartheta - \theta^0(\zeta)\right),$$

*locally uniformly as* $\lambda \to 0$*. That is, the frictional value function* $v^\lambda(\zeta, \vartheta)$ *has the expansion*

$$v^\lambda(\zeta, \vartheta) = v^0(\zeta) - \lambda^{1/2}(u(\zeta) + \varpi\left(\zeta, \vartheta - \theta^0(\zeta)\right) + o(\lambda^{1/2}).$$

The lengthy proof of this result is postponed to Section 6.

REMARK 4.4. In view of the explicit formula in Lemma 4.1, the penalty for deviations of the initial portfolio $\vartheta$ from the frictionless target $\theta^0$ is given by

$$\lambda^{1/2}\varpi\left(\zeta, \vartheta - \theta^0(\zeta)\right) = \lambda^{1/2}\frac{\partial_x v^0(\zeta)}{\sqrt{2R(\zeta)}}(\vartheta - \theta^0(\zeta))^\top \left(\left(\Lambda^{1/2}(\Lambda^{-1/2}\sigma_S\sigma_S^\top\Lambda^{-1/2})^{1/2}(\zeta)\right)(\vartheta - \theta^0(\zeta)\right).$$

Hence, it is negligible at the leading order $O(\lambda^{1/2})$ for initial positions $\vartheta$ sufficiently close to the frictionless optimizer $\theta^0(\zeta)$.

REMARK 4.5. By Lemma 4.1, the term $a$ from the First Corrector Equation (3.19) is nonnegative. Hence, if the regularity conditions of Karatzas and Shreve (1991, remark 5.7.8) or, more generally of Friedman (1964, chapter I) are satisfied, a smooth classical solution of the Second Corrector Equation (3.20) exists. It admits the Feynman-Kac representation

$$\begin{aligned} u(\zeta) &= \mathbb{E}\left[\int_t^T a\left(r, S_r^{t,s,y}, Y_r^{t,y}, X_r^{\zeta,\theta^0}\right) dr\right], \\ &= \mathbb{E}\left[\int_t^T \left(\frac{\partial_x v^0}{\sqrt{2R}} \operatorname{Tr}\left[\frac{d\langle\theta^0\rangle_r}{dr}\Lambda^{1/2}(\Lambda^{1/2}\sigma_S\sigma_S^\top\Lambda^{1/2})^{1/2}\Lambda^{1/2}\right]\right)(r, S_r^\zeta, Y_r^\zeta, X_r^{\zeta,\theta^0})dr\right]. \end{aligned} \tag{4.5}$$

Here, $X^{\zeta,\theta^0}$ denotes the optimal frictionless wealth process and $R(\zeta) := -\partial_x v^0(\zeta)/\partial_{xx}v^0(\zeta)$ represents the risk tolerance of the frictionless indirect utility function; the second equality in (4.5) follows from the explicit formula for $a$ in Lemma 4.1.

Conversely, if the frictionless solution and in turn (4.5) are sufficiently regular, then the probabilistic representation (4.5) provides a solution of the Second Corrector Equation (3.20). This is exploited in Section 8.

REMARK 4.6. As is well known, the dual minimizer for the frictionless version of the problem is typically the density process of a dual martingale measure $\mathbb{Q}$ (the "marginal pricing measure"). It is given by $\partial_x v^0(r, S_r, Y_r, X_r)/\partial_x v^0(t, s, y, x)$, the normalized wealth-derivative of the corresponding value function, evaluated along the optimal frictionless wealth process (see, e.g., Section 8 for a simple example; compare Schachermayer (2001) for a general setting). If the initial portfolio equals the frictionless target, $\vartheta = \theta^0(\zeta)$, Theorem 4.3, (4.5), and a first-order Taylor expansion therefore show that

$$v^\lambda(t, s, y, x, \vartheta) = v^0(t, s, y, x - \mathrm{CE}(t, s, y, x)) + o(\lambda^{1/2}),$$

where

$$\mathrm{CE}(\zeta) = \mathbb{E}_{\mathbb{Q}}\left[\lambda^{1/2}\int_t^T \frac{\mathrm{Tr}\left[\frac{d\langle\theta^0\rangle_r}{dr}\Lambda^{1/2}(\Lambda^{1/2}\sigma_S\sigma_S^\top\Lambda^{1/2})^{1/2}\Lambda^{1/2}\right]}{\sqrt{2R}}(r, S_r^\zeta, Y_r^\zeta, X_r^{\zeta,\theta^0})dr\right].$$

Hence, the certainty equivalent loss CE due to small price impact is given by the above $\mathbb{Q}$-expectation. This is the amount of initial endowment the investor would give up to trade without frictions. For a single risky asset, Formula (1.2) from the introduction obtains.

Under the sufficient Condition B for the abstract Assumption A provided in Section 7, we can also produce an "almost optimal" policy that achieves the leading-order optimal performance in Theorem 4.3:

THEOREM 4.7. **(Almost Optimal Policy)** *Suppose Assumptions 3.3 and B are satisfied. Then, the feedback control*

$$\dot\theta^\Lambda(\zeta, \vartheta) = \lambda^{-1/2}\left(\frac{\Lambda^{-1/2}(\Lambda^{-1/2}\sigma_S\sigma_S^\top\Lambda^{-1/2})^{1/2}\Lambda^{1/2}}{(2R)^{1/2}}\right)(\zeta)(\theta^0(\zeta) - \vartheta), \quad \zeta \in \mathfrak{D},\ \vartheta \in \mathbb{R}^d,$$

*is optimal at the leading order $O(\lambda^{1/2})$, where $R(\zeta) = -\partial_x v^0(\zeta)/\partial_{xx} v^0(\zeta)$ denotes the risk tolerance of the frictionless value function $v^0$. For a single risky asset ($d = 1$), this formula simplifies to*

$$\dot\theta^\Lambda(\zeta, \vartheta) = \sqrt{\left(\frac{\sigma_S^2}{2\lambda\Lambda R}\right)(\zeta)}(\theta^0(\zeta) - \vartheta),$$

*in accordance with* (1.1).

This result is proved in Section 7.

## 5. INTERPRETATION AND APPLICATION

In this section, we discuss the interpretation of our main results, their connections to the extant literature on portfolio choice with market frictions, and how they can be applied to determine utility-based option prices and hedging strategies. For simplicity, we mostly focus on the case of a single risky asset ($d = 1$), and refer the interested reader to Guasoni and Weber (2014) for a detailed discussion of portfolio choice in a multivariate Black–Scholes model with price impact.

### 5.1. Connections to Other Portfolio Choice Models with Price Impact

Let us first place our results in context by comparing them to the most closely related studies from the extant literature.

Garleanu and Pedersen (2013) consider investors with an infinite horizon and *local* mean-variance preferences, who consume trading gains immediately. These investors trade several risky assets driven by arithmetic Brownian motion with returns following a stationary Markovian state variable. In this setting, and also for time-varying risk aversion or volatility, the optimal policy is characterized by the solution of a multidimensional nonlinear ordinary differential equation (henceforth ODE). The latter can be solved in closed form if the state variable is of Ornstein–Uhlenbeck-type, risk aversion and volatility are constant, and price impact is proportional to the assets' covariance matrix.[21]

Like Garleanu and Pedersen, Almgren and Li (2011) also focus on local mean-variance preferences. For a single risky asset following arithmetic Brownian motion, traded with constant linear price impact, they study the hedging of European options. Explicit formulas for the optimal trading rate obtain under the assumption that the option's "Gamma" is constant.

Guasoni and Weber (2014, 2017) study a global optimization problem, namely an investor with constant relative risk aversion who maximizes utility from terminal wealth over a long horizon. For asset prices following geometric Brownian motions and price impact inversely proportional to the (representative) investor's wealth, they characterize the optimal policy and the corresponding welfare by the solution of an Abel ODE. In the limit for small trading costs, explicit formulas obtain, which are found to provide an excellent approximation of the exact solution.

The above studies differ with respect to preferences (local vs. global criteria, constant absolute vs. constant relative risk aversion), asset dynamics (arithmetic vs. geometric Brownian motions), price impacts (proportional to number of shares vs. proportional to amount of wealth traded), and time horizons (infinite vs. finite). For small price impact parameters, the broad conclusions nevertheless are the same in each model. Indeed, consider a single risky asset for simplicity.[22] Then, for small trading costs, the trading rate—interpreted appropriately in each model—is linear in i) the displacement from the frictionless target position and ii) a constant determined by the constant market, cost, and preference parameters.

The present study extends and unifies these results. Our optimal policy in Theorem 4.7 shows that—asymptotically—this structure indeed applies universally, even for general Markovian dynamics of asset prices, factors, and costs, as well as for arbitrary preferences over terminal wealth. In each case, the optimal trading rate (in numbers of shares traded) is given by

$$\dot{\theta}_t^{\Lambda} = \sqrt{\frac{(\sigma_t^S)^2}{2\Lambda_t R_t}} \left(\theta_t - \theta_t^{\Lambda}\right). \tag{5.1}$$

If the driving Brownian motion is arithmetic, the asset's local variance $(\sigma_t^S)^2$ is constant, so that a constant trading rate obtains for a constant price impact $\Lambda$ proportional to the number of shares traded, and constant risk tolerance $R$, in line with the results

[21] More generally, explicit solutions in a class of policies linear in the state variable are studied by Collin-Dufresne et al. (2012).

[22] The discussion for several risky assets is analogous, but the formulas are more involved and harder to interpret.

of Garleanu and Pedersen (2013) as well as Almgren and Li (2011). If the driving Brownian motion is geometric, as in Guasoni and Weber (2014, 2017), then $(\sigma_t^S)^2 = \sigma^2 S_t^2$ is proportional to the squared asset price. Hence, a constant trading rate (in terms of relative wealth turnover $\dot{\theta}_t^\Lambda S_t / X_t^{\theta^\Lambda}$) obtains if risk tolerance $R_t$ is proportional to current wealth $X_t^{\theta^\Lambda}$ (i.e., if relative risk aversion is constant), and price impact is proportional to the square of the current stock price and inversely proportional to current wealth, $\Lambda_t = \lambda S_t^2 / X_t^{\theta^\Lambda}$ as in Guasoni and Weber (2014, 2017).

For more general preferences as well as price and cost dynamics, the same policy remains optimal if variance, risk tolerance, and impact costs are updated dynamically. These inputs are all "myopic," in the sense that they are determined by the frictionless problem and the current state of the model. In particular, the same leading-order corrections obtain for local preferences (as in Garleanu and Pedersen 2013; Almgren and Li 2011) and for global maximization problems (like in Guasoni and Weber 2014, 2017 and the present study). This parallels the situation for proportional transaction costs, where local and global preferences also lead to the same leading-order corrections for small costs (Soner and Touzi 2013; Kallsen and Muhle-Karbe 2013; Martin 2012; Kallsen and Li 2013).

## 5.2. Connections to the Optimal Execution Literature

The optimal trading rate (5.1) can also be connected to the optimal execution literature, which studies how to split up a single, exogenously given order efficiently.

Indeed, the key parameter—the square root of variance, times risk aversion, divided by two times the trading cost—also plays a pivotal role in the analysis of Almgren and Chriss (2001) as well as Schied and Schöneborn (2009). This can be related to the present model for dynamic portfolio choice as follows. Suppose that the investor currently holds a position $\theta_t^\Lambda$. In the absence of frictions ($\lambda = 0$), she would immediately trade toward the optimal frictionless allocation $\theta_t^0$. With price impact ($\lambda > 0$), she instead trades toward the latter at the finite absolutely continuous rate $\dot{\theta}_t^\Lambda$ from (5.1). *Locally*, the latter corresponds to the optimal initial execution rate for the order $\theta_t^\Lambda - \theta_t^0$ determined by Almgren and Chriss (2001) as well as Schied and Schöneborn (2009).[23] The same remains true in a multidimensional setting, where optimal execution has been studied by Schied et al. (2010) as well as Schöneborn (2011).

On each infinitesimally short time interval, the dynamic portfolio choice policy therefore corresponds to the Almgren–Chriss execution path toward the frictionless target position. That is, for small price impacts, the local trade scheduling is the same, with market, price impact, and preference parameters updated dynamically over time. The key difference is that there is not a single buy or sell order to be executed here; instead one tracks a moving target that evolves dynamically over time.

## 5.3. Application to Utility-Based Option Pricing and Hedging

Suppose that the investor under consideration has constant absolute risk aversion $\eta > 0$, i.e., an exponential utility function $U(x) = -e^{-\eta x}$. Then, it is well known that a random endowment $H$ at the terminal time $T$ can be absorbed by a change of measure.

[23] Almgren and Chriss (2001) consider mean-variance preferences, whereas Schied and Schöneborn (2009) extend their analysis to general von Neumann-Morgenstern utilities.

To wit, defining

$$\frac{d\mathbb{P}^H}{d\mathbb{P}} = \frac{e^{-\eta H}}{\mathbb{E}[e^{-\eta H}]},$$

the investor's problem is then equivalent to the pure investment problem without random endowment under the equivalent probability $\mathbb{P}^H$. If the change of measure leaves the structure of the model intact, random endowments can therefore be dealt with without additional difficulties.

In the present setting, suppose the investor has sold a European option with payoff $h(S_T)$ at time $T$ for a premium $p$. Then, $H = p - h(S_T)$, so that the change of measure is governed by the Radon–Nikodym derivative $d\mathbb{P}^H/d\mathbb{P} = e^{\eta h(S_T)}/\mathbb{E}[e^{\eta h(S_T)}]$. Given sufficient regularity, the Markov property implies that the corresponding density process $Z_t^H = \mathbb{E}[\frac{d\mathbb{P}^H}{d\mathbb{P}}|\mathcal{F}_t]$ is given by a function $f(t, S_t, Y_t)$ of time, the underlying, and the state variable, which can be determined from Itô's formula and the martingale property of $Z^H$. The model dynamics under $\mathbb{P}^H$ can in turn be computed with Girsanov's theorem by adjusting the drift rates of prices and state variables accordingly. If $f$ and its derivatives are sufficiently regular to satisfy Condition B also under $\mathbb{P}^H$, then our main results, Theorems 4.3 and 4.7, still apply. In particular, this shows that the trading rate of Theorem 4.7 is universal, in that it applies both for pure investment problems (as in Garleanu and Pedersen 2013; Guasoni and Weber 2014, 2017), and option hedging (as in Almgren and Li 2011). The only change is the frictionless target strategy. The expansion of the value function from Theorem 4.3 in turn enables us to compute first-order approximations of utility-indifference prices à là Hodges and Neuberger (1989) as well as Davis et al. (1993).[24]

## 5.4. Connections to Models with Proportional and Fixed Transaction Costs

In the above sections, we have argued that the trading rate (5.1) is ubiquitous in all kinds of optimization problems with small linear price impact. Now, we want to compare this policy to its counterparts for other market frictions, namely proportional and fixed transaction costs.

At first glance, the respective policies are radically different. With linear price impact, one always trades toward the frictionless target at a finite, absolutely continuous rate. In contrast, proportional and fixed transaction costs both lead to a "no-trade region" around the frictionless optimizer. In this region, investors remain inactive, and only trade once its boundaries are breached. This different "fine structure" is a consequence of the different penalizations of trades of various sizes: the quadratic trading costs induced by linear price impact are low for small trades, so that it is optimal to trade at all times. Conversely, they are prohibitively high for large orders, so that bulk trades (as for fixed costs) or "local-time-type" reflection (like for proportional costs) cannot be implemented, and the displacement from the frictionless target cannot be kept uniformly small. Compared to quadratic costs, proportional trading costs punish small trades more severely, leading to a no-trade region. However, as larger trades are penalized less, the position can always be kept inside this region by reflection at the boundaries ("pushing

[24]For proportional transaction costs, a number of corresponding results have been obtained, formally (Whalley and Wilmott 1997; Kallsen and Muhle-Karbe 2015) and rigorously (Bichuch 2014; Bouchard et al. 2013; Possamai and Royer 2014).

at an infinite rate"). With fixed costs, all trades are penalized alike. Whence, infinitely many small trades become infeasible and positions are immediately rebalanced to the frictionless target once the boundaries of the no-trade region are breached.

Despite these fundamental differences, all three market frictions nevertheless induce a surprisingly similar "coarse structure" as we now argue informally.[25] Indeed, with proportional transaction costs $\Lambda_t$, investors always keep their actual position in a no-trade region around the frictionless target, whose halfwidth can be determined explicitly for small costs (Martin 2012; Soner and Touzi 2013; Kallsen and Muhle-Karbe 2013, 2015; Kallsen and Li 2013). In the interior of this region, the investor's portfolio evolves uncontrolled, with instantaneous reflection at the boundaries. At the leading order, the distribution of such diffusion processes can be approximated by the uniform stationary law for reflected Brownian motion (Rogers 2004; Janeček and Shreve 2004; Goodman and Ostrov 2010; Kallsen and Muhle-Karbe 2013, 2015; Kallsen and Li 2013). Hence, the average squared deviation of the actual position from the frictionless target is given by one-third of the halfwidth of the corresponding no-trade region:

$$\frac{1}{\sqrt[3]{12}}\left(\frac{R_t\Lambda_t}{(\sigma_t^S)^2}\right)^{2/3}\left(\sigma_t^{\theta^0}\right)^{4/3},$$

where $\sigma_t^{\theta^0}=\sqrt{d\langle\theta^0\rangle_t/dt}$ is the volatility of the frictionless optimizer $\theta^0$.

For fixed transaction costs, the portfolio again moves uncontrolled inside a no-trade region, but is rebalanced directly to the frictionless target position once its boundaries are breached. At the leading order, this leads to a deviation with probability density given by a "hat function," which arises as the stationary law for Brownian motion killed and restarted at the origin upon hitting the boundaries of a symmetric interval. As a result, the variance of the corresponding deviation from the frictionless optimizer equals one-sixth of the halfwidth of the respective no-trade region:

$$\frac{1}{\sqrt{3}}\left(\frac{R_t\Lambda_t}{(\sigma_t^S)^2}\right)^{1/2}\sigma_t^{\theta^0}.$$

Up to the change of powers and a constant, the optimal policy is therefore determined by the same quantities in each case.

The optimal trading rate (1.1) with linear price impact leads to a deviation $\Delta_t = \theta_t^\Lambda - \theta_t^0$ following a mean-reverting diffusion process:

$$d\Delta_t = -\sqrt{\frac{(\sigma_t^S)^2}{2\Lambda_t R_t}}\Delta_t dt + d\theta_t^0.$$

For small price impact ($\Lambda \sim 0$) this is locally an Ornstein-Uhlenbeck process (globally, if the frictionless target strategy follows Brownian motion and the mean-reversion speed is constant), with Gaussian stationary law and leading-order variance

$$\sqrt{2}\left(\frac{R_t\Lambda_t}{(\sigma_t^S)^2}\right)^{1/2}(\sigma_t^{\theta^0})^2.$$

Again, the specific friction contributes the respective powers and a universal constant. In contrast, the input parameters and the corresponding comparative statics are universal:

[25] These arguments could be made rigorous similarly as in Kallsen and Li (2013).

the effect of a small friction is large if market risk is high compared to the investor's risk tolerance, if trading costs are substantial, or if the frictionless target strategy prescribes a lot of rebalancing.

In summary, even though different trading costs lead to fundamentally different optimal policies on a "microscopic" level, the "macroscopic" picture turns out to be surprisingly robust.

## 6. PROOF OF THEOREM 4.3

This section contains the proof of our first main result, the asymptotic expansion of the value function $v^\lambda$ for small price impacts $\lambda\Lambda(\cdot) \sim 0$ from Theorem 4.3. Throughout, we write[26]

$$\lambda = \varepsilon^4 \quad \text{and} \quad \Lambda(\zeta) = E(\zeta)^4,$$

to avoid the use of fractional powers. With a slight abuse of notation, we also index all quantities associated to the problem with price impact by $\varepsilon$. For example, we write $v^\varepsilon$ for the frictional value function $v^\lambda$, denote the corresponding optimal portfolio $\theta^\Lambda$ by $\theta^\varepsilon$, etc.

The strategy for the proof of Theorem 4.3 is as follows: Lemma 4.1 together with the results of Section 6.3 (see Propositions 6.3, 6.4, and 6.5) and Assumption (A3) yield

$$\bar{u}_*(\zeta, \theta^0(\zeta)) \geq u(\zeta) \geq \bar{u}^*(\zeta, \theta^0(\zeta)), \quad \text{forall } \zeta \in \mathfrak{D}.$$

On the other hand, we show in Proposition 6.6 (the functions $u_*$ and $u^*$ therein are defined in Section 6.2) that, for all $(\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d$:

$$\bar{u}_*(\zeta, \theta^0(\zeta)) \leq \bar{u}_*(\zeta, \vartheta) - \varpi \circ \xi\xi_1(\zeta, \vartheta) \leq \bar{u}^*(\zeta, \vartheta) - \varpi \circ \xi\xi_1(\zeta, \vartheta) \leq \bar{u}^*(\zeta, \theta^0(\zeta)).$$

Together, these two estimates prove Theorem 4.3.

### 6.1. Remainder Estimate

The first—and the most tedious—step is to estimate the remainders of the expansion in Theorem 4.3. This parallels Soner and Touzi (2013, remark 3.4, section 4.2); see also Bouchard, Moreau, and Soner (2013, lemma 4.4).

LEMMA 6.1. *Suppose Assumption (A1) is satisfied, and recall* $\boldsymbol{\xi}_\varepsilon(\zeta, \vartheta) = (\vartheta - \theta^0(\zeta))/\varepsilon$. *Fix* $\varepsilon > 0$, *two* $C^{1,2}(\mathfrak{D} \times \mathbb{R}^d)$*-functions* $\phi$ *and* $w$, *and define*

$$\psi^\varepsilon : (\zeta, \vartheta) \longmapsto v^0(\zeta) - \varepsilon^2\phi(\zeta, \vartheta) - \varepsilon^4 w^\varepsilon(\zeta, \vartheta), \quad \text{with } w^\varepsilon(\zeta, \vartheta) := w \circ \boldsymbol{\xi}_\varepsilon(\zeta, \vartheta) = w(\zeta, \boldsymbol{\xi}_\varepsilon(\zeta, \vartheta)).$$

*Set* $D_\varepsilon^\iota := \{\partial_x \psi^\varepsilon > 0\} \cap \{\varepsilon^2 \partial_x(\phi + \varepsilon^2 w^\varepsilon)/\partial_x v^0 \leq \iota\}$ *for some* $\iota < 1$. *Then:*

$$\begin{aligned}
\mathcal{L}^\vartheta \psi^\varepsilon &= \varepsilon^2 \left( \frac{1}{2} \left| \boldsymbol{\xi}_\varepsilon^\top \sigma_S \right|^2 \partial_{xx} v^0 - \mathcal{L}^{\theta^0} \phi - \frac{1}{2} \mathrm{Tr}\left[ c_{\theta^0} D^2_{\xi\xi} w \right] + \mathcal{R}^\varepsilon_{\mathcal{L}} \right), \\
\mathcal{H}^\varepsilon \psi^\varepsilon &= \varepsilon^2 \left( \frac{(D_\xi w \circ \boldsymbol{\xi}_\varepsilon)^\top E^{-4} D_\xi w \circ \xi\xi_\varepsilon}{4\partial_x v^0} + \mathcal{R}^\varepsilon_{\mathcal{H}} \right) + \hat{\mathcal{L}}^\varepsilon \phi, \quad \text{on } D^\iota_\varepsilon,
\end{aligned}$$

[26] Here, $E$ is the unique symmetric, positive definite matrix for which this representation holds true.

*with*

$$\hat{\mathcal{L}}^{\varepsilon}\phi := \frac{(D_\vartheta\phi)^\top E^{-4}(D_\vartheta\phi + 2\varepsilon D_\xi w \circ \boldsymbol{\xi}_\varepsilon)}{4\partial_x v^0} + \frac{\varepsilon^2\partial_x\phi}{4(\partial_x v^0)^2}(D_\vartheta\phi)^\top E^{-4} D_\vartheta\phi, \tag{6.1}$$

$\theta^0$ *defined as in* (3.5)*, and where* $\mathcal{R}^\varepsilon_{\mathcal{L}}$ *and* $\mathcal{R}^\varepsilon_{\mathcal{H}}$ *are continuous maps defined on* $D^\iota_\varepsilon$ *such that:*

*(Ri) For each bounded set* $B \subset \mathfrak{D} \times \mathbb{R}^d \times \mathbb{R}^d$*, there exists* $\varepsilon_B > 0$ *such that*

$$\left\{\varepsilon^{-1}\left(\left|\mathcal{R}^\varepsilon_{\mathcal{L}}\right| + \left|\mathcal{R}^\varepsilon_{\mathcal{H}}\right|\right)(\zeta,\vartheta) : (\zeta,\vartheta,\boldsymbol{\xi}_\varepsilon(\zeta,\vartheta)) \in B, \varepsilon \in (0,\varepsilon_B]\right\}$$

*is bounded;*

*(Rii) Let* $B \subset \mathfrak{D}$ *be a bounded set. Assume that* $\phi \in C^\infty_b(B \times \mathbb{R}^d)$ *and that* $w$ *satisfies* (4.4)*. Then, there exist* $\varepsilon_B > 0$ *and* $C_B > 0$ *such that*

$$\left|\mathcal{R}^\varepsilon_{\mathcal{L}}(\zeta,\vartheta)\right| + \left|\mathcal{R}^\varepsilon_{\mathcal{H}}(\zeta,\vartheta)\right| \le C_B\left(1 + \varepsilon\,|\xi\xi_\varepsilon| + \varepsilon^2\,|\xi\xi_\varepsilon|^2\right),$$

*for all* $\varepsilon \in (0,\varepsilon_B]$ *and* $(\zeta,\vartheta) \in B \times \mathbb{R}^d$*.*

*Proof.* For the sake of clarity, write

$$\bar{\mu}^0_\vartheta := \begin{pmatrix} 0 \\ 0 \\ \vartheta^\top\mu_S \end{pmatrix} \quad \text{and} \quad \bar{\sigma}^0_\vartheta := \begin{pmatrix} 0 \\ 0 \\ \vartheta^\top\sigma_S \end{pmatrix},$$

for any $\vartheta \in \mathbb{R}^d$. We work on $D^\iota_\varepsilon$ and omit the corresponding arguments for brevity.

*Step 1: expand the linear operator.* First, use $\vartheta = \theta^0 + \varepsilon\boldsymbol{\xi}_\varepsilon$, obtaining

$$\begin{aligned}
\mathcal{L}^\vartheta v^0 &= \mathcal{L}^{\theta^0} v^0 + \bar{\mu}^0_{\varepsilon\boldsymbol{\xi}_\varepsilon} D_\zeta v^0 + \mathrm{Tr}\left[\sigma_{\theta^0}\left(\bar{\sigma}^0_{\varepsilon\boldsymbol{\xi}_\varepsilon}\right)^\top D^2_\zeta v^0\right] + \frac{1}{2}\varepsilon^2\left|\boldsymbol{\xi}_\varepsilon^\top\sigma_S\right|^2\partial_{xx}v^0 \\
&= \mathcal{L}^{\theta^0} v^0 + (\varepsilon\boldsymbol{\xi}_\varepsilon)^\top\left(\mu_S\partial_x v^0 + \sigma_S\bar{\sigma}_0^\top D_{(s,y)}\left(\partial_x v^0\right) + \sigma_S\sigma_S^\top\theta^0\partial_{xx}v^0\right) + \frac{1}{2}\varepsilon^2\left|\xi\xi_\varepsilon^\top\sigma_S\right|^2\partial_{xx}v^0 \\
&= \frac{1}{2}\varepsilon^2\left|\xi\xi_\varepsilon^\top\sigma_S\right|^2\partial_{xx}v^0,
\end{aligned}$$

by the frictionless DPE (3.3) and the first-order condition (3.5) for the frictionless optimizer $\theta^0$, which hold due to Assumption (A1). The same calculation also yields $\mathcal{L}^\vartheta(\varepsilon^2\phi) = \varepsilon^2\mathcal{L}^{\theta^0}\phi + \varepsilon^2\mathcal{R}^\varepsilon_1$, with

$$\mathcal{R}^\varepsilon_1 := (\varepsilon\boldsymbol{\xi}_\varepsilon)^\top\left(\mu_S\partial_x\phi + \sigma_S\bar{\sigma}_0^\top D_{(s,y)}(\partial_x\phi) + \sigma_S\sigma_S^\top\theta^0\partial_{xx}\phi\right) + \frac{1}{2}\varepsilon^2\left|\boldsymbol{\xi}_\varepsilon^\top\sigma_S\right|^2\partial_{xx}\phi.$$

Now, observe $\boldsymbol{\xi}_\varepsilon = \boldsymbol{\xi}_1/\varepsilon$ so that, by definition of $\boldsymbol{\xi}_1$ and $w^\varepsilon$:

$$\begin{aligned}
D_\zeta w^\varepsilon &= D_\zeta w - \tfrac{1}{\varepsilon}D_\zeta\theta^0 D_\xi w, \\
D^2_{\zeta\zeta}w^\varepsilon &= \tfrac{1}{\varepsilon^2}D_\zeta\theta^0 D^2_{\xi\xi}w\, D^\top_\zeta\theta^0 - \tfrac{1}{\varepsilon}\left(D_\zeta\theta^0 D^\top_\zeta(D_\xi w) + D_\zeta(D_\xi w)D^\top_\zeta\theta^0 + D^2_{\zeta\zeta}\theta^0 D^\top_\xi w\right) + D^2_{\zeta\zeta}w.
\end{aligned}$$

As a result (recall (3.18)):

$$\mathcal{L}^\vartheta(\varepsilon^4 w^\varepsilon) = \varepsilon^2\frac{1}{2}\mathrm{Tr}\left[D^\top_\zeta\theta^0\sigma_{\theta^0}\sigma^\top_{\theta^0}D_\zeta\theta^0 D^2_{\xi\xi}w\right] + \varepsilon^2\mathcal{R}^\varepsilon_2, \tag{6.2}$$

with

$$\mathcal{R}_2^\varepsilon := \varepsilon^2\partial_t w^\varepsilon + \varepsilon^2\mu_{\theta^0+\varepsilon\boldsymbol{\xi}_\varepsilon}\cdot D_\zeta w^\varepsilon + \varepsilon^2\frac{1}{2}\mathrm{Tr}\left[\sigma_{\theta^0+\varepsilon\boldsymbol{\xi}_\varepsilon}\sigma_{\theta^0+\varepsilon\boldsymbol{\xi}_\varepsilon}^\top D_{\zeta\zeta}^2 w^\varepsilon - \frac{1}{\varepsilon^2}D_\zeta^\top\theta^0\sigma_{\theta^0}\sigma_{\theta^0}^\top D_\zeta\theta^0 D_{\xi\xi}^2 w\right]$$

$$\begin{aligned}
&= \varepsilon^2\partial_t w - \varepsilon D_t\theta^0\cdot D_\xi w + \varepsilon^2\mu_{\theta^0+\varepsilon\boldsymbol{\xi}_\varepsilon}\cdot D_\zeta w - \varepsilon\mu_{\theta^0+\varepsilon\xi\xi_\varepsilon}\cdot D_\zeta\theta^0 D_\xi w\\
&\quad + \frac{1}{2}\mathrm{Tr}\left[\left(\sigma_{\theta^0}\bar{\sigma}_{\varepsilon\boldsymbol{\xi}_\varepsilon}^{0\top} + \bar{\sigma}_{\varepsilon\xi\xi_\varepsilon}^{0\top}\sigma_{\theta^0}^\top + \bar{\sigma}_{\varepsilon\xi\xi_\varepsilon}^{0}\bar{\sigma}_{\varepsilon\xi\xi_\varepsilon}^{0\top}\right)D_\zeta\theta^0 D_{\xi\xi}^2 w\, D_\zeta^\top\theta^0\right]\\
&\quad - \mathrm{Tr}\left[\sigma_{\theta^0+\varepsilon\xi\xi_\varepsilon}\sigma_{\theta^0+\varepsilon\xi\xi_\varepsilon}^\top\left(\varepsilon\left(D_\zeta\theta^0 D_\zeta^\top(D_\xi w) + D_\zeta(D_\xi w)D_\zeta^\top\theta^0 + D_{\zeta\zeta}^2\theta^0 D_\xi^\top w\right) - \varepsilon^2 D_{\zeta\zeta}^2 w\right)\right].
\end{aligned}$$

The asserted estimates for $\mathcal{R}_\mathcal{L}^\varepsilon := \mathcal{R}_1^\varepsilon + \mathcal{R}_2^\varepsilon$ now follow from Assumption (A1), (4.4), and the continuity of the coefficients of the SDEs (2.1), (2.2), (2.4), and (2.5).

*Step 2: expand the nonlinear operator.* First, observe that $\partial_x\psi^\varepsilon > 0$ on $D_\varepsilon^\iota$; whence (recall Remark 3.4):

$$\mathcal{H}^\varepsilon\psi^\varepsilon = \frac{(D_\vartheta\psi^\varepsilon)^\top E^{-4} D_\vartheta\psi^\varepsilon}{4\varepsilon^4\partial_x v^0}\times\frac{1}{1-\varepsilon^2\partial_x(\phi+\varepsilon^2 w^\varepsilon)/\partial_x v^0}.$$

A first-order expansion of the right-hand side in turn gives

$$\mathcal{H}^\varepsilon\psi^\varepsilon = \frac{(D_\vartheta\psi^\varepsilon)^\top E^{-4} D_\vartheta\psi^\varepsilon}{4\varepsilon^4\partial_x v^0}\left(1+\varepsilon^2\frac{\partial_x\phi}{\partial_x v^0}\right) + \varepsilon^2\mathcal{R}_3^\varepsilon,$$

with

$$\begin{aligned}
\left|\mathcal{R}_3^\varepsilon\right| &\le \frac{(D_\vartheta\psi^\varepsilon)^\top E^{-4} D_\vartheta\psi^\varepsilon}{4\varepsilon^6\partial_x v^0}\times\left(\varepsilon^4\frac{\partial_x w^\varepsilon}{\partial_x v^0} + \frac{2}{(1-\iota)^3}\times\frac{\varepsilon^4\left|\partial_x(\phi+\varepsilon^2 w^\varepsilon)\right|^2}{(\partial_x v^0)^2}\right)\\
&= \frac{(D_\vartheta\phi+\varepsilon^2 D_\xi w)^\top E^{-4}(D_\vartheta\phi+\varepsilon^2 D_\xi w)}{4\partial_x v^0}\\
&\quad\times\left(\frac{\varepsilon^2\partial_x w - \varepsilon\partial_x\theta^0\cdot D_\xi w}{\partial_x v^0} + \frac{2\varepsilon^2\left|\partial_x\phi - \varepsilon\partial_x\theta^0\cdot D_\xi w + \varepsilon^2\partial_x w\right|^2}{(1-\iota)^3(\partial_x v^0)^2}\right),
\end{aligned}$$

where we have used for the first estimate that we are working on $D_\varepsilon^\iota$. Thus, we compute

$$\begin{aligned}
\mathcal{H}^\varepsilon\psi^\varepsilon &= \frac{\varepsilon^2(D_\xi w)^\top E^{-4} D_\xi w + (D_\vartheta\phi)^\top E^{-4}(D_\vartheta\phi + 2\varepsilon D_\xi w)}{4\partial_x v^0}\\
&\quad + \frac{\varepsilon^2\partial_x\phi}{4(\partial_x v^0)^2}(D_\vartheta\phi)^\top E^{-4} D_\vartheta\phi + \varepsilon^2(\mathcal{R}_3^\varepsilon + \mathcal{R}_4^\varepsilon),
\end{aligned}$$

with

$$\mathcal{R}_4^\varepsilon := \frac{2\varepsilon\partial_x\phi(D_\vartheta\phi)^\top E^{-4} D_\xi w + \varepsilon^2(D_\xi w)^\top E^{-4} D_\xi w}{4(\partial_x v^0)^2}.$$

Again, the asserted estimates for $\mathcal{R}_\mathcal{H}^\varepsilon := \mathcal{R}_3^\varepsilon + \mathcal{R}_4^\varepsilon$ now follow from the continuity of the involved functions, Assumption (A1), and (4.4). Together with Step 1, this completes the proof. $\square$

### 6.2. The Adjusted Relaxed Semi-Limits $u^*$, $u_*$

Unlike for models with proportional (Soner and Touzi 2013; Possamai et al. 2013; Bouchard et al. 2013) or fixed transaction costs (Altarovici et al. 2015), the relaxed semilimits of $\bar{u}^\varepsilon = (v^0 - v^\varepsilon)/\varepsilon^2$ do depend on the number of shares in the investor's portfolio for the present price impact model. As a result, the crucial simplification offered by homogenization apparently breaks down: the number of variables in the first-order correction term is the same as in the original frictional value function, rather than being reduced to the variables of its frictionless counterpart as in Soner and Touzi (2013); Possamai et al. (2013); Bouchard et al. (2013); Altarovici et al. (2015).

However—crucially—the heuristic arguments from Section 3.3 suggest that $\bar{u}^\varepsilon$ only depends on the initial number of risky shares $\vartheta$ through the quadratic function $\varpi$ determined by the first corrector equation. For intermediate times, this follows from the expansion of the frictional DPE, at the terminal time this is a consequence of the definition of the liquidation penalty in (2.6). In fact, the latter is chosen precisely so that a simple quadratic function does the job here, see Remark 2.4.

After subtracting this penalty term, the remaining first-order correction becomes independent of the current portfolio like for proportional and fixed costs.

To proceed, define for all $\varepsilon > 0$ the map $u^\varepsilon : \mathfrak{D} \times \mathbb{R}^d \to \mathbb{R}$ by

$$u^\varepsilon := \bar{u}^\varepsilon - \varepsilon^2 \varpi \circ \boldsymbol{\xi}_\varepsilon, \tag{6.3}$$

where the normalized deviation $\boldsymbol{\xi}_\varepsilon(\zeta, \vartheta) = (\vartheta - \theta^0(\zeta))/\varepsilon$ from the frictionless target $\theta^0$ is defined as in (3.11) and $\varpi(\xi)$ is the solution of the first corrector equation constructed in Lemma 4.1. In analogy with (4.2), the corresponding relaxed semilimits are then defined as

$$u^*(\zeta, \vartheta) := \limsup_{\varepsilon \to 0, (\zeta', \vartheta') \to (\zeta, \vartheta)} u^\varepsilon(\zeta', \vartheta'), \qquad u_*(\zeta, \vartheta) := \liminf_{\varepsilon \to 0, (\zeta', \vartheta') \to (\zeta, \vartheta)} u^\varepsilon(\zeta', \vartheta').$$

Evidently, the families $\{\bar{u}^\varepsilon : \varepsilon > 0\}$ and $\{u^\varepsilon : \varepsilon > 0\}$ do not have the same relaxed semilimits. Indeed, $\bar{u}^*$ and $\bar{u}_*$ are not independent of the $\vartheta$-variable, as is immediately apparent for $t = T$. In contrast, we shall see that $u^*$ and $u_*$ do not depend on the $\vartheta$-variable (this is again evident for $t = T$). This will be verified a posteriori, contrary to Soner and Touzi (2013), where this can be checked a priori for the relaxed semilimits $\bar{u}^*$ and $\bar{u}_*$, and is crucially used to establish the main result.

Define, for all $\varepsilon > 0$ and $(\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d$,

$$u_*^\varepsilon(\zeta, \vartheta) := \frac{v^0(\zeta) - v^{\varepsilon *}(\zeta, \vartheta)}{\varepsilon^2} \quad \text{and} \quad u^{\varepsilon *}(\zeta, \vartheta) := \frac{v^0(\zeta) - v_*^\varepsilon(\zeta, \vartheta)}{\varepsilon^2},$$

where $v^{\varepsilon *}$ and $v_*^\varepsilon$ denote the upper and lower semicontinuous envelopes of $v^\varepsilon$, respectively, and observe that

$$u^*(\zeta, \vartheta) = \limsup_{\varepsilon \to 0, (\zeta', \vartheta') \to (\zeta, \vartheta)} u^{\varepsilon *}(\zeta', \vartheta'), \qquad u_*(\zeta, \vartheta) = \liminf_{\varepsilon \to 0, (\zeta', \vartheta') \to (\zeta, \vartheta)} u_*^\varepsilon(\zeta', \vartheta'). \tag{6.4}$$

The following is a simple consequence of Assumptions (A2), (A1), as well as Lemma 4.1:

LEMMA 6.2. *Suppose Assumptions (A2) and (A1) are satisfied. Then, for all* $(\zeta_o, \vartheta_o) \in \mathfrak{D} \times \mathbb{R}^d$*, there are* $r_o, \varepsilon_o > 0$ *such that*

$$-\infty < u_*^\varepsilon \le u^{\varepsilon *} < +\infty, \quad \text{on } B_{r_o}(\zeta_o, \vartheta_o) \cap \mathfrak{D}, \text{ for all } \varepsilon \in (0, \varepsilon_o].$$

*In particular, the relaxed semilimits* $u_*$ *and* $u^*$ *are locally bounded.*

## 6.3. PDE Characterization Along the Frictionless Optimizer

In this section, we show that $\zeta \in \mathfrak{D} \longmapsto u^*(\zeta, \theta^0(\zeta)) = \bar{u}^*(\zeta, \theta^0(\zeta))$ and $\zeta \in \mathfrak{D} \longmapsto u_*(\zeta, \theta^0(\zeta)) = \bar{u}_*(\zeta, \theta^0(\zeta))$ are viscosity sub- and supersolutions, respectively, of the Second Corrector Equation (3.20), where $(a, \varpi)$ is the solution of the First Corrector Equation (3.19) constructed in Lemma 4.1.

### *6.3.1.* **Viscosity subsolution property**.

PROPOSITION 6.3. *Suppose Assumptions 3.3 and A are satisfied. Then,* $\zeta \in \mathfrak{D} \longmapsto u^*(\zeta, \theta^0(\zeta)) = \bar{u}^*(\zeta, \theta^0(\zeta))$ *is a viscosity subsolution of the Second Corrector Equation* (3.20) *on* $\mathfrak{D}_<$.

*Proof.* Consider $\zeta_o \in \mathfrak{D}_<$ and $\varphi \in C^{1,2}(\mathfrak{D}_<)$ such that

$$\max_{\zeta \in \mathfrak{D}_<}(\text{strict})(u^*(\zeta, \theta^o(\zeta)) - \varphi(\zeta)) = u^*(\zeta_o, \vartheta_o) - \varphi(\zeta_o) = 0, \tag{6.5}$$

where $\vartheta_o := \theta^0(\zeta_o)$. We have to show that $-\mathcal{L}^{\theta^0}\varphi(\zeta_o) \le a(\zeta_o)$.

*Step 1: provide a localizing sequence.*

By (6.4) and continuity of $\varphi$, there exist $(\zeta^\varepsilon, \vartheta^\varepsilon)_{\varepsilon>0} \subset \mathfrak{D}_< \times \mathbb{R}^d$ such that

$$(\zeta^\varepsilon, \vartheta^\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} (\zeta_o, \vartheta_o), \quad u^{\varepsilon *}(\zeta^\varepsilon, \vartheta^\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} u^*(\zeta_o, \vartheta_o), \quad \text{and} \quad p^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{6.6}$$

where

$$p^\varepsilon := u^{\varepsilon *}(\zeta^\varepsilon, \vartheta^\varepsilon) - \varphi(\zeta^\varepsilon). \tag{6.7}$$

Now, on the one hand, Lemma 6.2 guarantees the existence of $r_o, \varepsilon_0 > 0$ such that, with $B_o := B_{r_o}(\zeta_o) \times B_{r_o}(\vartheta_o)$,[27] we have $b^* := \sup\{u^{\varepsilon *}(\zeta, \vartheta)\ ,\ (\zeta, \vartheta) \in B_o\ ,\ \varepsilon \in (0, \varepsilon_0]\} < \infty$. On the other hand, by Assumption (A1), there exists $\alpha \in (0, r_o]$ for which

$$\theta^0 \in \bar{B}_{\frac{r_o}{4}}(\vartheta_o), \quad \text{on } \bar{B}_\alpha(\zeta_o), \tag{6.8}$$

and, for some $\iota > 0$:

$$2/\iota > -\partial_{xx} v^0 \wedge \partial_x v^0 > \iota, \quad \text{on } \bar{B}_\alpha(\zeta_o). \tag{6.9}$$

Now, choose $\mathbf{d} > 0$ such that:

$$\left|\zeta - \zeta'\right|^4 \ge \mathbf{d}, \quad \text{for all } (\zeta, \zeta') \in \left(\bar{B}_\alpha(\zeta_o) \backslash B_{\alpha/2}(\zeta_o)\right) \times \bar{B}_{\alpha/4}(\zeta_o).$$

[27] Here and in the following viscosity proofs, we always choose $r_o$ sufficiently small to guarantee that the respective neighborhoods are contained in $\mathfrak{D}_<$ resp. $\mathfrak{D}$.

By continuity of $\varphi$, we have $\sup\{2+b^* - \varphi(\zeta)\;;\;\zeta \in \bar{B}_\alpha(\zeta_o)\} =: M < +\infty$, and we in turn define the constant $c_o := M/(\mathbf{d} \wedge (\frac{r_o}{4})^4)$ . In view of (6.6), Assumption (A1), as well as Lemma 4.1, and reducing $\varepsilon_o > 0$ if necessary, we obtain:

$$\begin{aligned} |\zeta^\varepsilon - \zeta_o| \vee |\vartheta^\varepsilon - \vartheta_o| &\leq \tfrac{\alpha}{4}\,, \quad \left|\vartheta^\varepsilon - \theta^0(\zeta^\varepsilon)\right|^4 \leq 1/3c_o, \\ |p^\varepsilon| &\leq 1, \quad \text{and}\, \varpi \circ \xi\xi_1(\zeta^\varepsilon, \vartheta^\varepsilon) \leq 1/3, \quad \text{for all } \varepsilon \in (0, \varepsilon_o]. \end{aligned} \tag{6.10}$$

Then, with $B_\alpha := B_\alpha(\zeta_o) \times B_{r_o}(\vartheta_o)$, observe that we still have

$$u^{\varepsilon *}(\zeta, \vartheta) \leq b^*, \quad \text{for all}\, (\zeta, \vartheta) \in \bar{B}_\alpha \text{ and } \varepsilon \in (0, \varepsilon_o].$$

*Step 2: construct a test function for $v_*^\varepsilon$ and a sequence of local interior minimizers.* For each $\varepsilon \in (0,1)$, define

$$\phi^\varepsilon : (\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d \longmapsto c_o \left( |\zeta - \zeta^\varepsilon|^4 + \left|\vartheta - \theta^0(\zeta)\right|^4 \right)$$

and introduce the following subset of $\bar{B}_\alpha$:

$$B_{o,\alpha} := \bar{B}_{\alpha/2}(\zeta_o) \times \bar{B}_{r_0/2}(\vartheta_o).$$

Recalling (6.8), (6.10), and the choice of $c_o$, it follows that

$$\phi^\varepsilon(\zeta, \vartheta) \geq 2 + b^* - \varphi(\zeta), \quad \text{for all}\, \varepsilon \leq \varepsilon_o \text{ and } (\zeta, \vartheta) \in \bar{B}_\alpha \backslash B_{o,\alpha}. \tag{6.11}$$

On the other hand, the last estimate in the first line of (6.10) gives:

$$\phi^\varepsilon(\zeta^\varepsilon, \vartheta^\varepsilon) \leq 1/3. \tag{6.12}$$

We now define, for all $\varepsilon, \eta \in (0, 1]$, the function

$$\psi^{\varepsilon,\eta} := v^0 - \varepsilon^2\,(p^\varepsilon + \varphi + \phi^\varepsilon) - \varepsilon^4(1+\eta)\varpi \circ \boldsymbol{\xi}_\varepsilon,$$

and show that $v_*^\varepsilon - \psi^{\varepsilon,\eta}$ (or equivalently $I^{\varepsilon,\eta} := (v_*^\varepsilon - \psi^{\varepsilon,\eta})/\varepsilon^2$) admits an interior local minimizer. By definition of $u^\varepsilon$ in (6.3),

$$I^{\varepsilon,\eta} = -u^{\varepsilon *} + (p^\varepsilon + \varphi + \phi^\varepsilon) + \eta \varpi \circ \boldsymbol{\xi}_1.$$

Combining the definition of $p^\varepsilon$ with (6.12) and the last term in (6.10), we first notice that, for all $(\varepsilon, \eta) \in (0, \varepsilon_o] \times (0, 1]$:

$$\inf_{\bar{B}_\alpha} I^{\varepsilon,\eta} \leq \inf_{B_{o,\alpha}} I^{\varepsilon,\eta} \leq I^{\varepsilon,\eta}(\zeta^\varepsilon, \vartheta^\varepsilon) \leq 2/3.$$

On the other hand, because $\varpi \geq 0$ by Lemma 4.1, it follows from (6.10) and (6.11) that

$$I^{\varepsilon,\eta}(\zeta, \vartheta) \geq 1, \quad \text{for all}\, (\zeta, \vartheta) \in \bar{B}_\alpha \backslash B_{o,\alpha} \text{ and } \varepsilon \in (0, \varepsilon_o].$$

Hence, by lower-semicontinuity of $I^{\varepsilon,\eta}$ and compactness of $B_{o,\alpha}$, there exists a minimizer $(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon) \in \bar{B}_{o,\alpha} \subset \bar{B}_\alpha$. (The latter also depends on $\eta$, but we do not explicitly note this dependence as it is of no importance here.) This minimizer satisfies, for all $\varepsilon \in (0, \varepsilon_o]$ and $\eta \in (0, 1)$:

$$I^{\varepsilon,\eta}\left(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon\right) \leq 0 \quad \text{and} \quad \left|\varepsilon \boldsymbol{\xi}_\varepsilon(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon)\right| \vee \left|\tilde{\zeta}^\varepsilon - \zeta_o\right| \leq r_1, \tag{6.13}$$

for some constant $r_1 > 0$, where we recall that $\varepsilon \boldsymbol{\xi}_\varepsilon(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon) = \tilde{\vartheta}^\varepsilon - \theta^0(\tilde{\zeta}^\varepsilon)$.

*Step 3: show that for each* $\eta \in (0, 1]$, *there is* $C_\eta > 0$ *such that* $|\boldsymbol{\xi}_\varepsilon(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon)| \le C_\eta$, $\forall \varepsilon \in (0, \varepsilon_o]$. As $(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon)$ are interior local minimizers of $v_*^\varepsilon - \psi^{\varepsilon,\eta}$ by Step 2, the viscosity supersolution property of $v^\varepsilon$ for (3.6) yields

$$-\left(\mathcal{L}^{\tilde{\vartheta}^\varepsilon} + \mathcal{H}^\varepsilon\right)\psi^{\varepsilon,\eta}\left(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon\right) \ge 0. \tag{6.14}$$

Observe from (6.9) and (6.13) that, after possibly reducing $\varepsilon_o > 0$, we have $\partial_x \psi^{\varepsilon,\eta} > 0$ and $\varepsilon^2 \partial_x(\phi + \varepsilon^2 w^\varepsilon) \le \iota \partial_x v^0$, for $\varepsilon \in (0, \varepsilon_o]$. Hence, the requirements of (Ri) in Lemma 6.1 are satisfied so that, for all $\varepsilon \in (0, \varepsilon_o]$:

$$\begin{aligned}
\mathcal{L}^{\tilde{\vartheta}^\varepsilon}\psi^{\varepsilon,\eta}(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon) &= \varepsilon^2\left(\frac{1}{2}\left|\xi\xi_\varepsilon^\top \sigma_S\right|^2 \partial_{xx}v^0 - \mathcal{L}^{\theta^0}\bar{\phi}^\varepsilon - \frac{1}{2}(1+\eta)\mathrm{Tr}\left[c_{\theta^0} D^2_{\xi\xi}\varpi\right]\right)(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon) \\
&\quad + \varepsilon^2 \mathcal{R}^\varepsilon_{\mathcal{L}}(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon), \mathcal{H}^\varepsilon \psi^{\varepsilon,\eta}(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon) \\
\mathcal{H}^\varepsilon\psi^{\varepsilon,\eta}(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon) &= \varepsilon^2\left(\frac{(1+\eta)^2(D_\xi\varpi \circ \boldsymbol{\xi}_\varepsilon)^\top E^{-4} D_\xi \varpi \circ \boldsymbol{\xi}_\varepsilon}{4\partial_x v^0} + \frac{\hat{\mathcal{L}}^\varepsilon \bar{\phi}^\varepsilon}{\varepsilon^2}\right)(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon) + \varepsilon^2 \mathcal{R}^\varepsilon_{\mathcal{H}}.
\end{aligned}$$

(6.15)

Here (recall (6.7)),

$$\bar{\phi}^\varepsilon := p^\varepsilon + \varphi + \phi^\varepsilon \tag{6.16}$$

and $\mathcal{R}^\varepsilon := \mathcal{R}^\varepsilon_{\mathcal{L}} + \mathcal{R}^\varepsilon_{\mathcal{H}}$, which satisfies

$$|\mathcal{R}^\varepsilon|(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon) \le c_1, \qquad \text{for all } \varepsilon \in (0, \varepsilon_o], \tag{6.17}$$

for some constant $c_1 > 0$. Now, rewrite $\mathcal{L}^{\tilde{\vartheta}^\varepsilon}\psi^{\varepsilon,\eta}$ above using that $\varpi$ is a solution of the First Corrector Equation (3.19). For all $\varepsilon \in (0, \varepsilon_o]$, Estimate (6.14) then leads to:

$$\begin{aligned}
&\left\{\frac{\eta}{2}\left|\boldsymbol{\xi}_\varepsilon^\top \sigma_S\right|^2 \partial_{xx}v^0 + \mathcal{L}^{\theta^0}\bar{\phi}^\varepsilon + (1+\eta)a - \mathcal{R}^\varepsilon + \frac{(1+\eta)(D_\xi\varpi\circ\boldsymbol{\xi}_\varepsilon)^\top E^{-4} D_\xi\varpi\circ\boldsymbol{\xi}_\varepsilon}{4\partial_x v^0}\right. \\
&\quad \left. - \frac{(1+\eta)^2(D_\xi\varpi\circ\boldsymbol{\xi}_\varepsilon)^\top E^{-4} D_\xi\varpi\circ\boldsymbol{\xi}_\varepsilon}{4\partial_x v^0} - \frac{\hat{\mathcal{L}}^\varepsilon\bar{\phi}^\varepsilon}{\varepsilon^2}\right\}(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon) \ge 0.
\end{aligned} \tag{6.18}$$

Observe that, as $E$ is positive-definite and $\eta \ge 0$:

$$\frac{[1+\eta-(1+\eta)^2](D_\xi\varpi\circ\boldsymbol{\xi}_\varepsilon)^\top E^{-4} D_\xi\varpi\circ\boldsymbol{\xi}_\varepsilon}{4\partial_x v^0} \le 0.$$

We prove in Step 4 below that there is a constant $c_2 > 0$ such that, for $\varepsilon \in (0, \varepsilon_o]$:

$$-\frac{\hat{\mathcal{L}}^\varepsilon\bar{\phi}^\varepsilon}{\varepsilon^2}(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon) \le c_2. \tag{6.19}$$

Combining this with (6.18), (6.9), (6.17), and the Ellipticity Condition (2.3) gives

$$c_1 + c_2 + \left\{(1+\eta)a + \mathcal{L}^{\theta^0}\bar{\phi}^\varepsilon\right\}(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon) \ge (\iota\eta\gamma_o/2)\left|\boldsymbol{\xi}_\varepsilon\right|^2(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon), \qquad \text{for all } \varepsilon \in (0, \varepsilon_o],$$

for some $\gamma_o > 0$. The assertion of Step 3 now follows by taking into account the continuity of $a$ and $\mathcal{L}^{\theta^0}\bar{\phi}^\varepsilon$ as well as (6.10) and (6.13).

*Step 4: prove* (6.19). Recall the definition of $\hat{\mathcal{L}}^\varepsilon$ in (6.1); as $E$ and $k_2$ are positive-definite, it follows that

$$
\begin{aligned}
-\frac{\hat{\mathcal{L}}^\varepsilon \bar{\phi}^\varepsilon}{\varepsilon^2} &\leq -\frac{(1+\eta)(D_\vartheta \bar{\phi}^\varepsilon)^\top E^{-4}(D_\xi \varpi \circ \boldsymbol{\xi}_\varepsilon)}{2\varepsilon \partial_x v^0} - \frac{\partial_x \bar{\phi}^\varepsilon}{4(\partial_x v^0)^2}(D_\vartheta \bar{\phi}^\varepsilon)^\top E^{-4} D_\vartheta \bar{\phi}^\varepsilon \\
&\leq -\frac{(1+\eta)4c_o|\boldsymbol{\xi}_1|^2 \boldsymbol{\xi}_1^\top E^{-4} k_2 \boldsymbol{\xi}_1}{\varepsilon^2 \partial_x v^0} - \frac{\partial_x \bar{\phi}^\varepsilon}{4(\partial_x v^0)^2}(D_\vartheta \bar{\phi}^\varepsilon)^\top E^{-4} D_\vartheta \bar{\phi}^\varepsilon \\
&\leq -\frac{\partial_x \bar{\phi}^\varepsilon}{4(\partial_x v^0)^2}(D_\vartheta \bar{\phi}^\varepsilon)^\top E^{-4} D_\vartheta \bar{\phi}^\varepsilon,
\end{aligned}
$$

where the second inequality follows from direct computations based on the definition of $\bar{\phi}^\varepsilon$ in (6.16) and the construction of $\varpi$ in Lemma 4.1. By construction of $\bar{\phi}^\varepsilon$, as well as (6.13) and (6.9), this yields the desired upper bound $c_2$ at $(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon)$.

*Step 5: conclude the proof of the proposition.* By the previous step, $(\tilde{\zeta}^\varepsilon, \boldsymbol{\xi}_\varepsilon(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon))_{\varepsilon \in (0, \bar{\varepsilon}_\eta]}$ is uniformly bounded. Hence, there is $(\bar{\zeta}, \bar{\xi})$ such that, possibly along a subsequence, $(\tilde{\zeta}^\varepsilon, \boldsymbol{\xi}_\varepsilon(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon)) \to (\bar{\zeta}, \bar{\xi})$ as $\varepsilon \to 0$. Moreover, by (6.5), classical arguments in the theory of viscosity solutions give $\bar{\zeta} = \zeta_o$, see, e.g., Crandall, Ishii, and Lions (1992). (Observe that $\bar{\xi}$ depends on $\eta$, but we shall see below that this dependence is harmless.) By (6.14),

$$
\lim_{\varepsilon \to 0} -\frac{1}{\varepsilon^2}\left(\mathcal{L}^{\tilde{\vartheta}^\varepsilon} + \mathcal{H}^\varepsilon\right)\psi^{\varepsilon,\eta}\left(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon\right) \geq 0.
$$

Using (6.15), we further deduce that

$$
\begin{aligned}
\lim_{\varepsilon \to 0}\Bigg(&-\frac{1}{2}\left|\boldsymbol{\xi}_\varepsilon^\top \sigma_S\right|^2 \partial_{xx} v^0 + \mathcal{L}^{\theta^0}\varphi + \mathcal{L}^{\theta^0}\phi^\varepsilon + \frac{1+\eta}{2}\mathrm{Tr}\left[c_{\theta^0} D^2_{\xi\xi}\varpi \circ \boldsymbol{\xi}_\varepsilon\right] \\
&-\frac{(1+\eta)^2 (D_\xi \varpi \circ \boldsymbol{\xi}_\varepsilon)^\top E^{-4} D_\xi \varpi \circ \boldsymbol{\xi}_\varepsilon}{4\partial_x v^0} + \mathcal{R}^\varepsilon - \frac{\hat{\mathcal{L}}^\varepsilon \phi^\varepsilon}{\varepsilon^2}\Bigg)\left(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon\right) \geq 0,
\end{aligned}
$$

where, by (Ri) in Lemma 6.1: $\mathcal{R}^\varepsilon(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon) \to 0, \quad$ as $\varepsilon \to 0$. By definition of $\phi^\varepsilon$ and Step 3, $(\mathcal{L}^{\theta^0}\phi^\varepsilon - \frac{\hat{\mathcal{L}}^\varepsilon \phi^\varepsilon}{\varepsilon^2})(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence, also taking into account that $\varpi$ is a solution of the First Corrector Equation (3.19):

$$
\left(\mathcal{L}^{\theta^0}\varphi + \frac{\eta}{2}\mathrm{Tr}\left[c_{\theta^0} D^2_{\xi\xi}\varpi(\cdot, \bar{\xi})\right] - \frac{(2\eta + \eta^2)(D_\xi \varpi \circ (\cdot, \bar{\xi}))^\top E^{-4} D_\xi \varpi(\cdot, \bar{\xi})}{4\partial_x v} + a\right)(\zeta_o) \geq 0. \tag{6.20}
$$

Now, note that

$$
\frac{(2\eta + \eta^2)(D_\xi \varpi \circ (\zeta_o, \bar{\xi}))^\top E^{-4} D_\xi \varpi(\cdot, \bar{\xi})}{4\partial_x v(\zeta_o)} \geq 0
$$

due to (6.9). Together with (6.20), this shows

$$
\left(\mathcal{L}^{\theta^0}\varphi + \frac{\eta}{2}\mathrm{Tr}\left[c_{\theta^0} D^2_{\xi\xi}\varpi(\cdot, \bar{\xi})\right] + a\right)(\zeta_o) \geq 0.
$$

Finally, note that $\frac{\eta}{2}\mathrm{Tr}[c_{\theta^0} D^2_{\xi\xi}\varpi(\zeta_o, \bar{\xi})] = \eta \mathrm{Tr}[c_{\theta^0} k_2(\zeta_o)]$ does not depend on $\bar{\xi}$. We now send $\eta$ to zero to arrive at $-\mathcal{L}^{\theta^0}\varphi(\zeta_o) \leq a(\zeta_o)$. This completes the proof. $\square$

*6.3.2.* **Viscosity supersolution property**.

PROPOSITION 6.4. *Suppose Assumptions 3.3 and A are satisfied. Then,* $\zeta \in \mathfrak{D} \longmapsto u_*(\zeta, \theta^0(\zeta)) = \bar{u}_*(\zeta, \theta^0(\zeta))$ *is a viscosity supersolution of the Second Corrector Equation* (3.20) *on* $\mathfrak{D}_<$.

*Proof.* Consider $\zeta_o \in \mathfrak{D}_<$ and $\varphi \in C^{1,2}(\mathfrak{D}_<)$ such that

$$\min_{\zeta \in \mathfrak{D}_<} (\mathrm{strict})(u_*(\zeta, \theta^o(\zeta)) - \varphi(\zeta)) = u_*(\zeta_o, \vartheta_o) - \varphi(\zeta_o) = 0, \tag{6.21}$$

where $\vartheta_o := \theta^0(\zeta_o)$. We have to show $-\mathcal{L}^{\theta^0}\varphi(\zeta_o) \geq a(\zeta_o)$. By (6.4) and continuity of $\varphi$, there exist $(\zeta^\varepsilon, \vartheta^\varepsilon)_{\varepsilon>0} \subset \mathfrak{D}_< \times \mathbb{R}^d$ such that

$$(\zeta^\varepsilon, \vartheta^\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} (\zeta_o, \vartheta_o), \quad u_*^\varepsilon(\zeta^\varepsilon, \vartheta^\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} u_*(\zeta_o, \vartheta_o), \quad \text{and } p^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

where $p^\varepsilon := u_*^\varepsilon(\zeta^\varepsilon, \vartheta^\varepsilon) - \varphi(\zeta^\varepsilon)$. By Assumption (A1) and Lemma 4.1, there are $r_o > 0$ and $\varepsilon_o \in (0, 1]$ satisfying

$$|\zeta^\varepsilon - \zeta_o| \leq \frac{r_o}{2}, \quad |p^\varepsilon| \leq 1, \quad \text{and} \quad \varpi \circ \boldsymbol{\xi}_1(\zeta^\varepsilon, \vartheta^\varepsilon) \leq 1/3, \quad \text{for all } \varepsilon \leq \varepsilon_0. \tag{6.22}$$

Moreover, Assumption (A1) ensures the existence of $\iota > 0$ such that

$$2/\iota > -\partial_{xx} v^0 \wedge \partial_x v^0 > 2\iota, \quad \text{on } \bar{B}_{r_o}(\zeta_o). \tag{6.23}$$

*Step 1: for each* $\varepsilon \in (0, \bar{\varepsilon}]$, *provide a penalization function* $\phi^\varepsilon$, *in order to construct a convenient test function for* $v^\varepsilon$ *in Steps 2 and 3. Also provide a constant* $\xi^*$, *independent of* $\varepsilon$, *that will be used in Steps 5 and 6.*

As $\varphi$ is smooth, there exists a constant $M < \infty$ such that

$$\sup\left\{\varphi(\zeta)\ ;\ \zeta \in \bar{B}_{r_o}(\zeta_o)\right\} \leq M - 4. \tag{6.24}$$

In view of (6.22), there is a finite $\mathbf{d} > 0$ so that $|\zeta - \zeta^\varepsilon|^4 \geq \mathbf{d}$ for all $\zeta \in \partial B_{r_o}(\partial_o)$, and we choose $c_o > 0$ such that $c_o\mathbf{d} \geq M$. With this notation, define

$$\phi^\varepsilon(\zeta) := \varphi(\zeta) + p^\varepsilon - c_o|\zeta - \zeta^\varepsilon|^4,$$

and observe from (6.22), (6.24), and the choice of $c_o$ that

$$\phi^\varepsilon(\zeta) \leq -3, \quad \text{for all } \zeta \in \partial B_{r_o}(\zeta_o) \text{ and } \varepsilon \in (0, \varepsilon_o]. \tag{6.25}$$

Recall the definition of $p^\varepsilon$ and the last term in (6.22), and observe for later use that

$$-\bar{u}_*^\varepsilon(\zeta^\varepsilon, \vartheta^\varepsilon) + \phi^\varepsilon(\zeta^\varepsilon) \geq -1/3, \quad \text{for all } \varepsilon \in (0, \varepsilon_o]. \tag{6.26}$$

Now, on the one hand, combining (6.23) with the positive-definiteness of $k_2 E^{-4} k_2$ yields the existence of $\gamma_E > 0$ such that

$$\frac{\mathbf{x}^\top (k_2 E^{-4} k_2)(\zeta)\mathbf{x}}{4\partial_x v(\zeta)} \geq \gamma_E\, |\mathbf{x}|^2, \quad \text{for all } (\zeta, \mathbf{x}) \in \bar{B}_{r_o}(\zeta_o) \times \mathbb{R}^d. \tag{6.27}$$

On the other hand, (6.23) together with the continuity of $E^{-4}$ and $k_2$ ensures that there is $K_E > 0$ such that

$$\frac{\left|E^{-4}\right| |k_2|^2(\zeta)}{4\partial_x v(\zeta)} \leq K_E, \quad \text{for all } \zeta \in \bar{B}_{r_o}(\zeta_o). \tag{6.28}$$

Also denote for later use by $K_0$, $K_2$, $K_{\theta^0} > 0$ three finite constants such that

$$2|k_2(\zeta)| \leq K_2\,, \quad |c_{\theta^0}(\zeta)| \leq 2K_{\theta^0}, \ \text{ and } \ \left|\mathcal{L}^{\theta^0}\phi^0(\zeta)\right| \leq K_0, \ \text{ for all } \zeta \in \bar{B}_{r_o}(\zeta_o), \tag{6.29}$$

where $\phi^0(\zeta) := \varphi(\zeta) - c_o|\zeta - \zeta_o|^4$. By a slight adaptation of Possamai, Soner, and Touzi (2013, lemma 5.4), there exist $(h^\eta)_{\eta\in(0,1]} \subset C^\infty(\mathbb{R}^d; [0,1])$ and $(a_\eta)_{\eta\in(0,1]} \subset (1,\infty)$ satisfying

$$\begin{aligned} &h^\eta = 1, \quad \text{on } \bar{B}_1(0)\,, \quad h^\eta = 0, \quad \text{on } \bar{B}^c_{a_\eta}(0),\\ &|\mathbf{x}|\,|D_{\mathbf{x}}h^\eta(\mathbf{x})| \leq \eta \quad \text{and} \quad |\mathbf{x}|^2\left|D^2_{\mathbf{xx}}h^\eta(\mathbf{x})\right| \leq C^*, \end{aligned} \tag{6.30}$$

for all $\mathbf{x} \in \mathbb{R}^d$ and some constant $C^* > 0$ independent of $\eta$. Finally, for each $\delta \in (0,1]$, we choose $\xi^{*,\delta} > 0$ satisfying

$$(\xi^{*,\delta})^2 = 1 + \frac{2[K_0 + K_{\theta^0}K_2(6 + C^*)}{\gamma_E(2\delta - \delta^2)}.$$

*Step 2: construct a "first draft" of a test function for $v^\varepsilon$, that will be used to construct the "true" test function in Step 3.*

For every $(\varepsilon, \eta, \delta) \in (0, \varepsilon_o] \times (0,1)^2$, define

$$\psi^{\varepsilon,\eta,\delta} := v^0 - \varepsilon^2\phi^\varepsilon - \varepsilon^4(\varpi H^{\eta,\delta}) \circ \boldsymbol{\xi}_\varepsilon,$$

where

$$H^{\eta,\delta} : \xi \in \mathbb{R}^d \longmapsto (1-\delta)h^\eta\left(\frac{\xi}{\xi^{*,\delta}}\right),$$

the normalized deviation $\boldsymbol{\xi}_\varepsilon$ is defined as in (3.11), and $\varpi$ is the solution of the first corrector equation from Lemma 4.1. We want to construct a local maximizer of $v^{\varepsilon *} - \psi^{\varepsilon,\eta,\delta}$ (or equivalently $I^{\varepsilon,\eta,\delta} := \frac{1}{\varepsilon^2}(v^{\varepsilon *} - \psi^{\varepsilon,\eta,\delta})$). However, it will turn out below that $\psi^{\varepsilon,\eta,\delta}$ needs to be modified further to make this possible. Indeed, consider

$$I^{\varepsilon,\eta,\delta} = -\bar{u}^\varepsilon_* + \phi^\varepsilon + \varepsilon^2(\varpi H^{\eta,\delta}) \circ \boldsymbol{\xi}_\varepsilon.$$

By (6.26) and because $\varpi H^{\eta,\delta} \geq 0$,

$$I^{\varepsilon,\eta,\delta}(\zeta^\varepsilon, \vartheta^\varepsilon) \geq -1/3. \tag{6.31}$$

On the other hand, the construction of $\varpi$ in Lemma 4.1 together with (4.1), (6.22), (6.29) $\eta, \delta \in (0,1)$, and $0 \leq H^{\eta,\delta}(\xi) \leq \mathbf{1}_{\{|\xi| \leq a_\eta \xi^*\}}$ implies that, for all $(\zeta, \vartheta) \in \bar{B}_{r_o}(\zeta_o) \times \mathbb{R}^d$:

$$\begin{aligned} I^{\varepsilon,\eta,\delta}(\zeta,\vartheta) &\leq \phi^\varepsilon(\zeta) + K_2\varepsilon^2|\boldsymbol{\xi}_\varepsilon|^2\mathbf{1}_{\{|\boldsymbol{\xi}_\varepsilon| \leq a_\eta\xi^{*,\delta}\}}(\zeta,\vartheta)\\ &\leq \phi^\varepsilon(\zeta) + K_2\varepsilon^2(a_\eta\xi^{*,\delta})^2\\ &\leq \phi^\varepsilon(\zeta) + 1, \qquad \text{for all } \varepsilon \leq \varepsilon_{\eta,\delta}, \end{aligned} \tag{6.32}$$

where $\varepsilon_{\eta,\delta} := \varepsilon_o \wedge (K_2^{1/2} a_\eta \xi^{*,\delta})^{-1}$. Observe that in (6.32), unlike in the proof of the subsolution property in Proposition 6.3, deviations of $\vartheta$ from $\theta^0(\zeta)$ are not penalized by $\phi^\varepsilon$. Hence, the supremum—even if it is finite—is not necessarily attained.

Define the set $\mathcal{Q}_o := \{(\zeta, \vartheta) \in \mathfrak{D}_< \times \mathbb{R}^d : \zeta \in \bar{B}_{r_o}(\zeta_o)\}$, and observe from (6.32) that

$$\sup_{(\zeta,\vartheta)\in\mathcal{Q}_o} I^{\varepsilon,\eta,\delta}(\zeta, \vartheta) \le \sup_{\zeta\in\bar{B}_{r_o}(\zeta_o)} \{\phi^\varepsilon(\zeta) + 1\}, \qquad \text{for all } \varepsilon \le \varepsilon_{\eta,\delta}.$$

Hence, by compactness of $\bar{B}_{r_o}(\zeta_o)$, continuity of $\phi^\varepsilon$, (6.22), and the fact that $\varepsilon_{\eta,\delta} \le \varepsilon_o$, we have:

$$\mathcal{I}^{\varepsilon,\eta,\delta} := \sup_{(\zeta,\vartheta)\in\mathcal{Q}_o} I^{\varepsilon,\eta,\delta}(\zeta, \vartheta) < \infty, \quad \forall \varepsilon \le \varepsilon_{\eta,\delta}.$$

As a result, for each $\varepsilon \in (0, \varepsilon_{\eta,\delta}]$, there exists $(\hat{\zeta}^{\varepsilon,\eta,\delta}, \hat{\vartheta}^{\varepsilon,\eta,\delta}) \in \mathrm{Int}(\mathcal{Q}_o)$ satisfying

$$I^{\varepsilon,\eta,\delta}\big(\hat{\zeta}^{\varepsilon,\eta,\delta}, \hat{\vartheta}^{\varepsilon,\eta,\delta}\big) \ge \mathcal{I}^{\varepsilon,\eta,\delta} - \frac{\varepsilon^2}{2}. \tag{6.33}$$

*Step 3: for each $\eta, \delta \in (0, 1)$ and $\varepsilon \in (0, \varepsilon_{\eta,\delta}]$, finally provide a test function $\bar{\psi}^{\varepsilon,\eta,\delta}$ and a test point $(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}) \in Int(\mathcal{Q}_o)$, satisfying*

$$\max_{\mathcal{Q}_o}(v^{\varepsilon*} - \bar{\psi}^{\varepsilon,\eta,\delta}) = (v^{\varepsilon*} - \bar{\psi}^{\varepsilon,\eta,\delta})(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}).$$

Introduce an even real-valued function $f \in C_b^\infty(\mathbb{R})$ satisfying $0 \le f \le 1$, $f(0) = 1$ and $f(x) = 0$ whenever $|x| \ge 1$. Also fix $\eta, \delta \in (0, 1)$ and $\varepsilon \in (0, \varepsilon_{\eta,\delta}]$. Consider

$$\bar{\psi}^{\varepsilon,\eta,\delta}(\cdot, \vartheta) := \psi^{\varepsilon,\eta,\delta}(\cdot, \vartheta) - \varepsilon^4 f\left(\left|\vartheta - \hat{\vartheta}^{\varepsilon,\eta,\delta}\right|\right)$$

as well as

$$\bar{I}^{\varepsilon,\eta,\delta}(\cdot, \vartheta) := \frac{1}{\varepsilon^2}\left(v^{\varepsilon*} - \bar{\psi}^{\varepsilon,\eta,\delta}\right)(\cdot, \vartheta) = I^{\varepsilon,\eta,\delta}(\cdot, \vartheta) + \varepsilon^2 f\left(\left|\vartheta - \hat{\vartheta}^{\varepsilon,\eta,\delta}\right|\right).$$

By (6.33) and $f(0) = 1$,

$$\bar{I}^{\varepsilon,\eta,\delta}\big(\hat{\zeta}^{\varepsilon,\eta,\delta}, \hat{\vartheta}^{\varepsilon,\eta,\delta}\big) = I^{\varepsilon,\eta,\delta}\big(\hat{\zeta}^{\varepsilon,\eta,\delta}, \hat{\vartheta}^{\varepsilon,\eta,\delta}\big) + \varepsilon^2 \ge \mathcal{I}^{\varepsilon,\eta,\delta} + \frac{\varepsilon^2}{2}. \tag{6.34}$$

Moreover, by definition of $f$, if $\vartheta \in \mathbb{R}^d$ satisfies $|\vartheta - \hat{\vartheta}^{\varepsilon,\eta,\delta}| > 1$ then

$$\bar{I}^{\varepsilon,\eta,\delta}(\zeta, \vartheta) = I^{\varepsilon,\eta,\delta}(\zeta, \vartheta).$$

Hence, setting $\mathcal{Q}_1^\varepsilon := \{(\zeta, \vartheta) \in \mathcal{Q}_o : |\vartheta - \hat{\vartheta}^{\varepsilon,\eta,\delta}| \le 1\}$ and because $(\hat{\zeta}^{\varepsilon,\eta,\delta}, \hat{\vartheta}^{\varepsilon,\eta,\delta}) \in \mathcal{Q}_1^\varepsilon$, this equality combined with (6.34) implies

$$\sup_{\mathcal{Q}_1^\varepsilon} \bar{I}^{\varepsilon,\eta,\delta} > \sup_{\mathcal{Q}_o} I^{\varepsilon,\eta,\delta} \ge \sup_{\mathcal{Q}_o\setminus\mathcal{Q}_1^\varepsilon} I^{\varepsilon,\eta,\delta} = \sup_{\mathcal{Q}_o\setminus\mathcal{Q}_1^\varepsilon} \bar{I}^{\varepsilon,\eta,\delta}.$$

As a result:

$$\sup_{(\zeta,\vartheta)\in\mathcal{Q}_o} \bar{I}^{\varepsilon,\eta,\delta}(\zeta, \vartheta) = \sup_{(\zeta,\vartheta)\in\mathcal{Q}_1^\varepsilon} \bar{I}^{\varepsilon,\eta,\delta}(\zeta, \vartheta).$$

Thus, by upper-semicontinuity of $\bar{I}^{\varepsilon,\eta,\delta}$ and compactness of $\mathcal{Q}_1^\varepsilon$, there exists $(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}) \in \mathcal{Q}_o$ maximizing $\bar{I}^{\varepsilon,\eta,\delta}$. In fact, $(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}) \in \mathrm{Int}(\mathcal{Q}_o)$, because (6.22), (6.31), $f \geq 0$, and $\varepsilon \in (0, \varepsilon_{\eta,\delta}]$ give

$$\bar{I}^{\varepsilon,\eta,\delta}\left(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}\right) \geq \bar{I}^{\varepsilon,\eta,\delta}(\zeta^\varepsilon, \vartheta^\varepsilon) \geq I^{\varepsilon,\eta,\delta}(\zeta^\varepsilon, \vartheta^\varepsilon) = 0,$$

whereas (6.25), (6.32), $f \leq 1$, and $\varepsilon \in (0, \varepsilon_{\eta,\delta}]$ with $\varepsilon_{\eta,\delta} \leq 1$ imply

$$\bar{I}^{\varepsilon,\eta,\delta} \leq I^{\varepsilon,\eta,\delta} \leq -2 + \varepsilon^2 < 0, \quad \text{on } \partial\mathcal{Q}_o.$$

*Step 4: show that, for each $\eta, \delta \in (0,1)$, $\{\boldsymbol{\xi}_\varepsilon(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta})\ ;\ \varepsilon \in (0, \bar{\varepsilon}_{\eta,\delta}]\}$ is uniformly bounded and therefore converges along a subsequence toward some $\bar{\xi}^{\eta,\delta} \in \mathbb{R}^d$ as $\varepsilon \to 0$.*

By the previous step and Proposition 3.6,

$$-\left(\mathcal{L}^{\tilde{\vartheta}^{\varepsilon,\eta,\delta}} + \mathcal{H}^\varepsilon\right)\bar{\psi}^{\varepsilon,\eta,\delta}(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}) \leq 0.$$

Moreover, by (6.23), construction of $H^{\eta,\delta}$, as $\xi^*$ does not depend on $\varepsilon$ and $f \in C_b^\infty(\mathbb{R})$, possibly diminishing $\varepsilon_{\eta,\delta} > 0$ yields $\partial_x\bar{\psi}^{\varepsilon,\eta,\delta}(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}) > 0$ and $\varepsilon^2\partial_x(\phi + \varepsilon^2(\varpi H^{\eta,\delta}) \circ \xi\xi_\varepsilon) \leq \iota\partial_x v^0$. Applying (Rii) in Lemma 6.1 then gives

$$\begin{aligned}&\left\{-\frac{1}{2}\left|\boldsymbol{\xi}_\varepsilon^\top\sigma_S\right|^2\partial_{xx}v^0 + \mathcal{L}^{\theta^0}\bar{\phi}^\varepsilon + \frac{1}{2}\mathrm{Tr}\left[c_{\theta^0}D^2_{\xi\xi}(\varpi H^{\eta,\delta}) \circ \boldsymbol{\xi}_\varepsilon\right]\right.\\ &\left.-\mathcal{R}^\varepsilon_{\mathcal{L}} - \frac{(D_\vartheta\bar{\psi}^{\varepsilon,\eta,\delta})^\top E^{-4}D_\vartheta\bar{\psi}^{\varepsilon,\eta,\delta}}{4\varepsilon^6\partial_x\bar{\psi}^{\varepsilon,\eta,\delta}}\right\}(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}) \leq 0,\end{aligned} \tag{6.35}$$

where $\bar{\phi}^\varepsilon(\cdot, \vartheta) := \phi^\varepsilon - \varepsilon^2 f(|\vartheta - \hat{\vartheta}^{\varepsilon,\eta,\delta}|)$ and, for some constant $C > 0$ and all $\varepsilon \in (0, \varepsilon_{\eta,\delta}]$:

$$|\mathcal{R}^\varepsilon_{\mathcal{L}}|(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}) \leq C\left(\varepsilon + \left|\varepsilon\boldsymbol{\xi}_\varepsilon\right| + \left|\varepsilon\boldsymbol{\xi}_\varepsilon\right|^2\right)(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}).$$

Assume now that $\{\boldsymbol{\xi}_\varepsilon(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta})\ ;\ \varepsilon \in (0, \bar{\varepsilon}_{\eta,\delta}]\}$ is *not* uniformly bounded along some subsequence. Then, by construction of $H^{\eta,\delta}$ and as $\xi^{*,\delta}$ does not depend on $\varepsilon$, it follows that $(\varpi H^{\eta,\delta}) \circ \boldsymbol{\xi}_\varepsilon$ and all of its derivatives vanish. On the other hand, $f \in C_b^\infty(\mathbb{R})$ implies that $|(D_\vartheta\bar{\psi}^{\varepsilon,\eta,\delta})^\top E^{-4}D_\vartheta\bar{\psi}^{\varepsilon,\eta,\delta}| \leq \varepsilon^8 c_f$ for some constant $c_f$. Finally, by construction of $\bar{\phi}^{\varepsilon,\eta,\delta}$ and $\tilde{\zeta}^{\varepsilon,\eta,\delta} \in \bar{B}_{r_o}(\zeta_o)$, we conclude that

$$\frac{(D_\vartheta\bar{\psi}^{\varepsilon,\eta,\delta})^\top E^{-4}D_\vartheta\bar{\psi}^{\varepsilon,\eta,\delta}}{4\varepsilon^6\partial_x\bar{\psi}^{\varepsilon,\eta,\delta}}(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}) \to 0, \quad \text{as } \varepsilon \to 0.$$

After possibly increasing $C > 0$, it follows that

$$\left\{-\frac{1}{2}\left|\boldsymbol{\xi}_\varepsilon^\top\sigma_S\right|^2\partial_{xx}v^0 + \mathcal{L}^{\theta^0}\bar{\phi}^\varepsilon\right\}(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}) \leq C\left(1 + \left|\varepsilon\boldsymbol{\xi}_\varepsilon\right| + \left|\varepsilon\boldsymbol{\xi}_\varepsilon\right|^2\right)(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}).$$

Denote by $\gamma > 0$ the constant in (2.3) corresponding to the set $\bar{B}_{r_o}(\zeta_o)$. Combining (6.23) with the continuity of $\mathcal{L}^{\theta^0}\bar{\phi}_\varepsilon$ and $\tilde{\zeta}^{\varepsilon,\eta,\delta} \in \bar{B}_{r_o}(\zeta_o)$, we then obtain

$$\gamma\iota|\boldsymbol{\xi}_\varepsilon|^2(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}) \leq C\left(1 + \left|\varepsilon\boldsymbol{\xi}_\varepsilon\right| + \left|\varepsilon\boldsymbol{\xi}_\varepsilon\right|^2\right)(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}).$$

This contradicts the assumption that $\{\boldsymbol{\xi}_\varepsilon(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta})\ ;\ \varepsilon \in (0, \bar{\varepsilon}_{\eta,\delta}]\}$ is unbounded. In particular, along a subsequence, $(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \boldsymbol{\xi}_\varepsilon(\tilde{\zeta}^{\varepsilon,\eta,\delta}, \tilde{\vartheta}^{\varepsilon,\eta,\delta}))$ therefore converges toward some finite $(\bar{\zeta}^{\eta,\delta}, \bar{\xi}^{\eta,\delta}) \in \mathfrak{D}_< \times \mathbb{R}^d$ as $\varepsilon \to 0$.

*Step 5: show that, for each* $\delta \in (0, 1)$, *there is* $\bar{\eta}_\delta \in (0, 1)$ *such that* $\{\bar{\xi}^{\eta,\delta}\ ;\ \eta \in (0, \bar{\eta}_\delta]\} \subset B_{\xi^{*,\delta}}(0)$ *and therefore converges, possibly along a subsequence, to a point* $\hat{\xi}^\delta \in B_{\xi^{*,\delta}}(0)$.

First, notice that the previous step implies that the requirements of (Ri) in Lemma 4.1 are satisfied, so that the remainder $\mathcal{R}^\varepsilon_{\mathcal{L}}(\tilde{\zeta}^{\varepsilon,\eta}, \tilde{\vartheta}^{\varepsilon,\eta})$ in (6.35) converges to zero as $\varepsilon \to 0$. By continuity of all the involved functions, sending $\varepsilon \to 0$ in (6.35) gives

$$\begin{aligned}&\left\{-\frac{1}{2}\left|(\bar{\xi}^{\eta,\delta})^\top \sigma_S\right|^2 \partial_{xx} v^0 - \frac{[D_\xi(H^{\eta,\delta}\varpi)]^\top E^{-4} D_\xi(H^{\eta,\delta}\varpi)}{4\partial_x v^0}\right\}(\bar{\zeta}^{\eta,\delta}, \bar{\xi}^{\eta,\delta}) \\ &\leq \left\{\left|\mathcal{L}^{\theta^0}\phi^0\right| + \frac{1}{2}\left|\mathrm{Tr}\left[c_{\theta^0} D^2_{\xi\xi}(H^{\eta,\delta}\varpi)\right]\right|\right\}(\bar{\zeta}^{\eta,\delta}, \bar{\xi}^{\eta,\delta}).\end{aligned} \tag{6.36}$$

We focus first on the right-hand side of this inequality. As $(\bar{\zeta}^{\eta,\delta})_{(\eta,\delta)\in(0,1)^2} \subset \bar{B}_{r_o}(\zeta_o)$, combining Lemma 4.1 with (6.29) and the last term in (6.30) gives, for all $(\eta, \delta) \in (0, 1)^2$:

$$\left\{\left|\mathcal{L}^{\theta^0}\phi^0\right| + \frac{1}{2}\left|\mathrm{Tr}\left[c_{\theta^0} D^2_{\xi\xi}(H^{\eta,\delta}\varpi)\right]\right|\right\}(\bar{\zeta}^{\eta,\delta}, \bar{\xi}^{\eta,\delta}) \leq K_0 + K_{\theta^0}(6K_2 + C^* K_2). \tag{6.37}$$

Consider now the left-hand side in (6.36) and omit the parameters $(\bar{\zeta}^{\eta,\delta}, \bar{\xi}^{\eta,\delta})$ to ease notation. As $0 \leq \left|H^{\eta,\delta}\right| \leq (1 - \delta)$ and $E^{-4}$ is positive definite, we have

$$\begin{aligned}&-\frac{1}{2}\left|(\bar{\xi}^{\eta,\delta})^\top \sigma_S\right|^2 \partial_{xx} v^0 - \frac{[D_\xi(H^{\eta,\delta}\varpi)]^\top E^{-4} D_\xi(H^{\eta,\delta}\varpi)}{4\partial_x v^0} \\ &\geq -\frac{1}{2}\left|(\bar{\xi}^{\eta,\delta})^\top \sigma_S\right|^2 \partial_{xx} v^0 - (1-\delta)^2\frac{[D_\xi \varpi]^\top E^{-4} D_\xi \varpi}{4\partial_x v^0}\end{aligned} \tag{6.38}$$

$$-\frac{2(1-\delta)H^{\eta,\delta}\varpi\,\frac{1}{\xi^{*,\delta}}[D_\xi\varpi]^\top E^{-4} D_{\mathbf{x}} h\left(\frac{\cdot}{\xi^{*,\delta}}\right)}{4\partial_x v^0} \tag{6.39}$$

$$-\frac{(1-\delta)^2\varpi^2\left(\frac{1}{\xi^{*,\delta}}\right)^2[D_{\mathbf{x}} h\left(\frac{\cdot}{\xi^{*,\delta}}\right)]^\top E^{-4} D_{\mathbf{x}} h\left(\frac{\cdot}{\xi^{*,\delta}}\right)}{4\partial_x v^0}. \tag{6.40}$$

Because $\varpi$ solves the First Corrector Equation (3.19), the terms in (6.38) satisfy

$$\begin{aligned}-\frac{1}{2}\left|(\bar{\xi}^{\eta,\delta})^\top \sigma_S\right|^2 \partial_{xx} v^0 - (1-\delta)^2\frac{[D_\xi \varpi]^\top E^{-4} D_\xi \varpi}{4\partial_x v^0} &= (2\delta - \delta^2)\frac{[D_\xi \varpi]^\top E^{-4} D_\xi \varpi}{4\partial_x v^0} \\ &\geq (2\delta - \delta^2)\gamma_E\left|\bar{\xi}^{\eta,\delta}\right|^2,\end{aligned}$$

where the second inequality follows from (6.27) and Lemma 4.1, recall that $\bar{\zeta}^{\eta,\delta} \in \bar{B}_{r_o}(\zeta_o)$. Next, Lemma 4.1, (6.28), (6.30), and $\bar{\zeta}^{\eta,\delta} \in \bar{B}_{r_o}(\zeta_o)$ imply the following estimate for (6.39):

$$-\frac{2(1-\delta)H^{\eta,\delta}\varpi\,\frac{1}{\xi^{*,\delta}}[D_\xi\varpi]^\top E^{-4} D_{\mathbf{x}} h\left(\frac{\cdot}{\xi^{*,\delta}}\right)}{4\partial_x v^0} \geq -4(1-\delta)\eta K_E\left|\bar{\xi}^{\eta,\delta}\right|^2.$$

Likewise, for (6.40), we have

$$-\frac{(1-\delta)^2\varpi^2\left(\frac{1}{\xi^{*,\delta}}\right)^2\left[D_{\mathbf{x}}h\left(\frac{\cdot}{\xi^{*,\delta}}\right)\right]^\top E^{-4}D_{\mathbf{x}}h\left(\frac{\cdot}{\xi^{*,\delta}}\right)}{4\partial_x v^0} \geq -(1-\delta)^2\eta^2 K_E\left|\bar{\xi}^{\eta,\delta}\right|^2.$$

Together, these three inequalities give

$$\begin{aligned}
&-\frac{1}{2}\left|(\bar{\xi}^{\eta,\delta})^\top\sigma_S\right|^2\partial_{xx}v^0 - \frac{[D_\xi(H^{\eta,\delta}\varpi)]^\top E^{-4}D_\xi(H^{\eta,\delta}\varpi)}{4\partial_x v^0}\\
&\geq \left|\bar{\xi}^{\eta,\delta}\right|^2\left[(2\delta-\delta^2)\gamma_E - K_E(1-\delta)\eta\,(4+(1-\delta)\eta)\right].
\end{aligned}$$

Now, notice that $(2\delta-\delta^2)\gamma_E > 0$ for all $\delta \in (0,1)$. Hence, for each $\delta \in (0,1)$, there exists $\bar{\eta}_\delta \in (0,1)$ such that $-K_E(1-\delta)\eta(4+(1-\delta)\eta) \geq -(2\delta-\delta^2)\gamma_E/2$ and in turn

$$-\frac{1}{2}\left|(\bar{\xi}^{\eta,\delta})^\top\sigma_S\right|^2\partial_{xx}v^0 - \frac{[D_\xi(H^{\eta,\delta}\varpi)]^\top E^{-4}D_\xi(H^{\eta,\delta}\varpi)}{4\partial_x v^0} \geq \frac{(2\delta-\delta^2)\gamma_E}{2}\left|\bar{\xi}^{\eta,\delta}\right|^2. \tag{6.41}$$

Finally, combining (6.36) with (6.37) and (6.41) gives

$$\left|\bar{\xi}^{\eta,\delta}\right|^2 \leq \frac{2\,[K_0 + K_{\theta^0}(6K_2 + C^* K_2)]}{(2\delta-\delta^2)\gamma_E} < (\xi^{*,\delta})^2,$$

completing Step 5.

*Step 6: conclude the proof of the proposition.* First, observe that $\left|\bar{\xi}^{\eta,\delta}\right| < \xi^{*,\delta}$, for all $\eta \in (0,\bar{\eta}_\delta]$, together with the definition of $H^{\eta,\delta}$ gives that $H^{\eta,\delta}(\bar{\xi}^{\eta,\delta}) = 1-\delta$ and that its derivatives vanish for all $(\delta,\eta) \in (0,1)\times(0,\bar{\eta}_\delta]$. Let $(\hat{\zeta}^\delta, \hat{\xi}^\delta)$ denote the limits of the (sub)sequence $(\bar{\zeta}^{\eta,\delta}, \bar{\xi}^{\eta,\delta})$ as $\eta \to 0$. By classical arguments in the theory of viscosity solutions (cf., e.g., Crandall et al. 1992), (6.21) implies that $\hat{\zeta}^\delta = \zeta_o$. Combining (6.35) with the fact that $\varpi$ solves the First Corrector Equation (3.19) in turn yields

$$\begin{aligned}
0 &\geq \left\{(2\delta-\delta^2)\frac{(D_\xi\varpi)^\top E^{-4}D_\xi(\varpi)}{4\partial_x v} + \mathcal{L}^{\theta^0}\varphi + (1-\delta)a\right\}(\zeta_o,\hat{\xi}^\delta)\\
&\geq \mathcal{L}^{\theta^0}\varphi(\zeta_o) + (1-\delta)a(\zeta_o).
\end{aligned}$$

Here, the last inequality follows directly from $\delta \in (0,1)$, Lemma 4.1, (6.23), and the positive-definiteness of $E^{-4}$. As $a(\zeta_o)$ does not depend on $\delta$, sending $\delta \to 0$ completes the proof of the proposition. □

### 6.3.3. **Terminal condition.**

PROPOSITION 6.5. *Suppose Assumptions 3.3 and A are satisfied. Then,*

$$u^*(\zeta,\theta^0(\zeta)) = u_*(\zeta,\theta^0(\zeta)) = 0, \quad \textit{for all } \zeta \in \partial_T\mathfrak{D}.$$

*Proof.* By definition, we have $u^*(\zeta,\theta^0(\zeta)) \geq u_*(\zeta,\theta^0(\zeta)) \geq 0$. Hence, it suffices to show $u^*(\zeta,\theta^0(\zeta))) \leq 0$, for all $\zeta \in \partial_T\mathfrak{D}$. Assume to the contrary that there is $(\zeta_o,\delta) \in \partial_T\mathfrak{D}\times(0,\infty)$ such that, with $\vartheta_o := \theta^0(\zeta_o)$:

$$u^*(\zeta_o,\vartheta_o) \geq 5\delta > 0. \tag{6.42}$$

*Step 1: provide a test function $\psi^\varepsilon$ for $v_*^\varepsilon$ and a local minimizer of $v_*^\varepsilon - \psi^\varepsilon$.*
By (6.4), there exist $(\zeta_\varepsilon, \vartheta_\varepsilon)_{\varepsilon>0} \subset \mathfrak{D} \times \mathbb{R}^d$ such that

$$(\zeta_\varepsilon, \vartheta_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} (\zeta_o, \vartheta_o) \quad \text{and} \quad u^{\varepsilon *}(\zeta_\varepsilon, \vartheta_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} u^*(\zeta_o, \vartheta_o). \tag{6.43}$$

Assume that, possibly along a subsequence, $\zeta_\varepsilon \in \partial_T \mathfrak{D}$. Then, the terminal conditions in Assumption 3.3 and Proposition 3.1 combined with $\varpi \geq 0$ (cf. Lemma 4.1) yield

$$u^{\varepsilon *}(\zeta_\varepsilon, \vartheta_\varepsilon) = (\bar{u}^{\varepsilon *} - \varpi \circ \xi \xi_1)(\zeta_\varepsilon, \vartheta_\varepsilon) \leq 0,$$

which contradicts (6.42) for small $\varepsilon$. Therefore we can assume without loss of generality that

$$\zeta_\varepsilon \in \mathfrak{D}_<. \tag{6.44}$$

By similar arguments as in the proof of Proposition 6.3, Assumptions (A1) and (A2) combined with (6.42) and (6.43) enable us to find $r_o \geq \alpha > 0$, $c_o > 0$, $\iota > 0$, and $\varepsilon_o > 0$ such that, for all $\varepsilon \in (0, \varepsilon_o]$:

$$(\zeta_\varepsilon, \vartheta_\varepsilon) \in B_{o,\alpha}\ , \quad \left|\vartheta_\varepsilon - \theta^0(\zeta_\varepsilon)\right|^2 \leq \delta / c_o, \quad \text{and} \quad u^{\varepsilon *}(\zeta_\varepsilon, \vartheta_\varepsilon) \geq 4\delta,$$

$$\partial_x v^0 \wedge (-\partial_{xx} v^0) \geq 2\iota \qquad \text{and} \quad \varpi \circ \boldsymbol{\xi}_1 \leq \delta \quad \text{on } \bar{B}_\alpha\ , \tag{6.45}$$

$$u^{\varepsilon *} - \bar{\phi}(\cdot; \zeta_\varepsilon) < 0 \quad \text{on } B_\alpha \backslash B_{o,\alpha}, \tag{6.46}$$

where $B_\alpha := (B_\alpha(\zeta_o) \cap \mathfrak{D}) \times B_{r_o}(\vartheta_o)$ as well as

$$B_{o,\alpha} := \left\{(\zeta, \vartheta) \in \bar{B}_\alpha : \zeta \in \bar{B}_{\frac{\alpha}{2}}(\zeta_o) \text{ and } \vartheta \in \bar{B}_{\frac{r_o}{2}}(\vartheta_o)\right\},$$

$$\bar{\phi} : (\zeta, \vartheta; \zeta') \in \mathfrak{D} \times \mathbb{R}^d \times \mathfrak{D} \longmapsto c_o \left(\left|\zeta - \zeta'\right|^4 + \left|\vartheta - \theta^0(\zeta)\right|^2\right).$$

By positive-definiteness and continuity of $E^{-4}$ combined with Assumption (A1), there exists $\gamma_E > 0$ such that

$$\frac{\xi^\top E^{-4} \xi}{\partial_x v^0}(\zeta) \geq \gamma_E \left|\xi\right|^2, \quad \text{for all } \xi \in \mathbb{R}^d \text{ and all } \zeta \in \bar{B}_\alpha. \tag{6.47}$$

On the other hand, continuity of $\sigma_S$ and Assumption (A1) imply that there is $\bar{\gamma} > 0$ such that

$$-\frac{1}{2}\left|\xi^\top \sigma_S\right|^2 \partial_{xx} v^0 \leq \bar{\gamma} |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^d \text{ and all } \zeta \in \bar{B}_\alpha. \tag{6.48}$$

Hence, we can choose the constant $c_o$ in the definition of $\bar{\phi}$ large enough to satisfy

$$\bar{\gamma} - c_o^2 \gamma_E \leq 0. \tag{6.49}$$

Define

$$\phi^\varepsilon : (\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d \longmapsto \delta \frac{T - t}{T - t_\varepsilon} + \bar{\phi}(\zeta, \vartheta; \zeta_\varepsilon).$$

Then, by Assumption (A1) and (6.44), the function $\psi^\varepsilon := v^0 - \varepsilon^2 \phi^\varepsilon$ is smooth. The lower-semicontinuity of $v_*^\varepsilon$ in turn allows to deduce from (6.46) that, on $\bar{B}_\alpha$, the function $v_*^\varepsilon -$

$\psi^\varepsilon$ has a local minimizer $(\tilde{\zeta}^\varepsilon, \tilde{\vartheta}^\varepsilon) \in B_{o,\alpha} \subset \mathrm{Int}(B_\alpha)$. Moreover, by (6.45), this minimizer satisfies $u^{\varepsilon *}(\tilde{\zeta}_\varepsilon, \tilde{\vartheta}_\varepsilon) \geq \delta$, and repeating the arguments leading to (6.44) shows $\tilde{\zeta}_\varepsilon \in \mathfrak{D}_<$.

*Step 2: conclude the proof.* In view of the previous step and Assumption 3.3, we have

$$-\left(\mathcal{L}^{\tilde{\vartheta}_\varepsilon} + \mathcal{H}^\varepsilon\right) \psi^\varepsilon(\tilde{\zeta}_\varepsilon, \tilde{\vartheta}_\varepsilon) \geq 0, \quad \text{for all } \varepsilon \in (0, \varepsilon_o].$$

By construction of $\psi^\varepsilon$ and because $(\tilde{\zeta}_\varepsilon, \tilde{\vartheta}_\varepsilon) \in \bar{B}_\alpha$, possibly reducing $\varepsilon_o$ gives

$$(\tilde{\zeta}_\varepsilon, \tilde{\vartheta}_\varepsilon) \in \{\partial_x \psi^\varepsilon > 0\} \cap \{\varepsilon^2 \partial_x(\phi + \varepsilon^2 w^\varepsilon) \leq \iota \partial_x v^0\},$$

so that (Ri) holds. Hence, Lemma 6.1 yields

$$\left\{-\frac{1}{2}\left|\boldsymbol{\xi}_\varepsilon^\top \sigma_S\right|^2 \partial_{xx} v^0 + \mathcal{L}^{\theta^0}\phi^\varepsilon - \frac{(D_\vartheta \bar{\phi})^\top E^{-4} D_\vartheta \bar{\phi}}{4\varepsilon^2 \partial_x v^0} - \frac{\partial_x \bar{\phi}}{4(\partial_x v^0)^2}(D_\vartheta \bar{\phi})^\top E^{-4} D_\vartheta \bar{\phi} + \mathcal{R}^\varepsilon\right\}(\tilde{\zeta}_\varepsilon, \tilde{\vartheta}_\varepsilon) \geq 0,$$

where $\mathcal{R}^\varepsilon(\tilde{\zeta}_\varepsilon, \tilde{\vartheta}_\varepsilon)$ is uniformly bounded for $\varepsilon \in (0, \varepsilon_o]$. Thus, by Assumption (A1) and construction of $\psi^\varepsilon$, there is a constant $C > 0$ independent of $\varepsilon$ such that:

$$\left\{-\frac{\delta}{T - t_\varepsilon} - \frac{1}{2}\left|\boldsymbol{\xi}_\varepsilon^\top \sigma_S\right|^2 \partial_{xx} v^0 - \frac{4c_o^2 \left|\boldsymbol{\xi}_\varepsilon\right|^\top E^{-4}\left|\boldsymbol{\xi}_\varepsilon\right|}{4\partial_x v^0}\right\}(\tilde{\zeta}_\varepsilon, \tilde{\vartheta}_\varepsilon) \geq -C, \quad \text{for all } \varepsilon \in (0, \varepsilon_o].$$

Recall that $(\tilde{\zeta}_\varepsilon, \tilde{\vartheta}_\varepsilon) \in \bar{B}_\alpha$; therefore, (6.47-6.49) yield

$$\left\{-\frac{1}{2}\left|\boldsymbol{\xi}_\varepsilon^\top \sigma_S\right|^2 \partial_{xx} v^0 - \frac{4c_o^2 \left|\boldsymbol{\xi}_\varepsilon\right|^\top E^{-4}\left|\boldsymbol{\xi}_\varepsilon\right|}{4\partial_x v^0}\right\}(\tilde{\zeta}_\varepsilon, \tilde{\vartheta}_\varepsilon) \leq (\bar{\gamma} - c_o^2 \gamma_E)\boldsymbol{\xi}_\varepsilon^2(\tilde{\zeta}_\varepsilon, \tilde{\vartheta}_\varepsilon) \leq 0.$$

As a result: $\delta/(T - t_\varepsilon) \leq C$, for all $\varepsilon \in (0, \varepsilon_o]$. Note that the time component of $\zeta_o$ is $T$, because $\zeta_o \in \partial_T \mathfrak{D}$. In contrast, the time component of $\zeta_\varepsilon$ is $t_\varepsilon$. For small $\varepsilon$, this contradicts (6.43), completing the proof. □

## 6.4. The Eikonal Equation

This section is devoted to the proof of the following result, which is crucially used in the proof of our Main Theorem 4.3.

PROPOSITION 6.6. *Suppose Assumptions 3.3, (A1) and (A2) are satisfied. Then,*

$$u_*(\zeta, \theta^0(\zeta)) \leq u_*(\zeta, \vartheta) \leq u^*(\zeta, \vartheta) \leq u^*(\zeta, \theta^0(\zeta)), \quad \text{for all } (\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d.$$

For notational convenience, define

$$\mathfrak{n} : (\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d \longmapsto -2\partial_x v^0 \partial_{xx} v^0 \left|\boldsymbol{\xi}_1^\top \sigma_S\right|^2 (\zeta, \vartheta). \tag{6.50}$$

By Assumption (A1), this is a nonnegative smooth function.

LEMMA 6.7. *Suppose Assumptions 3.3, (A1) and (A2) are satisfied. Then, $\bar{u}^*$ and $\bar{u}_*$ are (discontinuous) viscosity sub- and supersolutions, respectively, of the Eikonal equation*

$$(D_\vartheta \bar{u}^*)^\top E^{-4} D_\vartheta \bar{u}^* \leq \mathfrak{n}, \quad \textit{respectively} \quad (D_\vartheta \bar{u}_*)^\top E^{-4} D_\vartheta \bar{u}_* \geq \mathfrak{n}, \quad \textit{on } \mathfrak{D}_< \times \mathbb{R}^d.$$

*Proof.* We focus on the subsolution property; the supersolution property is obtained similarly. Consider $(\zeta_o, \vartheta_o) \in \mathfrak{D}_< \times \mathbb{R}^d$ and a smooth function $\varphi$ such that

$$\max_{\mathfrak{D}_< \times \mathbb{R}^d} (\text{strict})(\bar{u}^* - \varphi) = (\bar{u}^* - \varphi)(\zeta_o, \vartheta_o) = 0.$$

By definition of $\bar{u}^*$, there exist $(\zeta_\varepsilon, \vartheta_\varepsilon)_{\varepsilon>0} \subset \mathfrak{D}_< \times \mathbb{R}^d$, for which

$$(6.51) \qquad \begin{array}{l} (\zeta_\varepsilon, \vartheta_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} (\zeta_o, \vartheta_o) , \quad \bar{u}^{\varepsilon *}(\zeta_\varepsilon, \vartheta_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \bar{u}^*(\zeta_o, \vartheta_o), \\ \text{and} \quad p^\varepsilon := \bar{u}^{\varepsilon *}(\zeta_\varepsilon, \vartheta_\varepsilon) - \varphi(\zeta_\varepsilon, \vartheta_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0. \end{array}$$

By Assumptions (A1), (A2), and (6.51), there are $r_o, \varepsilon_o, \iota > 0$ such that

$$(6.52) \qquad \begin{array}{l} 2/\iota \geq -\partial_{xx} v \wedge \partial_x v \geq \iota \text{ on } B_o , \quad |p^\varepsilon| \leq 1 , \quad (\zeta_\varepsilon, \vartheta_\varepsilon) \in B_{r_o}(\zeta_o, \vartheta_o), \\ \text{and} \quad b^* := \sup \{\bar{u}^{\varepsilon *}(\zeta, \vartheta) : (\zeta, \vartheta) \in B_o \ , \ \varepsilon \in (0, \varepsilon_o]\} < \infty, \end{array}$$

where $B_o := B_{4r_o}(\zeta, \vartheta_o)$. The last estimate implies the existence of $\mathbf{d} > 0$ for which

$$|\zeta - \zeta_\varepsilon|^4 + |\vartheta - \vartheta_\varepsilon|^4 \geq \mathbf{d}, \quad \text{for all } (\zeta, \vartheta) \in \partial B_o \text{ and } \varepsilon \in (0, \varepsilon_o].$$

On the other hand, continuity of $\varphi$ yields $1 \vee \sup\{2 + b^* - \varphi(\zeta, \vartheta) : (\zeta, \vartheta) \in B_o\} =: M < +\infty$, so that we can choose a constant $c_o \geq M/\mathbf{d} > 0$, independent of $\varepsilon$. It follows that

$$(6.53) \qquad \phi^\varepsilon(\zeta, \vartheta) \geq 2 + b^* - \varphi(\zeta, \vartheta), \quad \text{for all } (\zeta, \vartheta) \in \partial B_o \text{ and } \varepsilon \in (0, \varepsilon_o],$$

where

$$\phi^\varepsilon : (\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d \longmapsto c_o \left(|\zeta - \zeta_\varepsilon|^4 + |\vartheta - \vartheta_\varepsilon|^4\right).$$

Now, define $\psi^\varepsilon := v^0 - \varepsilon^2(p^\varepsilon + \varphi + \phi^\varepsilon)$ and $I^\varepsilon := (v_*^\varepsilon - \psi^\varepsilon)/\varepsilon^2$. Then, on the one hand, we have $I^\varepsilon(\zeta_\varepsilon, \vartheta_\varepsilon) = 0$. On the other hand, by definition of $p^\varepsilon$, $\bar{u}^{\varepsilon *}$, and $\phi^\varepsilon$, as well as (6.52) and (6.53): $I^\varepsilon(\zeta, \vartheta) \geq 1$ for all $(\zeta, \vartheta) \in \partial B_o$. By upper-semicontinuity of $I^\varepsilon$, it follows that $I^\varepsilon$ admits an interior minimizer $(\tilde{\zeta}_\varepsilon, \tilde{\vartheta}_\varepsilon)$ on $B_o$. Moreover, classical arguments (Crandall et al. 1992) show $(\tilde{\zeta}_\varepsilon, \tilde{\vartheta}_\varepsilon) \to (\zeta_o, \vartheta_o)$ as $\varepsilon \to 0$. Hence, the viscosity supersolution property in Assumption 3.3 implies $-(\mathcal{L}^{\tilde{\vartheta}_\varepsilon} + \mathcal{H}^\varepsilon)\psi^\varepsilon(\tilde{\eta}_\varepsilon, \tilde{\vartheta}_\varepsilon) \geq 0,$ for all $\varepsilon \in (0, \varepsilon_o]$. After possibly reducing $\varepsilon_o > 0$, we obtain $\partial_x \psi^\varepsilon(\tilde{\zeta}_\varepsilon, \tilde{\vartheta}_\varepsilon) > 0$. Hence, Lemma 6.1, continuity of $\varphi$, and the fact that $\phi^\varepsilon$ as well as its derivatives vanish as $\varepsilon \to 0$ yield

$$\left(-\frac{1}{2}\left|\boldsymbol{\xi}_1^\top \sigma_S\right|^2 \partial_{xx} v^0 + \varepsilon^2 \mathcal{R}_\varepsilon - \frac{(D_\vartheta \varphi)^\top E^{-4} D_\vartheta \varphi}{4\partial_x \phi^\varepsilon}\right)(\tilde{\zeta}_\varepsilon, \tilde{\vartheta}_\varepsilon) \geq 0,$$

where $\varepsilon^2 \mathcal{R}_\varepsilon \to 0$ as $\varepsilon \to 0$. Sending $\varepsilon \to 0$ in turn gives

$$-\frac{1}{2}\left|\boldsymbol{\xi}_1^\top \sigma_S\right|^2 \partial_{xx} v^0(\zeta_o, \vartheta_o) \geq \frac{(D_\vartheta \varphi)^\top E^{-4} D_\vartheta \varphi}{4\partial_x v^0}(\zeta_o, \vartheta_o),$$

which proves the asserted viscosity subsolution property. □

Next, we show that $\bar{u}^*$ and $\bar{u}_*$ satisfy a generalized terminal condition as in Crandall, Ishii, and Lions (1992, definition 7.4):

LEMMA 6.8. *Suppose Assumptions 3.3, (A1) and (A2) are satisfied. Then, $\bar{u}^*$ and $\bar{u}_*$ are (discontinuous) viscosity sub- and supersolutions, respectively, of*

$$\min\left\{\bar{u}^* - \boldsymbol{\xi}_1^\top k_2 \boldsymbol{\xi}_1 \;;\; (D_\vartheta \bar{u}^*)^\top E^{-4} D_\vartheta \bar{u}^* - \mathfrak{n}\right\} \le 0, \quad \text{on } \partial_T \mathfrak{D} \times \mathbb{R}^d,$$
$$\text{and} \quad \max\left\{\bar{u}_* - \boldsymbol{\xi}_1^\top k_2 \boldsymbol{\xi}_1 \;;\; (D_\vartheta \bar{u}_*)^\top E^{-4} D_\vartheta \bar{u}_* - \mathfrak{n}\right\} \ge 0, \quad \text{on } \partial_T \mathfrak{D} \times \mathbb{R}^d.$$

*Proof.* Consider $(\zeta_o, \vartheta_o) \in \partial_T \mathfrak{D} \times \mathbb{R}^d$ and a smooth function $\varphi$ such that

$$0 = (\bar{u}^* - \varphi)(\zeta_o, \vartheta_o) = \max_{\mathfrak{D} \times \mathbb{R}^d} (\text{strict})(\bar{u}^* - \varphi).$$

Assume that there is $\delta > 0$ for which $\bar{u}^*(\zeta_o, \vartheta_o) - \boldsymbol{\xi}_1(\zeta_o, \vartheta_o)^\top k_2(\zeta_o) \boldsymbol{\xi}_1(\zeta_o, \vartheta_o) \ge \delta$. Repeating the arguments of Proposition 6.5 then gives

$$-\frac{1}{2}\left|\boldsymbol{\xi}_1^\top \sigma_S\right|^2 \partial_{xx} v^0(\zeta_o, \vartheta_o) \ge \frac{(D_\vartheta \varphi)^\top E^{-4} D_\vartheta \varphi}{4 \partial_x v^0}(\zeta_o, \vartheta_o),$$

and the subsolution property follows. The supersolution property is obtained similarly. $\square$

Next, we show that $\bar{u}^*$, $\bar{u}_*$ also solve the Eikonal equation if the $\zeta$-variable is fixed and they are considered as functions of the $\vartheta$-variable only:

LEMMA 6.9. *Suppose Assumptions 3.3, (A1) and (A2) are satisfied. Then, for any $\zeta_o \in \mathfrak{D}_<$, the functions $\vartheta \longmapsto \bar{u}^*(\zeta_o, \vartheta)$ and $\vartheta \longmapsto \bar{u}_*(\zeta_o, \vartheta)$ are viscosity sub- and supersolutions, respectively, of*

$$\begin{cases} (D_\vartheta \varphi)^\top E^{-4} D_\vartheta \varphi = \mathfrak{n}, & \text{on } \mathbb{R}^d \backslash \{\theta^0(\zeta_o)\}, \\ \varphi \ge \bar{u}_*(\zeta_o, \cdot) \; (\textit{resp.} \le \bar{u}^*(\zeta_o, \theta^0(\zeta_o))), & \text{on} \{\vartheta = \theta^0(\zeta_o)\}. \end{cases}$$

*For any $\zeta_o \in \partial_T \mathfrak{D}$, the functions $\vartheta \longmapsto \bar{u}^*(\zeta_o, \vartheta)$ and $\vartheta \longmapsto \bar{u}_*(\zeta_o, \vartheta)$ are viscosity sub- and supersolutions, respectively, of*

$$\min\left\{\bar{u}^*(\zeta_o, \cdot) - \boldsymbol{\xi}_1^\top k_2(\zeta_o) \boldsymbol{\xi}_1(\zeta_o, \cdot), (D_\vartheta \bar{u}^*)^\top E^{-4}(\zeta_o) D_\vartheta \bar{u}^*(\zeta_o, \cdot) - \mathfrak{n}(\zeta_o, \cdot)\right\} \le 0,$$
$$\max\left\{\bar{u}_*(\zeta_o, \cdot) - \boldsymbol{\xi}_1^\top k_2(\zeta_o) \boldsymbol{\xi}_1(\zeta_o, \cdot), (D_\vartheta \bar{u}_*)^\top E^{-4}(\zeta_o) D_\vartheta \bar{u}_*(\zeta_o, \cdot) - \mathfrak{n}(\zeta_o, \cdot)\right\} \ge 0.$$

*Proof.* We focus on the viscosity supersolution property on $\mathbb{R}^d \backslash \{\theta^0(\zeta_o)\}$ for $\zeta_o \in \mathfrak{D}_<$; the other properties are either evident, or obtained similarly (compare Lemma 6.8).

Fix an arbitrary $\zeta_o \in \mathfrak{D}_<$, and consider a smooth function $\varphi$ and $\vartheta_o \in \mathbb{R}^d \backslash \{\theta^0(\zeta_o)\}$ such that

$$0 = \bar{u}_*(\zeta_o, \vartheta_o) - \varphi(\vartheta_o) = \min_{\mathbb{R}^d \backslash \{\vartheta_o\}} (\text{strict})(\bar{u}_*(\zeta_o, \cdot) - \varphi(\cdot)). \tag{6.54}$$

For each $n \in \mathbb{N}$, define

$$\psi^n : (\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d \longmapsto \varphi(\vartheta) - n\left|\zeta - \zeta_o\right|^2,$$
$$\text{and} \quad I^n : (\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d \longmapsto \bar{u}_*(\zeta, \vartheta) - \psi^n(\zeta, \vartheta).$$

By Lemma 6.2, there are $r_o > 0$ and $b_o \ge 0$ for which

$$\bar{u}_* \ge -b_o, \quad \text{on } B_o, \tag{6.55}$$

where $B_o := \bar{B}_{r_o}(\zeta_o, \vartheta_o)$ and $r_o$ is chosen so that $B_o \subset \mathfrak{D}_<$. By compactness of $B_o$ and lower-semicontinuity of $I^n$, there is $(\zeta_n, \vartheta_n) \in B_o$ minimizing $I^n$ on $B_o$ for each $n \in \mathbb{N}$. Moreover, there exist $(\zeta^*, \vartheta^*) \in B_o$ such that $(\zeta_n, \vartheta_n) \to (\zeta^*, \vartheta^*)$ as $n \to +\infty$, possibly along a subsequence. Now, on the one hand, the minimality of $I^n(\zeta_n, \vartheta_n)$ on $B_o$ implies that $I^n(\zeta_n, \vartheta_n) \le I^n(\zeta_o, \vartheta_o) = \bar{u}_*(\zeta_o, \vartheta_o) - \varphi(\vartheta_o)$, which is finite and does not depend on $n$. On the other hand, if $\zeta^* \neq \zeta_o$, (6.55) gives $I^n(\zeta_n, \vartheta_n) \to +\infty$ as $n \to +\infty$. Hence, $\zeta^* = \zeta_o$.

Observe now that $\bar{u}_*(\zeta_o, \vartheta_o) - \varphi(\vartheta_o) = I^n(\zeta_o, \vartheta_o) \ge I^n(\zeta_n, \vartheta_n)$ implies

$$\bar{u}_*(\zeta_o, \vartheta_o) - \varphi(\vartheta_o) \ge \liminf_{n\to+\infty} I^n(\zeta_n, \vartheta_n) \ge \bar{u}_*(\zeta_o, \vartheta^*) - \varphi(\vartheta^*).$$

Therefore, $\vartheta^* = \vartheta_o$ by the strict minimum property in (6.54). Hence, $(\zeta_n, \vartheta_n) \in \mathrm{Int}(B_o)$ for sufficiently large $n$ so that, by construction, $(\zeta_n, \vartheta_n)$ is a local minimum of $I^n$. Lemma 6.7 in turn yields $(D_\vartheta \psi^n)^\top E^{-4} D_\vartheta \psi^n(\zeta_n, \vartheta_n) \ge \mathfrak{n}(\zeta_n, \vartheta_n)$. As a result, sending $n \to +\infty$ finally proves the assertion after recalling from Lemma 4.1 that $\mathfrak{n}$ is continuous.

In view of Lemma 6.9 and Proposition 6.5 define, for each $\zeta \in \mathfrak{D}$, the following subsets of $\mathbb{R}^d$:

$$\mathcal{O}^{\zeta *} := \left\{\vartheta \in \mathbb{R}^d : (D_\vartheta \bar{u}^*)^\top E^{-4} D_\vartheta \bar{u}^*(\zeta, \vartheta) \le \mathfrak{n}(\zeta, \vartheta)\right\} \setminus \{\theta^0(\zeta)\},$$
$$\mathcal{O}^{\zeta}_* := \left\{\vartheta \in \mathbb{R}^d : (D_\vartheta \bar{u}_*)^\top E^{-4} D_\vartheta \bar{u}_*(\zeta, \vartheta) \ge \mathfrak{n}(\zeta, \vartheta)\right\} \setminus \{\theta^0(\zeta)\}.$$

(Here, the inequalities have to be understood in the viscosity sense.) By construction, $\bar{u}^*$ and $\bar{u}_*$ are viscosity sub- resp. supersolutions of the Eikonal equation

$$(D_\vartheta \varphi)^\top E^{-4} D_\vartheta \varphi(\zeta, \cdot) = \mathfrak{n}(\zeta, \cdot),$$

on $\mathcal{O}^{\zeta *}$ resp. $\mathcal{O}^{\zeta}_*$. Observe from the first part of Lemma 6.9 that, for all $\zeta \in \mathfrak{D}_<$ (i.e., before the terminal time), we have the following simplification: $\mathcal{O}^{\zeta *} = \mathcal{O}^{\zeta}_* = \mathbb{R}^d \setminus \{\theta^0(\zeta)\}$, or equivalently $(\mathcal{O}^{\zeta *})^c = (\mathcal{O}^{\zeta}_*)^c = \{\theta^0(\zeta)\}$. Hence, we have the following estimate for all $\zeta \in \mathfrak{D}_<$:

$$\begin{aligned} &\bar{u}^*(\zeta, \cdot) \le \bar{u}^*(\zeta, \theta^0(\zeta)) + \boldsymbol{\xi}_1(\zeta, \cdot)^\top k_2(\zeta) \boldsymbol{\xi}_1(\zeta, \cdot), \quad \text{on } (\mathcal{O}^{\zeta *})^c\ , \\ &\bar{u}_*(\zeta, \cdot) \ge \bar{u}_*(\zeta, \theta^0(\zeta)) + \boldsymbol{\xi}_1(\zeta, \cdot)^\top k_2(\zeta) \boldsymbol{\xi}_1(\zeta, \cdot), \quad \text{on } (\mathcal{O}^{\zeta *})^c\ . \end{aligned} \tag{6.56}$$

For $\zeta \in \partial_T \mathfrak{D}$, such a simplification of $\mathcal{O}^{\zeta *}$ or $\mathcal{O}^{\zeta}_*$ is not available. However, combining the second part of Lemma 6.9 with Proposition 6.5, we find that (6.56) holds for all $\zeta \in \partial_T \mathfrak{D}$ as well, and hence for all $\zeta \in \mathfrak{D}$.

For later use, also note the following. For any $\zeta \in \mathfrak{D}$, we have $\theta^0(\zeta) \notin \mathcal{O}^{\zeta *} \cup \mathcal{O}^{\zeta}_*$. Hence, Assumption (A1) and the ellipticity of $\sigma_S \sigma_S^\top$ imply the following estimate for the function $\mathfrak{n}$ defined in (6.50):

$$\mathfrak{n}(\zeta, \vartheta) > 0 \quad \text{on} \quad \mathcal{O}^{\zeta *} \cup \mathcal{O}^{\zeta}_*.$$

Now introduce, for any $\zeta \in \mathfrak{D}$, the operator

$$H^\zeta : (\vartheta, r, q) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longmapsto -\mathfrak{n}(\zeta, \vartheta) r^2 + q^\top E^{-4}(\zeta) q.$$

Also define, for $M > 0$, the class $\mathcal{C}^-_M$ of negative functions $\mathbb{R}^d \to \mathbb{R}$ bounded from below by $-M$. We can then establish the comparison property for $H^\zeta$ on $\mathcal{C}^-_M$:

LEMMA 6.10. *Suppose Assumption (A1) is satisfied. For any $\zeta \in \mathfrak{D}$, let $\mathcal{O}^\zeta$ be a subset of $\mathbb{R}^d$ for which $\mathfrak{n}(\zeta,\cdot) > 0$ on $\mathcal{O}^\zeta$, and let $\mathfrak{v}^{1\zeta}$, $\mathfrak{v}^{2\zeta}$, $\mathfrak{v}^{3\zeta} \in \mathcal{C}_M^-$ (for some $M > 0$) be lower-semicontinuous, smooth, and upper-semicontinuous functions, satisfying (in the viscosity sense for $\mathfrak{v}^{1\zeta}$ and $\mathfrak{v}^{3\zeta}$):*

$$H^\zeta(\cdot,\mathfrak{v}^{1\zeta}, D_\vartheta \mathfrak{v}^{1\zeta}) \geq 0\,, \quad H^\zeta(\cdot,\mathfrak{v}^{2\zeta}, D_\vartheta \mathfrak{v}^{2\zeta}) = 0, \quad \text{and} \quad H^\zeta(\cdot,\mathfrak{v}^{3\zeta}, D_\vartheta \mathfrak{v}^{3\zeta}) \leq 0, \quad \text{on } \mathcal{O}^\zeta. \tag{6.57}$$

*Then if $\mathfrak{v}^{1\zeta} \geq \mathfrak{v}^{2\zeta} \geq \mathfrak{v}^{3\zeta}$ on $\mathbb{R}^d \backslash \mathcal{O}^\zeta$, we have $\mathfrak{v}^{1\zeta} \geq \mathfrak{v}^{2\zeta} \geq \mathfrak{v}^{3\zeta}$ on $\mathbb{R}^d$.*

*Proof.* Fix $\zeta \in \mathfrak{D}$ and drop it from the notation for clarity. We focus on the inequality $\mathfrak{v}^1 \geq \mathfrak{v}^2$; the other one is obtained analogously. For $\mathfrak{v}^1$ and $\mathfrak{v}^2$ as in the statement of the lemma, assume that there are $\bar\vartheta \in \mathcal{O}$ and $\alpha > 0$ such that

$$\mathfrak{v}^1(\bar\vartheta) - \mathfrak{v}^2(\bar\vartheta) \leq -\alpha < 0, \tag{6.58}$$

and work toward a contradiction. Choose $\beta \in C^\infty(\mathbb{R}^d)$, satisfying $0 \leq \beta \leq 1$, $\beta(0) = 1$, $D_\vartheta \beta(0) = 0$ and $\beta(x) = 0$ for all $x \in \mathbb{R}^d \backslash \bar{B}_1(0)$, and define, for all $\eta > 0$:

$$\Phi_\eta : \vartheta \in \mathbb{R}^d \longmapsto (\mathfrak{v}^1 - \mathfrak{v}^2 - 2M\beta_\eta(\cdot - \bar\vartheta))(\vartheta), \quad \text{where } \beta_\eta(x) := \beta(x/\eta).$$

By definition of $\mathcal{C}_M^-$ and boundedness of $\beta_\eta$, we have $\inf_{\mathbb{R}^d} \Phi_\eta > -\infty$. Hence, for each $\delta > 0$, there is $\vartheta_\delta \in \mathbb{R}^d$ such that

$$\Phi_\eta(\vartheta_\delta) \leq \inf_{\mathbb{R}^d} \Phi_\eta + \delta. \tag{6.59}$$

Pick a function $\chi \in C^\infty(\mathbb{R}^d)$ satisfying

$$0 \leq \chi \leq 1, \quad \chi(0) = 1, \quad \chi(x) = 0 \text{ if } |x|^2 > 1, \quad \text{and} \quad |D_\vartheta \chi| \leq c,$$

for a constant $c > 0$ independent of $\delta$. For each $\delta > 0$, let $\chi_\delta := \chi(\cdot - \vartheta_\delta)$ Then, for all $\delta > 0$:

$$0 \leq \chi_\delta \leq 1, \quad \chi_\delta(\vartheta_\delta) = 1, \quad \chi_\delta(\vartheta) = 0 \text{ if } |\vartheta - \vartheta_\delta|^2 > 1, \quad \text{and} \quad |D_\vartheta \chi_\delta| \leq c.$$

Now define, for every $\eta, \delta > 0$:

$$\Psi_{\eta,\delta} : \vartheta \in \mathbb{R}^d \longmapsto (\Phi_\eta - 2\delta\chi_\delta)(\vartheta) = (\mathfrak{v}^1 - \mathfrak{v}^2 - 2M\beta_\eta(\cdot - \bar\vartheta) - 2\delta\chi_\delta)(\vartheta).$$

On the one hand, (6.59) in turn enables us to deduce that, for all $\eta, \delta > 0$,

$$\Psi_{\eta,\delta}(\vartheta_\delta) = \Phi(\vartheta_\delta) - 2\delta \leq \inf_{\mathbb{R}^d} \Phi_\eta - \delta < \inf_{\mathbb{R}^d} \Phi_\eta.$$

On the other hand:

$$\Psi_{\eta,\delta}(\vartheta) = \Phi_\eta(\vartheta) \geq \inf_{\mathbb{R}^d} \Phi_\eta, \quad \text{for all } \vartheta \in \mathbb{R}^d \text{ such that } |\vartheta - \vartheta_\delta|^2 > 1.$$

As a result, the lower-semicontinuity of $\Psi_{\eta,\delta}$ yields that we can find a minimizing sequence $(\hat\vartheta_{\eta,\delta})_{\eta,\delta>0}$ for $\Psi_{\eta,\delta}$. Moreover, $\chi_\delta \geq 0$, (6.58), and the definition of $\Psi_{\eta,\delta}$ give

$$\Psi_{\eta,\delta}(\hat\vartheta_{\eta,\delta}) \leq \Psi_{\eta,\delta}(\bar\vartheta) \leq -\alpha - 2M. \tag{6.60}$$

As $\beta$, $\chi_\delta \leq 1$, it follows that $(\mathfrak{v}^1 - \mathfrak{v}^2)(\hat{\vartheta}_{\eta,\delta}) \leq -\alpha + 2\delta < 0$, for all $\delta < \alpha/2$. Hence, $\hat{\vartheta}_{\eta,\delta} \in \mathcal{O}$ for all such small $\delta$. As $\mathfrak{v}^1, \mathfrak{v}^2 \in \mathcal{C}_M^-$ and $\chi_\delta \leq 1$,

$$\Psi_{\eta,\delta}(\hat{\vartheta}_{\eta,\delta}) \geq -M - 2M\beta_\eta(\hat{\vartheta}_{\eta,\delta} - \bar{\vartheta}) - 2\delta.$$

Combined with (6.60), this leads to

$$2M\beta_\eta(\hat{\vartheta}_{\eta,\delta} - \bar{\vartheta}) \geq M - 2\delta > 0, \qquad \text{for all } (\eta, \delta) \in (0, \infty) \times (0, M/2).$$

By definition of $\beta_\eta$, it in turn follows that $\hat{\vartheta}_{\eta,\delta} \in \bar{B}_\eta(\bar{\vartheta})$ for all $(\eta, \delta) \in (0, \infty) \times (0, M/2)$.

Because $\hat{\vartheta}_{\eta,\delta} \in \mathcal{O}$, (6.57) yields, for all $(\eta, \delta) \in (0, \infty) \times (0, M/2 \wedge \alpha/2)$:

$$H(\cdot, \mathfrak{v}^1, D_\vartheta(\mathfrak{v}^2 + 2M\beta_\eta(\cdot - \bar{\vartheta}) + 2\delta\chi_\delta))(\hat{\vartheta}_{\eta,\delta}) \geq 0 \quad \text{and} \quad H(\cdot, \mathfrak{v}^2, D_\vartheta \mathfrak{v}^2)(\hat{\vartheta}_{\eta,\delta}) = 0.$$

As $\mathfrak{n} > 0$ on $\mathcal{O}$, this gives

$$[(\mathfrak{v}^1)^2 - (\mathfrak{v}^2)^2](\hat{\vartheta}_{\eta,\delta}) - \frac{[D_\vartheta^\top \varrho E^{-4} D_\vartheta \varrho(\hat{\vartheta}_{\eta,\delta})]^2 - [D_\vartheta^\top \mathfrak{v}^2 E^{-4} D_\vartheta \mathfrak{v}^2(\hat{\vartheta}_{\eta,\delta})]^2}{\mathfrak{n}(\hat{\vartheta}_{\eta,\delta})} \leq 0,$$

with $\varrho := (\mathfrak{v}^2 + 2M\beta_\eta(\cdot - \bar{\vartheta}) + 2\delta\chi_\delta)$. As we have seen above that $\hat{\vartheta}_{\eta,\delta} \in \bar{B}_\eta(\bar{\vartheta})$, there exists $\bar{\vartheta}_\eta \in \bar{B}_\eta(\bar{\vartheta})$ such that $\hat{\vartheta}_{\eta,\delta} \to \bar{\vartheta}_\eta$ as $\delta \to 0$, possibly along a subsequence, and in turn $\bar{\vartheta}_\eta \to \bar{\vartheta}$ as $\eta \to 0$. Hence, taking into account Assumption (A1), continuity of $\mathfrak{v}^2$ and its gradient, $D_\vartheta \beta(0) = 0$, and $|D_\vartheta \chi_\delta| \leq c$ independent of $\delta$, the following limit obtains after sending first $\delta \to 0$ and then $\eta \to 0$:

$$\liminf_{\delta,\eta\to 0} (\mathfrak{v}^1)^2(\hat{\vartheta}_{\eta,\delta}) - (\mathfrak{v}^2)^2(\bar{\vartheta}) \leq 0.$$

Because $\vartheta \longmapsto (\mathfrak{v}^1)^2(\vartheta)$ is lower-semicontinuous, it follows that $(\mathfrak{v}^1 + \mathfrak{v}^2)(\mathfrak{v}^1 - \mathfrak{v}^2)(\bar{\vartheta}) \leq 0$, As $\mathfrak{v}^1 + \mathfrak{v}^2 < 0$ because $\mathfrak{v}^1, \mathfrak{v}^2 \in \mathcal{C}_M^-$, this contradicts (6.58) and thereby proves the assertion. □

Now, for all $(\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d$, define the mappings $\bar{\mathfrak{u}}^*, \bar{\mathfrak{u}}_*, \tilde{\mathfrak{u}}^*, \tilde{\mathfrak{u}}_* : \mathfrak{D} \times \mathbb{R}^d \to \mathbb{R}$ as follows:

$$\bar{\mathfrak{u}}^*(\zeta, \vartheta) = -e^{-\bar{u}^*(\zeta,\vartheta)}, \qquad \tilde{\mathfrak{u}}^*(\zeta, \vartheta) = -e^{-(\bar{u}^*(\zeta,\theta^0(\zeta)) + \boldsymbol{\xi}_1^\top(\zeta,\vartheta)k_2(\zeta)\boldsymbol{\xi}_1(\zeta,\vartheta))},$$

$$\bar{\mathfrak{u}}_*(\zeta, \vartheta) = -e^{-\bar{u}_*(\zeta,\vartheta)}, \qquad \tilde{\mathfrak{u}}_*(\zeta, \vartheta) = -e^{-(\bar{u}_*(\zeta,\theta^0(\zeta)) + \boldsymbol{\xi}_1^\top(\zeta,\vartheta)k_2(\zeta)\boldsymbol{\xi}_1(\zeta,\vartheta))}.$$

One readily verifies that this change of variable produces bounded solutions to the Eikonal equation from Lemma 6.10, for which a comparison principle holds on the class of bounded functions by Lemma 6.10:

LEMMA 6.11. *Suppose Assumptions 3.3, (A1) and (A2) are satisfied. Then, for all $\zeta_o \in \mathfrak{D}$, the mappings $\bar{\mathfrak{u}}^*(\zeta_o, \cdot)$, $\tilde{\mathfrak{u}}^*(\zeta_o, \cdot)$, $\bar{\mathfrak{u}}_*(\zeta_o, \cdot)$, and $\tilde{\mathfrak{u}}_*(\zeta_o, \cdot)$ are viscosity subsolution, classical solution, viscosity supersolution, and classical solution, respectively, of*

$$H^{\zeta_o}(\cdot, \bar{\mathfrak{u}}^*, D_\vartheta \bar{\mathfrak{u}}^*) \leq 0, \quad H^{\zeta_o}(\cdot, \tilde{\mathfrak{u}}^*, D_\vartheta \tilde{\mathfrak{u}}^*) = 0, \quad \text{on } \mathcal{O}^{\zeta_o *},$$

$$H^{\zeta_o}(\cdot, \bar{\mathfrak{u}}_*, D_\vartheta \bar{\mathfrak{u}}_*) \geq 0, \quad \text{and} \quad H^{\zeta_o}(\cdot, \tilde{\mathfrak{u}}_*, D_\vartheta \tilde{\mathfrak{u}}_*) = 0. \quad \text{on } \mathcal{O}_*^{\zeta_o}.$$

*Moreover, $\bar{\mathfrak{u}}^* = \tilde{\mathfrak{u}}^*$ on $(\mathcal{O}^{\zeta_o *})^c$ and $\bar{\mathfrak{u}}_* = \tilde{\mathfrak{u}}_*$ on $(\mathcal{O}_*^{\zeta_o})^c$.*

Putting together all the previous results, we can now prove Proposition 6.6:

PROPOSITION 6.6. *First observe from* (4.1), (4.2)*, and the definition of* $\bar{\mathrm{u}}^*$ *and* $\bar{\mathrm{u}}_*$ *that* $-1 \leq \bar{\mathrm{u}}_* \leq \bar{\mathrm{u}}^* < 0$ *so that* $\bar{\mathrm{u}}^*, \bar{\mathrm{u}}_* \in \mathcal{C}_1^-$. *Lemmata 6.10 and 6.11 in turn yield that, for any* $(\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d$:

$$\tilde{\mathrm{u}}_*(\zeta, \vartheta) \leq \bar{\mathrm{u}}_*(\zeta, \vartheta) \quad \text{and} \quad \bar{\mathrm{u}}^*(\zeta, \vartheta) \leq \tilde{\mathrm{u}}^*(\zeta, \vartheta).$$

*As* $\bar{\mathrm{u}}_* \leq \bar{\mathrm{u}}^*$ *by definition, this yields*

$$\begin{aligned}\bar{u}_*(\zeta, \theta^0(\zeta)) + \boldsymbol{\xi}_1^\top(\zeta, \vartheta) k_2(\zeta) \boldsymbol{\xi}_1(\zeta, \vartheta) \leq \bar{u}_*(\zeta, \vartheta) \leq \bar{u}^*(\zeta, \vartheta) \leq \bar{u}^*(\zeta, \theta^0(\zeta)) \\ + \boldsymbol{\xi}_1^\top(\zeta, \vartheta) k_2(\zeta) \boldsymbol{\xi}_1(\zeta, \vartheta).\end{aligned}$$

*Proposition 6.6 now follows from the definition of* $u_*$ *and* $u^*$ *in* (6.3). □

## 7. SUFFICIENT CONDITIONS FOR ASSUMPTION A

In this section, we provide a set of sufficient conditions for the abstract Assumption A under which our Main Theorem 4.3 holds. These sufficient conditions are typical for verification theorems (compare, e.g., Touzi 2013), and can be readily verified in concrete models, see Section 8. Moreover, under these conditions, the policy from Theorem 4.7 is indeed optimal at the leading order for small price impact costs.

Throughout, we assume that the frictionless value function $v^0$ and the corresponding optimal policy $\theta^0$ are given. The function $v^0$ satisfies $\partial_x v^0 \vee (-\partial_{xx} v^0) > 0$ and is a classical $C^{1,2}$-solution of the frictionless DPE (3.3). The policy $\theta^0$ is characterized by the First-Order Condition (3.5) and belongs to $C^{1,2}$. In particular, Assumption (A1) is satisfied.[28]

For any positive function $f : \mathfrak{D} \to \mathbb{R}$, we denote by $\mathcal{C}^f$ the class of functions $g$ dominated by $f$ in the following sense (here, $\partial\mathfrak{D}$ denotes the spatial boundary of $\mathfrak{D}$):

$$\limsup_{\zeta \to \partial\mathfrak{D}} \frac{|g|(\zeta)}{1 + |f|(\zeta)} = 0. \tag{7.1}$$

With this notation, the sufficient conditions for the validity of Assumption A read as follows:

ASSUMPTION B.

- *(B1) There is a nonnegative function* $\chi \in C^{1,2}$ *satisfying* $-\mathcal{L}^{\theta^0}\chi > 0$ *on* $\mathfrak{D}_<$;
- *(B2) There exists a classical* $C^{1,2}$*-solution* $\hat{u}$ *of the Second Corrector Equation* (3.20)*, where the pair* $(a, \varpi)$ *is the solution of the First Corrector Equation* (3.19) *from Lemma 4.1;*
- *(B3)* $\hat{u}$ *and the function* $u$ *defined though the Probabilistic Representation* (4.5) *belong to* $\mathcal{C}^\chi$;
- *(B4) The feedback policy*

$$\dot{\theta}^\varepsilon(\zeta, \vartheta) := -\frac{[E^{-4} D_\xi \varpi] \circ \boldsymbol{\xi}_\varepsilon(\zeta, \vartheta)}{2\varepsilon \partial_x v^0(\zeta)} = \frac{E^{-2}(E^{-2}\sigma_S \sigma_S^\top E^{-2})^{1/2} E^2}{\varepsilon^2(-2\partial_x v^0/\partial_{xx} v^0)^{1/2}}(\zeta) \times (\theta^0(\zeta) - \vartheta),$$

*from Theorem 4.7 is an admissible control.*

[28]These assumptions are satisfied if a *classical* frictionless verification theorem applies, cf., e.g., Touzi (2013) and the references therein. In particular, they typically hold in the concrete models that can be solved explicitly.

(B5) *Set* $\hat{v}^\varepsilon := v^0 - \varepsilon^2 \hat{u} - \varepsilon^4 \varpi \circ \boldsymbol{\xi}_\varepsilon$. *For every* $\varepsilon > 0$, *there is a function* $\gamma^\varepsilon$ *such that* $|\hat{v}^\varepsilon| \leq \gamma^\varepsilon$ *on* $\mathfrak{D} \times \mathbb{R}$ *and, for all* $(\zeta, \vartheta, \varepsilon) \in \mathfrak{D} \times \mathbb{R} \times (0, \infty)$:

$$\sup_{t \leq r \leq T} \gamma^\varepsilon \left(r, S_r^\zeta, Y_r^\zeta, X_r^{\zeta,\vartheta,\varepsilon}, \theta_r^{t,\vartheta,\varepsilon}\right) \in L^1.$$

(B6) *The remainder* $\mathcal{R}_{\mathcal{L}}^\varepsilon$ *of Lemma 6.1, computed for* $\psi^\varepsilon = \hat{v}^\varepsilon$, *satisfies:*

$$\mathbb{E}\left[\int_t^T \left|\mathcal{R}_{\mathcal{L}}^\varepsilon + \tilde{\mathcal{R}}\right| \left(r, S_r^\zeta, Y_r^\zeta, X_r^{\zeta,\vartheta,\varepsilon}, \theta_r^{t,\vartheta,\varepsilon}\right) dr\right] \leq \varepsilon \beta(\zeta, \vartheta),$$

*for some continuous function* $\beta : \mathfrak{D} \times \mathbb{R}^d \to \mathbb{R}$, *where, for all* $(\zeta, \vartheta) \in \mathfrak{D} \times \mathbb{R}^d$:

$$\tilde{\mathcal{R}}(\zeta, \vartheta) := \frac{[(D_\xi \varpi)^\top E^{-4} D_\xi \varpi] \circ \boldsymbol{\xi}_1}{4(\partial_x v^0)^2} \left(\partial_x \hat{u} - \partial_x \theta^0 D_\xi \varpi \circ \boldsymbol{\xi}_1 + \partial_x \varpi \circ \boldsymbol{\xi}_1\right)(\zeta, \vartheta).$$

REMARK 7.1. Assumption (1) requires extra integrability of the candidate strategy from Assumption (B4). This enables us to apply dominated convergence along a sequence of localizing stopping times in the verification argument in the proof of Proposition 7.2 below.

Under Assumption (B5), the remainder of the asymptotic expansion can be controlled along the candidate almost optimal strategy. Indeed, this remainder is then of order $\varepsilon^3$, allowing us not only to recover Assumption (A2) but also to prove that the proposed strategy is optimal at the leading order $O(\varepsilon^2)$. In concrete settings, these two assumptions can be verified using estimates on the diffusions driving the control $\dot{\theta}^\varepsilon$ of Assumption (B4), compare Section 8.

PROPOSITION 7.2. *Assumption B implies Assumption (A2), Assumption (A3), with* $\mathcal{C} = \mathcal{C}^\chi$, *and* $u_* = u^* = u = \hat{u}$.

*Proof. Step 1: prove Assumption* (A2).

Fix $(\zeta, \vartheta, \varepsilon) \in \mathfrak{D}_< \times \mathbb{R}^d \times (0, \infty)$, set $(X, \theta) := (X^{\zeta,\vartheta,\varepsilon}, \theta^{t,\vartheta,\varepsilon})$ and $\Upsilon := (S^\zeta, Y^\zeta, X^{\zeta,\vartheta,\varepsilon}, \theta^{t,\vartheta,\varepsilon})$ to ease notation, and define the stopping times

$$\tau_n^\varepsilon := T \wedge \inf\{u \geq t : \Upsilon_u \notin B_n(\zeta, \vartheta)\}, \quad n \geq 1.$$

By smoothness of $v^0$, $\theta^0$, and Assumption (B2), we have $\hat{v}^\varepsilon \in C^{1,2}(\mathfrak{D} \times \mathbb{R}^d)$. Itô's formula in turn yields

$$\hat{v}^\varepsilon(\zeta, \vartheta) = \mathbb{E}\left[\hat{v}^\varepsilon\left(\tau_n^\varepsilon, \Upsilon_{\tau_n^\varepsilon}\right) - \int_t^{\tau_n^\varepsilon} \left(\mathcal{L}^{\theta^\varepsilon} v^\varepsilon + \varepsilon^2 \frac{[(D_\xi \varpi)^\top E^{-4} D_\xi \varpi] \circ \xi \xi_\varepsilon}{4 \partial_x v^0} + \varepsilon^2 \tilde{\mathcal{R}}\right)(u, \Upsilon_u)\, du\right].$$

In view of Lemma 6.1,

$$\mathcal{L}^\vartheta \hat{v}^\varepsilon(\zeta, \vartheta) = \left\{\mathcal{L}^{\theta^0} v^0 + \varepsilon^2 \left(\frac{1}{2}\left|\boldsymbol{\xi}_\varepsilon^\top \sigma_S\right|^2 \partial_{xx} v^0 - \mathcal{L}^{\theta^0} \hat{u} - \frac{1}{2} \mathrm{Tr}\left[c_{\theta^0} D_{\xi\xi}^2 \varpi \circ \boldsymbol{\xi}_\varepsilon\right] + \hat{\mathcal{R}}_{\mathcal{L}}^\varepsilon\right)\right\}(\zeta, \vartheta).$$

Now, use the frictionless DPE (3.4) for $v^0$, the Second Corrector Equation (3.20) for $\hat{u}$ (which holds by Assumption (B2)), and the definition of $\varpi$ (cf. Lemma 4.1), obtaining

$$\begin{aligned}\hat{v}^\varepsilon(\zeta,\vartheta) = \; & \mathbb{E}\left[\hat{v}^\varepsilon\left(\tau_n^\varepsilon, \Upsilon_{\tau_n^\varepsilon}\right) - \varepsilon^2 \int_t^{\tau_n^\varepsilon} \left(\hat{\mathcal{R}}_{\mathcal{L}}^\varepsilon + \tilde{\mathcal{R}}^\varepsilon\right)(u, \Upsilon_u)\, du\right] \\ & \le \mathbb{E}\left[\hat{v}^\varepsilon\left(\tau_n^\varepsilon, \Upsilon_{\tau_n^\varepsilon}\right)\right] + \varepsilon^3 \beta(\zeta,\vartheta),\end{aligned}$$

where the inequality follows from (B5). In view of (1) and the terminal condition $\hat{u}(T,\cdot) = 0$, dominated convergence in turn yields

$$\hat{v}^\varepsilon(\zeta,\vartheta) \le \mathbb{E}\left[U\left(X_T^{\zeta,\vartheta,\varepsilon}\right) - U'\left(X_T^{\zeta,\vartheta,\varepsilon}\right)\mathfrak{P}(T,\Upsilon_T)\right] + \varepsilon^3\beta(\zeta,\vartheta) \le v^\varepsilon(\zeta,\vartheta) + \varepsilon^3\beta(\zeta,\vartheta), \tag{7.2}$$

as $n \to \infty$. Here, the last inequality follows from admissibility of the wealth process $X^{\zeta,\vartheta,\varepsilon}$ (cf. Assumption (B4)) and the definition of the frictional value function (2.6). By definition of $\bar{u}^\varepsilon$ in (4.1), (7.2) gives

$$\bar{u}^\varepsilon(\zeta,\vartheta) \le (\hat{u} + \varepsilon\beta + \varpi \circ \boldsymbol{\xi}_1)(\zeta,\vartheta). \tag{7.3}$$

Assumption (A2) in turn follows from the continuity of $\hat{u}$, $\beta$, and $\varpi$.

*Step 2: show that Assumption (A3) holds, and $u_* = u^* = u = \hat{u}$.*

Let $\tilde{u} \in C^{1,2}(\mathfrak{D}) \cap \mathcal{C}^\chi$ be a classical solution of (3.20), and let $u_1 \in \mathcal{C}^\chi$ (resp. $u_2 \in \mathcal{C}^\chi$) be a lower-(resp. upper-) semicontinuous viscosity supersolution (resp. subsolution) of (3.20) such that $u_1 \ge \tilde{u} \ge u_2$ on $\partial_T\mathfrak{D}$. We prove that $u_1 \ge \tilde{u}$ on $\mathfrak{D}$; the inequality $\tilde{u} \le u_2$ is obtained similarly.

Assume to the contrary that there is $\hat{\zeta} \in \mathfrak{D}_<$ such that $(u_1 - \tilde{u})(\hat{\zeta}) < 0$. For $\kappa > 0$ small enough, we then have $(u_1 - \tilde{u} + \kappa\chi)(\hat{\zeta}) < 0$. As, moreover, the definition of $\mathcal{C}^\chi$ in (7.1) implies $(u_1 - \tilde{u} + \kappa\chi) > 0$ near the spatial boundary of $\mathfrak{D}$, it follows that there is $\zeta_\kappa \in \mathfrak{D}$ such that

$$\min_{\mathfrak{D}}(u_1 - \tilde{u} + \kappa\chi) = (u_1 - \tilde{u} + \kappa\chi)(\zeta_\kappa) \le (u_1 - \tilde{u} + \kappa\chi)(\hat{\zeta}) < 0.$$

As $u_1 \ge \tilde{u}$ on $\partial_T\mathfrak{D}$, $\zeta_\kappa \in \partial_T\mathfrak{D}$ would imply $\chi(\zeta_\kappa) < 0$, which contradicts $\chi \ge 0$ in (B1). Therefore, $\zeta_k$ is an interior minimum of $u_1 - (\tilde{u} - \kappa\chi)$, and the viscosity supersolution property of $u_1$ gives $-\mathcal{L}^{\theta^0}(\tilde{u} - \kappa\chi)(\zeta_\kappa) \ge a$. Because $\tilde{u}$ is a classical solution of $-\mathcal{L}^{\theta^0}\tilde{u} = a$, it follows that $\mathcal{L}^{\theta^0}\chi \ge 0$, which contradicts (B1). Thus, $u_1 \ge \tilde{u}$ on $\mathfrak{D}$ as claimed.

Applying (7.3) to any subsequence $(\zeta_\varepsilon, \vartheta_\varepsilon)$ and using $\hat{u} \in \mathcal{C}^\chi$ (cf. (B3)) yields $u_*, u^* \in \mathcal{C}^\chi$. As the classical solution $\hat{u}$ is also a viscosity solution of (3.20), Propositions 6.3, 6.4, and the comparison result established above show that $u_* \ge \hat{u} \ge u^*$. As $u^* \ge u_*$ by definition, this shows $\hat{u} = u_* = u^*$. □

The function $u$ defined in (4.5) is locally bounded because $u \in \mathcal{C}^\chi$ and $\chi \in C^{1,2}$. Hence, $u$ is a viscosity solution of (3.20), and it follows as above that $u = \hat{u} = u^* = u_*$.

As a corollary, we obtain our second main result, Theorem 4.7:

COROLLARY 7.3. *Under Assumptions 3.3 and B, the investment strategy $\dot{\theta}^\varepsilon$ defined in (B4) is optimal at the leading order $O(\varepsilon^2)$. That is, for each compact subset $B$ of $\mathfrak{D} \times \mathbb{R}^d$ and $\varepsilon > 0$, there is a constant $K_B^\varepsilon > 0$ such that $K_B^\varepsilon \to 0$ as $\varepsilon \to 0$ and*

$$v^\varepsilon(\zeta,\vartheta) - \varepsilon^2 K_B^\varepsilon \le \mathbb{E}\left[U\left(X_T^{\zeta,\vartheta,\varepsilon}\right) - U'\left(X_T^{\zeta,\vartheta,\varepsilon}\right)\mathfrak{P}\left(T, S_T^\zeta, Y_T^\zeta, X_T^{\zeta,\vartheta,\varepsilon}\right)\right], \quad \text{for all } (\zeta,\vartheta) \in B \text{ and } \varepsilon > 0,$$

*where $(X^{\zeta,\vartheta,\varepsilon}, \theta^{t,\vartheta,\varepsilon})$ is defined as in (B4).*

*Proof.* In the proof of Proposition 7.2, we have shown (7.2):

$$v^0(\zeta) - \varepsilon^2 u(\zeta) - \varepsilon^2 \varpi \circ \boldsymbol{\xi}_1(\zeta, \vartheta) - \varepsilon^3 \beta(\zeta, \vartheta) \le \mathbb{E}\left[U\left(X_T^{\zeta,\vartheta,\varepsilon}\right) - U'\left(X_T^{\zeta,\vartheta,\varepsilon}\right)\mathfrak{P}\left(T, S_T^{\zeta}, Y_T^{\zeta}, X_T^{\zeta,\vartheta,\varepsilon}\right)\right].$$

This corollary thus follows from the local uniform convergence of $\bar{u}^\lambda$ shown in Theorem 4.3. □

## 8. EXAMPLES

In this section we show how all of our technical assumptions can be verified in concrete settings. For the sake of clarity, we do not strive for minimal assumptions. Throughout, we consider an investor with an exponential utility function $-e^{-\eta x}$ with constant absolute risk aversion $\eta > 0$.

### 8.1. Portfolio Choice

First we focus on a portfolio choice problem. There is a single risky asset with dynamics[29]

$$dS_t = \mu_S(Y_t)dt + \sigma_S(Y_t)dW_t^1,$$

driven by a one-dimensional autonomous diffusion:

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)d\left(\rho W_t^1 + \sqrt{1-\rho^2}\,W_t^2\right).$$

Here, $W = (W^1, W^1)$ is a two-dimensional standard Brownian motion, $\rho \in [-1, 1]$, and the mappings $\mu_S, \mu_Y, \sigma_S, \sigma_Y : \mathbb{R} \longmapsto \mathbb{R}$ all are bounded and smooth, with bounded derivatives of all orders, and the volatilities $\sigma_S, \sigma_Y$ are bounded away from zero. Then, $Y$ and in turn $S$ are well defined and it follows similarly as in Zariphopoulou (2001) that the frictionless value function $v^0$ is a classical solution of the frictionless DPE, which can be transformed into a linear, uniformly parabolic equation in this case. The value function $v^0$ can be written as

$$v^0(t, y, x) = e^{-\eta x} w^0(t, y), \tag{8.1}$$

and the corresponding optimal policy is given by

$$\theta_t^0 = \theta^0(t, Y_t) = \frac{\mu_S(Y_t)}{\eta\sigma_S^2(Y_t)} + \frac{\rho\sigma_Y(Y_t)}{\eta\sigma_S(Y_t)}\frac{\partial_y w^0(t, Y_t)}{w^0(t, Y_t)}.$$

Similarly as in Zariphopoulou (2001, theorem 3.1), one verifies that $w^0, \theta^0$ are also bounded and smooth, with bounded derivatives of all orders.[30] In particular, all regularity assumptions imposed on the frictionless problem in Section 7 are satisfied. Moreover, it follows from Novikov's condition and Girsanov's theorem that

[29] This specification allows for predictable returns as in De Lataillade et al. (2012); Martin (2012); Garleanu and Pedersen (2013, 2013); Collin-Dufresne et al. (2012). To ensure enough integrability for a rigorous verification theorem, we truncate large values of the state variable by assuming boundedness of all coefficients. Nonlinear dynamics and stochastic volatility can be handled without difficulties.

[30] For $w^0$, this follows from the corresponding Feynman-Kac representation. As all coefficients are smooth, one can then differentiate the PDE for $w^0$ and argue analogously for all of its derivatives.

$\partial_x v^0(t, Y_t, X_t^{\theta^0})/\partial_x v^0(0, y, x)$ is the density process of an equivalent martingale measure $\mathbb{Q}$, the dual minimizer for the optimization problem at hand.

Now, consider constant linear price impact, $\Lambda_t = \lambda = \varepsilon^4 > 0$. Then, all of our technical assumptions hold and we have the following result:

THEOREM 8.1. *In the setting of Section 8, Assumptions 3.3 and B are satisfied, so that Theorems 4.3 and 7.3 are applicable. As a consequence, a leading-order optimal policy with small constant price impact $\Lambda_t = \lambda = \varepsilon^4$ is given in feedback form as*

$$\dot{\theta}_t^\varepsilon = \sqrt{\frac{\eta\sigma_S^2(Y_t)}{2\varepsilon^4}}\left(\theta_t^0 - \theta_t^\varepsilon\right). \tag{8.2}$$

*The corresponding first-order correction of the value function reads as:*

$$v^\varepsilon(t, y, x, \vartheta) = v^0\Big(t, y, x - \mathrm{CE}(t, y, \vartheta)\Big) + o(\varepsilon^2),$$

*where*

$$\mathrm{CE}(t, y, \vartheta) = \frac{\varepsilon^2}{\sqrt{2\eta}}\left(\mathbb{E}_\mathbb{Q}\left[\int_t^T \left(\partial_y\theta^0(Y_r^{t,y})^2\sigma_Y(Y_r^{t,y})^2\sigma_S(Y_r^{t,y})\right)dr\right] + \sigma_S(y)(\theta^0(0, y) - \vartheta)^2\right).$$

*Proof.* Because no state constraints are needed for exponential utility, (weak) dynamic programming and in turn the viscosity solution property of the frictional value function (Assumption 3.3) can be derived along the lines of Bouchard and Touzi (2011).

Let us now verify Assumption B. First, note that—due to boundedness and smoothness of all coefficient functions—it follows from dominated convergence and Itô's formula that the probabilistic representation (4.5) is a classical solution of the Second Corrector Equation (3.20). In particular, (B2) is satisfied. Next, one readily verifies that (B1) and (B3) also hold with $\chi(t, y, x) = e^{-at}(e^{-y} + e^y + v^0(t, y, x)^2)$, if $a$ is chosen sufficiently large. The feedback policy $\dot{\theta}^\varepsilon$ from (8.2) implies that the corresponding number $\theta^\varepsilon$ of risky shares solves a (random) linear ODE. It is therefore given explicitly by

$$\theta^{t,\vartheta,\varepsilon} = e^{-\int_t^\cdot \sqrt{\eta\sigma_S^2(Y_r)/2\varepsilon^4}dr}\left(\vartheta + \int_t^\cdot \left(e^{\int_t^r \sqrt{\eta\sigma_S^2(Y_s)/2\varepsilon^4}ds}\sqrt{\eta\sigma_S^2(Y_r)/2\varepsilon^4}\theta^0(r, Y_r)\right)dr\right).$$

Hence, $\theta^\varepsilon$ is well defined and uniformly bounded. As a result, the corresponding wealth process (2.5) is well defined, too, and the corresponding utility (2.7) is integrable by Novikov's condition and the boundedness of $\theta^\varepsilon$, $\theta^0$, $\mu_S$, and $\sigma_S$. Moreover, dominated convergence shows that the corresponding wealth process can be approximated by simple strategies as in Biagini and Černý (2011). In summary, (B4) is satisfied.

Now, turn to (B5). By (8.1), (4.5), and Lemma 4.1, we can choose $\gamma^\varepsilon(x) = Ge^{-\eta x}$ for a suitable constant $G > 0$, because the quadratic trading cost, the risky asset's volatility, the investor's absolute risk aversion, the frictionless reduced value function $w^0$, and the quadratic variation of the frictionless trading strategy $\theta^0$ are all uniformly bounded. The frictional wealth process $X_t^{\zeta,\vartheta,\varepsilon}$ is an Itô process with bounded drift and diffusion coefficients. Hence, it follows from Novikov's condition and Doobs maximal inequality that it's running supremum has exponential moments of all orders, verifying Assumption (B5).

(B6) is derived along the same lines by using that $\mathbb{E}[\int_t^T |\theta_r^0 - \theta_r^\varepsilon|^2/\varepsilon^2 dr]$ is uniformly bounded in $\varepsilon > 0$. To see this, first notice that

$$\theta^0 - \theta^\varepsilon = e^{-\varepsilon^{-2}\int_t^\cdot \sqrt{\eta\sigma_S^2(Y_r)/2}dr}(\theta^0(\zeta) - \vartheta) + \int_t^\cdot e^{-\varepsilon^{-2}\int_r^\cdot \sqrt{\eta\sigma_S^2(Y_s)/2}ds}d\theta_r^0,$$

by (8.2) and the explicit formula for solutions of linear SDEs (cf., e.g., Protter 2005, Theorem V.52). Recall that the drift and diffusion coefficients of the frictionless optimizer $\theta^0$ are uniformly bounded by constants $M, \Sigma > 0$, and that $\sqrt{\eta\sigma_S^2(\cdot)/2}$ is uniformly bounded away from zero by some constant $C > 0$. Hence it follows from the algebraic inequality $(x+y)^2 \leq 2x^2 + 2y^2$, Jensen's inequality, the Itô isometry, and a simple integration that

$$\mathbb{E}\left[\int_t^T \frac{|\theta_r^0 - \theta_r^\varepsilon|^2}{\varepsilon^2} dr\right] \leq \frac{|\theta^0(\zeta) - \vartheta|^2}{C} + \frac{2(M^2T^2 + \Sigma^2 T)}{C},$$

establishing the claimed uniform bound in $\varepsilon > 0$. In summary, Assumption B is satisfied and the leading-order optimality of the trading rate (8.2) follows from Theorem 7.3. The representation for the leading-order correction of the corresponding value function is a consequence of Theorem 4.3, Proposition 7.2, as well as Taylor expansion and the definition of $\mathbb{Q}$. □

### 8.2. Random Endowments

Similar arguments can be used to verify the regularity assumptions needed to apply the general argument from Section 5.3 to deal with random endowments. To illustrate this, consider the Bachelier model

$$dS_t = \mu dt + \sigma dW_t,$$

for a standard Brownian motion $W$, and a European option with payoff function $H = h(S_T)$. If the function $h : \mathbb{R}_+ \to \mathbb{R}$ is bounded and smooth, with bounded and smooth derivative of all orders,[31] then it follows from the Markov property that the density process $Z_t^H$ generated by the Radon-Nikodym derivative $d\mathbb{P}^H/d\mathbb{P} = e^{-\eta h(S_T)}/\mathbb{E}[e^{-\eta h(S_T)}]$ is given by a smooth function $f(t, S_t)$ which solves

$$\partial_t f(t,s) + \mu\partial_s f(t.s) + \frac{\sigma^2}{2}\partial_{ss} f(t,s) = 0, \quad f(T,s) = \frac{e^{-\eta h(s)}}{\int_{-\infty}^{\infty} e^{-\eta h(\mu T + \sigma\sqrt{T}s')}\phi(s')ds']},$$

where $\phi$ denotes the density function of the standard Normal distribution. Due to our assumptions on $h$, the function $f$ is smooth, bounded, and bounded away from zero; by the dominated convergence theorem, the same holds for all of its derivatives. As a result, Itô's formula shows that the dynamics of the density process $Z_t^H$ are given by $dZ_t^H/Z_t^H = (\partial_s f(t, S_t)/f(t, S_t))\sigma dW_t$. Girsavov's theorem in turn yields the dynamics of the risky asset $S$ under the measure $\mathbb{P}^H$:

$$dS_t = \left(\mu + \frac{\partial_s f(t, S_t)}{f(t, S_t)}\sigma^2\right) dt + \sigma dW_t^H,$$

for a $\mathbb{P}^H$-Brownian motion $W^H$. Due to the regularity of $f$ and its derivatives, the regularity assumptions of Section 8.1 are satisfied. As a consequence, the portfolio choice problem with the random endowment $H = h(S_T)$ is equivalent to the pure investment

[31] Weakening these regularity assumptions to European call and put options, for example, is an open problem even in simpler models with proportional transaction costs (Bichuch 2014; Possamai and Royer 2014).

problem under the measure $\mathbb{P}^H$, whose solution is provided by Theorem 8.1. Utility-based prices and hedging strategies can in turn be computed using the indifference argument of Hodges and Neuberger (1989).

## REFERENCES


ALFONSI, A., A. FRUTH, and A. SCHIED (2010): Optimal Execution Strategies in Limit Order Books with General Shape Functions, *Quant. Finance* 10(2), 143–157.

ALMGREN, R. F., and N. CHRISS (2001): Optimal Execution of Portfolio Transactions, *J. Risk* 3, 5–40.

ALMGREN, R. F., and T. M. LI (2011): A Fully-Dynamic Closed-Form Solution for $\Delta$-Hedging with Market Impact, Preprint.

ALMGREN, R. F., C. THUM, E. HAUPTMANN, and H. LI (2005): Direct Estimation of Equity Market Impact, *RISK* July, 57–62.

ALTAROVICI, A., J. MUHLE-KARBE, and H. M. SONER (2015): Asymptotics for Fixed Transaction Costs, *Finance Stoch.* 19(2), 363–414.

BERTSIMAS, D., and A. W. LO (1998): Optimal Control of Execution Costs, *J. Finan. Markets* 1(1), 1–50.

BIAGINI, S., and A. ČERNÝ (2011): Admissible Strategies in Semimartingale Portfolio Selection, *SIAM J. Control Optim.* 49(1), 42–72.

BICHUCH, M. (2014): Pricing a Contingent Claim Liability Using Asymptotic Analysis for Optimal Investment in Finite Time with Transaction Costs, *Finance Stoch.* 18(3), 651–694.

BOUCHARD, B., L. MOREAU, and H. M. SONER (2013): Hedging under an Expected Loss Constraint with Small Transaction Costs, Preprint.

BOUCHARD, B., and M. NUTZ (2012): Weak Dynamic Programming for Generalized State Constraints, *SIAM J. Control Optim.* 50(6), 3344–3373.

BOUCHARD, B., and N. TOUZI (2011): Weak Dynamic Programming Principle for Viscosity Solutions, *SIAM J. Control Optim.* 49(3), 948–962.

BREEN, W. J., L. S. HODRICK, and R. A. KORAJCZYK (2002): Predicting Equity Liquidity, *Manage. Sci.* 48(4), 470–483.

COLLIN-DUFRESNE, P., K. DANIEL, C. MOALLEMI, and M. SAGLAM (2012): Strategic Asset Allocation with Predictable Returns and Transaction Costs, Preprint.

CRANDALL, M., H. ISHII, and P. LIONS (1992): User's Guide to Viscosity Solutions of Second Order Partial Differential Equations, *Bull. Amer. Math. Soc. (N.S.)* 27(1), 1–67.

DAVIS, M. H. A. (1997): Option Pricing in Incomplete Markets, in *Mathematics of Derivative Securities*, Cambridge: Cambridge University Press, pp. 216–226.

DAVIS, M. H. A., V. G. PANAS, and T. ZARIPHOPOULOU (1993): European Option Pricing with Transaction Costs, *SIAM J. Control Optim.* 31(2), 470–493.

DeLATAILLADE, J., C. DEREMBLE, M. POTTERS, and J.-P. BOUCHAUD (2012): Optimal Trading with Linear Costs, Preprint.

ENGLE, R., R. FERSTENBERG, and J. RUSSELL (2008): Measuring and Modeling Execution Cost and Risk, Preprint.

EVANS, L. (1992): Periodic Homogenisation of Certain Fully Nonlinear Partial Differential Equations, *Proc. Roy. Soc. Edinburgh A* 120(3-4), 245–265.

FLEMING, W. H., and H. M. SONER (2006): *Controlled Markov Processes and Viscosity Solutions*, 2nd ed., New York: Springer.

FRIEDMAN, A. (1964): *Partial Differential Equations of Parabolic Type*, Englewood Cliffs, NJ: Prentice-Hall Inc.

GARLEANU, N., and L. H. PEDERSEN (2013): Dynamic Portfolio Choice with Frictions, Preprint.

GARLEANU, N., and L. H. PEDERSEN (2013): Dynamic Trading with Predictable Returns and Transaction Costs, *J. Finance* 68(6), 2309–2340.

GATHERAL, J. (2010): No-Dynamic-Arbitrage and Market Impact, *Quant. Finance* 10(7), 749–759.

GOODMAN, J., and D. N. OSTROV (2010): Balancing Small Transaction Costs with Loss of Optimal Allocation in Dynamic Stock Trading Strategies, *SIAM J. Appl. Math.* 70(6), 1977–1998.

GUASONI, P., and M. RÁSONYI (2014): Hedging, Arbitrage, and Optimality with Superlinear Frictions, *Ann. Appl. Probab.*, to appear.

GUASONI, P., and M. WEBER (2014): Rebalancing Multiple Assets with Mutual Price Impact, Preprint.

GUASONI, P., and M. WEBER (2017): Dynamic Trading Volume, *Math. Finance* 27(2), 313–349.

HODGES, S., and A. NEUBERGER (1989): Optimal Replication of Contingent Claims under Transaction Costs, *Rev. Futures Markets* 8, 222–239.

HUBERMAN, G., and W. STANZL (2005): Optimal Liquidity Trading, *Rev. Finance* 9(2), 165–200.

ISHII, H. (1987): A Simple, Direct Proof of Uniqueness for Solutions of the Hamilton-Jacobi Equations of Eikonal Type, *Proc. Amer. Math. Soc.* 100(2), 247–251.

JANEČEK, K., and S. E. SHREVE (2004): Asymptotic Analysis for Optimal Investment and Consumption with Transaction Costs, *Finance Stoch.* 8(2), 181–206.

KALLSEN, J., and S. LI (2013): Portfolio Optimization under Small Transaction Costs: A Convex Duality Approach, Preprint.

KALLSEN, J., and J. MUHLE-KARBE (2013): The General Structure of Optimal Investment and Consumption with Small Transaction Costs, *Math. Finance*, to appear.

KALLSEN, J., and J. MUHLE-KARBE (2015): Option Pricing and Hedging with Small Transaction Costs, *Math. Finance* 25(4), 702–723.

KARATZAS, I., and S. G. KOU (1996): On the Pricing of Contingent Claims under Constraints, *Ann. Appl. Probab.* 6(2), 321–369.

KARATZAS, I., and S. E. SHREVE (1991): *Brownian Motion and Stochastic Calculus*, 2nd ed., New York: Springer.

KORN, R. (1998): Portfolio Optimisation with Strictly Positive Transaction Costs and Impulse Control, *Finance Stoch.* 2(2), 85–114.

KRAMKOV, D., and M. SIRBU (2007): Asymptotic Analysis of Utility-Based Hedging Strategies for Small Number of Contingent Claims, *Stoch. Process. Appl.* 117(11), 1606–1620.

KRUŽKOV, S. N. (1975): Generalized Solutions of Hamilton-Jacobi Equations of Eikonal Type. I. Statement of the Problems; Existence, Uniqueness and Stability Theorems; Certain Properties of the Solutions, *Mat. Sb. (N.S.)* 98(140)(3(11)), 450–493, 496.

KYLE, A. S., and A. A. OBIZHAEVA (2011): Market Microstructure Invariants: Empirical Evidence from Portfolio Transitions, Preprint.

LILLO, F., J. D. FARMER, and R. N. MANTEGNA (2003): Master Curve for Price-Impact Function, *Nature* 421, 129–130.

MARTIN, R. (2012): Optimal Multifactor Trading under Proportional Transaction Costs, Preprint.

OBIZHAEVA, A. A., and J. WANG (2013): Optimal Trading Strategy and Supply/Demand Dynamics, *J. Finan. Markets* 16(1), 1–32.

POSSAMAI, D., and G. ROYER (2014): General Indifference Pricing with Small Transaction Costs, Preprint.

POSSAMAI, D., H. M. SONER, and N. TOUZI (2013): Homogenization and Asymptotics for Small Transaction Costs: The Multidimensional Case, *Commun. Part. Diff. Eq.*, to appear.

PROTTER, P. E. (2005): *Stochastic Integration and Differential Equations*, 2nd ed., Berlin: Springer.

ROCH, A., and H. M. SONER (2013): Resilient Price Impact of Trading and the Cost of Illiquidity, *Int. J. Theor. Appl. Finan.* 16(6), 1350037.

ROGERS, L. C. G. (2004): Why Is the Effect of Proportional Transaction Costs $O(\delta^{2/3})$?, in *Mathematics of Finance*, Providence, RI: American Mathematical Society, pp. 303–308.

SCHACHERMAYER, W. (2001): Optimal Investment in Incomplete Markets When Wealth May Become Negative, *Ann. Appl. Probab.* 11(3), 694–734.

SCHIED, A., and T. SCHÖNEBORN (2009): Risk Aversion and the Dynamics of Optimal Liquidation Strategies in Illiquid Markets, *Finance Stoch.* 13(2), 181–204.

SCHIED, A., T. SCHÖNEBORN, and M. TEHRANCHI (2010): Optimal Basket Liquidation for CARA Investors Is Deterministic, *Appl. Math. Finance* 17(6), 471–489.

SCHÖNEBORN, T. (2011): Adaptive Basket Liquidation, Preprint.

SONER, H. M., and N. TOUZI (2013): Homogenization and Asymptotics for Small Transaction Costs, *SIAM J. Control Optim.* 51(4), 2893–2921.

SONER, H. M., and M. VUKELJA (2014): Expected Utility from Terminal Wealth in an Illiquid Market in Continuous Time, Preprint.

TÓTH, B., Y. LEMPÉRIÈRE, C. DEREMBLE, J. DELATAILLADE, J. KOCKELKOREN, and J.-P. BOUCHAUD (2011): Anomalous Price Impact and the Critical Nature of Liquidity in Financial Markets, *Phys. Rev. X* 1, 021006.

TOUZI, N. (2013): *Optimal Stochastic Control, Stochastic Target Problems, and Backward SDE*, New York: Springer.

WHALLEY, A. E., and P. WILMOTT (1997): An Asymptotic Analysis of an Optimal Hedging Model for Option Pricing with Transaction Costs, *Math. Finance* 7(3), 307–324.

ZARIPHOPOULOU, T. (2001): A Solution Approach to Valuation with Unhedgeable Risks, *Finance Stoch.* 5(1), 61–82.