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A Primer on Portfolio Choice with Small Transaction Costs

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Abstract

This review is an introduction to asymptotic methods for portfolio choice problems with small transaction costs. We outline how to derive the corresponding dynamic programming equations and how to simplify them in the small-cost limit. This allows one to obtain explicit solutions in a wide range of settings, which we illustrate for a model with mean-reverting expected returns and proportional transaction costs. For more complex models, we present a policy iteration scheme that allows one to numerically compute the solution.

1. INTRODUCTION

Starting with the work of Magill & Constantinides (1976), Constantinides (1986), Davis & Norman (1990), Dumas & Luciano (1991), and Shreve & Soner (1994), models with transaction costs have been the subject of intensive research. For example, much effort has been devoted to understanding liquidity premia in asset pricing (Constantinides 1986, Jang et al. 2007, Lynch & Tan 2011, Dai et al. 2016) and how transaction costs shape the trading volume in financial markets (Scheinkman & Xiong 2003; Lo, Mamaysky & Wang 2004; Gerhold et al. 2014). On a more practical level, transaction costs play a crucial role in the design and implementation of trading strategies in the asset management industry (see, e.g., Grinold 2006, Martin & Schöneborn 2011, De Lataillade et al. 2012, Martin 2014).

However, the quantitative analysis of models with trading costs is difficult. Unlike in frictionless models, the position in each asset becomes a state variable because it can no longer be adjusted immediately and for free. As a consequence, explicit solutions are no longer available even in the simplest models with constant market and preference parameters and an infinite planning horizon (Taksar, Klass & Assaf 1988; Davis & Norman 1990; Dumas & Luciano 1991). The difficulty is compounded in more complex models with random and time-varying investment opportunities. However, transaction costs become crucially important in precisely such settings where, for example, prices exhibit momentum or mean reversion (Martin & Schöneborn 2011; Collin-Dufresne et al. 2012; De Lataillade et al. 2012; Garleanu & Pedersen 2013, 2016; Martin 2014), there is switching between different regimes (Jang et al. 2007), or investors are exposed to idiosyncratic endowment shocks (Lo, Mamaysky & Wang 2004; Lynch & Tan 2011). In such settings, investors can no longer "accommodate large transaction costs by drastically reducing the frequency and volume of trade" (Constantinides 1986, p. 859), as in simple models where portfolio rebalancing is the only motive to trade. Instead, striking the right balance between adjustments to optimize performance and the induced implementation costs then becomes a central issue.

To obtain tractable results in complex models with transaction costs, it is often useful to take an asymptotic perspective and view transaction costs as small perturbations of a frictionless benchmark model. The goal is to obtain explicit asymptotic formulas for optimal trading policies and for the associated welfare effects of small transaction costs. Results of this kind were first obtained in simple concrete models that can be solved explicitly in the frictionless case (Dixit 1991; Shreve & Soner 1994; Whalley & Wilmott 1997; Janeček & Shreve 2004; Lo, Mamaysky & Wang 2004; Bichuch 2012; Bichuch & Shreve 2013; Gerhold et al. 2014). In the past few years, much progress has been made in extending these sensitivity analyses to more general settings (Soner & Touzi 2013; Bichuch 2014; Martin 2014; Rosenbaum & Tankov 2014; Ahrens 2015; Altarovici, Muhle-Karbe & Soner 2015; Kallsen & Muhle-Karbe 2015a,b; Possamaï, Soner & Touzi 2015; Bouchard, Moreau & Soner 2016; Cai, Rosenbaum & Tankov 2017a,b; Feodoria 2016; Moreau, Muhle-Karbe & Soner 2017). These results have been obtained using a range of different methods, from analytic studies of the dynamic programming equation (Soner & Touzi 2013; Martin 2014; Altarovici, Muhle-Karbe & Soner 2015; Possamaï, Soner & Touzi 2015; Moreau, Muhle-Karbe & Soner 2017) to convex duality arguments (Ahrens 2015) and weak-convergence techniques (Cai, Rosenbaum & Tankov 2017a,b).

In this survey, we review the homogenization approach put forward by Soner & Touzi (2013) for models with proportional transaction costs. This approach, based on partial differential equations, is very flexible and can be readily adapted to many variations of the model, e.g., different cost structures (Altarovici, Muhle-Karbe & Soner 2015; Altarovici, Reppen & Soner 2016; Moreau, Muhle-Karbe & Soner 2017) or preferences (Bouchard, Moreau & Soner 2016;

Melnyk, Muhle-Karbe & Seifried 2017). Hence, like the classical dynamic approach to frictionless control problems (for an overview, see, e.g., Fleming & Soner 2006), this method has the potential to be a key tool for the analysis of a wide range of complex models.[1] This review is written as a user's guide to the application of this method. We explain in detail both the basic, underlying ideas and each step of their application to a concrete problem. The goal is to provide a blueprint that will allow the reader to apply the approach to a wide range of related problems.

To illustrate the effects of transaction costs on active investment in a concrete example, we also provide a detailed discussion of a model with mean-reverting returns. Starting from the frictionless solution of Kim & Omberg (1996), we show how both the leading-order optimal trading policy and the corresponding performance loss due to the trading friction can be computed explicitly. We show that—as is true for models with constant investment opportunities (Constantinides 1986)—this leads to wide no-trade regions and a severe reduction in trading volume. However, the corresponding welfare losses are greatly amplified—the opportunity cost of not being able to "time the market" is substantial. This underlines the importance of correcting the performance of active investment strategies for trading costs (see Bajgrowicz & Scaillet 2012 and the references therein), for which the tractable asymptotic formulas reviewed here provide a convenient analytical tool.

This review is organized as follows. We introduce our continuous-time model without and with transaction costs in Section 2. We then derive the corresponding frictionless and frictional dynamic programming equations in Section 3. These partial differential equations for the value functions of the problems at hand are the starting point of the subsequent analysis. In Section 4, we outline the homogenization approach and discuss in detail how to apply it to models with proportional transaction costs. This method allows one to reduce the complexity of the problem by reducing the number of state variables, simplifying the underlying state dynamics, and postponing finite time horizons to infinity. In Section 5, we explain how this allows one to obtain explicit solutions in the model of Kim & Omberg (1996), where asset prices exhibit momentum. Section 6 provides references to several extensions of the homogenization results to more general asset dynamics, preferences, and cost structures. Finally, in Section 7, we discuss a scheme that permits the numerical computation of the solution to the simpler homogenized problem using a policy iteration algorithm. So that we can focus on the main ideas and computational issues, mathematical formalism is treated liberally throughout this survey. Rigorous verification theorems for the results presented here can be found in the work of Soner & Touzi (2013); Rosenbaum & Tankov (2014); Ahrens (2015); Altarovici, Muhle-Karbe & Soner (2015); Possamaï, Soner & Touzi (2015); Cai, Rosenbaum & Tankov (2017a,b); Feodoria (2016); Moreau, Muhle-Karbe & Soner (2017); and Melnyk & Seifried (2017).

We close this introduction by explaining some notation used in this article. Write $D_x\varphi(t,x,y,f)$, $D_{xf}\varphi(t,x,y,f)$, etc. for the partial derivatives of a multivariate function $\varphi(t,x,y,f)$. When there is no confusion, we use the more compact notation φ_x, φ_{xf}, etc. As is customary in asymptotic analysis, $\mathcal{O}(\delta)$ denotes any function satisfying $|\mathcal{O}(\delta)| \leq C\delta$ for a constant $C > 0$ and all $\delta \in [0,1]$. For any integrable random variable ξ and a time point $t \geq 0$, $E_t[\xi]$ denotes the expectation of ξ conditional on the information up to time t.

[1] Martingale methods based on the duality theory introduced by Cvitanić & Karatzas (1996) allow one to go beyond the Markovian paradigm (Ahrens 2015; Kallsen & Muhle-Karbe 2015a,b). However, they are more difficult to adapt to new models and are not applicable if the problem at hand is not convex, e.g., because of the presence of fixed trading costs.

2. MODEL

2.1. Financial Market

We consider a financial market with one safe and one risky asset with dynamics modulated by a general factor process. More precisely, the safe asset follows

$$\frac{dB_s}{B_s} = r(F_s)ds,$$

and the risky dynamics are

$$\frac{dS_s}{S_s} = [r(F_s) + \mu_S(F_s)]ds + \sigma_S(F_s)dW_s^S. \qquad 1.$$

Here, $(W_s^S)_{s\in[0,T]}$ is a standard Brownian motion; the safe rate $r(f)$, the expected excess return $\mu(f)$, and the volatility $\sigma(f)$ are sufficiently smooth deterministic functions of the factor process $(F_s)_{s\in[0,T]}$. The factor process follows an autonomous diffusion:

$$dF_s = \mu_F(F_s)ds + \sigma_F(F_s)dW_s^F. \qquad 2.$$

Here, $(W_s^F)_{s\in[0,T]}$ is another standard Brownian motion that has constant correlation $\rho \in [-1, 1]$ with the Brownian motion $(W_s^S)_{s\in[0,T]}$ driving the risky returns. Both $\mu_F(f)$ and $\sigma_F(f)$ are sufficiently regular deterministic functions.

All tractable models from the literature fit into this framework. Examples include:

1. The Black–Scholes model: The standard example for the asset dynamics is the Black–Scholes model, where the safe rate, the expected risky return, and the volatility are all constants: $r(f) \equiv r$, $\mu_S(f) \equiv \mu_S$, and $\sigma_S(f) \equiv \sigma_S$.
2. The Kim–Omberg model: To study the effects of transaction costs in a model where investment opportunities vary randomly over time, we consider the model of Kim & Omberg (1996).[2] This means that the safe rate and volatility remain constant [$r(f) \equiv r, \sigma_S(f) \equiv \sigma_S$], but the expected excess returns follow an Ornstein–Uhlenbeck process[3]: $\mu_S(f) = f$ and
$$dF_s = \kappa(\bar{F} - F_s)ds + \sigma_F dW_s^F, \qquad 3.$$
for constants κ, $\bar{F}$, and σ_F describing the mean-reversion speed, the mean-reversion level, and the volatility of the expected excess return.
3. Heston-type models: In this widely used class of models, the volatility is assumed to be a mean-reverting process. For example, Heston (1993) proposes a constant interest rate r and excess return μ_S, as well as a stochastic volatility $\sigma_S(f) = \sqrt{f}$, where the factor F is a square root process. Liu (2007) instead sets $\mu_S(f) = \alpha f$ for some constant α, retaining the other specifications of Heston. Chacko & Viceira (2005) keep Heston's constant r and μ_S, but their volatility is $\sigma_S(f) = \sqrt{1/f}$.

[2]This is a standard model for the predictability of asset returns, which has been discussed extensively in the empirical literature (Welch & Goyal 2008, Cochrane 2008). The importance of transaction costs in environments that require one to time the market is evident and is discussed by, for example, Martin & Schöneborn (2011); Collin-Dufresne et al. (2012); De Lataillade et al. (2012); Garleanu & Pedersen (2013, 2016); Martin (2014); and Moreau, Muhle-Karbe & Soner (2017).

[3]The framework of Kim & Omberg (1996) also allows other choices of μ^S and σ^S for which the Sharpe ratio μ^S/σ^S remains an Ornstein–Uhlenbeck process; this is because all of these models span the same frictionless payoff spaces. As this invariance breaks down with transaction costs, we focus on the present specification here.

2.2. Trading and Optimization

We now turn to trading and optimization in the above financial market. For concreteness, we focus on a specific portfolio choice problem where consumption takes place only at the terminal time. Extensions to more general settings do not pose any essential difficulties and are discussed in Section 6. We first briefly recall the frictionless case and then turn to models with transaction costs.

2.2.1. Frictionless case. Starting with an initial endowment of x_0 dollars in the safe account and a risky position worth y_0 dollars, an agent can trade the safe and the risky asset continuously on $[0, T]$. Without trading costs, positions can be changed freely over time, so the amount of money Y_t invested in the risky asset is therefore a suitable control variable. The wealth dynamics generated by such a strategy are obtained by simply weighting the safe and risky returns according to the corresponding investments:

$$\begin{aligned} \mathrm{d}Z_s^{t,z,f} &= Y_s\left(\left[r(F_s^{t,f}) + \mu_S(F_s^{t,f})\right]\mathrm{d}s + \sigma_S(F_s^{t,f})\mathrm{d}W_s^S\right) + (Z_s^{Y,x_0+y_0} - Y_s)r(F_s^{t,f})\mathrm{d}s, \\ s &\geq t, \quad F_t^{t,f} = f, \quad Z_t^{t,z,f} = z = x_0 + y_0, \end{aligned} \qquad 4.$$

where the superscripts in our notation refer to the initial conditions $Z_t^{t,z,f} = z$ and $F_t^{t,f} = f$.

Without trading costs, the wealth z is the only state variable we need to keep track of, apart from the factor f. In contrast, the decomposition of z into the risky position y and the safe position $x = z - y$ is irrelevant because it can be changed instantly and without cost by updating the control. If agents choose their trading strategies to maximize expected utility from terminal wealth at time T for some utility function U, we therefore expect the value function to be a deterministic function of the current time t, the current wealth z, and the current value f of the factor only:

$$v(t, z, f) = \sup_{(Y_s)_{s\in[t,T]}} E_t\left[U(Z_T^{t,z,f})\right]. \qquad 5.$$

2.2.2. Proportional transaction costs. Now suppose that trades incur a cost λ proportional to the value traded. The decomposition of the total wealth then matters, as this ratio can no longer be adjusted for free. As a consequence, we need to keep track of the evolution of the safe and risky positions separately. These quantities now both become state variables that can be adjusted only gradually. To wit, agents now choose nondecreasing adapted processes L and M that describe the cumulative transfers from the safe to the risky account and vice versa. The corresponding dynamics of the safe account are[4]

$$\begin{aligned} \mathrm{d}X_s^{t,x,y,f} &= r(F_s^{t,f})X_s^{t,x,y,f}\mathrm{d}s - (1+\lambda)\mathrm{d}L_s + (1-\lambda)\mathrm{d}M_s, \quad s \geq t, \\ X_t^{t,x,y,f} &= x. \end{aligned} \qquad 6.$$

The dynamics of the risky account are

$$\begin{aligned} \mathrm{d}Y_s^{t,x,y,f} &= Y_s^{t,x,y,f}\left(\left[r(F_s^{t,f}) + \mu_S(F_s^{t,f})\right]\mathrm{d}s + \sigma_S(F_s^{t,f})\mathrm{d}W_s^S\right) + \mathrm{d}L_s - \mathrm{d}M_s, \quad s \geq t, \\ Y_t^{t,x,y,f} &= y. \end{aligned} \qquad 7.$$

[4]This means that transaction costs are always deducted from the safe account, for both purchases and sales of the risky asset.

If agents maximize expected utility from terminal paper wealth,[5] the corresponding frictional value function will in turn depend on the current values of the safe and risky account in addition to time and the current value of the factor process:

$$v^{\lambda}(t,x,y,f) = \sup_{(L_s,M_s)_{s\in[t,T]}} E_t\Big[U(X_T^{t,x,y,f} + Y_T^{t,x,y,f})\Big]. \tag{8}$$

2.2.3. Fixed and proportional transaction costs.

In this survey, we focus on the above model with proportional transaction costs in order to most clearly describe the main ideas of the homogenization approach for small transaction costs. However, the methods outlined here can be readily adapted to more general settings (Altarovici, Muhle-Karbe & Soner 2015; Altarovici, Reppen & Soner 2016; Moreau, Muhle-Karbe & Soner 2017). For example, in a model with proportional costs λ_p and additional fixed costs λ_f per trade, the cash dynamics in Equation 6 change to

$$X_s^{t,x,y,f} = x + \int_t^s r(F_\tau^{t,f})X_\tau^{t,x,y,f}\,\mathrm{d}\tau - L_s + M_s - \lambda_p(L_s + M_s) - \lambda_f J_s(L,M), \quad s \geq t,$$

where $J_s(L,M)$ is the total number of jumps of L and M up to time s. Because one pays a fixed, nonzero amount for each jump, $J_s(L,M)$ must be finite for any admissible strategy, unlike for proportional costs. Nevertheless, the value function depends on the same variables as in Equation 8.

3. DYNAMIC PROGRAMMING

The key concept for studying the value functions in Equations 5 and 8 and the corresponding optimal trading strategies is to describe them in terms of a partial differential equation derived from the dynamic programming principle of stochastic optimal control. Loosely speaking, this states that if we have already determined the optimal policy on $[t+\mathrm{d}t, T]$, then fixing this policy and optimizing over the choice at time t leads to the same solution as optimizing over the entire interval $[t,T]$. In discrete time, this means that the optimal policy can be computed by backward induction; in the continuous-time limit, a partial differential equation is obtained.[6]

3.1. Frictionless Case

Let us illustrate this idea by first briefly recalling the frictionless case. The dynamic programming principle suggests that

$$v(t,z,f) = \sup_{Y_t} E_t\Big[v\Big(t+\mathrm{d}t, z+\mathrm{d}Z_t^{t,z,f}, f+\mathrm{d}F_t^{t,f}\Big)\Big].$$

Now apply Itô's formula, insert the state dynamics (Equations 1 and 2), and cancel the stochastic integrals (because they are martingales and therefore have zero expectation if the integrands are sufficiently integrable). Finally, divide by $\mathrm{d}t$ and send $\mathrm{d}t$ to zero. Dropping the arguments to ease notation, this leads to the following equation:

$$0 = v_t + \mu_F v_f + \frac{1}{2}\sigma_F^2 v_{ff} + \sup_{Y_t}\left(\Big[Y_t(r+\mu_S) + (z-Y_t)r\Big]v_z + \frac{1}{2}Y_t^2\sigma_S^2 v_{zz} + Y_t\rho\sigma_S\sigma_F v_{zf}\right). \tag{9}$$

[5]One could also consider expected utility from liquidation wealth $X_T^{t,x,y,f} + (1-\lambda)Y_T^{t,x,y,f}$, but this does not affect the asymptotic results at the leading order.

[6]Here we provide only a heuristic derivation of the dynamic programming equations. For a mathematically rigorous treatment, we refer the reader to the monograph by Fleming & Soner (2006); for more recent results, see Bouchard & Touzi (2011); Bouchard & Nutz (2012); El Karoui & Tan (2013); Altarovici, Muhle-Karbe & Soner (2015); and Altarovici, Reppen & Soner (2016).

This equation incorporates the evolution of the system through the value function. Hence, the problem at time t is reduced to a simple optimization over instantaneous controls. In this particular case, we obtain a pointwise quadratic problem in Y_t whose solution is given by

$$Y_t = \theta(t, z, f) := -\frac{\mu_S(f)}{\sigma_S^2(f)} \frac{v_z(t, z, f)}{v_{zz}(t, z, f)} - \frac{\rho\sigma_F(f)}{\sigma_S(f)} \frac{v_{zf}(t, z, f)}{v_{zz}(t, z, f)}. \qquad 10.$$

Plugging this expression back into the dynamic programming equation 9 yields

$$\mathcal{A}v := v_t + \mu_F v_f + \frac{1}{2}\sigma_F^2 v_{ff} + rzv_z + \mu_S\theta v_z + \frac{1}{2}\sigma_S^2\theta^2 v_{zz} + \theta\sigma_S\sigma_F\rho v_{zf} = 0. \qquad 11.$$

This is the dynamic programming equation for the value function v. As the optimal portfolio in Equation 10 depends on v and its derivatives, this partial differential equation is fully nonlinear. Nevertheless, it can be solved in closed form for standard utility functions of power or exponential form and for a number of particular asset dynamics (Merton 1971, Kim & Omberg 1996, Chacko & Viceira 2005, Liu 2007). This, in turn, leads to explicit expressions for the optimal trading policy in Equation 10. This is illustrated in Section 5 for the model with mean-reverting returns first studied by Kim & Omberg (1996); the closed-form solution for the corresponding dynamic programming equation 11 is provided in Section 8.

3.2. Dynamic Programming with Proportional Transaction Costs

Let us now pass to the value function (Equation 8) with proportional transaction costs. In this case, the value function depends not only on the total wealth z but rather on the full portfolio decomposition (x, y), where x is the cash position and y is the dollar amount invested in the risky asset. The dynamic programming principle then takes the following form:

$$v^\lambda(t, x, y, f) = \sup_{dM_t, dL_t} E_t\left[v^\lambda\left(t + dt,\ x + dX_t^{t,x,y,f},\ y + dY_t^{t,x,y,f},\ f + dF_t^{t,f}\right)\right].$$

As in the frictionless case, apply Itô's formula, insert the state dynamics (Equations 2, 6, and 7), cancel the stochastic integrals (because they are martingales and therefore have zero expectation if the integrands are sufficiently integrable), divide by dt, and send dt to zero. This leads to

$$\begin{aligned}0 &= v_t^\lambda + \mu_F v_f^\lambda + y(r + \mu_S)v_y^\lambda + xrv_x^\lambda + \frac{\sigma_F^2}{2}v_{ff}^\lambda + \frac{\sigma_S^2 y^2}{2}v_{yy}^\lambda + \rho\sigma_S\sigma_F y v_{yf}^\lambda \\ &\quad + \sup_{dM_t, dL_t}\left\{\left[v_y^\lambda - (1+\lambda)v_x^\lambda\right]\frac{dL_t}{dt} + \left[(1-\lambda)v_x^\lambda - v_y^\lambda\right]\frac{dM_t}{dt}\right\}. \end{aligned} \qquad 12.$$

Hence, if the marginal utility of increasing the risky position is neither too high nor too low at a point (t, x, y, f),

$$(1+\lambda)v_x^\lambda(t, x, y, f) > v_y^\lambda(t, x, y, f) > (1-\lambda)v_x^\lambda(t, x, y, f),$$

then it is not optimal to transact at all ($dM_t = dL_t = 0$). We call the set of all such positions (t, x, y, f) the no-trade region. In view of the dynamic programming equation 12, it follows that the following standard linear partial differential equation is satisfied inside it:

$$0 = v_t + \mu_F v_f + y(r + \mu_S)v_y + xrv_x^\lambda + \frac{\sigma_F^2}{2}v_{ff} + \frac{\sigma_S^2 y^2}{2}v_{yy} + \rho\sigma_S\sigma_F y v_{yf}. \qquad 13.$$

Outside the no-trade region, Equation 12 shows that the right side of this equation is less than or equal to zero, as one can always let the portfolio evolve uncontrolled. Moreover, if the marginal utility of increasing the risky position is high enough, $v_y \geq (1+\lambda)v_x$, then the optimal control is to buy

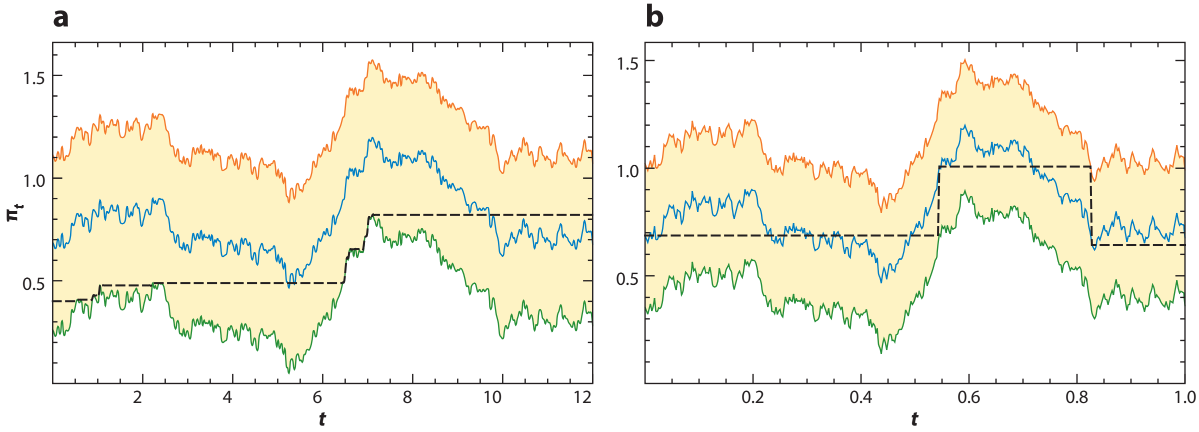


Figure 1

Simulation of the frictionless portfolio weight $\pi_t = Y_t/(X_t + Y_t)$ and the boundaries of the corresponding (asymptotic) no-trade regions in the Kim–Omberg model (*solid lines*). Dashed lines depict the paths of optimal frictional portfolios for (*a*) proportional transaction costs of 1% and (*b*) fixed costs of \$1 for an investor with an initial wealth of \$5,000. The parameters are $\gamma = 3$, $r = 0.0168$, $\sigma_S = 0.151$, $\kappa = 0.271$, $\bar{F} = 0.041$, and $\sigma_F = 0.0343$.

risky shares at an infinite rate ($dL_t/dt = \infty$). Conversely, it is optimal to sell risky shares at an infinite rate ($dM_t/dt = \infty$) whenever this marginal utility is low enough, $v_y \leq (1-\lambda)v_x$. This means that it is optimal to perform the minimal amount of trading that keeps the portfolio within the no-trade region; for an illustration of this, see **Figure 1*a***. Moreover, it follows that in order to satisfy the dynamic programming equation 12 with equality, we must have $v_y = (1+\lambda)v_x$ in the buying region and $v_y = (1-\lambda)v_x$ in the selling region. Together with Equation 13, this leads to the following variational inequality for the value function (Equation 8):

$$0 = \max\Bigg[v_t^\lambda + \mu_F v_f^\lambda + y(r+\mu_S)v_y^\lambda + xrv_x^\lambda + \frac{\sigma_F^2}{2}v_{ff}^\lambda + \frac{\sigma_S^2 y^2}{2}v_{yy}^\lambda + \rho\sigma_S\sigma_F y v_{yf}^\lambda; \quad (1+\lambda)v_x^\lambda - v_y^\lambda;\ v_y^\lambda - (1-\lambda)v_x^\lambda \Bigg]. \qquad 14.$$

Explicit solutions for this equation are not available even in the simplest models. The key difficulty is that the transaction costs increase the number of state variables by one and introduce a free boundary—the no-trade region is unknown and needs to be determined as part of the solution. In the Black–Scholes model, this requires us to solve for a time-dependent smooth curve (Dai & Yi 2009).[7] In the Kim–Omberg model, the tractability issue is further compounded, as the trading boundaries additionally depend on the mean-reverting expected return process. This lack of analytical tractability can be overcome by passing to the small-cost limit, which we discuss in Section 4.

[7]In the most tractable infinite-horizon models, the trading boundaries are constant and can be characterized by a scalar nonlinear equation (Taksar, Klass & Assaf 1988; Dumas & Luciano 1991; Gerhold, Muhle-Karbe & Schachermayer 2013; Gerhold et al. 2014); for a survey of this literature, see Guasoni & Muhle-Karbe (2013).

3.3. Dynamic Programming with Fixed Costs

Before turning to small-cost asymptotics, we briefly sketch how to adapt the above derivations for models with additional fixed costs. As only finitely many trades $\Delta L_t > 0$ or $\Delta M_t > 0$ are possible, we have

$$v^\lambda(t, x, y, f) \geq v^\lambda\Big[t,\ x - \Delta L_t + \Delta M_t - \lambda_p(\Delta L_t + \Delta M_t) - \lambda_f,\ y + \Delta L_t - \Delta M_t,\ f\Big].$$

With

$$H\Big[t, x, y, v^\lambda(t, \cdot, \cdot, f)\Big] := \sup_{\ell, m \geq 0}\Big[v^\lambda(t,\ x - \ell + m - \lambda_p(\ell + m) - \lambda_f,\ y + \ell - m,\ f)\Big],$$

it follows that the frictional value function v^λ satisfies the inequality

$$v^\lambda(t, x, y, f) \geq H\Big[t, x, y, v^\lambda(t, \cdot, \cdot, f)\Big].$$

The states for which $v^\lambda > H$ again form a no-trade region, where the same argument as in Section 3.2 shows that the frictional value function solves the linear partial differential equation 13. Combining this with the nonlinear constraint $v^\lambda = H$, which is binding outside the no-trade region, we obtain the following variational inequality for the frictional value function:

$$0 = \max\Bigg[v_t^\lambda + \mu_F v_f^\lambda + y(r + \mu_S)v_y^\lambda + xrv_x^\lambda + \frac{\sigma_F^2}{2}v_{ff}^\lambda + \frac{\sigma_S^2 y^2}{2}v_{yy}^\lambda + \rho\sigma_S\sigma_F y v_{yf}^\lambda;$$
$$H\Big[t, x, y, v^\lambda(t, \cdot, \cdot, f)\Big] - v^\lambda(t, x, y, f)\Bigg].$$

With only fixed costs ($\lambda_p = 0$), all trades are penalized equally, so the corresponding optimal policy rebalances all the way back to the frictionless target, as shown in **Figure 1*b***. Models with both fixed and proportional costs are intermediate between this regime and that of proportional costs from Section 3.2 in the sense that the optimal policy is to trade to a point in between the boundary of the no-trade region and the frictionless target portfolio (for a detailed description, see Korn 1998; Altarovici, Reppen & Soner 2016).

4. HOMOGENIZATION

We now turn to the asymptotic analysis of models with small transaction costs. To simplify the exposition, we focus on a single risky asset traded with purely proportional costs. A multi-asset model with proportional costs is discussed by Possamaï, Soner & Touzi (2015); fixed and proportional costs are treated by Altarovici, Muhle-Karbe & Soner (2015) and Altarovici, Reppen & Soner (2016); and a study of quadratic trading costs is performed by Moreau, Muhle-Karbe & Soner (2017).

As pointed out above, explicit solutions for portfolio choice problems with transaction costs are not available even in those settings that can be solved in closed form in the frictionless case. To overcome this lack of tractability, it is natural to study small transaction costs as a perturbation of the frictionless benchmark. The goal is for our setting to "reveal the salient features of the problem while remaining a good approximation to the full but more complicated model" (Whalley & Wilmott 1997, p. 308).

The method developed by Soner & Touzi (2013), which we present here, has its roots in the homogenization literature (Kozlov 1979, Papanicolaou & Varadhan 1981). This class of problems contains an ergodic fast variable, and the theory studies the limit problem as this variable oscillates more and more quickly. This leads to a homogenized equation. Interestingly, this homogenized equation is not simply the ergodic average of the original equation. Instead, it is obtained by a nontrivial coupling with a so-called corrector equation (sometimes also called the cell equation). Models with small transaction costs are only loosely in analogy with these problems, as the dependence on the portfolio composition disappears in the limit and does not immediately offer a fast variable. However, Soner & Touzi (2013) observe that, after a suitable rescaling, the deviation of the portfolio from the target position (ξ in Equation 15 below) plays the same role as the fast variable. This observation allows one to employ formal calculations similar to those in homogenization theory to characterize the asymptotic solution. Moreover, the powerful perturbed test function method of Evans (1992, 1989) can be modified to obtain rigorous convergence results (Soner & Touzi 2013; Possamaï, Soner & Touzi 2015; Altarovici, Muhle-Karbe & Soner 2015; Altarovici, Reppen & Soner 2016; Bouchard, Moreau & Soner 2016; Moreau, Muhle-Karbe & Soner 2017).

4.1. Identifying the Correct Scalings

The starting point for the asymptotic analysis is an appropriate ansatz for the value function v^λ with small transaction costs λ. To this end, a key observation of Janeček & Shreve (2004) and Rogers (2004) is that two competing effects need to be balanced. On the one hand, a narrower no-trade region leads to more frequent trading and thus also higher direct transaction costs. On the other hand, a wider no-trade region leads to larger oscillations around the frictionless target portfolio and thus higher indirect losses resulting from displacement from the optimal risk-return trade-off.

With proportional transaction costs, the amount of trading required to remain inside a small no-trade region with width Δ scales with the inverse of Δ (this is a property of reflected Brownian motion and the local time it accumulates at the boundaries). Locally around the frictionless optimum, the first-order condition implies that the value function is quadratic, so the displacement loss should scale with the squared width Δ^2 of the no-trade region. In summary, this suggests that Δ needs to minimize a total loss of the form

$$C_1\Delta^2 + \frac{C_2\lambda}{\Delta}.$$

As a consequence, the optimal no-trade region should be of order $\mathcal{O}(\lambda^{1/3})$, with a corresponding minimal utility loss of order $\mathcal{O}(\lambda^{2/3})$.[8]

4.2. Ansatz for the Asymptotic Frictional Value Function

As discussed in Section 3.2, the frictional value function v^λ depends on time t, the current value f of the factor process, and the current safe and risky positions x and y. As the transaction cost λ tends to zero, both the risky and safe positions converge to their frictionless counterparts. In order to obtain nontrivial limiting quantities for the asymptotic analysis, we therefore reparametrize the model by switching from x and y to the frictionless state variable

$$z = x + y$$

[8]For fixed costs, the argument is similar: A trade is initiated whenever the portfolio reaches the boundary of the no-trade region. At such a point, the portfolio is rebalanced to the frictionless portfolio. The time it takes Brownian motion to reach the boundary of the no-trade region is proportional to Δ^2, so the number of trades per unit of time is proportional to $1/\Delta^2$. The corresponding utility loss should therefore be of the form $C_1\Delta^2 + C_2\lambda/\Delta^2$, which has a minimizer of order $\mathcal{O}(\lambda^{1/4})$ and a minimal value of order $\mathcal{O}(\lambda^{1/2})$; for more details, see Altarovici, Muhle-Karbe & Soner (2015).

and the normalized deviation

$$\xi = \frac{y - \theta(t, z, f)}{\lambda^{1/3}} \qquad 15.$$

of the risky position from its frictionless target (Equation 10).[9] In view of the discussion in Section 4.1, we then expect Equation 15 to converge to a finite limit as $\lambda \to 0$. To avoid fractional powers in the calculations below, set

$$\epsilon = \lambda^{1/3}.$$

With this notation and the above change of variables, the frictional value function can be written as

$$v^{\lambda}(t, x, y, f) =: v^{\epsilon}(t, z, \xi, f).$$

The considerations in Section 4.1 suggest that the leading-order term in the asymptotic expansion of the value function is of order $\mathcal{O}(\epsilon^2)$. Because the impact of a single trade is of higher order $\mathcal{O}(\epsilon^3)$, this term should not depend on the deviation (Equation 15) and should thus be a function $\epsilon^2 u(t, z, f)$ of only the frictionless state variables. However, the deviation of the frictionless optimizer (i.e., the position in the no-trade region) plays a key role in determining the optimal trading policy (i.e., when to start trading). To take this into account, motivated by the homogenization literature, we introduce a second term $\epsilon^4 w(t, z, \xi, f)$ in the asymptotic expansion. It is negligible at the leading order in the value expansion, but via Equation 15 its derivatives play a crucial role in determining the optimal trading policy from the variational inequality 14.

In summary, our ansatz for the asymptotic value function reads as follows:

$$v^{\epsilon}(t, z, \xi, f) = v(t, z, f) - \epsilon^2 u(t, z, f) - \epsilon^4 w(t, z, \xi, f) + \mathcal{O}(\epsilon^3). \qquad 16.$$

The goal now is to determine u and w by plugging this expansion into the dynamic programming equation 14 and matching terms sorted in powers of the asymptotic parameter $\lambda = \epsilon^3$.

4.3. Asymptotic Dynamic Programming and Corrector Equations

In order to recast the variational inequality 14 in terms of the new variables z and ξ instead of x and y, we need to rewrite the corresponding differential operators. For an arbitrary function Ψ of (t, x, y, f) or, equivalently, (t, z, ξ, f), we have

$$D_x \Psi(t, x, y, f) = D_z \Psi(t, z, \xi, f) - \frac{\theta_z(t, z, f)}{\epsilon} D_\xi \Psi(t, z, \xi, f),$$
$$D_y \Psi(t, x, y, f) = D_z \Psi(t, z, \xi, f) + \frac{1 - \theta_z(t, z, f)}{\epsilon} D_\xi \Psi(t, z, \xi, f),$$

and, in turn,

$$D_{yy} \Psi(t, x, y, f) = D_z \left(D_z \Psi(t, z, \xi, f) + \frac{1 - \theta_z(t, z, f)}{\epsilon} D_\xi \Psi(t, z, \xi, f) \right) + \frac{1 - \theta_z(t, z, f)}{\epsilon} D_\xi \left(D_z \Psi(t, z, \xi, f) + \frac{1 - \theta_z(t, z, f)}{\epsilon} D_\xi \Psi(t, z, \xi, f) \right).$$

[9] The definition of this fast variable depends on the scaling appropriate for the problem at hand. For example, for problems with fixed rather than proportional costs, one needs to divide by $\lambda^{1/4}$ (Altarovici, Muhle-Karbe & Soner 2015).

Likewise,

$$D_f\Psi(t,x,y,f) = D_f\Psi(t,z,\xi,f) - \frac{\theta_f}{\epsilon} D_\xi \Psi(t,z,\xi,f),$$

$$\begin{aligned} D_{ff}\Psi(t,x,y,f) &= D_f\left(D_f\Psi(t,z,\xi,f) - \frac{\theta_f(t,z,f)}{\epsilon} D_\xi\Psi(t,z,\xi,f)\right) \\ &\quad - \frac{\theta_f(t,z,f)}{\epsilon} D_\xi\left(D_f\Psi(t,z,\xi,f) - \frac{\theta_f(t,z,f)}{\epsilon} D_\xi\Psi(t,z,\xi,f)\right), \end{aligned}$$

$$\begin{aligned} D_{yf}\Psi(t,x,y,f) &= D_z\left(D_f\Psi(t,z,\xi,f) - \frac{\theta_f(t,z,f)}{\epsilon} D_\xi\Psi(t,z,\xi,f)\right) \\ &\quad + \frac{1-\theta_z(t,z,f)}{\epsilon} D_\xi\left(D_f\Psi(t,z,\xi,f) - \frac{\theta_f(t,z,f)}{\epsilon} D_\xi\Psi(t,z,\xi,f)\right). \end{aligned}$$

Note the terms of order $\mathcal{O}(\epsilon^{-2})$ arising in some of these expressions. These are the reason $\epsilon^4 w(t,z,\xi,f)$ cannot be absorbed in $\mathcal{O}(\epsilon^3)$ in Equation 16 but needs to be treated separately.

With the above expressions, our ansatz (Equation 16) implies

$$\begin{aligned} D_x v^\epsilon &= D_z v - \epsilon^2 D_z u + \epsilon^3 \theta_z w_\xi + \mathcal{O}(\epsilon^4) = D_z v - \epsilon^2 D_z u + \mathcal{O}(\epsilon^3), \\ D_y v^\epsilon &= D_z v - \epsilon^2 D_z u - \epsilon^3 (1-\theta_z) w_\xi + \mathcal{O}(\epsilon^4) = D_z v - \epsilon^2 D_z u + \mathcal{O}(\epsilon^3), \\ D_{yy} v^\epsilon &= D_{zz} v - \epsilon^2 [D_z z u + (1-\theta_z)^2 D_{\xi\xi} w] + \mathcal{O}(\epsilon^3), \\ D_f v^\epsilon &= D_f v - \epsilon^2 D_f u + \mathcal{O}(\epsilon^3), \\ D_{ff} v^\epsilon &= D_{ff} v - \epsilon^2 (D_{ff} u + \theta_f^2 D_{\xi\xi} w) + \mathcal{O}(\epsilon^3), \\ D_{yf} v^\epsilon &= D_{zf} v - \epsilon^2 [u - (1-\theta_z)\theta_f D_{\xi\xi} w] + \mathcal{O}(\epsilon^3). \end{aligned} \qquad 17.$$

We want to use these expressions to expand the frictional dynamic programming equation 14. To this end, recall its frictionless counterpart, Equation 11,

$$\mathcal{A}v := v_t + \mu_F v_f + \frac{1}{2}\sigma_F^2 v_{ff} + rzv_z + \mu_S \theta v_z + \frac{1}{2}\sigma_S^2 \theta^2 v_{zz} + \theta \sigma_S \sigma_F \rho v_{zf} = 0,$$

where the corresponding optimal risky position is

$$\theta = \frac{\mu_S v_z + \rho \sigma_S \sigma_F v_{fz}}{-\sigma_S^2 v_{zz}}. \qquad 18.$$

Together with Equation 17, using $y = \theta + \epsilon\xi = \theta + \mathcal{O}(\epsilon)$ to replace $\epsilon^2 y$ with $\epsilon^2\theta + \mathcal{O}(\epsilon^3)$, the partial differential equation 13 in the no-trade region can now be expanded as follows:

$$\begin{aligned} \mathcal{L}v^\epsilon &:= v_t^\epsilon + \mu_F v_f^\epsilon + y\left[v_y^\epsilon(\mu_S + r) - v_x^\epsilon r\right] + rzv_x^\epsilon + \frac{1}{2}\sigma_S^2 y^2 v_{yy}^\epsilon + \sigma_S y \sigma_F \rho v_{fy}^\epsilon + \frac{1}{2}\sigma_F^2 v_{ff}^\epsilon \\ &= \underbrace{\mathcal{A}v}_{\text{Eq. 11}=0} + \underbrace{(y-\theta)\mu_S v_z + \frac{1}{2}\sigma_S^2 (y^2-\theta^2) v_{zz} + \sigma_S (y-\theta)\sigma_F \rho v_{zf}}_{\mathrm{I}} \\ &\quad - \epsilon^2 \underbrace{\left(u_t + rzu_z + \mu_S y u_z + \mu_F u_f + \frac{1}{2}\sigma_F^2 u_{ff} + \frac{1}{2}\sigma_S^2 y^2 u_{zz} + \sigma_S y \sigma_F \rho u_{zf}\right)}_{\mathrm{II}} \\ &\quad - \epsilon^2 w_{\xi\xi} \frac{1}{2} \underbrace{\left[\sigma_S^2\theta^2(1-\theta_z)^2 - 2\sigma_S\sigma_F\rho\theta(1-\theta_z)\theta_f + \sigma_F^2\theta_f^2\right]}_{\alpha^2} + \mathcal{O}(\epsilon^3). \end{aligned} \qquad 19.$$

By Equation 18,

$$\mathrm{I} = \left(\mu_S v_z + \sigma_S\sigma_F\rho v_{fz}\right)(y-\theta) + \frac{1}{2}\sigma_S^2(y^2-\theta^2) = -\sigma_S^2 v_{zz}\theta(y-\theta) + \frac{1}{2}\sigma_S^2(y^2-\theta^2)$$
$$= \frac{1}{2}\sigma_S^2 v_{zz}(y-\theta)^2 = \frac{1}{2}\sigma_S^2 v_{zz}\epsilon^2\xi^2.$$

Next, note that $y = \theta + \epsilon\xi = \theta + \mathcal{O}(\epsilon)$ implies[10]

$$\mathrm{II} = \mathcal{A}u + \mathcal{O}(\epsilon).$$

In summary, Equation 19 simplifies to the following asymptotic expansion of the dynamic programming equation in the no-trade region:

$$\mathcal{L}v^\epsilon = -\epsilon^2\left(-\frac{1}{2}\sigma_S^2\xi^2 v_{zz} + \mathcal{A}u + \frac{1}{2}\alpha^2 w_{\xi\xi}\right) + \mathcal{O}(\epsilon^3).$$

It remains to derive expansions in the buy and sell regions. To this end, we rewrite the gradient constraint from Equation 14,

$$v_x^\epsilon - (1-\epsilon^3)v_y^\epsilon,$$

using the expressions from Equation 17, obtaining

$$v_x^\epsilon - (1-\epsilon^3)v_y^\epsilon = \epsilon^3 v_y + \underbrace{(v_x - v_y)}_{\epsilon^3 w_\xi + \mathcal{O}(\epsilon^4)} = \epsilon^3(v_z + w_\xi) + \mathcal{O}(\epsilon^4).$$

Analogously, the second gradient constraint in Equation 14 can be expanded as follows:

$$v_y^\epsilon - (1-\epsilon^3)v_x^\epsilon = \epsilon^3(v_z - w_\xi) + \mathcal{O}(\epsilon^4).$$

Altogether, the asymptotic dynamic programming equation is

$$\min\left\{\epsilon^2\left(-\frac{1}{2}\sigma_S^2\xi^2 v_{zz} + \mathcal{A}u + \frac{1}{2}\alpha^2 w_{\xi\xi}\right);\ \epsilon^3(v_z + w_\xi);\ \epsilon^3(v_z - w_\xi)\right\} = 0.$$

Because factoring out ϵ^2 and ϵ^3 does not change this equation, it is equivalent to

$$\min\left\{-\frac{1}{2}\sigma_S^2\xi^2 v_{zz} + \mathcal{A}u + \frac{1}{2}\alpha^2 w_{\xi\xi};\ v_z + w_\xi; v_z - w_\xi\right\} = 0. \qquad 20.$$

The variational inequality 20, with two unknowns, w and u, turns out to effectively consist of two separate equations. To see why, write

$$a(t,z,f) := \mathcal{A}u(t,z,f).$$

Then Equation 20 can be rewritten as an equation for w and a:

$$\min\left\{-\frac{1}{2}\sigma_S^2\xi^2 v_{zz} + a + \frac{1}{2}\alpha^2 w_{\xi\xi};\ v_z + w_\xi;\ v_z - w_\xi\right\} = 0, \qquad 21.$$

where

$$\alpha^2 := \sigma_S^2\theta^2(1-\theta_z)^2 - 2\sigma_S\sigma_F\rho\theta(1-\theta_z)\theta_f + \sigma_F^2\theta_f^2 \qquad 22.$$

[10]Recall that $\mathcal{A}$ is a nonlinear operator in the frictionless dynamic programming equation 11, as the frictionless control θ depends on the solution v of the equation. In contrast, θ is already determined in terms of v here, so $\mathcal{A}$ acts as a linear operator on u.

is determined by the model parameters and the frictionless optimizer. For each value of t, z, and f, Equation 21 has a solution $\xi \mapsto w(t,z,\xi,f)$ for precisely one value of $a = a(t,z,f)$. Thus, by finding the solution (w,a) to this equation, we have obtained a unique function $a(t,z,f)$, from which we can in turn determine u as the solution of a linear partial differential equation:

$$\mathcal{A}u = a.$$

The key to this separation is the uniqueness of the solution (w,a) to Equation 21.[11] Any other choice of function u' would give another value $a' = \mathcal{A}u'$, for which, by uniqueness of (w,a), there would not exist a solution w'.

In summary, the asymptotic expansion 16 of the frictional value function is determined by the following equations:

Definition 4.1 (First corrector equation). For any (t,z,f), the first corrector equation for the pair $[w(t,z,\cdot,f), a(t,z,f)]$ is

$$\min\left\{\underbrace{-\frac{1}{2}\sigma_S^2\xi^2 v_{zz} + a + \frac{1}{2}\alpha^2 w_{\xi\xi}}_{\text{no-trade region}};\ \underbrace{v_z + w_\xi;\ v_z - w_\xi}_{\text{trade regions}}\right\} = 0. \qquad 23.$$

Because any constant can be added to a solution w to obtain another solution, we impose the normalization $w(t,z,0,f) = 0$, which affects neither the value expansion nor the optimal policy at the leading asymptotic order.

Definition 4.2 (Second corrector equation). Given a solution (w,a) of the first corrector equation, the second corrector equation for $u(\cdot,\cdot,\cdot)$ is

$$\mathcal{A}u(t,z,f) = a(t,z,f), \qquad u(T,z,f) = 0,$$

where $\mathcal{A}$ is the generator of the frictionless optimal wealth process, appearing in Equation 11.

The intuition for this separation into two equations is the following. In the first corrector equation, only the deviation ξ of the portfolio from its frictionless target is a variable. In contrast, the frictionless state variables (t,z,f) are treated as constants because they vary much more slowly than ξ for small transaction costs. Conversely, the fast variable ξ is averaged out in the second corrector equation determining the leading order utility loss u, in that it does not enter directly but only through the function $a(t,z,f)$ computed from the first corrector equation.

4.4. Explicit Solutions

In the present setting, the first corrector equation can be solved explicitly. This allows us to understand the comparative statics of the asymptotically optimal no-trade region and the corresponding leading-order welfare effect of small transaction costs. As a byproduct, the calculations below also explain why there is only a single value of a for which the first corrector equation has a solution.

[11]For any solution (w,a), $(w+C,a)$ is also a solution for any constant C, so the uniqueness only concerns a. However, for a given choice of normalization, e.g., $w(z,0) = 0$, w is also uniquely determined. The choice of normalization affects neither the equation for u nor the policy generated from w. A precise formulation of the uniqueness result is presented by Possamaï, Soner & Touzi (2015, theorem 3.1). Similar uniqueness results are proven by Hynd (2012) and Menaldi & Robin (2013).

To find the solution, fix (t, z, f) and make the following ansatz: (*a*) The no-trade region is a symmetric interval $(-\Delta\xi, \Delta\xi)$ around the frictionless optimizer ($\xi = 0$), and (*b*) the asymptotic value function is of the following form[12]:

$$w(\xi) = w(t, z, \xi, f) = \begin{cases} c_4\xi^4 + c_2\xi^2 & \text{if } |\xi| \leq \Delta\xi, \\ w(\Delta\xi) + (\xi - \Delta\xi) & \text{if } \xi \geq \Delta\xi, \\ w(\Delta\xi) - (\xi - \Delta\xi) & \text{if } \xi \leq \Delta\xi. \end{cases}$$

Here, c_4 and c_2 are parameters to be determined along with a and $\Delta\xi$. In the no-trade region, plugging this ansatz into Equation 23 leads to

$$a = \frac{1}{2}\sigma_S^2\xi^2 v_{zz} - \alpha^2(6c_4\xi^2 + c_2) = \left(\frac{1}{2}\sigma_S^2 v_{zz} - 6\alpha^2 c_4\right) - \alpha^2 c_2.$$

Because this needs to hold for any value $\xi \in (-\Delta\xi, \Delta\xi)$, comparison of the coefficients of ξ^2 and 1 yields

$$c_4 = \frac{\sigma_S^2 v_{zz}}{12\alpha^2} \quad \text{and} \quad \mathrm{c}_2 = -\frac{\mathrm{a}}{\alpha^2}. \tag{24.}$$

To pin down a and $\Delta\xi$, we impose the requirement that the value function be not only continuous but also twice continuously differentiable across the trading boundaries $\pm\Delta\xi$ (these are the smooth pasting conditions of Beneš, Shepp & Witsenhausen 1980; Dumas 1991). By symmetry, this leads to the following two additional conditions:

$$0 = 12c_4(\Delta\xi)^2 + 2c_2,$$
$$v_z = 4c_4(\Delta\xi)^3 + 2c_2\Delta\xi.$$

These equations readily yield

$$a = \frac{\sigma_S^2 v_{zz}}{2}\Delta\xi^2, \tag{25.}$$

with

$$\Delta\xi = \Delta\xi(t, z, f) = \left(\frac{-v_z}{v_{zz}}\frac{3\alpha^2}{2\sigma_S^2}\right)^{1/3}. \tag{26.}$$

Together with Equation 24, this leads to a closed-form solution of the first corrector equation 23 in terms of model parameters and inputs from the frictionless optimization problem.

4.5. Asymptotically Optimal Policy

Recalling that $\lambda = \epsilon^3$, we find that the asymptotically optimal no-trade region corresponding to the leading-order variational inequality 20 is

$$\mathrm{NT}_\lambda \approx \left\{(t, x, y, f) : |y - \theta(t, x + y, f)| \leq \lambda^{1/3}\Delta\xi(t, z, f)\right\}. \tag{27.}$$

In view of Equation 26 and the representation for α^2 (Equation 22), this asymptotic no-trade region is determined by (*a*) the diffusion coefficients σ_S, σ_F of the risky asset and the factor process, (*b*) the frictionless optimal portfolio θ and its derivatives θ_z, θ_f, and (*c*) the risk tolerance $-v_z/v_{zz}$ of the frictionless value function. The comparative statics of this formula for general utilities

[12]This is the lowest-order symmetric polynomial in the deviation ξ with enough degrees of freedom to ensure value matching and smooth pasting at the trading boundaries $\pm\Delta\xi$.

are discussed by Kallsen & Muhle-Karbe (2015b). Here, we focus on the case most relevant for applications: power utilities with constant relative risk aversion.

As is well known (see, e.g., Zariphopoulou 2001), power utilities $U(x) = x^{1-\gamma}/(1-\gamma)$ imply that the optimal fraction of wealth invested in a risky asset, $\pi(t, f) := \theta(t, z, f)/z$, is independent of the wealth level. Moreover, the value function inherits the homotheticity $v(t, z, f) = z^{1-\gamma}v(t, 1, f)$, so $-(v_{zz}/v_z)(t, z, f) = \gamma/z$. In view of Equations 26 and 22, the asymptotic no-trade region can therefore be written in terms of risky weights (in contrast to monetary amounts) as

$$\mathrm{NT}_\lambda \approx \left\{(t, x, y, f) : \left|\frac{y}{x+y} - \pi(t, f)\right| \leq \lambda^{1/3}\Delta\pi(t, f)\right\}, \tag{28}$$

where

$$\Delta\pi = \left[\frac{3}{2\gamma}\left(\pi^2(1-\pi)^2 - 2\pi(1-\pi)\pi_f\frac{\sigma_F}{\sigma_S} + \pi_f^2\frac{\sigma_F^2}{\sigma_S^2}\right)\right]^{1/3}. \tag{29}$$

Hence, in this case, the half-width of the asymptotically optimal no-trade region is fully determined by the volatilities σ_S, σ_F, the risk aversion γ, and the frictionless portfolio weight π and sensitivity π_f with respect to the state variable f.

4.6. Welfare Loss

With the explicit solution of the first corrector equation from Section 4.4, we can also say more about the leading-order term $\epsilon^2 u(t, z, f)$ in the expansion of the frictional value function (Equation 16). To this end, recall that $\mathcal{A}u = a$, where a is given by Equation 25 and the differential operator $\mathcal{A}$ is defined in Equation 11. Note that $\mathcal{A}$ is the infinitesimal generator of the frictionless optimal wealth process Z_t. Hence, the Feynman–Kac formula and Equation 25 show that

$$\begin{aligned} u(t, z, f) &= E_t\left[\int_t^T -a(s, Z_s, F_s)\mathrm{d}s\right] \\ &= v_z(t, z, f)E_t^Q\left[\int_t^T -\frac{v_{zz}(s, Z_s, F_s)}{v_z(s, Z_s, F_s)}\frac{\sigma_S(F_s)^2}{2}\Delta\xi(s, Z_s, F_s)^2\mathrm{d}s\right], \end{aligned} \tag{30}$$

where Q is the frictionless investor's marginal pricing measure, whose density process is proportional to the marginal indirect utility v_z by the first-order condition of convex duality (Karatzas & Kou 1996, Davis 1997). (In complete markets, Q is simply the unique equivalent martingale measure. In any case, the corresponding density process is known explicitly if the value function of the problem at hand can be computed in closed form.)

With the representation in Equation 30 for u, Taylor's theorem allows us to rewrite the expansion 16 as

$$v^\lambda(t, z, f) = v\left[t, z - \mathrm{CEL}^\lambda(t, x, f), f\right] + \mathcal{O}(\lambda). \tag{31}$$

Here,

$$\mathrm{CEL}^\lambda(t, z, f) = \lambda^{2/3}E_t^Q\left[\int_t^T -\frac{v_{zz}(s, Z_s, F_s)}{v_z(s, Z_s, F_s)}\frac{\sigma_S(F_s)^2}{2}\Delta\xi(s, Z_s, F_s)^2\mathrm{d}s\right] \tag{32}$$

is the certainty equivalent loss resulting from small transaction costs, i.e., the amount of initial endowment an investor would forgo in order to trade the risky asset without transaction costs. Like the asymptotic no-trade region, this measure for the welfare effect of the trading costs is determined by (*a*) the diffusion coefficients of the risky asset and the factor process, (*b*) the frictionless optimal policy and its sensitivities, and (*c*) the risk tolerance of the frictionless value function. As all of these quantities are generally random and time dependent, they are averaged with respect to both time and states.

4.6.1. Constant relative risk aversion. For power utilities with constant relative risk aversion $\gamma > 0$, Equation 32 can be simplified:

$$\mathrm{CEL}^{\lambda}(t, z, f) = z\lambda^{2/3} E_t^{\tilde{P}}\left[\int_t^T \frac{\gamma}{2}\sigma_S(F_s)^2 \Delta\pi(s, F_s)^2 \mathrm{d}s\right], \tag{33.}$$

where the expectation is computed under the measure $\tilde{P}$ whose density process is proportional to the value function $v(s, Z_s, F_s)$ evaluated along the frictionless optimal wealth process (this measure also plays an important role in the asymptotic analysis of small unhedgeable risks; see Kramkov & Sirbu 2006). By scaling out the current wealth z, this leads to the relative certainty equivalent loss, an appealing scale-invariant measure for the welfare effect of transaction costs that is also used in, for example, the numerical work of Balduzzi & Lynch (1999). Here, its small-cost approximation is obtained by averaging the frictionless optimizer, its sensitivities, and the volatilities of the risky asset and the factor process in a suitable way. These are all functions of the factor process F. To find the dynamics of F under the measure $\tilde{P}$, use Itô's formula to compute the dynamics of the density process:

$$\frac{\mathrm{d}v(s, Z_s, F_s)}{v(s, Z_s, F_s)} = \frac{\mathcal{A}v(s, Z_s, F_s)}{v(s, Z_s, F_s)}\mathrm{d}s + \frac{v_z(s, Z_s, F_s)}{v(s, Z_s, F_s)}\mathrm{d}Z_s + \frac{v_f(s, Z_s, F_s)}{v(s, Z_s, F_s)}\mathrm{d}F_s =: \mathrm{d}L_t. \tag{34.}$$

The $\mathrm{d}s$ term vanishes by the frictionless dynamic programming equation 11. It follows that the density process of $\tilde{P}$ is a stochastic exponential $\mathcal{E}(L)_t = \exp(L_t - \frac{1}{2}\langle L\rangle_t)$, and the $\tilde{P}$ dynamics of F can be readily computed using Girsanov's theorem. If the frictionless value function is known in closed form, the corresponding change of drift is once more fully explicit.

4.6.2. Long investment horizons. The relative certainty equivalent loss (Equation 33) is a function of only time and the state variable. If the planning horizon T is long, the time variable averages out and even more tractable formulas obtain. For large T, the frictionless policy $\pi(t, f)$ typically converges quickly to a steady-state value $\bar{\pi}(f)$ that depends only on the state variable (but not the current time), and the frictionless value function approximately scales as follows (Guasoni & Robertson 2012):

$$v(t, z, f) \approx \frac{z^{1-\gamma}}{1-\gamma} e^{(1-\gamma)(T-t)\mathrm{ESR}}. \tag{35.}$$

Here, the equivalent safe rate ESR is a fictitious interest rate that, in the long run, yields the same growth rate of utility as trading in the original market. With small transaction costs, Equations 31, 33, and 35 show that the corresponding leading-order expression is[13]

$$v^{\lambda}(t, x, y) \approx \frac{z^{1-\gamma}}{1-\gamma} e^{(1-\gamma)(T-t)(\mathrm{ESR}-\Delta\mathrm{ESR}_t^{\lambda})}.$$

Here, the equivalent safe rate loss resulting from small transaction costs is

$$\Delta\mathrm{ESR}_t^{\lambda} = \lambda^{2/3}\frac{\gamma}{2}\frac{1}{(T-t)} E_t^{\tilde{P}}\left[\int_t^T \Delta\pi(s, F_s)^2 \sigma_S(F_s)^2 \mathrm{d}s\right].$$

[13]This requires that the effect of transaction costs be small even when compounded over a long horizon. To make this argument precise, one can directly consider the infinite-horizon problem as in the work of Gerhold et al. (2014), Kallsen & Muhle-Karbe (2015b), and Melnyk & Seifried (2017).

On any finite time horizon T, this quantity is a function of time t and the value f of the state variable. However, as the horizon grows, the ergodic theorem implies that if the state variable F has a stationary distribution $\nu_F^{\tilde{P}}(\mathrm{d}f)$ under the measure $\tilde{P}$, then the equivalent safe rate loss converges to a constant, like the frictionless equivalent safe rate[14]:

$$\Delta \mathrm{ESR}^{\lambda} \approx \lambda^{2/3} \frac{\gamma}{2} \int_{-\infty}^{\infty} \Delta\bar{\pi}(f)^2 \sigma(f)^2 \nu_F^{\tilde{P}}(\mathrm{d}f). \qquad 36.$$

In summary, the welfare effect in infinite-horizon models with small transaction costs can be computed by performing a simple numerical quadrature.

5. EXAMPLES

We now illustrate the asymptotic results from Section 4 in two concrete examples. As a sanity check, we first consider the Black–Scholes model and verify that the formulas derived above indeed coincide with the expressions directly obtained for this simple model (cf. Janeček & Shreve 2004, Bichuch 2012, Soner & Touzi 2013, Gerhold et al. 2014). Afterward, we turn to the Kim–Omberg model and discuss how the results change with stochastic investment opportunities.

5.1. Black–Scholes Model

For an investor with constant relative risk aversion $\gamma > 0$, the value function in the Black–Scholes model is

$$v(t, z) = \frac{z^{1-\gamma}}{1-\gamma} \exp\left[(1-\gamma)\left(r + \frac{\mu^2}{2\gamma\sigma^2}\right)(T-t)\right],$$

and the corresponding risky weight is constant: $\pi_{\mathrm{BS}} = \mu/\gamma\sigma^2$. As a consequence, the half-width (Equation 29) of the asymptotic no-trade region simplifies to

$$\Delta\pi_{\mathrm{BS}} = \left(\frac{3}{2\gamma}\pi_{\mathrm{BS}}^2(1-\pi_{\mathrm{BS}})^2\right)^{1/3}.$$

The corresponding formula for the leading-order equivalent safe rate loss is

$$\Delta \mathrm{ESR}^{\lambda} = \frac{\gamma\sigma^2}{2}\left(\frac{3\lambda}{2\gamma}\pi_{\mathrm{BS}}^2(1-\pi_{\mathrm{BS}})^2\right)^{2/3}.$$

Both of these expressions vanish if zero or full investment is optimal in the frictionless model. If this is the case, the respective frictionless optimal strategies never trade, and no transaction costs need to be paid. In contrast, transaction costs play a more important role if the frictionless target weight is close to 1/2, but even then the quantitative effects are rather small (Constantinides 1986). If a leveraged portfolio is optimal ($\pi_{\mathrm{BS}} > 1$), the welfare loss can be more substantial (Gerhold et al. 2014).

5.2. Kim–Omberg Model

Let us now sketch how the above results change in the Kim–Omberg model, where the expected excess return follows an Ornstein–Uhlenbeck process with dynamics as in Equation 3. For a power

[14]We denote by $\Delta\bar{\pi}(f)$ the half-width of the stationary no-trade region obtained from the long-run portfolio $\bar{\pi}(f)$ via Equation 29.

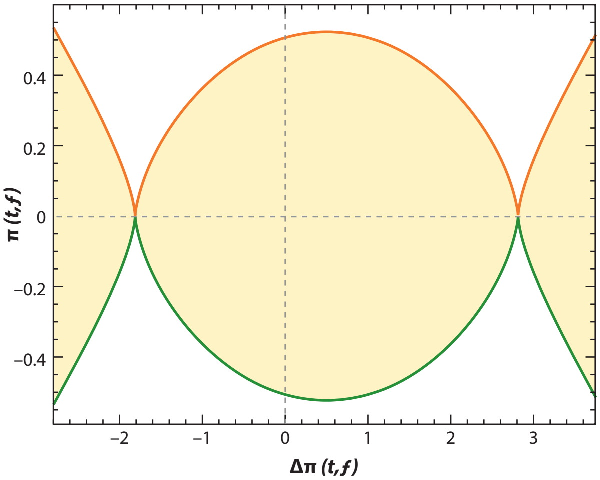


Figure 2

Half-width of the no-trade region in the Kim–Omberg model plotted against the optimal frictionless risky weight. Parameters (measured yearly) are $T = 40$, $\gamma = 3$, $r = 0.0168$, $\sigma_S = 0.151$, $\kappa = 0.271$, $\bar{F} = 0.041$, and $\sigma_F = 0.0343$ as in the study of Barberis (2000).

utility function with constant relative risk aversion $\gamma > 1$, the frictionless value function v has the following closed-form expression (Kim & Omberg 1996):

$$v(s, z, f) = \frac{z^{1-\gamma}}{1-\gamma} \exp\left(A(s) + B(s) f + \frac{1}{2} C(s) f^2 \right), \quad 37.$$

where the functions A, B, and C are explicit solutions of Riccati equations, as recalled in Section 8. The corresponding optimal risky weight is linear in the state variable:

$$\pi_{\mathrm{KO}}(s, F_s) = \frac{F_s}{\gamma \sigma_S^2} + \frac{\rho \sigma_F}{\gamma \sigma_S} [B(s) + C(s) F_s].$$

In view of Equation 29, this formula immediately yields a closed-form expression for the half-width $\Delta\pi_{\mathrm{KO}}$ of the asymptotic no-trade region with transaction costs. **Figure 1** shows a simulated sample path of the frictionless portfolio π_{KO} and the boundaries of the no-trade region $\pi_{\mathrm{KO}} \pm \Delta\pi_{\mathrm{KO}}$.

Because the frictionless risky weight is sensitive to changes in the state variable, the no-trade region no longer vanishes if the frictionless risky weight is zero or one, but instead does so at two other levels determined by the model parameters. For the parameters estimated from a long equity time series by Barberis (2000), this is illustrated in **Figure 2**, where the half-width of the no-trade region is plotted against the optimal frictionless risky weight for different values of the expected excess return.

This is complemented by **Figure 3**, which shows how the optimal frictionless portfolio and the corresponding no-trade region converge to their stationary long-run limits as the planning horizon grows. This stationary policy $\bar{\pi}_{\mathrm{KO}}$ corresponds to the stationary points $\bar{B}$ and $\bar{C}$ of the Riccati equations for $B(s)$ and $C(s)$. The corresponding no-trade region is in turn derived from Equation 29.

The long-horizon convergence of the functions $B(s)$ and $C(s)$ can also be used to simplify the computation of the leading-order relative certainty equivalent loss (Equation 36). In particular, Girsanov's theorem and the long-run convergence show that the long-term dynamics of the factor

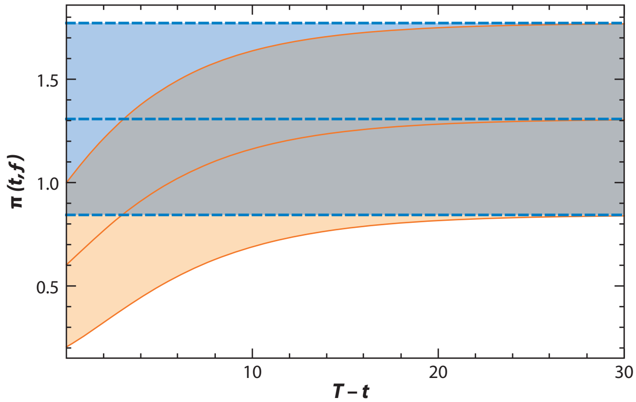


Figure 3

The frictionless position and no-trade region (*solid lines*) as a function of time (measured in years) plotted alongside infinite-horizon values (*dashed lines*). Parameters (measured yearly) are $\gamma = 3$, $r = 0.0168$, $\sigma_S = 0.151$, $\kappa = 0.271$, $\bar{F} = 0.041$, and $\sigma_F = 0.0343$ as in the study of Barberis (2000).

process F under the measure $\tilde{P}$ with density process as in Equation 34 are

$$\begin{aligned} \mathrm{d}F_s &= \kappa(\bar{F} - F_s)\mathrm{d}s + \sigma_F \mathrm{d}W^F \\ &= \left\{\kappa(\bar{F} - F_s) + \sigma_F(1-\gamma)\pi_{\mathrm{KO}}(s, F_s)\sigma_S\rho + [B(s) + C(s)F_s]\sigma_F^2\right\}\mathrm{d}s + \sigma_F \mathrm{d}\tilde{W}^F \\ &\approx \left[\kappa(\bar{F} - F_s) + \sigma_F(1-\gamma)\bar{\pi}_{\mathrm{KO}}(F_s)\sigma_S\rho + (\bar{B} + \bar{C}F_s)\sigma_F^2\right]\mathrm{d}s + \sigma_F \mathrm{d}\tilde{W}^F \\ &=: \tilde{\kappa}(\tilde{F} - F_s)\mathrm{d}s + \sigma_F \mathrm{d}\tilde{W}^F, \end{aligned}$$

for a $\tilde{P}$-Brownian motion $\tilde{W}^F$ and suitably chosen constants $\tilde{\kappa}$ and $\tilde{F}$. Hence, for a long planning horizon, the factor process still has Ornstein–Uhlenbeck dynamics under the auxiliary measure $\tilde{P}$, and its stationary law is $\nu_F^{\tilde{P}} \sim \mathcal{N}(\tilde{F}, \sigma_F^2/2\tilde{\kappa})$. This allows computation of the relative certainty equivalent loss (Equation 36) resulting from small transaction costs as

$$\Delta \mathrm{ESR}_{\mathrm{KO}}^{\lambda} \approx \lambda^{2/3} \frac{\gamma}{\sqrt{2(\sigma_F^2/\tilde{\kappa})^2\pi}} \int_{-\infty}^{\infty} \left[\Delta\bar{\pi}_{\mathrm{KO}}(f)^2 \sigma(f)^2 \exp\left(-\frac{(f-\tilde{F})^2}{2(\sigma_F^2/2\tilde{\kappa})^2}\right)\right] \mathrm{d}f. \qquad 38.$$

Because the integrand is known in closed form, Equation 38 is easily evaluated by numerical quadrature. This is illustrated in **Figure 4**, which plots the relative certainty equivalence loss as a function of the proportional transaction cost. Compared to a Black–Scholes model with the same expected excess return, we observe that the welfare effect of transaction costs is indeed increased substantially by having to react to time-varying investment opportunities.

6. EXTENSIONS

So far, we have focused on the application of the homogenization approach to a portfolio choice problem with proportional transaction costs for purchases and sales of a single risky asset with Markovian dynamics. Performance has been measured in terms of expected utility from terminal wealth only, i.e., intertemporal consumption has been absent. This choice was made for concreteness and ease of exposition. In this section, we survey results from the recent literature that show that the findings outlined in Section 4 remain true much more generally.

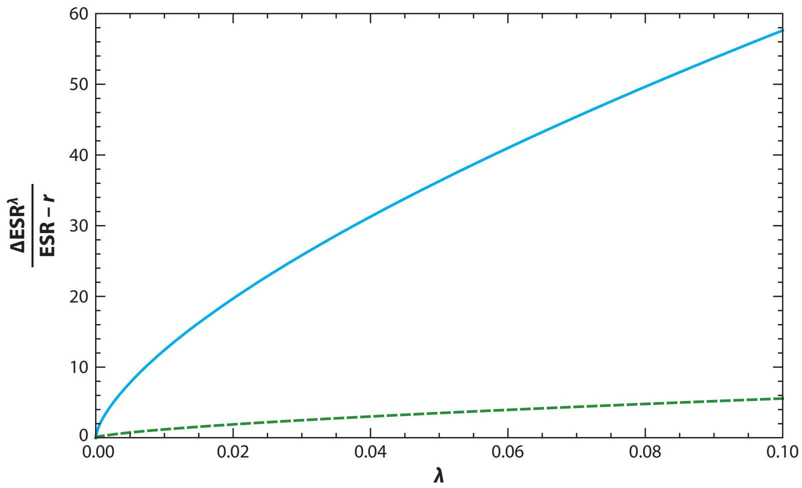


Figure 4

Relative loss in equivalent safe rate resulting from transaction costs for the Kim–Omberg model (*solid line*) and a Black–Scholes model (*dotted line*) with $\mu_S \equiv \bar{F}$, plotted against the size of the cost. Parameters (measured yearly) are $\gamma = 3$, $r = 0.0168$, $\sigma_S = 0.151$, $\kappa = 0.271$, $\bar{F} = 0.041$, and $\sigma_F = 0.0343$ as in Barberis (2000).

6.1. More General Preferences

The results of the previous sections can be readily extended to models with intermediate consumption. For example, Soner & Touzi (2013) study an infinite-horizon model with preferences of the form

$$E\left[\int_0^\infty e^{-\delta t} U(c_t)\mathrm{d}t\right] \to \max!,$$

where $\delta > 0$ is the investor's discount rate and $U(c_t)$ measures the utility from consumption rate c_t at time t. The asymptotic no-trade region for this optimization criterion turns out to be of the same form as in Equation 27—intermediate consumption is only reflected through the frictionless optimal policy. This remains true for more general additive preferences of the form

$$E\left[\int_0^T U_1(t, c_t)\mathrm{d}t + U_2(Z_T)\right] \to \max!, \qquad 39.$$

where $U_1(t, c_t)$ is the utility from intermediate consumption at time $t \in [0, T]$ and $U_2(Z_T)$ is the utility from terminal wealth Z_T at time T (for details, see Ahrens 2015, Kallsen & Muhle-Karbe 2015b). Despite this robustness result, substantial intertemporal consumption can have a nonnegligible effect if the trading costs are not small enough. The intuition is that because costs are paid from a savings account, investors are willing to accept smaller risky positions before rebalancing (Davis & Norman 1990). In infinite-horizon Black–Scholes models, this manifests itself through a downward shift of the no-trade region at the second asymptotic order $O(\lambda^{2/3})$ (Janeček & Shreve 2004; Lo, Mamaysky & Wang 2004; Gerhold, Muhle-Karbe & Schachermayer 2012). For more general models, such results are not available.

The effects of the transaction costs on optimal consumption policy are of the simplest conceivable form: It is asymptotically optimal to simply adjust the frictionless rule for the (typically lower) wealth with transaction costs (Kallsen & Muhle-Karbe 2015b). For power utility, this implies that the consumption/wealth ratio is unaffected by the trading costs (Kallsen & Muhle-Karbe 2015b, section 4).

Recently it was shown (Melnyk, Muhle-Karbe & Seifried 2017) that even the additive structure in the preferences in Equation 39 is not crucial, in that the same leading-order results also remain true for recursive preferences as in the work of Duffie & Epstein (1992) and for models with habit formation as in the work of Hindy & Huang (1993). At the leading asymptotic order, the optimal trading strategy is again completely characterized by the local curvature of agents' preferences, measured by the risk tolerance of their indirect utilities, and the consumption/wealth ratio remains unchanged. The fine structure of the preferences at hand enters only at the next-to-leading order.

6.2. More General Asset Dynamics

In previous sections, we have assumed that the joint dynamics of the asset prices (described in Equations 1 and 2) and the factor process are Markovian; i.e., all drift and diffusion coefficients are deterministic functions of the current state of the system. This assumption allows us to apply partial differential equation techniques, but is not crucial for the validity of Equations 27 and 31. Indeed, these formulas are derived by freezing the frictionless state variables for the analysis of the first corrector equation—a procedure that readily generalizes to general, not necessarily Markovian systems where drift and diffusion coefficients can be arbitrary functionals of the information flow. Even in such more general models, Equation 27 for the asymptotically optimal no-trade region and Equation 31 for the leading-order effect of small transaction costs remain valid (for details, see Kallsen & Muhle-Karbe 2015a,b; Cai, Rosenbaum & Tankov 2017a,b).

Extending the approach presented here to models with jumps leads to the analysis of nonlocal integro-differential equations. Although this appears to be a daunting task, it is shown by Rosenbaum & Tankov (2014), using probabilistic techniques, that asymptotic results similar to Equations 27 and 31 can still be obtained.

6.3. More General Transaction Costs

Although we have focused here on proportional transaction costs, this structure, too, is not crucial. At least on a formal level, fixed costs (Altarovici, Muhle-Karbe & Soner 2015), fixed and proportional costs (Altarovici, Reppen & Soner 2016), and quadratic costs (Moreau, Muhle-Karbe & Soner 2017) can be treated similarly.

In each case, the fine structure of the optimal strategy crucially depends on the transaction cost considered. With proportional costs, one performs the minimal amount of trading to remain in a no-trade interval around the frictionless target. With fixed costs, it is no longer possible to implement a strategy involving infinitely many small trades; hence, one directly trades back to a target portfolio once the boundary of the no-trade region is reached. Conversely, quadratic costs lead to smaller penalties for very small trades but make large turnover rates prohibitively expensive. Thus, optimal strategies always trade toward the target at some finite, absolutely continuous rate.

Despite these apparent differences, the coarse structures of all these models are very similar: In each case, the distance from the target is a trade-off against the specific trading cost, balanced by an appropriate control. The corresponding expected displacement and average transaction costs display the same comparative statics in each case, up to a change of asymptotic convergence rates and constants. In particular, the implications of transaction costs for welfare and average trading volume are very similar in each case (for further details, see Moreau, Muhle-Karbe & Soner 2017).

6.4. Multiple Risky Assets

The extensions sketched so far eventually lead to explicit asymptotic formulas of complexity similar to that of the benchmark model discussed in Section 4. In contrast, less is known about the case of several risky assets. In this case, the homogenization approach still reduces the dimensionality of the problem (Possamaï, Soner & Touzi 2015), but the resulting corrector equations no longer admit an explicit solution. As a consequence, numerical methods such as the policy iteration scheme in Section 7 are needed even for the asymptotic analysis. Models with quadratic costs (Garleanu & Pedersen 2013, 2016; Guasoni & Weber 2016; Moreau, Muhle-Karbe & Soner 2017) and fixed costs (Atkinson & Wilmott 1995; Altarovici, Muhle-Karbe & Soner 2015) can still be solved in closed form in the multidimensional case, but the tractability issue is exacerbated for general nonlinear costs.

7. NUMERICAL SOLUTIONS IN MULTIPLE DIMENSIONS

The corrector equations obtained by passing to the small-cost limit are considerably simpler than the original dynamic programming equation. Even in situations with multiple risky assets (Possamaï, Soner & Touzi 2015) or nonlinear costs (Altarovici, Reppen & Soner 2016), where explicit solutions are not available, this considerably simplifies the numerical analysis. In this section, we present a variant of the classical policy iteration algorithm that is tailored to the control problem at hand and that works very well in practice (alternative methods based on partial differential equation techniques can be found in the work of Muthuraman & Kumar 2006, Dai & Zhong 2010). An attractive feature of this method is that both the value function and the corresponding no-trade region are approximated simultaneously.

The main idea of policy iteration is to start with a guess for the optimal control and to compute the corresponding payoff by solving a linear equation. The result of this computation is then used to generate an improved policy by, at each point, determining the best response to this payoff. These two steps are iterated until a fixed point is found (for an overview, see, e.g., Puterman 1994).

To apply this technique in the presence of proportional transaction costs, the first step is to recognize that the first corrector equation can be interpreted as the dynamic programming equation of an ergodic control problem. Instead of using a direct policy iteration scheme as in the work of Chancelier, Messaoud & Sulem (2007), we approximate the singular controls by smooth trading rates, capped at some finite level (compare Davis & Norman 1990, section 3; Witte & Reisinger 2012), to achieve better stability (see, e.g., Azimzadeh & Forsyth 2016). The resulting control problem can then be discretized and solved by a classical scheme. Approximation by capping is a special case of the approach presented by Altarovici, Reppen & Soner (2016) for problems with proportional and fixed costs, and it can readily be generalized to other settings.

Consider a d-dimensional model with a frictionless value function v and optimal risky positions θ. To simplify the notation, all assets are traded with the same proportional transaction cost. Arguments analogous to those in Section 4 then show that the d-dimensional version of the first corrector equation 23 is

$$\min_{i=1,\dots,d} \min\left\{ -\frac{1}{2}|\sigma_S^\top \xi|^2 v_{zz} + a + \frac{1}{2}\mathrm{Tr}\left[\alpha\alpha^\top w_{\xi\xi}\right];\ v_z + w_\xi \cdot e_i;\ v_z - w_\xi \cdot e_i \right\} = 0, \qquad 40.$$

where α is a model-dependent matrix analogous to Equation 22 that depends on θ and its derivatives.

The corrector equation 40 can be interpreted as the dynamic programming equation of an infinite-horizon control problem. To wit, let L^i, M^i, and $i = 1, \dots, d$ be nondecreasing controls for

$$\Xi_s^i = \xi^i + \sum_{j=1}^{d} \alpha^{i,j}(t,z,f) W_s^j + L_s^i - M_s^i.$$

Then Equation 40 is the dynamic programming equation for the ergodic control problem

$$a(t,z,f) := \inf_{L,M} J(t,z,f;L,M), \tag{41.}$$

corresponding to the following infinite-horizon goal functional:

$$J(t,z,f;L,M) := \limsup_{s\to\infty} \frac{1}{s} E\left[\frac{1}{2}\int_0^s -v_{zz}(t,z,f)\left|\sigma_S^\top \Xi_\tau\right|^2 \mathrm{d}\tau + v_z(t,z,f)\sum_{i=1}^{d}(L_s^i + M_s^i)\right].$$

This means that the controls are chosen so as to minimize the long-run average deviations of the controlled process Ξ from zero, subject to proportional adjustment costs. Note that in this problem, the state variables (t,z,f) of the original problem are frozen, so the uncontrolled Ξ is a Brownian motion.

To solve this problem numerically, we approximate the controls L, M by absolutely continuous trading rates of the form

$$L_t^s = \int_0^s \ell^i(\Xi_\tau)\mathrm{d}\tau \quad \text{and} \quad M_s^i = \int_0^s m^i(\Xi_\tau)\mathrm{d}\tau,$$

for functions ℓ and m bounded by a finite constant K. (As the artificial constraint K tends to infinity, we then expect to approach the solution of the original problem.) With this restriction, the dynamics of the controlled process are

$$\mathrm{d}\Xi_s^i = \nu^i(\Xi;\ell,m)\mathrm{d}s + \alpha\mathrm{d}W_s,$$

where $\nu^i(\xi;\ell,m) = (\ell^i - m^i)(\xi)$. The corresponding dynamic programming equation for the restricted version of Equation 41 is

$$\min_{\substack{i=1,\dots,d\\ \ell^i,m^i\in[0,K]}} \left[\mathcal{L}^{\ell,m}w(\xi) + f(\ell,m,\xi)\right] = -a, \quad \forall\xi\in\mathbb{R}^d,$$

where

$$\mathcal{L}^{\ell,m}w(\xi) = \nu(\xi;\ell,m)^\top \frac{\partial w}{\partial\xi}(\xi) + \frac{1}{2}\mathrm{Tr}\left[\alpha\alpha^\top \frac{\partial^2 w}{\partial(\xi^i)\partial(\xi^j)}(\xi)\right] \tag{42.}$$

and

$$f(\ell,m,\xi) = -\frac{1}{2}\left|\sigma_S^\top\xi\right|^2 v_{zz} + v_z\sum_{i=1}^{d}(\ell^i + m^i).$$

Now truncate the state space for the control problem to a large finite domain in $\mathbb{R}^d$ and consider a discretization $\mathcal{D}\subset\mathbb{R}^d$ of this set. The approximation $\mathcal{L}_{\mathcal{D}}^{\ell,m} : \mathcal{D}\mapsto\mathbb{R}$ of the operator $\mathcal{L}^{\ell,m}$ in Equation 42 outlined in Section 9 can in turn be interpreted as the transition rate matrix of a discrete control problem with dynamic programming equation

$$\min_{\substack{i=1,\dots,d\\ \ell^i,m^i\in[0,K]}} \left(\sum_{\xi'\in\mathcal{D}} \mathcal{L}_{\mathcal{D}}^{\ell,m}(\xi,\xi')w(\xi') + f(\ell,m,\xi)\right) = -a, \quad \forall\xi\in\mathcal{D}.$$

If the truncated domain is sufficiently large, the probability of Ξ reaching its boundary is small, so the corresponding boundary conditions can be chosen arbitrarily as long as the discretized differential operator can be interpreted as the transition rate matrix of some discrete control problem. The key advantage of this scheme is that the bound K ensures that the transition probabilities are bounded away from 0, enabling us to represent the problem as a continuous-time Markov decision process for which standard policy iteration techniques apply. More specifically, this discrete problem can be solved using the following policy iteration algorithm by choosing an initial policy (ℓ_0, m_0), e.g., $\ell_0, m_0 \equiv 0$, and then iterating the following steps:

1. Compute $(w_j, a_j) \in \mathbb{R}^{|\mathcal{D}|} \times \mathbb{R}^+$ as the solution of

$$\left(\sum_{\xi' \in \mathcal{D}} \mathcal{L}_{\mathcal{D}}^{\ell_j, m_j}(\xi, \xi') w_j(\xi') + f(\ell_j, m_j, \xi) \right) = -a, \quad \forall \xi \in \mathcal{D}.$$

   Note that these are $|\mathcal{D}|$ equations for $|\mathcal{D}| + 1$ unknowns. The missing equation is obtained by normalizing w as in Section 4.
2. Find solutions ℓ_{j+1} and m_{j+1} to the $|\mathcal{D}|$ minimization problems

$$\ell_{j+1}(\xi), m_{j+1}(\xi) \in \underset{\substack{i=1,\dots,d \\ \ell^i, m^i \in [0,K]}}{\arg\min} \left(\sum_{\xi' \in \mathcal{D}} \mathcal{L}_{\mathcal{D}}^{\ell, m}(\xi, \xi') w_j(\xi') + f(\ell, m, \xi) \right)$$

   and return to the previous step.

The iteration is terminated when the difference between a_j and a_{j-1} is small enough. It is known that this difference converges to 0 in finite time (see, e.g., Puterman 1994). Although this bound is very large for a general policy iteration scheme, it has been observed that policy iterations typically converge very quickly, often in fewer than 20 iterations (Santos & Rust 2004). The fast convergence is attained thanks to the scheme's close connections to Newton's method. (For more details on these connections, as well as the convergence rate, see Puterman & Brumelle 1979; Santos & Rust 2004; Bokanowski, Maroso & Zidani 2009.)

Solving the $|\mathcal{D}|$ optimization problems in the second step of each iteration may seem daunting at first glance. However, when trading is only conducted through the safe account but not directly between risky assets, the solution of this problem is in fact explicit. Also, in more general settings, the optimization problems are entirely independent of each other, so their solution can be fully parallelized.

For simplicity, the chosen model is a Black–Scholes market consisting of two risky assets, where an agent optimizes the power utility of consumption over an infinite horizon (with impatience parameter δ), as in Section 6.1 or in the work of Possamaï, Soner & Touzi (2015). This choice appears in the above problem only through v_{zz}, v_z, and α, and in this case

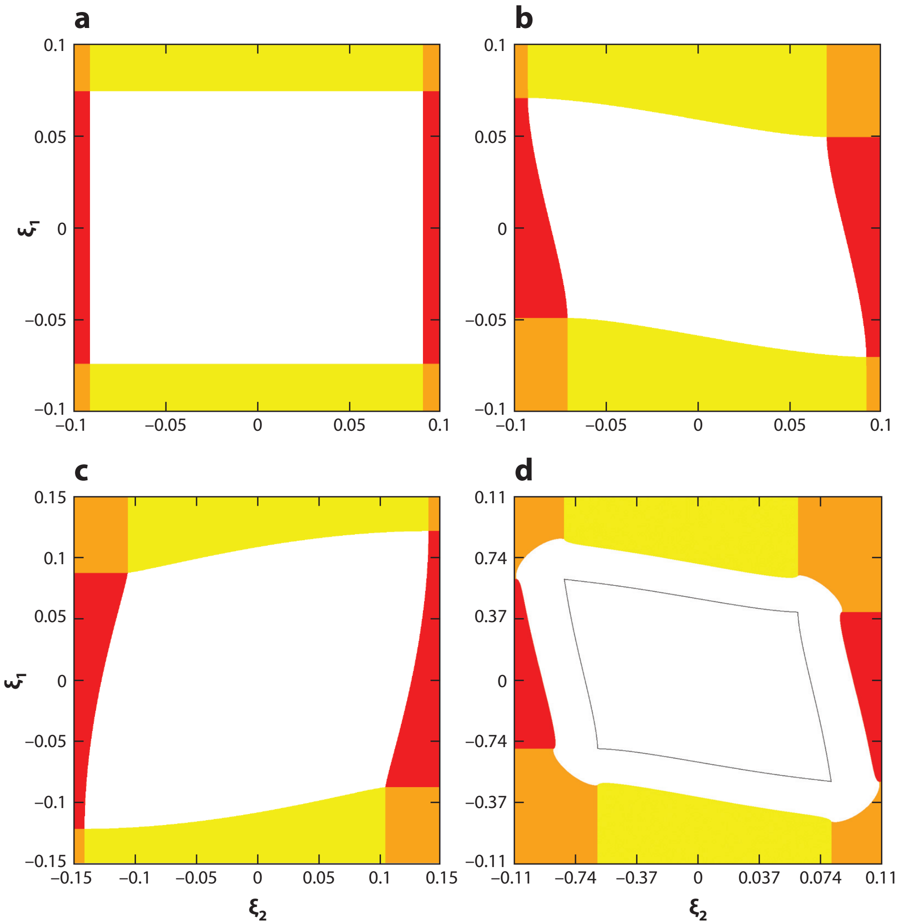
$$\alpha = (I_d - \theta_z^\top 1_d) \operatorname{diag}[\theta] \sigma_S,$$

where I_d is the d-dimensional identity matrix, $\operatorname{diag}[\theta]$ is the matrix with diagonal θ and other elements zero, and $1_d = (1, \dots, 1)^\top \in \mathbb{R}^d$. Optimal strategies computed using this algorithm are depicted in **Figure 5**.

The policy iteration scheme presented here can readily be generalized to more complex models. For example, details and justifications for such generalizations can be found in the work of Altarovici, Reppen & Soner (2016) for a model with proportional and fixed costs, as in Section 3.3. The output of the algorithm in this setting is illustrated in **Figure 5*d***. where the rebalancing target, to which the portfolio is readjusted once the boundaries of the no-trade region are breached, is represented visually.

8. APPENDIX: KIM–OMBERG VALUE FUNCTION

Consider a power utility function with constant relative risk aversion γ. As shown by Kim & Omberg (1996), the value function for the Kim–Omberg model (Equations 1 and 3) is then of the



Panel	δ	γ	r	μ_1	μ_2	σ_1	σ_2	ρ	λ_f	λ_p	z
a	1	2	0.03	0.08	0.08	0.4	0.33	0.00	0	3%	$50,000
b	1	2	0.03	0.08	0.08	0.4	0.33	0.30	0	3%	$50,000
c	1	2	0.03	0.08	0.08	0.4	0.33	−0.30	0	3%	$50,000
d	1	2	0.03	0.08	0.08	0.4	0.33	0.30	$1	3%	$50,000

Figure 5

Asymptotic no-trade regions (*white*) and regions in which there is trading in the assets (*yellow*, *red*, and *orange*) are shown for different parameters (μ_i and σ_i are the expected excess return and volatility of risky assets $i = 1, 2$; ρ is the correlation between their driving Brownian motions). Each axis represents the deviation of a risky weight from its frictionless target. The interpretation of the transaction cost is as in Equation 28. Except in the corners (*orange*), trading is performed in only one asset at a time, inducing vertical or horizontal movements of the portfolio position. For example, panel *a* has a no-trade region of half-width 0.091, meaning that it is optimal to trade risky asset 2 (only) when its current weight deviates from the frictionless optimizer by 9.1% of current wealth. (*d*) The area within the no-trade region that is enclosed in a solid line is the rebalancing target, to which the portfolio is readjusted once the boundaries of the no-trade region are breached.

form

$$v(u,z,f)=\frac{z^{1-\gamma}}{1-\gamma}\exp\left(A(u)+B(u)f+\frac{1}{2}C(u)f^2\right).$$

Define

$$b=2\left(\frac{1-\gamma}{\gamma}\frac{\sigma_F}{\sigma_S}\rho-\kappa\right),\quad \eta=\sqrt{b^2-4\frac{1-\gamma}{\gamma}\left(\frac{\sigma_F}{\sigma_S}\right)^2\left(1+\frac{1-\gamma}{\gamma}\rho\right)}.$$

In the empirically relevant case when $\gamma>1$ and $\rho<0$, the discriminant η is positive, so A, B, and C can be identified as the normal solution of Kim & Omberg (1996):

$$C(u)=\frac{1-\gamma}{\gamma}\frac{2}{\sigma_S^2}\frac{1-\exp[-\eta(T-u)]}{2\eta-(b+\eta)\left\{1-\exp[-\eta(T-u)]\right\}},$$

$$B(u)=4\frac{1-\gamma}{\gamma}\frac{\kappa\bar{F}}{\sigma_S^2}\frac{1-\exp\left[-\eta(T-u)/2\right]}{2\eta^2-\eta(b+\eta)\left\{1-\exp[-\eta(T-u)]\right\}},$$

$$\begin{aligned}A(u)=&\frac{1-\gamma}{\gamma}\left(\gamma r+\frac{2\kappa^2\bar{F}^2}{\sigma_S^2\eta^2}+\frac{\sigma_F^2}{\sigma_S^2(\eta-b)}\right)(T-u)\\&+\frac{1-\gamma}{\gamma}\frac{4\kappa^2\bar{F}^2}{\sigma_S^2}\frac{(2b+\eta)\exp[-\eta(T-u)]-4b\exp[-\eta(T-u)/2]+2b-\eta}{\eta^3\left(2\eta-(b+\eta)\left\{1-\exp[-\eta(T-u)]\right\}\right)}\\&+\frac{1-\gamma}{\gamma}\frac{2\sigma_F^2}{\sigma_S^2}\frac{\log\left|2\eta-(b+\eta)\left\{1-\exp[-\eta(T-u)]\right\}\right|}{2\eta(\eta^2-b^2)}.\end{aligned}$$

9. APPENDIX: DISCRETIZATION SCHEME FOR POLICY ITERATION

For the numerical computations in Section 7, the differential operator of the ergodic control problem needs to be discretized. To interpret the discretized operator as the transition rate matrix of some (continuous-time) Markov decision process, the following discretization scheme is used:

$$\frac{\partial w}{\partial(\xi^i)}(\xi)\approx\begin{cases}\dfrac{w(\xi+e_ih_i)-w(\xi)}{h_i} & \text{if } \nu^i(\xi;\ell,m)>0,\\[2ex] \dfrac{w(\xi)-w(\xi-e_ih_i)}{h_i} & \text{if } \nu^i(\xi;\ell)<0,\end{cases}$$

$$\frac{\partial^2 w}{\partial(\xi_i)^2}(\xi)\approx\frac{w(\xi+e_ih_i)-2w(\xi)+w(\xi-e_ih_i)}{h_i^2},$$

$$\frac{\partial^2 w}{\partial(\xi_i)\partial(\xi_i)}(\xi) \approx \begin{cases} \dfrac{2w(\xi)+w(\xi+e_ih_i+e_jh_j)+w(\xi-e_ih_i-e_jh_j)}{2h_ih_j} \\ \quad -\dfrac{w(\xi+e_ih_i)+w(\xi-e_ih_i)+w(\xi+e_jh_j)+w(\xi-e_jh_j)}{2h_ih_j} & \text{if } A_{i,j}>0, \\ -\dfrac{2w(\xi)+w(\xi+e_ih_i-e_jh_j)+w(\xi-e_ih_i+e_jh_j)}{2h_ih_j} \\ \quad +\dfrac{w(\xi+e_ih_i)+w(\xi-e_ih_i)+w(\xi+e_jh_j)+w(\xi-e_jh_j)}{2h_ih_j} & \text{if } A_{i,j}<0, \end{cases}$$

for $i, j = 1, \ldots, d$, $i \neq j$, and where h_i is the grid size in the ξ^i direction. With $A = \alpha\alpha^\top$, the approximation $\mathcal{L}^\ell_{\mathcal{D}} : \mathcal{D} \mapsto \mathbb{R}$ of the differential operator $\mathcal{L}^\ell$ is then given by

$$\begin{aligned}
\mathcal{L}^\ell_{\mathcal{D}}(\xi,\xi) &= -\sum_{i=1}^{d}\left(\frac{A_{i,i}}{h_i^2} - \frac{1}{2}\sum_{\substack{j=1,\\ j\neq i}}^{d}\frac{|A_{i,j}|}{h_ih_j}\right) - \sum_{i=1}^{d}\frac{|\nu^i(\xi;\ell)|}{h_i},\\
\mathcal{L}^\ell_{\mathcal{D}}(\xi,\xi+e_ih_i) &= \frac{1}{2}\left(\frac{A_{i,i}}{h_i^2} - \sum_{\substack{j=1,\\ j\neq i}}^{d}\frac{|A_{i,j}|}{h_ih_j}\right) + \sum_{i=1}^{d}\frac{\max\{0,\nu^i(\xi;\ell)\}}{h_i},\\
\mathcal{L}^\ell_{\mathcal{D}}(\xi,\xi-e_ih_i) &= \frac{1}{2}\left(\frac{A_{i,i}}{h_i^2} - \sum_{\substack{j=1,\\ j\neq i}}^{d}\frac{|A_{i,j}|}{h_ih_j}\right) + \sum_{i=1}^{d}\frac{\max\{0,-\nu^i(\xi;\ell)\}}{h_i},\\
\mathcal{L}^\ell_{\mathcal{D}}(\xi,\xi\pm e_ih_i\pm e_jh_j) &= \frac{\max\{0,A_{i,j}\}}{2h_ih_j},\\
\mathcal{L}^\ell_{\mathcal{D}}(\xi,\xi\pm e_ih_i\mp e_jh_j) &= \frac{\max\{0,-A_{i,j}\}}{2h_ih_j},
\end{aligned}$$

for $i, j = 1, \ldots, d$ and $i \neq j$. For a finite domain, we also need the condition

$$\sum_{\xi'\in\mathcal{D}}\mathcal{L}^{\ell_j}_{\mathcal{D}}(\xi,\xi') = 0.$$

This ensures that the Markov decision process stays within the domain at all times. As long as the domain is chosen large enough to contain the no-trade region, this is not a constraint, as the optimal strategy already ensures that the process does not exit the domain.

DISCLOSURE STATEMENT

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