Arch. Rational Mech. Anal.
Digital Object Identifier (DOI) 10.1007/s00205-017-1178-0

Constrained Optimal Transport

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Communicated by D. Kinderlehrer

Abstract

The classical duality theory of Kantorovich (C R (Doklady) Acad Sci URSS (NS) 37:199–201, 1942) and Kellerer (Z Wahrsch Verw Gebiete 67(4):399–432, 1984) for classical optimal transport is generalized to an abstract framework and a characterization of the dual elements is provided. This abstract generalization is set in a Banach lattice $\mathcal{X}$ with an order unit. The problem is given as the supremum over a convex subset of the positive unit sphere of the topological dual of $\mathcal{X}$ and the dual problem is defined on the bi-dual of $\mathcal{X}$. These results are then applied to several extensions of the classical optimal transport.

1. Introduction

The Kantorovich relaxation [24] of Monge's optimal transport problem [27] is to maximize

$$\eta(f) := \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y)\, \eta(dx, dy)$$

over all probability measures η that have given marginals μ and ν. Kantorovich proved that the convex dual of this problem is given by

$$\mathbf{D}_{ot}(f) := \inf \left\{ \mu(h) + \nu(g) \ : \ h(x) + g(y) \geqq f(x, y)\ \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \right\}.$$

Indeed, a standard application of the Fenchel–Moreau theorem shows that these two problems have the same value when f is continuous and bounded. We refer the reader to the lecture notes of Ambrosio [3], the classic books of Rachev and

* Research is partly supported by the ETH Foundation, the Swiss Finance Institute and a Swiss National Foundation Grant SNF 200021-153555.

Published online: 09 October 2017

Rüschendorf [29], Villani [35] and the references therein for more information, and to the recent article of Zaev [36], which provides a new approach to duality.

The above duality can be seen as a consequence of a pairing between the primal measures and a set of dual functions. Indeed, let $\mathcal{Q}_{ot}$ be the set of all probability measures with given marginals μ and ν and $\mathcal{H}_{ot}$ be the set of all continuous and bounded functions of the form

$$k(x, y) = (h(x) - \mu(h)) + (g(y) - \nu(h)), \quad (x, y) \in \mathbb{R}^d \times \mathbb{R}^d,$$

for some $h, g \in \mathcal{C}_b(\mathbb{R}^d)$. Then, we can rewrite the dual problem compactly as follows:

$$\mathbf{D}_{ot}(f) = \inf \left\{ c \in \mathbb{R} \ : \ \exists k \in \mathcal{H}_{ot} \ \text{ such that } \ c + k \geqq f \right\}.$$

Moreover, the dual functions $\mathcal{H}_{ot}$ and the primal measures $\mathcal{Q}_{ot}$ are in duality in the sense that $\mathcal{Q}_{ot}$ is the set of all probability measures that annihilate $\mathcal{H}_{ot}$.

We extend the classical optimal transport problem based on this duality between the dual and primal elements. Namely, we start with a with a convex subset $\mathcal{Q}$ of the positive unit sphere in the topological dual $\mathcal{X}'$ of a Banach lattice $\mathcal{X}$. We assume that there exists a closed subspace $\mathcal{H}_{\mathcal{Q}} \subset \mathcal{X}$ such that $\mathcal{Q}$ is given as the intersection of $\mathcal{H}_{\mathcal{Q}}^{\perp}$ (the annihilator of $\mathcal{H}_{\mathcal{Q}}$) with the positive unit sphere in the topological dual $\mathcal{X}'$. Then, a direct application of the classical Fenchel–Moreau Theorem yields

$$\begin{aligned} \mathbf{P}(f) &:= \sup_{\eta \in \mathcal{Q}} \eta(f) \\ &= \mathbf{D}(f) := \inf\{c \in \mathbb{R} \ : \ \exists\, h \in \mathcal{H}_{\mathcal{Q}} \ \text{s.t.} \ c\,\mathbf{e} + h \geqq f\}, \quad \forall f \in \mathcal{X}, \end{aligned} \tag{1.1}$$

where $\mathbf{e}$ is a unit oder in the Banach lattice $\mathcal{X}$. In fact, Corollary 4.2 proves this duality as a consequence of a general result.

The above result is proved by applying convex duality to the map $\mathbf{D} : \mathcal{X} \to \mathbb{R}$. When $\mathcal{X} = C_b(\Omega)$ with a given topological space Ω, one could also consider $\mathbf{D}$ as a Lipschitz, convex function on bounded and Borel measurable functions, $\mathcal{B}_b(\Omega)$. Then, the convex dual of this problem would be a function on the topological dual of $\mathcal{B}_b(\Omega)$, namely, the set of bounded and finitely additive functionals, $ba(\Omega)$. Hence, to have the duality for all bounded and measurable functions and not only for continuous ones, one needs to augment the primal measures by adding an appropriate subset of $ba(\Omega)$. Indeed, Example 8.1 of [8] shows that this extension to $ba(\Omega)$ is necessary if one fixes the dual elements $\mathcal{H}_{\mathcal{Q}}$ or equivalently the dual problem $\mathbf{D}$.

Instead, we start with the primal problem defined on the bidual given by

$$\mathbf{P} : \mathfrak{a} \in \mathcal{X}'' \quad \to \quad \mathbf{P}(\mathfrak{a}) := \sup_{\eta \in \mathcal{Q}} \mathfrak{a}(\eta),$$

where $\mathcal{Q}$ is a given closed convex set in the positive unit sphere of $\mathcal{X}'$. Since one may view $\mathcal{X}$ as a subset of its bidual, this approach includes the duality (1.1). Also, with appropriate choices of $\mathcal{X}$, one may embed $\mathcal{B}_b(\Omega)$ as a closed subset of $\mathcal{X}''$.

Once $\mathbf{P}$ is defined on a dual space, the duality can be proved by direct separation arguments. In particular, we prove in Theorem 4.1 that

$$\begin{aligned}\mathbf{P}(\mathfrak{a}) &:= \sup_{\eta \in \mathcal{Q}} \mathfrak{a}(\eta) \\ &= \mathbf{D}(\mathfrak{a}) := \inf\{c \in \mathbb{R} \ : \ \exists\, \mathfrak{z} \in \mathfrak{K}_{\mathcal{Q}} \ \text{ s.t. } \ c\,\mathbf{e} + \mathfrak{z} = \mathfrak{a}\}, \quad \forall \mathfrak{a} \in \mathcal{X}'',\end{aligned}$$

where the dual set $\mathfrak{K}_{\mathcal{Q}}$ is given by,

$$\mathfrak{K}_{\mathcal{Q}} := \{\mathfrak{z} \in \mathcal{X}'' \ : \ \mathfrak{z}(\eta) \leqq 0 \ \ \forall \eta \in \mathcal{Q}\}.$$

Moreover, by its definition, $\mathfrak{K}_{\mathcal{Q}}$ is a convex cone and is weak* closed. Yoshida [34] shows that such sets in a dual space are *regularly convex* as defined by by Krein & Šmulian [26]. The defining property of regular convexity is convex separation in the pre-dual (see Definition 3.4 below). This allows us to prove the stated duality in $\mathcal{X}''$ with a fixed primal set $\mathcal{Q}$ in $\mathcal{X}'$. In particular, we identify the dual elements *without* augmenting $\mathcal{Q}$ or equivalently without extending $\mathcal{Q}$ to $ba(\Omega)$. Additionally, Theorem 4.1 proves that on any closed subspace $\mathfrak{L}$ of $\mathcal{X}''$, the duality can hold only with the dual set $\mathfrak{L} \cap \mathfrak{K}_{\mathcal{Q}}$.

In the applications further characterization of $\mathfrak{K}_{\mathcal{Q}}$ is desired. Indeed, for the classical optimal transport with $\mathcal{X} = C_b(\Omega)$ and $\Omega = \mathbb{R}^d \times \mathbb{R}^d$, Proposition 5.4 proves that the dual set $\mathfrak{K}_{ot} := \mathfrak{K}_{\mathcal{Q}_{ot}}$ is given by $\mathfrak{H}_{ot} + \mathcal{X}''_{-}$, where $\mathfrak{H}_{ot} = (\mathcal{H}_{ot}^{\perp})^{\perp}$ is the annihilator of the subspace $\mathcal{H}_{ot}^{\perp}$. The set $\mathfrak{H}_{ot}$ is also characterized as the sum of two natural sets. Then, the duality in $\mathcal{B}_b(\Omega)$ is proved in Proposition 5.5 as a consequence of these results.

Our approach is quite different than the previous studies and is based on the notion of regular convexity as developed by Krein & Šmulian [26]. Indeed, as a general result proved in Lemma 3.5, the dual set $\mathfrak{K}_{\mathcal{Q}}$ is the weak* closure, or equivalently the regular convex envelope of $\mathfrak{H}_{\mathcal{Q}} + \mathcal{X}''_{-}$, where $\mathfrak{H}_{\mathcal{Q}} := (\mathcal{H}_{\mathcal{Q}}^{\perp})^{\perp}$. Then, to prove that these two sets are equal it suffices to show that $\mathfrak{H}_{\mathcal{Q}} + \mathcal{X}''_{-}$ is weak* closed. The main difficulty in proving this or in characterizing $\mathfrak{H}_{\mathcal{Q}}$ emanates from the fact that sum of unbounded regularly convex sets may not be regularly convex. We use two results from [26] to overcome this. One is the classical Krein & Šmulian Theorem. It states that a set is regularly convex if and only if its intersection with all bounded balls are regularly convex. Secondly, sums of bounded regularly convex sets is again regularly convex. Therefore, in applications, our method necessitates to prove uniform pointwise estimates of the decomposition of dual elements. Indeed, Lemma 5.3 proves this estimate for the optimal transport and allows for duality results Proposition 5.4 in $\mathcal{C}_b''(\Omega)$ and Proposition 5.5 in $\mathcal{B}_b(\Omega)$. A similar estimate for super-martingales is obtained in Step 4 of the proof of Proposition 7.2 and the inequality (8.3) in Theorem 8.2 proves it for the martingale optimal transport.

In Section 6, we successfully apply this technique to an extension of the optimal transport which we call *constrained optimal transport*. In this problem, the set of primal measures are further constrained by specifying their actions on a finite dimensional subset of $\mathcal{X}$. This class of problems was also considered by Rachev and Rüschendorf in [29][Section 4.6.3] for lower semi-continuous functions.

Proposition 6.3 proves the duality for this extension by the outlined method in the bidual and also in $\mathcal{B}_b(\Omega)$.

A motivating example of the abstract extension is the martingale optimal transport. In this problem, $\mathcal{Q}_{mot}$ is the set of probability measures in $\mathcal{Q}_{ot}$ that also annihilate all functions of the form $\gamma(x) \cdot (x - y)$. Indeed, let $\mathcal{H}_{mot}$ be the set of all linearly growing functions of the form

$$k(x, y) = (h(x) - \mu(h)) + (g(y) - \nu(g)) + \gamma(x) \cdot (x - y), \quad (x, y) \in \Omega,$$

for some linearly growing, continuous functions h, g and a bounded, continuous vector valued function γ. Then, $\mathcal{Q}_{mot}$ is the intersection of $\mathcal{H}_{mot}^{\perp}$ with the unit positive sphere. This problem can also be seen as a constrained optimal transport but the dual set is now enlarged with countably and not finitely many functions.

Martingale optimal transport is first introduced in discrete time by Beiglböck, Henry-Labordère and Penkner [6] and in continuous time by Galichon, Henry-Labordère and Touzi [17]. The main motivation for this extension comes from model-free finance or robust hedging results of Hobson [20] and Hobson and Neuberger [21]. Initial papers [6,17] also prove the duality for continuous functions. The duality is then further extended by Dolinsky and the second author [14–16] to the case when Ω is the Skorokhod space of càdlàg functions by discretization techniques and later Hou and Obłój [22] extended these by considering further constraints. Also, recent manuscripts [18,19] use the S-topology of Jakubowski in the Skorokhod space to study the properties of the martingale optimal transport.

Several other types of extensions of the martingale optimal transport duality are studied in the literature. Indeed, especially in financial applications, it is needed to relax the pointwise inequalities in the definition of the dual problem. The first relaxation is already given in the initial paper [17] by using the quasi-sure framework developed in [31,32]. In the other types of extensions, one keeps $\Omega = \mathbb{R}^d \times \mathbb{R}^d$ but studies the duality for all bounded measurable not only continuous or upper semi-continuous functions. This problem poses interesting new questions. In one space dimension, they are analyzed thoroughly in a recent paper by Beiglböck, Nutz and Touzi [8]. This paper contains, in addition to the characterization of the dual set, several motivating examples and counter-examples. A recent manuscript [28] studies the super-martingale couplings.

We study the problem of martingale optimal transport in the Banach lattice of linearly growing continuous functions. In Theorem 8.2, we obtain a complete characterization of the set $\mathfrak{H}_{mot} = (\mathcal{H}_{mot}^{\perp})^{\perp}$ that annihilates the primal signed measures in $\mathcal{H}_{mot}^{\perp}$. However, the set $\mathfrak{H}_{mot} + \mathcal{X}_{-}''$ is not equal to $\mathfrak{K}_{mot} = \mathfrak{K}_{\mathcal{Q}_{mot}}$ as shown in Example 8.4 and in Example 8.4 of [8]. Similarly, in Section 7, we study the related problem defined through martingale measures. These problems are very closely related to the convex functions and also to the classical result of Strassen [33]. These connections are made precise in Section 9.

In a series of papers, Bartl, Cheredito, Kupper and Tangpi [4,12,13] also develop a functional analytic framework for duality problems of these type. They, however, start with the dual problem and characterize the primal measures using semi-continuity assumptions on the dual functional. In a large variety of problems, including the financial markets with friction, they very efficiently obtain duality results for upper semi-continuous functions. Another related subject is the model-free fundamental theorem of asset pricing. In recent years, many interesting results in this direction have been proved [1,5,9–11]. These results essentially start with the dual elements and define the primal measures, $\mathcal{Q}$ as their annihilators. Then, their main concern is to prove that $\mathcal{Q}$ contains elements that are countably additive Borel measures. In this manuscript, we start with the set $\mathcal{Q}$ as a subset of $\mathcal{X}'$ and prove duality.

The paper is organized as follows. Section 2 introduces the notations and basic results used in the paper. Section 3 defines the abstract problem and introduces the regular convexity. The duality results are proved in the next section. Subsection 4.1 proves the first duality result in $\mathcal{X}''$ through $\mathfrak{K}_{\mathcal{Q}}$ and the complete duality in $\mathcal{X}$ is established in the subsection 4.2. The necessary and sufficient conditions for duality with a dual space of the form $\mathfrak{H}+\mathcal{X}''_{-}$ with a subspace $\mathfrak{H}$, is obtained in Theorem 4.4 in subsection 4.3. Subsection 4.4 proves a duality result in the quotient spaces. The classical optimal transport is studied in Section 5 and its extension to constrained optimal transport in Section 6. Section 7 defines and characterizes martingale measures. These results are used in Section 8 to study the multi-dimensional martingale optimal transport. The Final section states results for the convex functions defined on the bidual.

2. Preliminaries

For convex closed subsets of $X, Y \subset \mathbb{R}^d$, we denote $\Omega := X \times Y$, and for a Banach lattice $\mathcal{X}$, we use the following standard notations for which we refer to the classical books of Aliprantis & Border [2] and Yoshida [34] or to the lecture notes by Kaplan [23]:

- $\mathcal{X}'$ is the topological dual of $\mathcal{X}$ and $\mathcal{X}''$ is its bidual;
- $\mathcal{B}_b(\Omega)$ is the Banach space of bounded real valued Borel measurable functions with the supremum norm,
- $C(\Omega)$ is the set of all continuous real-valued functions;
- $C_b(\Omega)$ is the Banach lattice of all continuous real-valued bounded functions with the supremum norm.

For $h \in C_b(X)$ and $g \in C_b(Y)$, we set

$$(h \oplus g)(x, y) := h(x) + g(y), \quad \forall \omega = (x, y) \in \Omega.$$

For $\eta \in (C_b(\Omega))'$, its *marginals,* $\eta_x \in (C_b(X))'$ and $\eta_y \in (C_b(Y))'$ are given by

$$\eta_x(h) := \eta\,(h \oplus \mathbf{0}), \quad \eta_y(g) := \eta\,(\mathbf{0} \oplus g),$$

where $\mathbf{0}$ is the constant function identically equal to zero.

The Banach space $\mathcal{X}$ is embedded into $\mathcal{X}''$ by the canonical mapping

$$x \in \mathcal{X} \;\mapsto\; \mathfrak{I}(x) \in \mathcal{X}'' \quad \text{where} \quad \mathfrak{I}(x)(x') := x'(x), \;\forall x' \in \mathcal{X}'.$$

Clearly this notation of $\mathfrak{I}$ depends on the underlying space and we suppress this dependence in our notation.

On $\mathcal{X}'$, we use the order induced by the order of $\mathcal{X}$. Let $\mathcal{X}'_+$ be the set of all positive elements in $\mathcal{X}'$, i.e., $\eta \in \mathcal{X}'_+$ if $\eta(f) \geqq 0$ for every $f \in \mathcal{X}$ and $f \geqq 0$. We define $\mathcal{X}'_-$ similarly. On $\mathcal{X}''$ we use the order induced by $\mathcal{X}'$.

We always assume that the Banach lattice $\mathcal{X}$ is an AM-space endowed with the lattice norm induced by an order unit $\mathbf{e} \in \mathcal{X}_+$, i.e.,

$$\|f\|_{\mathcal{X}} = \inf\left\{c \in \mathbb{R} \ : \ -\mathbf{c} \leqq f \leqq \mathbf{c}\right\}, \quad \text{where} \quad \mathbf{c} := c\,\mathbf{e}. \tag{2.1}$$

Then, the bidual $\mathcal{X}''$ is also an AM-space with $\mathfrak{I}(\mathbf{e})$ as its order unit; [2] [Theorem 9.31]. Moreover,

$$\|\mathfrak{a}\|_{\mathcal{X}''} = \inf\left\{c \in \mathbb{R} \ : \ -\mathbf{c} \leqq f \leqq \mathbf{c}\right\}, \quad \text{where} \quad \mathbf{c} := c\,\mathfrak{I}(\mathbf{e}). \tag{2.2}$$

We also have

$$\eta(\mathbf{e}) = \|\eta^+\|_{\mathcal{X}'} - \|\eta^-\|_{\mathcal{X}'}, \quad \forall\, \eta \in \mathcal{X}'. \tag{2.3}$$

We denote the unit ball in $\mathcal{X}$ by B_1 and set $B_+ := B_1 \cap \mathcal{X}_+$. Similarly, we let B'_1 and B''_1 be the unit balls in $\mathcal{X}'$ and in $\mathcal{X}''$, respectively. We set $B'_+ := B'_1 \cap \mathcal{X}_+$, $B''_+ := B''_1 \cap \mathcal{X}_+$.

For a given a subset A of a Banach space $\mathcal{Z}$ and a subset $\Theta \subset \mathcal{Z}'$, the annihilator of A and the pre-annihilator of Θ are given by

$$A^{\perp} := \left\{\eta \in \mathcal{Z}' \ : \ \eta(a) = 0, \quad \forall a \in A\right\}, \quad \Theta_{\perp} := \{a \in \mathcal{Z} : \ \eta(a) = 0, \forall \eta \in \Theta\}.$$

It is clear that $A^{\perp}$ is weak* closed in $\mathcal{Z}'$ and $\Theta_{\perp}$ is weakly closed in $\mathcal{Z}$.

For a scalar c, $\mathbf{c}$ denotes the function equal to $c\,\mathbf{e}$. Clearly, this notation depends on the order unit but with an abuse of notation, we use the same notation in all domains.

Throughout the paper, we mostly use the Banach lattice $C_b(\Omega)$ or the space $C_\ell(\Omega)$ of linearly growing continuous functions defined by

$$C_\ell(\Omega) := \{\, f \in C(\Omega) \ : \ \|f\|_\ell < \infty\},$$

where, with $\ell_X(x) := 1 + |x|$, $\ell_Y(y) := 1 + |y|$ and $\ell(x, y) = \ell_X(x) + \ell_Y(y)$,

$$\|f\|_\ell := \sup_{\omega \in \Omega} \frac{|f(\omega)|}{\ell(\omega)}. \tag{2.4}$$

To simplify the presentation, we denote

$$\mathcal{C}_b := C_b(\Omega) \quad \text{and} \quad \mathcal{C}_\ell := C_\ell(\Omega).$$

It is clear that both of these spaces are AM-spaces and the order unit in $\mathcal{C}_b$ is $\mathbf{e} \equiv 1$. The space $\mathcal{C}_\ell$ has the order unit $\mathbf{e}(x, y) = \ell(x, y)$. Moreover, the weighted spaces $\mathcal{C}_{\ell_X}(X)$ and $\mathcal{C}_{\ell_Y}(Y)$ are also AM-spaces with order units ℓ_X and ℓ_Y, respectively.

We also use the notation $\mathcal{C} := C(\Omega)$ and view it as a Frechet space. Then, $\mathcal{C}'$ is equal to $ca_{r,c}(\Omega)$, all countably additive, regular measures that are compactly supported.

3. Abstract Problem

Let $\mathcal{X}$ be a Banach lattice with a lattice norm given by (2.1) and with an order unit $\mathbf{e}$. Recall the notation, $\mathbf{c} := c\,\mathbf{e}$ and with an abuse of notation, we use the same notation in the bidual as well. Let $\partial B_1'$ be the unit sphere in $\mathcal{X}'$. In view of (2.3), we have

$$\partial B_+' := \partial B_1' \cap \mathcal{X}_+' = \left\{\eta \in \mathcal{X}_+' \ : \ \eta(\mathbf{e}) = 1\right\}.$$

The starting point of our analysis is a closed convex set $\mathcal{Q} \subset \partial B_+'$. We make the following standing assumption:

Assumption 3.1. We assume that $\mathcal{Q}$ is a non-empty, closed, convex subset of $\mathcal{X}'$ and that there exists a closed subspace $\mathcal{H}_\mathcal{Q} \subset \mathcal{X}$ such that

$$\mathcal{Q} = \mathcal{H}_\mathcal{Q}^\perp \cap \partial B_+'.$$

Set

$$\mathcal{A}_\mathcal{Q} := \mathcal{H}_\mathcal{Q}^\perp, \qquad \mathcal{C}_\mathcal{Q} := \left\{\lambda\eta \ : \ \eta \in \mathcal{Q}, \ \ \lambda \geqq 0\right\} = \mathcal{H}_\mathcal{Q}^\perp \cap \mathcal{X}_+'.$$

Note that the closed linear span of $\mathcal{Q}$ is equal to $\mathcal{C}_\mathcal{Q} - \mathcal{C}_\mathcal{Q}$ and is always a subset of $\mathcal{A}_\mathcal{Q}$, but in general this inclusion could be strict.

3.1. Definitions

Given $\mathcal{Q}$, the *constrained optimal transport* is given by,

$$\mathbf{P}(\mathfrak{a}; \mathcal{Q}) := \sup_{\eta \in \mathcal{Q}} \mathfrak{a}(\eta), \quad \mathfrak{a} \in \mathcal{X}''.$$

On the dual side, we start with a cone $\mathfrak{K} \subset \mathcal{X}''$ satisfying

$$\mathfrak{z} \in \mathfrak{K} \ \text{and} \ \mathfrak{n} \in \mathcal{X}_-'' \quad \Rightarrow \quad \mathfrak{z} + \mathfrak{n} \in \mathfrak{K}. \tag{3.1}$$

Then, the *dual constrained optimal transport* problem is defined by

$$\mathbf{D}(\mathfrak{a}; \mathfrak{K}) := \inf \{\, c \in \mathbb{R} \ : \ \exists\, \mathfrak{z} \in \mathfrak{K} \ \text{such that} \ \mathbf{c} + \mathfrak{z} = \mathfrak{a}\}, \quad \mathfrak{a} \in \mathcal{X}''.$$

We always use the convention that the infimum over an empty set is plus infinity.

The chief concern of this paper is to relate these two problems. In particular, the following sets are relevant:

$$\mathfrak{H}_\mathcal{Q} := \mathcal{A}_\mathcal{Q}^\perp, \qquad \mathfrak{K}_\mathcal{Q} := \left\{\mathfrak{z} \in \mathcal{X}'' \ : \ \mathfrak{z}(\eta) \leqq 0, \ \ \forall\, \eta \in \mathcal{Q}\right\}. \tag{3.2}$$

It is then immediately apparent that $\mathfrak{K}_\mathcal{Q} = \left\{\mathfrak{z} \in \mathcal{X}'' \ : \ \mathfrak{z}(\eta) \leqq 0, \ \ \forall\, \eta \in \mathcal{C}_\mathcal{Q}\right\}.$

Remark 3.2. These two sets are closely related to each other as we always have the inclusion, $\mathfrak{H}_\mathcal{Q} + \mathcal{X}_-'' \subset \mathfrak{K}_\mathcal{Q}$. We also show in Lemma 3.5 that the weak* closure of $\mathfrak{H}_\mathcal{Q} + \mathcal{X}_-''$ is equal to $\mathfrak{K}_\mathcal{Q}$. On the other hand this inclusion might be strict as shown in Example 8.4 below. Indeed, $\mathfrak{K}_\mathcal{Q}$ is always weak* closed, while $\mathfrak{H}_\mathcal{Q} + \mathcal{X}_-''$ may not even be strongly closed. $\square$

3.2. *Properties of* **D**

We prove several easy properties of **D** for future reference. Let $\overline{\mathfrak{K}}$ be the closure of $\mathfrak{K}$ under the strong topology of $\mathcal{X}''$.

Lemma 3.3. *Suppose that $\mathfrak{K}$ satisfies (3.1). Then, for every $\mathfrak{a} \in \mathcal{X}''$,*

$$\mathbf{D}(\mathfrak{a}; \mathfrak{K}) = \mathbf{D}(\mathfrak{a}; \overline{\mathfrak{K}}) \leqq \|\mathfrak{a}\|_{\mathcal{X}''}.$$

Moreover, if $\mathbf{D}(\mathfrak{a}; \mathfrak{K}) > -\infty$, then there exists $\mathfrak{z}_{\mathfrak{a}} \in \overline{\mathfrak{K}}$ satisfying

$$\mathfrak{a} = \mathfrak{z}_{\mathfrak{a}} + \mathbf{D}(\mathfrak{a}; \overline{\mathfrak{K}})\ \mathbf{e}.$$

Proof. Fix $\mathfrak{a} \in \mathcal{X}''$. By (2.2), $\|\mathfrak{a}\|_{\mathcal{X}''}\ \mathbf{e} \geqq \mathfrak{a}$. Since $\mathcal{X}''_- \subset \mathfrak{K}$, $\mathbf{D}(\mathfrak{a}; \mathfrak{K}) \leqq \|\mathfrak{a}\|_{\mathcal{X}''}$.

Let $(c, \mathfrak{z}) \in \mathbb{R} \times \overline{\mathfrak{K}}$ be such that $\mathfrak{a} = c\ \mathbf{e} + \mathfrak{z}$. Let $\{\mathfrak{z}_n\}_n \subset \mathfrak{K}$ be a sequence that converges to $\mathfrak{z}$. In view of (2.2),

$$\mathfrak{a} = c\ \mathbf{e} + \mathfrak{z} \leqq [c + \|\mathfrak{z} - \mathfrak{z}_n\|_{\mathcal{X}''}]\,\mathbf{e} + \mathfrak{z}_n.$$

Hence, by (3.1),

$$\mathfrak{n}_n := \mathfrak{a} - [c + \|\mathfrak{z} - \mathfrak{z}_n\|_{\mathcal{X}''}]\,\mathbf{e} - \mathfrak{z}_n \in \mathcal{X}''_- \quad \Rightarrow \quad \mathfrak{z}_n + \mathfrak{n}_n \in \mathfrak{K}.$$

Moreover,

$$\mathfrak{a} = [c + \|\mathfrak{z} - \mathfrak{z}_n\|_{\mathcal{X}''}]\,\mathbf{e} + [\mathfrak{z}_n + \mathfrak{n}_n] \quad \Rightarrow \quad \mathbf{D}(\mathfrak{a}; \mathfrak{K}) \leqq c + \|\mathfrak{z} - \mathfrak{z}_n\|_{\mathcal{X}''}.$$

Since above holds for every pair $(c, \mathfrak{z}) \in \mathbb{R} \times \overline{\mathfrak{K}}$ satisfying $\mathfrak{a} = c\ \mathbf{e} + \mathfrak{z}$, we conclude that $\mathbf{D}(\mathfrak{a}; \mathfrak{K}) \leqq \mathbf{D}(\mathfrak{a}; \overline{\mathfrak{K}})$. The opposite inequality is immediate, since $\mathfrak{K} \subset \overline{\mathfrak{K}}$.

Suppose that $\mathbf{D}(\mathfrak{a}; \mathfrak{K}) > -\infty$. Then, there is a sequence $(c_n, \mathfrak{z}_n) \in \mathbb{R} \times \mathfrak{K}$ such that $\mathfrak{a} = c_n \mathbf{e} + \mathfrak{z}_n$ and c_n tends to $\mathbf{D}(\mathfrak{a}; \mathfrak{K})$ as n tends to infinity. Then,

$$\|\mathfrak{z}_n - \mathfrak{z}_m\|_{\mathcal{X}''} = |c_n - c_m|, \quad \forall\, n, m.$$

Hence, $\{\mathfrak{z}_n\}_n$ is a Cauchy sequence. Let $\mathfrak{z}_{\mathfrak{a}} \in \overline{\mathfrak{K}}$ be its limit point. Then,

$$\mathfrak{a} = \mathfrak{z}_{\mathfrak{a}} + \mathbf{D}(\mathfrak{a}; \overline{\mathfrak{K}})\ \mathbf{e}.$$

□

3.3. *Regular Convexity*

The notion of regular convexity defined in [26] by KREIN & ŠMULIAN is useful in this context as it allows for convex separation in the pre-dual.

Definition 3.4. (*Regular Convexity*) Let $\mathcal{Z}$ be a Banach space. A subset $\mathfrak{A} \subset \mathcal{Z}'$ is called regularly convex if for any $\mathfrak{b} \notin \mathfrak{A}$, there exists $\eta \in \mathcal{Z}$ such that

$$\sup_{\mathfrak{f} \in \mathfrak{A}} \mathfrak{f}(\eta) < \mathfrak{b}(\eta).$$

It is proved by Yoshida [34] that a set is regularly convex if and only of it is convex and is weak* closed. We also recall a condition for regular convexity in Appendix at section 10.

The sets $\mathfrak{H}_{\mathcal{Q}}$ and $\mathfrak{K}_{\mathcal{Q}}$ defined earlier are both weak* closed and convex. Hence, they are regularly convex. Moreover, due to general facts of functional analysis and the relation between $\mathcal{A}_{\mathcal{Q}}$ and $\mathcal{Q}$, we have the following connection between the spaces $\mathfrak{K}_{\mathcal{Q}}$ and $\mathfrak{H}_{\mathcal{Q}} + \mathcal{X}''_-$:

Lemma 3.5. *Under the Assumption 3.1, the weak* closure of* $\mathfrak{I}(\mathcal{H}_{\mathcal{Q}}) + \mathcal{X}''_-$ *and* $\mathfrak{H}_{\mathcal{Q}} + \mathcal{X}''_-$ *are equal to* $\mathfrak{K}_{\mathcal{Q}}$.

Proof. Let $\mathfrak{K}$ be the weak* closures of $\mathfrak{I}(\mathcal{H}_{\mathcal{Q}}) + \mathcal{X}''_-$. Since $\mathfrak{I}(\mathcal{H}_{\mathcal{Q}}) + \mathcal{X}''_- \subset \mathfrak{K}_{\mathcal{Q}}$, we also have $\mathfrak{K} \subset \mathfrak{K}_{\mathcal{Q}}$. For a contraposition argument, assume that there exists $\mathfrak{a}_0 \in \mathfrak{K}_{\mathcal{Q}} \setminus \mathfrak{K}$. Since $\mathfrak{K}$ is convex and weak* closed, it is regularly convex. Hence, there exists $\eta_0 \in \mathcal{X}'$ satisfying,

$$c_0 := \sup_{\mathfrak{z} \in \mathfrak{K}} \mathfrak{z}(\eta_0) < \mathfrak{a}_0(\eta_0). \tag{3.3}$$

Since $\mathfrak{K}$ is a cone and contains 0, we conclude that $c_0 = 0$. Therefore, $\mathfrak{I}(h)(\eta_0) \leqq 0$ for every $h \in \mathcal{H}_{\mathcal{Q}}$. By the fact $\mathcal{H}_{\mathcal{Q}}$ is linear, we conclude that $\eta_0 \in \mathcal{H}_{\mathcal{Q}}^{\perp}$. Also, since $\mathcal{X}''_- \subset \mathfrak{K}$, $\eta_0 \geqq 0$. In particular, $\eta_0 \in \mathcal{H}_{\mathcal{Q}}^{\perp} \cap \mathcal{X}'_+$ and by Assumption 3.1, and the definition $\mathcal{C}_{\mathcal{Q}}$, we conclude that $\eta_0 \in \mathcal{C}_{\mathcal{Q}}$. On the other hand, $\mathfrak{a}_0 \in \mathfrak{K}_{\mathcal{Q}}$. Hence, $\mathfrak{a}_0(\eta_0) \leqq 0$. This is in contradiction with (3.3) and the fact that $c_0 = 0$. This proves that the weak* closure of $\mathfrak{I}(\mathcal{H}_{\mathcal{Q}}) + \mathcal{X}''_-$ is equal to $\mathfrak{K}_{\mathcal{Q}}$.

Finally, since $\mathfrak{H}_{\mathcal{Q}} + \mathcal{X}''_- \subset \mathfrak{K}_{\mathcal{Q}}$, and since $\mathfrak{I}(\mathcal{H}_{\mathcal{Q}}) \subset (\mathcal{H}_{\mathcal{Q}}^{\perp})^{\perp} = \mathfrak{H}_{\mathcal{Q}}$, we have

$$\mathfrak{I}(\mathcal{H}_{\mathcal{Q}}) + \mathcal{X}''_- \subset \mathfrak{H}_{\mathcal{Q}} + \mathcal{X}''_- \subset \mathfrak{K}_{\mathcal{Q}}.$$

We have already shown that the weak* closure of the smallest set above is equal to $\mathfrak{K}_{\mathcal{Q}}$. Hence, the weak* closures are all above sets are equal to $\mathfrak{K}_{\mathcal{Q}}$. □

4. Duality

In this section, we prove several duality results and also necessary and sufficient conditions for certain types of duality.

4.1. Main Duality

The following is the main duality result:

Theorem 4.1. (Duality) *Suppose that* $\mathcal{Q}$ *is a convex subset of* $\mathcal{X}'$ *satisfying Assumption 3.1. Let* $\mathfrak{L}$ *be a strongly closed subspace of* $\mathcal{X}''$ *containing* $\mathbf{e}$ *and* $\mathfrak{K} \subset \mathfrak{L}$ *be a cone satisfying*

$$\mathfrak{z} \in \mathfrak{K} \text{ and } \mathfrak{n} \in \mathfrak{L} \cap \mathcal{X}''_- \;\Rightarrow\; \mathfrak{z} + \mathfrak{n} \in \mathfrak{K}.$$

Then, the duality

$$\mathbf{P}(\mathfrak{a};\,\mathcal{Q}) = \mathbf{D}(\mathfrak{a};\,\mathfrak{K}), \quad \forall\, \mathfrak{a} \in \mathfrak{L} \tag{4.1}$$

holds if and only if the strong closure $\overline{\mathfrak{K}}$ of $\mathfrak{K}$ is equal to $\mathfrak{K}_{\mathcal{Q}} \cap \mathfrak{L}$. Moreover, there is dual attainment in $\overline{\mathfrak{K}}$. Namely, for every $\mathfrak{a} \in \mathfrak{L}$ there exists $\mathfrak{z}_{\mathfrak{a}} \in \overline{\mathfrak{K}}$ satisfying

$$\mathbf{D}(\mathfrak{a};\,\mathfrak{K})\,\mathbf{e} + \mathfrak{z}_{\mathfrak{a}} = \mathfrak{a}. \tag{4.2}$$

Proof. Fix $\mathfrak{a} \in \mathfrak{L}$. Set $\mathfrak{L}_{\mathcal{Q}} := \mathfrak{L} \cap \mathfrak{K}_{\mathcal{Q}}$ and

$$\mathfrak{D}_{\mathcal{Q}} := \{(c, \mathfrak{z}) \in \mathbb{R} \times \mathfrak{L}_{\mathcal{Q}} \;:\; \mathbf{c} + \mathfrak{z} = \mathfrak{a}\,\}.$$

In view of (2.2),

$$\mathfrak{n} := -\|\mathfrak{a}\|_{\mathcal{X}''}\,\mathbf{e} + \mathfrak{a} \leqq 0.$$

Hence, trivially, $\mathfrak{n} \in \mathfrak{L}_{\mathcal{Q}}$. Therefore, $(\|\mathfrak{a}\|_{\mathcal{X}''}, \mathfrak{n}) \in \mathfrak{D}_{\mathcal{Q}}$. In particular, $\mathfrak{D}_{\mathcal{Q}}$ is non-empty. Suppose that $(c, \mathfrak{z}) \in \mathfrak{D}_{\mathcal{Q}}$. Let $\eta \in \mathcal{Q}$. Then, $\eta(\mathbf{e}) = 1$, $\eta \geqq 0$ and $\mathfrak{z}(\eta) \leqq 0$ for every $\mathfrak{z} \in \mathfrak{L}_{\mathcal{Q}}$. Consequently,

$$\mathfrak{a}(\eta) = \eta(\mathbf{c}) + \mathfrak{z}(\eta) \leqq c.$$

This proves that $\mathbf{P}(\mathfrak{a};\,\mathcal{Q}) \leqq \mathbf{D}(\mathfrak{a};\,\mathfrak{L}_{\mathcal{Q}})$.

Since $\mathcal{Q}$ is non-empty and $\eta(\mathbf{e}) = 1$ for every $\eta \in \mathcal{Q}$, we also conclude that

$$-\|\mathfrak{a}\|_{\mathcal{X}''} \leqq \mathbf{P}(\mathfrak{a};\,\mathcal{Q}) \leqq \mathbf{D}(\mathfrak{a};\,\mathfrak{L}_{\mathcal{Q}}).$$

Hence, $\mathbf{D}(\mathfrak{a};\,\mathfrak{L}_{\mathcal{Q}})$ is finite and there exists a sequence $(c_n, \mathfrak{z}_n) \in \mathfrak{D}_{\mathcal{Q}}$ so that c_n monotonically converges to $\mathbf{D}(\mathfrak{a};\,\mathfrak{L}_{\mathcal{Q}})$. Then,

$$\|\mathfrak{z}_n - \mathfrak{z}_m\|_{\mathcal{X}''} = |c_n - c_m|$$

for each n, m. This implies that the strong limit $\mathfrak{z}_{\mathfrak{a}}$ of the sequence $\mathfrak{z}_n$ exists and satisfies (4.2). It is clear that $\mathfrak{K}_{\mathcal{Q}}$ is closed in the weak* topology and hence is also closed in the strong topology. Since $\mathfrak{L}$ is closed by hypothesis, $\mathfrak{z}_{\mathfrak{a}} \in \mathfrak{L}_{\mathcal{Q}} = \mathfrak{L} \cap \mathfrak{K}_{\mathcal{Q}}$. Set

$$c^* := \mathbf{P}(\mathfrak{z}_{\mathfrak{a}};\,\mathcal{Q}).$$

Since $\mathfrak{z}_{\mathfrak{a}} \in \mathfrak{L}_{\mathcal{Q}}$, $c^* \leqq 0$. Moreover, $\mathfrak{z}_{\mathfrak{a}} - \mathbf{c}^* \in \mathfrak{L}_{\mathcal{Q}}$ and

$$\left[\mathbf{D}(\mathfrak{a};\,\mathfrak{L}_{\mathcal{Q}}) + c^*\right]\mathbf{e} + \left[\mathfrak{z}_{\mathfrak{a}} - \mathbf{c}^*\right] = \mathfrak{a} \quad \Rightarrow \quad \left(\mathbf{D}(\mathfrak{a};\,\mathfrak{L}_{\mathcal{Q}}) + c^*, \mathfrak{z}_{\mathfrak{a}} - \mathbf{c}^*\right) \in \mathfrak{D}_{\mathcal{Q}}.$$

Since $\mathbf{D}(\mathfrak{a};\,\mathfrak{L}_{\mathcal{Q}})$ is the minimum over all constants c so that there is $\mathfrak{z} \in \mathfrak{L}_{\mathcal{Q}}$ satisfying $(c, \mathfrak{z}) \in \mathfrak{D}_{\mathcal{Q}}$ and since $c^* \leqq 0$, we conclude that $c^* = 0$. Then, $\mathfrak{a} = \mathfrak{z}_{\mathfrak{a}} + \mathbf{D}(\mathfrak{a};\,\mathfrak{L}_{\mathcal{Q}})\,\mathbf{e}$ and consequently,

$$\mathbf{P}(\mathfrak{a};\,\mathfrak{L}_{\mathcal{Q}}) = \sup_{\eta \in \mathcal{Q}} \left[\mathfrak{z}_{\mathfrak{a}} + \mathbf{D}(\mathfrak{a};\,\mathfrak{L}_{\mathcal{Q}})\,\mathbf{e}\right](\eta) = \mathbf{P}(\mathfrak{z}_{\mathfrak{a}};\,\mathfrak{L}_{\mathcal{Q}}) + \mathbf{D}(\mathfrak{a};\,\mathfrak{L}_{\mathcal{Q}}) = \mathbf{D}(\mathfrak{a};\,\mathfrak{L}_{\mathcal{Q}}).$$

Hence the duality on $\mathfrak{L}$ holds when $\mathfrak{K} = \mathfrak{L}_{\mathcal{Q}}$.

We continue by proving the opposite implication. Suppose that the duality (4.1) holds for every $\mathfrak{a} \in \mathfrak{L}$. Set $\overline{\mathfrak{K}}$ be the strong closure of $\mathfrak{K}$. Then, by Lemma 3.3,

$\mathbf{D}(\cdot;\mathfrak{K}) = \mathbf{D}(\cdot;\overline{\mathfrak{K}})$. We first claim that $\overline{\mathfrak{K}}$ is contained in $\mathfrak{L}_{\mathcal{Q}}$. Fix $\mathfrak{z}_0 \in \overline{\mathfrak{K}}$. By the definition of the dual problem, $\mathbf{D}(\mathfrak{z}_0;\overline{\mathfrak{K}}) \leqq 0$. Since the duality holds,

$$\sup_{\eta\in\mathcal{Q}} \mathfrak{z}_0(\eta) = \mathbf{P}(\mathfrak{z}_0;\mathcal{Q}) = \mathbf{D}(\mathfrak{z}_0;\mathfrak{K}) = \mathbf{D}(\mathfrak{z}_0;\overline{\mathfrak{K}}) \leqq 0.$$

Hence, $\mathfrak{z}_0 \in \mathfrak{K}_{\mathcal{Q}}$.

To prove the opposite inclusion, let $\mathfrak{z}^* \in \mathfrak{L}_{\mathcal{Q}}$. Then, $\mathfrak{z}^*(\eta) \leqq 0$ for every $\eta \in \mathcal{Q}$. Since, by hypothesis, the duality holds, we conclude that

$$c_0 := \mathbf{D}(\mathfrak{z}^*;\mathfrak{K}) = \mathbf{P}(\mathfrak{z}^*;\mathcal{Q}) = \sup_{\eta\in\mathcal{Q}} \mathfrak{z}^*(\eta) \leqq 0.$$

By Lemma 3.3, there are $\mathfrak{a}^* \in \overline{\mathfrak{K}}$ such that $\mathfrak{z}^* = c_0\,\mathbf{e} + \mathfrak{a}^*$. Since $c_0 \leqq 0$, we have $c_0\,\mathbf{e} \in \mathcal{X}''_-$. Since $\overline{\mathfrak{K}}$ satisfies (3.1), we conclude that $\mathfrak{z}^* \in \overline{\mathfrak{K}}$ and consequently, $\mathfrak{L}_{\mathcal{Q}} \subset \overline{\mathfrak{K}}$. Therefore, $\overline{\mathfrak{K}} = \mathfrak{L}_{\mathcal{Q}}$ whenever the duality holds. □

4.2. Duality in $\mathcal{X}$

We continue by proving the duality in $\mathcal{X}$. For the optimal transport and its several extensions, Zaev [36] also provides a proof of this duality when $\mathcal{X} = C_b(\Omega)$.

For any $f \in \mathcal{X}$, with an abuse of notation, we write $\mathbf{P}(f;\mathcal{Q})$ instead of $\mathbf{P}(\mathfrak{I}(f);\mathcal{Q})$ and $\mathbf{D}(f;\mathcal{H}_{\mathcal{Q}} + \mathcal{X}_-)$ instead of $\mathbf{D}(\mathfrak{I}(f);\mathfrak{I}(\mathcal{H}_{\mathcal{Q}} + \mathcal{X}_-))$. Next, we use Theorem 4.1 with $\mathfrak{L} = \mathfrak{I}(\mathcal{X})$ to prove duality in $\mathcal{X}$. This result can also be proved as a direct consequence of Theorem 7.51 of [2].

Recall that the sets $\mathcal{H}_{\mathcal{Q}}$ and $\mathcal{C}_{\mathcal{Q}}$ are defined in Section 3.

Corollary 4.2. *(Duality in $\mathcal{X}$) Under Assumption* (3.1)*,*

$$\mathbf{P}(f;\mathcal{Q}) = \mathbf{D}(f;\mathcal{H}_{\mathcal{Q}} + \mathcal{X}_-), \quad \forall f \in \mathcal{X}.$$

Proof. Let $\mathcal{K}$ be the strong closure of $\mathcal{H}_{\mathcal{Q}} + \mathcal{X}_-$. Since $\mathfrak{I}(\mathcal{X})$ is a closed set, in view of Theorem 4.1, it suffices to show that $\mathcal{K}$ is equal to

$$\mathcal{K}_{\mathcal{Q}} := \left\{k \in \mathcal{X} \;:\; \mathfrak{I}(k) \in \mathfrak{K}_{\mathcal{Q}}\right\}.$$

Towards a contraposition, assume that there is $f_0 \in \mathcal{K}_{\mathcal{Q}} \setminus \mathcal{K}$. Since $\mathcal{K}$ is closed by its definition, by Hahn-Banach, there exists $\eta_0 \in \mathcal{X}'$ satisfying

$$c_0 := \sup_{f\in\mathcal{K}} \eta_0(f) < \eta_0(f_0).$$

Since $\mathcal{K}$ contains $\mathcal{H}_{\mathcal{Q}} + \mathcal{X}_-$, $c_0 = 0$ and also $\eta_0 \geqq 0$. Consequently, $\eta_0 \in \mathcal{H}_{\mathcal{Q}}^{\perp} \cap \mathcal{X}'_+$ and this set is equal to $\mathcal{C}_{\mathcal{Q}}$.

Since $\mathfrak{I}(f_0) \in \mathfrak{K}_{\mathcal{Q}}$ and $\eta_0 \in \mathcal{C}_{\mathcal{Q}}$, $\eta_0(f_0) = \mathfrak{I}(f_0)(\eta_0) \leqq 0$. This contradicts the contraposition hypothesis. This proves that $\mathcal{K}_{\mathcal{Q}} = \mathcal{K} = \overline{\mathcal{H}_{\mathcal{Q}} + \mathcal{X}_-}$ and consequently,

$$\mathbf{P}(f;\mathcal{Q}) = \mathbf{D}(f;\overline{\mathcal{H}_{\mathcal{Q}} + \mathcal{X}_-}).$$

We now conclude by using Lemma 3.3. □

Remark 4.3. It is clear that the dual attainment is equivalent to the closedness of the set $\mathcal{H}_{\mathcal{Q}}+\mathcal{X}_-$. However, in general, this set may not be closed. In such situations, the duality holds without dual attainment.

The corollary above shows why the duality is usually easier to prove in $\mathcal{X}$. Indeed, as shown in Lemma 3.3, thanks to the *a priori* regularity of the value of the dual problem with respect to the lattice norm, a hedging set and its strong closure gives the same value for the dual problem. This invariance with respect to strong closure is exactly the crucial ingredient used above to prove the duality in $\mathcal{X}$.

An alternate approach to duality in $\mathcal{X}$ is developed in a series of papers [4, 12, 13]. Indeed, these papers also establish duality for continuous functions very efficiently for a very general class. Then, they extend their results to upper semi-continuous functions by analytic approximation techniques.

4.3. Duality with Lower Subspaces

We call a set in $\mathcal{X}''$ a *lower subspace* if it is of the form $\mathfrak{H} + \mathcal{X}''_-$ for some subspace $\mathfrak{H}$. In this section, we investigate when the duality holds with these types of dual sets.

Recall that $\mathcal{A}_{\mathcal{Q}}$, $\mathcal{C}_{\mathcal{Q}}$, $\mathfrak{K}_{\mathcal{Q}}$, $\mathfrak{H}_{\mathcal{Q}}$ are defined in Section 3. Futher let $\hat{\mathcal{A}}_{\mathcal{Q}}$ be the linear span of $\mathcal{Q}$. Then, $\hat{\mathcal{A}}_{\mathcal{Q}} = \mathcal{C}_{\mathcal{Q}} - \mathcal{C}_{\mathcal{Q}}$. Set

$$\hat{\mathfrak{h}}_{\mathcal{Q}} := \hat{\mathcal{A}}_{\mathcal{Q}}^{\perp}, \qquad \mathfrak{A}_{\mathcal{Q}} := \overline{\hat{\mathfrak{H}}_{\mathcal{Q}} + \mathcal{X}''_-}.$$

For any a set B in the dual of a Banach space, $\overline{B}^*$ is the weak* closure of B.

Theorem 4.4. *Under the Assumption 3.1, the following are equivalent:*

(1) *There exists a subspace $\mathfrak{H}$ of $\mathcal{X}''$ such that the duality on $\mathcal{X}''$ holds with $\mathfrak{H}+\mathcal{X}''_-$, i.e.,*

$$\mathbf{P}(\mathfrak{a};\, \mathcal{Q}) = \mathbf{D}(\mathfrak{a};\, \mathfrak{H} + \mathcal{X}''_-), \quad \forall \mathfrak{a} \in \mathcal{X}''.$$

(2) $\mathfrak{A}_{\mathcal{Q}} = \mathfrak{K}_{\mathcal{Q}}$.

(3) *The duality with $\mathfrak{A}_{\mathcal{Q}}$ holds on $\mathcal{X}''$, i.e.,*

$$\mathbf{P}(\mathfrak{a};\, \mathcal{Q}) = \mathbf{D}(\mathfrak{a};\, \mathfrak{A}_{\mathcal{Q}}), \quad \forall \mathfrak{a} \in \mathcal{X}''.$$

(4) $\overline{\mathfrak{I}(\hat{\mathcal{A}}_{\mathcal{Q}})}^{\,*} \cap \mathcal{X}'''_+ = \overline{\mathfrak{I}(\mathcal{C}_{\mathcal{Q}})}^{\,*}$.

Moreover, when (1) *holds, then, $\mathfrak{H}$ is a subset of $\hat{\mathfrak{H}}_{\mathcal{Q}}$ and the strong closure of $\mathfrak{H} + \mathcal{X}''_-$ is equal to $\mathfrak{K}_{\mathcal{Q}}$.*

Proof. (1) $\Rightarrow$ (2). For any $\mathfrak{a} \in \mathfrak{H}$ and $\eta \in \mathcal{Q}$,

$$\mathfrak{a}(\eta) \leqq \mathbf{P}(\mathfrak{a};\, \mathcal{Q}) = \mathbf{D}(\mathfrak{a};\, \mathfrak{H} + \mathcal{X}''_-) \leqq 0.$$

Since $\mathfrak{H}$ is a subspace, the above implies that $\mathfrak{H} \subset \hat{\mathfrak{H}}_{\mathcal{Q}}$. Let $\mathfrak{K}$ be the strong closure of $\mathfrak{H} + \mathcal{X}''_-$. Then, $\mathfrak{K} \subset \mathfrak{A}_{\mathcal{Q}} \subset \mathfrak{K}_{\mathcal{Q}}$.

Let $\mathfrak{a} \in \mathfrak{K}_{\mathcal{Q}}$. Then, by the definition of $\mathfrak{K}_{\mathcal{Q}}$, $\mathbf{P}(\mathfrak{a}; \mathcal{Q}) \leqq 0$. Hence,

$$\mathbf{D}(\mathfrak{a}; \mathfrak{K}) = \mathbf{P}(\mathfrak{a}; \mathcal{Q}) \leqq 0.$$

Since $\mathfrak{K}$ is closed and has the property (3.1), there is $\mathfrak{z}_0 \in \mathfrak{K}$ so that

$$\mathbf{P}(\mathfrak{a}; \mathcal{Q})\mathbf{e} + \mathfrak{z}_0 = \mathfrak{a}.$$

We use the property (3.1) once more to conclude that $\mathfrak{a} \in \mathfrak{K}$. So we have proved that $\mathfrak{K} = \mathfrak{K}_{\mathcal{Q}}$. Since $\mathfrak{K} \subset \mathfrak{A}_{\mathcal{Q}} \subset \mathfrak{K}_{\mathcal{Q}}$, this also proves that $\mathfrak{A}_{\mathcal{Q}} = \mathfrak{K}_{\mathcal{Q}}$.

(2) $\Rightarrow$ (3). This follows directly from Theorem 4.1.

(3) $\Rightarrow$ (1). In view Lemma 3.3, the primal problem with $\hat{\mathfrak{H}}_{\mathcal{Q}} + \mathcal{X}''_-$ and with its closure $\mathfrak{A}_{\mathcal{Q}}$ have the same value.

(4) $\Rightarrow$ (2). Towards a contraposition, suppose that $\mathfrak{A}_{\mathcal{Q}}$ is not equal to $\mathfrak{K}_{\mathcal{Q}}$. Let $\mathfrak{a}_0 \in \mathfrak{K}_{\mathcal{Q}} \setminus \mathfrak{A}_{\mathcal{Q}}$. By Hahn-Banach, there exists $\aleph_0 \in \mathcal{X}'''$ such that

$$c_0 = \sup_{\mathfrak{a} \in \mathfrak{A}_{\mathcal{Q}}} \aleph_0(\mathfrak{a}) < \aleph_0(\mathfrak{a}_0).$$

As was argued before in similar situations, we conclude that $\aleph_0 \in \hat{\mathfrak{H}}_{\mathcal{Q}}^{\perp} \cap \mathcal{X}'''_+$ and $c_0 = 0$. Since $\hat{\mathfrak{H}}_{\mathcal{Q}}^{\perp}$ is equal to the weak* closure of $\mathfrak{I}(\hat{\mathcal{A}}_{\mathcal{Q}})$, $\aleph_0 \in \overline{\mathfrak{I}(\hat{\mathcal{A}}_{\mathcal{Q}})}^{\,*} \cap \mathcal{X}'''_+$ and hence, $\aleph_0 \in \overline{\mathfrak{I}(\mathcal{C}_{\mathcal{Q}})}^{\,*}$. It is clear that

$$\aleph(\mathfrak{z}) \leqq 0, \quad \forall \mathfrak{z} \in \mathfrak{K}_{\mathcal{Q}}, \quad \aleph \in \overline{\mathcal{C}_{\mathcal{Q}}}^{\,*}.$$

Hence, $\aleph_0(\mathfrak{a}_0) \leqq 0$. This contradicts with the contraposition hypothesis. Hence, $\mathfrak{A}_{\mathcal{Q}} = \mathfrak{K}_{\mathcal{Q}}$.

(2) $\Rightarrow$ (4). Suppose that the contrary of (4) holds. Then, there is

$$\beth_0 \in \overline{\mathfrak{I}(\hat{\mathcal{A}}_{\mathcal{Q}})}^{\,*} \cap \mathcal{X}'''_+ \setminus \overline{\mathfrak{I}(\mathcal{C}_{\mathcal{Q}})}^{\,*}.$$

Since $\overline{\mathfrak{I}(\mathcal{C}_{\mathcal{Q}})}^{\,*}$ is regularly convex, there exists $\mathfrak{a}_0 \in \mathcal{X}''$ satisfying,

$$c_0 = \sup_{\beth \in \overline{\mathfrak{I}(\mathcal{C}_{\mathcal{Q}})}^{\,*}} \beth(\mathfrak{a}_0) < \beth_0(\mathfrak{a}_0).$$

Then, it is clear that $c_0 = 0$ and consequently, $\mathfrak{a}_0 \in \mathfrak{K}_{\mathcal{Q}}$. Then, by (2), $\mathfrak{a}_0 \in \mathfrak{A}_{\mathcal{Q}}$ and there exists a sequence $\mathfrak{a}_n = \mathfrak{h}_n + \mathfrak{z}_n \in \hat{\mathfrak{H}}_{\mathcal{Q}} + \mathcal{X}''_-$ converging to $\mathfrak{a}_0$ in the strong topology of $\mathcal{X}''$. For each n, since $\mathfrak{h}_n \in \hat{\mathfrak{H}}_{\mathcal{Q}}$, $\beth_0(\mathfrak{h}_n) = 0$ and since $\mathfrak{z}_n \leqq 0$, $\beth_0(\mathfrak{z}_n) \leqq 0$. Therefore, $\beth_0(\mathfrak{a}_n) \leqq 0$ for each n and by letting n tend to infinity, we conclude that $\beth_0(\mathfrak{a}_0) \leqq 0$. This contradicts with the choice of $\beth_0$, namely, $\beth_0(\mathfrak{a}_0) > c_0 = 0$. □

Remark 4.5. One could prove the implication (4) $\Rightarrow$ (1) by considering the duality in the space $\mathcal{X}''$ and then applying Corollary 4.2. However, for structural reasons, this approach requires the condition (4).

Example 8.4 below shows that, in general neither $\mathfrak{H}_{\mathcal{Q}} + \mathcal{X}''_-$ nor $\hat{\mathfrak{H}}_{\mathcal{Q}} + \mathcal{X}''_-$ are equal to $\mathfrak{K}_{\mathcal{Q}}$. Example 8.4 of [8] also demonstrates a similar phenomenon.

4.4. Factor Spaces

In our context, Theorem 12 of [26] states that if $\mathfrak{H}$ is a regularly convex subspace of $\mathcal{X}''$, then the dual of the subspace $\mathfrak{H}_\perp$ is equal to the quotient space $\mathcal{X}''/\mathfrak{H}$. This result provides a statement quite similar to the duality proved earlier but with two-sided inequalities.

Lemma 4.6. *Suppose that Assumption 3.1 holds. Then, $\mathcal{X}''/\mathfrak{H}_\mathcal{Q}$ is the topological dual of $\mathcal{A}_\mathcal{Q}$. Consequently,*

$$\sup_{\eta\in\mathcal{A}_\mathcal{Q}} \frac{|\mathfrak{a}(\eta)|}{\|\eta\|_{\mathcal{X}'}} = \inf_{\mathfrak{h}\in\mathfrak{H}_\mathcal{Q}} \|\mathfrak{a}-\mathfrak{h}\|_{\mathcal{X}''}, \quad \mathfrak{a}\in\mathcal{X}''.$$

Proof. Since by its definition $\mathcal{A}_\mathcal{Q}$ is closed, by the Lemma on page 573 in [26], we conclude that

$$\left(\mathfrak{H}_\mathcal{Q}\right)_\perp = \left(\mathcal{A}_\mathcal{Q}^\perp\right)_\perp = \mathcal{A}_\mathcal{Q}.$$

Hence, Theorem 12 of [26] implies the duality statement of the lemma. Also observe that for any $\mathfrak{a}\in\mathcal{X}''$,

$$\sup_{\eta\in\mathcal{A}_\mathcal{Q}} \frac{|\mathfrak{a}(\eta)|}{\|\eta\|_{\mathcal{X}'}} = \|\mathfrak{a}\|_{(\mathcal{A}_\mathcal{Q})'} = \|\mathfrak{a}\|_{\mathcal{X}''/\mathfrak{H}_\mathcal{Q}} = \inf_{\mathfrak{h}\in\mathfrak{H}_\mathcal{Q}} \|\mathfrak{a}-\mathfrak{h}\|_{\mathcal{X}''}.$$

□

Remark 4.7. One may interpret the left hand side of the above equation as a primal transport problem and the right hand side as its dual. Indeed, (2.2) implies the following duality with $\tilde{\mathcal{Q}} := \mathcal{H}_\mathcal{Q}^\perp \cap B_1''$,

$$\begin{aligned}\tilde{\mathbf{P}}(\mathfrak{a};\tilde{\mathcal{Q}}) &:= \sup_{\eta\in\tilde{\mathcal{Q}}} \mathfrak{a}(\eta)\\ &= \tilde{\mathbf{D}}(\mathfrak{a};\mathfrak{H}) := \inf\left\{\, c \geqq 0 \;:\; \exists\, \mathfrak{h}\in\mathfrak{H} \text{ such that } -\mathbf{c} \leqq \mathfrak{a}-\mathfrak{h} \leqq \mathbf{c}\right\}.\end{aligned}$$

Notice that $\tilde{\mathcal{Q}}$ is not a subset of $\mathcal{X}'_+$ and this is a crucial difference between the above identity and the duality (4.1).

5. Classical Optimal Transport

This section studies the classical duality result of KANTOROVICH [24] in this context. The optimal transport duality for general Borel measurable functions was proved by KELLERER [25] and a very general extension was recently given by BEIGLBÖCK, LEONARD and SCHACHERMAYER [7]. We also refer to the lecture notes of AMBROSIO [3] and the classical books of RACHEV and RÜSCHENDORF [29], VILLANI [35] and the references therein for more information.

5.1. Set-up

For two closed sets $X, Y \subset \mathbb{R}^d$ set $\Omega := X \times Y$,

$$\mathcal{C}_b = C_b(\Omega), \quad \mathcal{C}_x := C_b(X), \quad \mathcal{C}_y := C_b(Y).$$

The Banach lattice $\mathcal{X} = \mathcal{C}_b$ has the order unit $\mathbf{e} \equiv 1$. We fix

$$\mu \in \mathcal{M}_1(X), \quad \text{and} \quad \nu \in \mathcal{M}_1(Y),$$

where $\mathcal{M}_1(Z)$ is the set of all probability measures on a given Borel subset Z of a Euclidean space. Set

$$\mathcal{H}_{ot} := \left\{h \oplus g \;:\; h \in \mathcal{C}_x,\; g \in \mathcal{C}_y \quad \text{and} \quad \mu(h) = \nu(g) = 0\right\},$$
$$\mathcal{Q}_{ot} := \left\{\eta \in (\mathcal{C}_b')_+ \cap B_1' \;:\; \eta(f) = 0, \;\; \forall f \in \mathcal{H}_{ot}\right\}.$$

By its definition, $\mathcal{Q}_{ot}$ and $\mathcal{H}_{ot}$ satisfy Assumption 3.1. $\mathcal{Q}_{ot}$ also has the following well-known representation [25]. We provide its simple proof for completeness.

We first note that by its definition $\mathcal{Q}_{ot}$ is a subset of $\mathcal{C}_b'(\Omega)$ and any element $\varphi \in \mathcal{C}_b'(\Omega)$ is a *regular* bounded finitely additive measure on Ω. Moreover, φ is countably additive if and only if it is tight, i.e., for every $\epsilon > 0$, there is a compact $K_\epsilon \subset \Omega$ such that $|\varphi(\Omega \setminus K_\epsilon)| \leqq \epsilon$ (see [2]). Since the marginals μ and ν are countably additive measures, we show in the Lemma below that the elements of $\mathcal{Q}_{ot}$ are tight and consequently are countably additive probability measures.

Lemma 5.1. *Any $\eta \in \mathcal{C}_b'$ belongs to $\mathcal{A}_{ot} = \mathcal{H}_{ot}^\perp$ if and only if $\eta_x = \eta(\mathbf{1})\mu$ and $\eta_y = \eta(\mathbf{1})\nu$. Moreover, $\mathcal{Q}_{ot}$ is a non-empty subset of $\mathcal{M}_1(\Omega)$ and $\eta \in \mathcal{Q}_{ot}$ if and only if $\eta_x = \mu$ and $\eta_y = \nu$.*

Proof. Clearly $\mu \times \nu \in \mathcal{Q}_{ot}$ and hence $\mathcal{Q}_{ot}$ and therefore $\mathcal{H}_{ot}^\perp$ are non-empty.

Let $\eta \in \mathcal{H}_{ot}^\perp$. Then, for any $h \in \mathcal{C}_x$, $h \oplus \mathbf{0} - \mu(h) \in \mathcal{H}_{ot}$. Therefore,

$$0 = \eta(h \oplus \mathbf{0} - \mu(h)) = \eta_x(h) - \eta(\mathbf{1})\mu(h).$$

Hence, $\eta_x = \eta(\mathbf{1})\mu$. Similarly, $\eta_y = \eta(\mathbf{1})\nu$. The opposite implication is immediate.

Suppose $\eta \in \mathcal{Q}_{ot}$. Then, $\eta(\mathbf{1}) = 1$ and consequently, $\eta_x = \mu$ and $\eta_x = \nu$. It remains to show that η is in $ca_r(\Omega)$ or equivalently that it is countably additive.

For each $\epsilon > 0$ choose compact sets $\hat{K}_x^\epsilon \subset X$, $\hat{K}_y^\epsilon \subset Y$ so that

$$\mu\left(\hat{K}_x^\epsilon\right), \nu\left(\hat{K}_y^\epsilon\right) > 1 - \epsilon/2.$$

Then, there exist $h \in \mathcal{C}_x$, $g \in \mathcal{C}_y$ and compact sets $K_x^\epsilon \subset X$, $K_y^\epsilon \subset Y$ such that $0 \leqq h, g \leqq 1$, $h(x) = 1$ whenever $x \in \hat{K}_x^\epsilon$, $h(x) = 0$ for all $x \notin K_x^\epsilon$, and $g(y) = 1$ whenever $y \in \hat{K}_y^\epsilon$, $h(y) = 0$ for all $y \notin K_y^\epsilon$. Set $\Omega_\epsilon := K_x^\epsilon \times K_y^\epsilon$. Then, for any $\eta \in \mathcal{Q}_{ot}$,

$$\eta(\Omega_\epsilon) \geqq \eta(h \oplus g)/2.$$

Since $g, h \leqq 1$, $h(x), g(y) \geqq h(x)g(y)$. Hence,

$$h(x) + g(y) \geqq 2h(x)g(y) = h(x) + g(y) - h(x)(1 - g(y)) - h(y)(1 - g(x)) \\ \geqq h(x) + g(y) - (1 - g(y)) - (1 - h(x)) = 2[h(x) + g(y) - 1].$$

This implies that $(h \oplus g)/2 \geqq h \oplus \mathbf{0} + \mathbf{0} \oplus g - 1$. Combining all the above inequalities, we conclude the following for any $\eta \in \mathcal{Q}_{ot}$,

$$\eta(\Omega_\epsilon) \geqq \eta(h \oplus g)/2 \geqq \eta(h \oplus \mathbf{0}) + \eta(\mathbf{0} \oplus g) - 1 = \mu(h) + \nu(g) - 1 \\ \geqq \mu(\hat{K}_\epsilon) + \nu(\hat{K}_\epsilon) - 1 \geqq 1 - \epsilon.$$

Hence, any $\eta \in \mathcal{Q}_{ot}$ is tight. In addition $\eta \in \mathcal{C}_b'$ and therefore, it is regular and finitely additive. These imply that any $\eta \in \mathcal{Q}_{ot}$ is a countably additive. □

5.2. Dual Elements

Since $\mathcal{Q}_{ot}$ is non-empty, by Corollary 4.2, the duality holds for continuous functions, i.e.,

$$\begin{aligned}\mathbf{P}(f; \mathcal{Q}_{ot}) &:= \sup_{\eta \in \mathcal{Q}_{ot}} \eta(f) \\ &= \mathbf{D}(f; \mathcal{H}_{ot} + (\mathcal{C}_b(\Omega))_-) \\ &:= \inf \left\{ c \in \mathbb{R} \ : \ \exists h \in \mathcal{H}_{ot} \ \text{ such that } \ \mathbf{c} + h \geqq f \right\}, \quad f \in \mathcal{C}_b.\end{aligned}$$

We continue by studying the duality in the bidual and in $\mathcal{B}_b(\Omega)$. By Theorem 4.1, the duality on $\mathcal{C}_b''$ holds with

$$\mathfrak{K}_{ot} := \left\{ \mathfrak{z} \in \mathcal{C}_b'' \ : \ \mathfrak{z}(\eta) \leqq 0, \quad \forall\, \eta \in \mathcal{Q}_{ot} \right\}.$$

By Lemma 3.5, $\mathfrak{K}_{ot}$ is the weak* closure of $\mathfrak{H}_{ot} + (\mathcal{C}_b'')_-$, where

$$\mathfrak{H}_{ot} := \left\{ \mathfrak{h} \in \mathcal{C}_b'' : \ \mathfrak{h}(\eta) = 0, \ \forall\, \eta \in \mathcal{A}_{ot} = \mathcal{H}_{ot}^{\perp} \right\}.$$

We continue by obtaining a characterization of $\mathfrak{H}_{ot}$ and $\mathfrak{K}_{ot}$. We then use these results to prove the duality in $\mathcal{B}_b(\Omega)$.

Towards this goal, first observe that the projection maps

$$\Pi_x : \eta \in \mathcal{C}_b' \mapsto \eta_x \in \mathcal{C}_x', \quad \text{and} \quad \Pi_y : \eta \in \mathcal{C}_b' \mapsto \eta_y \in \mathcal{C}_y'$$

are bounded linear maps with operator norm equal to one. Also, for $\mathfrak{b} \in \mathcal{C}_x''$, $\mathfrak{c} \in \mathcal{C}_y''$, define $\mathfrak{b} \oplus \mathfrak{c}$ in $\mathcal{C}_b''$ by

$$(\mathfrak{b} \oplus \mathfrak{c})(\eta) := \mathfrak{b}(\eta_x) + \mathfrak{c}(\eta_y), \quad \forall\, \eta \in \mathcal{C}_b'.$$

We start by proving that certain relevant sets are regularly convex. Recall that $\mu \in \mathcal{M}_1(X)$ and $\nu \in \mathcal{M}_1(Y)$ are given probability measures. Set

$$\mathfrak{B} = \mathfrak{B}_\mu := \left\{ \mathfrak{a} \in \mathcal{C}_b'' \ : \ \exists \mathfrak{b} \in \mathcal{C}_x'', \ \text{ such that } \ \mathfrak{a} = \mathfrak{b} \oplus \mathbf{0} \ \text{ and } \ \mathfrak{b}(\mu) = 0 \right\},$$

$$\mathfrak{C} = \mathfrak{C}_\nu := \left\{ \mathfrak{a} \in \mathcal{C}_b'' \ : \ \exists \mathfrak{c} \in \mathcal{C}_y'', \ \text{ such that } \ \mathfrak{a} = \mathbf{0} \oplus \mathfrak{c} \ \text{ and } \ \mathfrak{c}(\nu) = 0 \right\}.$$

Lemma 5.2. *$\mathfrak{B}$ and $\mathfrak{C}$ are regularly convex.*

Proof. Since, both $\mathfrak{B}$ and $\mathfrak{C}$ are clearly convex, we need to prove that they are also weak* closed. Since the proofs for $\mathfrak{B}$ and $\mathfrak{C}$ are same, we prove only $\mathfrak{B}$. Let

$$\Pi_x' : \mathcal{C}_x'' \mapsto \mathcal{C}_b''$$

be the adjoint operator of Π_x. Then,

$$\Pi_x'(\mathfrak{b})(\eta) = \mathfrak{b}(\Pi_x(\eta)) = \mathfrak{b}(\eta_x) = \mathfrak{b} \oplus \mathbf{0}(\eta), \quad \forall \mathfrak{b} \in \mathcal{C}_x'', \ \eta \in \mathcal{C}_b'.$$

In particular,

$$\mathfrak{B} = \Pi_x' \left(\{\mathfrak{b} \in \mathcal{C}_x'' : \mathfrak{b}(\mu) = 0\} \right).$$

Additionally, the set $\{\mathfrak{b} \in \mathcal{C}_x'' : \mathfrak{b}(\mu) = 0\}$ is weak* closed. Therefore, by the closed range theorem [30][Theorem 4.14], the weak* closedness of $\mathfrak{B}$ is implied by the surjectivity of the maps Π_x (and hence the closedness of its range).

Indeed, fix $\beta' \in \mathcal{C}_y' \cap B_+'$. Then, for any $\alpha \in \mathcal{C}_x'$,

$$\Pi_x(\alpha \times \beta') = \alpha.$$

Hence, Π_x is surjective and hence, $\mathfrak{B}$ is regularly convex. □

We continue by characterizing $\mathfrak{K}_{ot}$. The following estimate is needed towards this result. For $R > 0$, set $\mathfrak{B}_R := \mathfrak{B} \cap B_R''$, $\quad \mathfrak{C}_R := \mathfrak{C} \cap B_R''$.

Lemma 5.3. *For any $R > 0$, we have,*

$$(\mathfrak{B} + \mathfrak{C}) \cap B_R'' \subset \mathfrak{B}_R + \mathfrak{C}_R,$$
$$\left(\mathfrak{B} + \mathfrak{C} + (\mathcal{C}_b'')_-\right) \cap B_R'' \subset \mathfrak{B}_{3R} + \mathfrak{C}_{3R} + \left[(\mathcal{C}_b'')_- \cap B_{7R}''\right].$$

Proof. The proof of the first statement is simpler, so we only prove the second estimate. Also by scaling it suffices to consider the case $R = 1$.

Set $\mathfrak{K} := \mathfrak{B} + \mathfrak{C} + (\mathcal{C}_b'')_-$ and fix $\mathfrak{z}_0 \in \mathfrak{K} \cap B_1''$. Then, there are $\mathfrak{b}_0 \in \mathcal{C}_x''$, $\mathfrak{c}_0 \in \mathcal{C}_y''$ and $\mathfrak{n}_0 \in (\mathcal{C}_b'')_-$ so that

$$\mathfrak{z}_0 = \mathfrak{b}_0 \oplus \mathfrak{c}_0 + \mathfrak{n}_0 \text{ and } \mathfrak{b}_0(\mu) = \mathfrak{c}_0(\nu) = 0.$$

Then, for any $\alpha \in \mathcal{M}_1(X)$ and $\beta \in \mathcal{M}_1(Y)$,

$$\mathfrak{b}_0(\alpha) = \mathfrak{b}_0(\alpha) + \mathfrak{c}_0(\nu) \geqq \mathfrak{z}_0(\alpha \times \nu) \geqq -1$$
$$\mathfrak{c}_0(\beta) = \mathfrak{b}_0(\mu) + \mathfrak{c}_0(\beta) \geqq \mathfrak{z}_0(\mu \times \beta) \geqq -1.$$

Hence, $\mathfrak{b}_0, \mathfrak{c}_0 \geqq -1$. Set,

$$\mathfrak{b}_1 := \mathfrak{b}_0 \wedge \mathbf{2}, \quad \mathfrak{b}_2 := \mathfrak{b}_1 - \mathfrak{b}_1(\mu)\mathbf{1},$$
$$\mathfrak{c}_1 := \mathfrak{c}_0 \wedge \mathbf{2}, \quad \mathfrak{c}_2 := \mathfrak{c}_1 - \mathfrak{c}_1(\nu)\mathbf{1}.$$

We know that $\mathfrak{z}_0 \leqq \mathfrak{b}_0 \oplus \mathfrak{c}_0$, $\mathfrak{z}_0 \leqq \mathbf{1}$ and $\mathfrak{b}_0, \mathfrak{c}_0 \geqq -\mathbf{1}$. Using all these we conclude that $\mathfrak{z}_0 \leqq \mathfrak{b}_1 \oplus \mathfrak{c}_1$. Also, $\mathfrak{b}_1(\mu) \leqq \mathfrak{b}_0(\mu) = 0$ and $\mathfrak{c}_1(\nu) \leqq \mathfrak{c}_0(\nu) = 0$. These imply that $\mathfrak{b}_1(\mu), \mathfrak{c}_1(\nu) \in [-1, 0]$. Therefore,

$$\mathfrak{z}_0 \leqq \mathfrak{b}_1 \oplus \mathfrak{c}_1 \leqq \mathfrak{b}_2 \oplus \mathfrak{c}_2 = \mathfrak{b}_2 \oplus \mathbf{0} + \mathbf{0} \oplus \mathfrak{c}_2.$$

Moreover,

$$\mathfrak{b}_2 = \mathfrak{b}_0 \wedge \mathbf{2} - \mathfrak{b}_1(\mu)\mathbf{1} \leqq \mathbf{2} + \mathbf{1} \leqq \mathbf{3}, \quad \mathfrak{b}_2 = \mathfrak{b}_0 \wedge \mathbf{2} - \mathfrak{b}_1(\mu)\mathbf{1} \geqq \mathfrak{b}_0 \wedge \mathbf{2} \geqq -\mathbf{1}.$$

Hence, $\mathfrak{b}_2 \oplus \mathbf{0} \in \mathfrak{B}_3$. Similarly, $\mathbf{0} \oplus \mathfrak{c}_2 \in \mathfrak{C}_3$.

Finally, set $\mathfrak{n}_2 := \mathfrak{z}_0 - \mathfrak{b}_2 \oplus \mathfrak{c}_2$. It is now clear that $0 \geqq \mathfrak{n}_2 \in B''_7$. □

5.3. *Duality in the Bidual*

We now obtain a complete characterization of the dual elements.

Proposition 5.4. *We have* $\mathfrak{H}_{ot} = \mathfrak{B} + \mathfrak{C}$ *and* $\mathfrak{K}_{ot} = \mathfrak{B} + \mathfrak{C} + (\mathcal{C}''_b)_- = \mathfrak{H}_{ot} + (\mathcal{C}''_b)_-$. *In particular, for* $\mathfrak{a} \in \mathcal{C}''_b$,

$$\begin{aligned}\mathbf{D}_{ot}(\mathfrak{a}) := \mathbf{D}(\mathfrak{a}; \mathfrak{K}_{ot}) &= \mathbf{D}\left(\mathfrak{a}; \mathfrak{B} + \mathfrak{C} + \left(\mathcal{C}''_b\right)_-\right) \\ &= \min\left\{ c \in \mathbb{R} \ : \ \exists\, \mathfrak{b} \in \mathfrak{B},\ \mathfrak{c} \in \mathfrak{C},\ \textit{such that}\ \ c\,\mathbf{1} + \mathfrak{b} \oplus \mathfrak{c} \geqq \mathfrak{a} \right\} \\ &= \mathbf{P}_{ot}(\mathfrak{a}) := \mathbf{P}(\mathfrak{a}; \mathcal{Q}_{ot})\end{aligned}$$

Proof. Set $\mathfrak{K} := \mathfrak{B} + \mathfrak{C} + (\mathcal{C}''_b)_-$. It is clear that $\mathfrak{K} \subset \mathfrak{K}_{ot}$.

Step 1. (*Regular convexity of* $\mathfrak{K}$).

In view of Lemma 5.3, for any $R > 0$,

$$\mathfrak{K} \cap B''_R \subset \mathfrak{K}_R := \mathfrak{B}_{3R} + \mathfrak{C}_{3R} + \left(\mathcal{C}''_b\right)_- \cap B''_{7R}.$$

It is shown in Lemma 5.2 that $\mathfrak{B}$ and $\mathfrak{C}$ are regularly convex. It is clear that $(\mathcal{C}''_b)_-$ is also regularly convex. Hence, $\mathfrak{B}_{3R}$ and $\mathfrak{C}_{3R}$ and as well as $\mathcal{C}''_- \cap B''_{7R}$ are regularly convex for each $R > 0$. By Theorem 7 of [26], $\mathfrak{K}_R$ is then regularly convex. Consequently, by Lemma 10.1, $\mathfrak{K}$ is regularly convex.

Step 2. ($\mathfrak{H}_{ot} = \mathfrak{B} + \mathfrak{C}$).

Let $\mathfrak{a}_0 \in \mathfrak{H}_{ot} = \mathcal{A}^{\perp}_{ot}$. For $\alpha \in \mathcal{C}'_x$, $\beta \in \mathcal{C}'_y$ define

$$\mathfrak{b}_0(\alpha) := \mathfrak{a}_0(\alpha \times \nu), \quad \mathfrak{c}_0(\beta) := \mathfrak{a}_0(\mu \times \beta).$$

Then, for any $\eta \in \mathcal{C}'_b$,

$$(\mathfrak{b}_0 \oplus \mathfrak{c}_0)(\eta) - \mathfrak{a}_0(\eta) = \mathfrak{a}_0(\tilde{\eta}), \quad \text{where} \quad \tilde{\eta} = \eta_x \times \nu + \mu \times \eta_y - \eta.$$

One can directly check that $\tilde{\eta}_x = \mu$ and $\tilde{\eta}_y = \nu$. Hence, by Lemma 5.1, $\tilde{\eta} \in \mathcal{A}_{ot}$ and consequently $\mathfrak{a}_0(\tilde{\eta}) = 0$. This shows that $\mathfrak{a}_0 = \mathfrak{b}_0 \oplus \mathfrak{c}_0$ or equivalently $\mathfrak{H}_{ot} \subset \mathfrak{B} + \mathfrak{C}$. The opposite inclusion is immediate.

Step 3. (*Conclusion*).

In view of Lemma 3.5, $\mathfrak{K}_{ot} = \mathfrak{K}_{\mathcal{Q}_{ot}}$ is the weak* closure of $\mathfrak{H}_{ot} + (\mathcal{C}''_b)_-$. Also by the second step, $\mathfrak{H}_{ot} + (\mathcal{C}''_b)_- = \mathfrak{B} + \mathfrak{C} + (\mathcal{C}''_b)_- =: \mathfrak{K}$. In the first step, it is shown that $\mathfrak{K}$ is regularly convex and consequently is weak* closed. Therefore, the weak* closure of $\mathfrak{K} = \mathfrak{H}_{ot} + (\mathcal{C}''_b)_-$ is equal to itself.

The duality statement follows from Theorem 4.1. □

5.4. Duality in $\mathcal{B}_b(\Omega)$

We close this section by proving the duality on $\mathcal{B}_b(\Omega)$, the set of all bounded, Borel measurable functions. Let $\mathcal{N}_{ot}$ be the set of all $\mathcal{Q}_{ot}$ polar sets, i.e., a Borel set N is $\mathcal{Q}_{ot}$ polar when $\eta(N) = 0$ for every $\eta \in \mathcal{Q}_{ot}$. Further let $\mathcal{Z}_{ot}$ be the set of all functions $\zeta \in \mathcal{B}_b(\Omega)$ such that $\{\zeta \neq 0\} \in \mathcal{N}_{ot}$. Finally, set

$$\mathcal{H}_{ot}^{\infty} := \{h \oplus g \ : \ h \in \mathcal{B}_b(X),\ g \in \mathcal{B}_b(Y) \quad \text{and} \quad \mu(h) = \nu(g) = 0\}$$
$$\hat{\mathcal{H}}_{ot}^{\infty} := \left\{\zeta + h \oplus g \ : \ \zeta \in \mathcal{Z}_{ot},\ h \oplus g \in \mathcal{H}_{ot}^{\infty}\right\}.$$

We follow the lecture notes of Kaplan [23] to characterize the dual set.

Proposition 5.5 (Duality in $\mathcal{B}_b(\Omega)$). *For any $\xi \in \mathcal{B}_b(\Omega)$,*

$$\mathbf{P}_{ot}(\xi) = \mathbf{D}\left(\xi; \hat{\mathcal{H}}_{ot}^{\infty} + \mathcal{B}_b(\Omega)_-\right)$$
$$= \min\left\{ c \in \mathbb{R} \ : \ \exists\, \zeta \in \mathcal{Z}_{ot},\ h \oplus g \in \mathcal{H}_{ot}^{\infty},\ \textit{such that}\ c\,\mathbf{1} + \zeta + h \oplus g \geqq \xi \right\}.$$

Proof. In view of Theorem 4.1, it suffices to show the following:

$$\mathcal{K}_{ot} := \left\{\xi \in \mathcal{B}_b(\Omega) : \mathbf{P}_{ot}(\xi) \leqq 0\right\} = \hat{\mathcal{H}}_{ot}^{\infty} + \mathcal{B}_b(\Omega)_-.$$

It is clear that $\hat{\mathcal{H}}_{ot}^{\infty} + \mathcal{B}_b(\Omega)_- \subset \mathcal{K}_{ot}$. We continue by proving the opposite inclusion.
Step 1. Fix $\xi \in \mathcal{K}_{ot}$. Then, $\mathfrak{I}(\xi) \in \mathfrak{K}_{ot}$ and in view of Proposition 5.4, there are $\mathfrak{b} \in \mathcal{C}_x''$, $\mathfrak{c} \in \mathcal{C}_x''$ satisfying, $\mathfrak{I}(\xi) \leqq \mathfrak{b} \oplus \mathfrak{c}$ and $\mathfrak{b}(\mu) = \mathfrak{c}(\nu) = 0$. Consider the map,

$$\mathcal{G}_{\mathfrak{b}} : H \in \mathcal{L}^1(\Omega, \mu) \mapsto \mathcal{G}_{\mathfrak{b}}(H) := \mathfrak{b}(\mu_H), \quad \text{where} \quad \mu_H(A) := \int_A H(x)\, d\mu(x).$$

It is immediate that $\mathcal{G}_{\mathfrak{b}}$ is a bounded linear map on $\mathcal{L}^1(\Omega, \mu)$. Hence, there exists $h_\xi \in \mathcal{L}^\infty(X, \mu)$ satisfying,

$$\mathcal{G}_{\mathfrak{b}}(H) = \int_X h_\xi(x) H(x) d\mu(x) = \mu_H(h_\xi).$$

We rewrite the above identity as follows,

$$\mathfrak{b}(\alpha) = \alpha(h_\xi), \quad \forall \alpha \in \mathcal{C}_x' \ \text{ and } \ |\alpha| \ll \mu.$$

Similarly there is $g_\xi \in \mathcal{L}^\infty(Y, \nu)$ such that

$$\mathfrak{c}(\beta) = \beta(g_\xi), \quad \forall \beta \in \mathcal{C}_y' \ \text{ and } \ |\beta| \ll \nu.$$

We fix pointwise representatives of h_ξ and g_ξ and set

$$\mathcal{N}_\xi := \left\{\omega \in \Omega \ : \ \zeta_\xi(\omega) > 0\right\}, \quad \text{where} \quad \zeta_\xi(\omega) := \left[\xi - h_\xi \oplus g_\xi\right](\omega).$$

Step 2. In this step, we show that $(\zeta_\xi)^+ \in \mathcal{Z}_{ot}$.

Fix $\eta \in \mathcal{Q}_{ot}$. Let $\{\nu(x, \cdot)\}_{x \in X} \subset \mathcal{M}_1(Y)$ and $\{\mu(y, \cdot)\}_{y \in Y} \subset \mathcal{M}_1(X)$ be Borel measurable families of probability measures satisfying

$$\int_\Omega \zeta d\eta = \int_X \left[\int_Y \zeta(x,y)\nu(x,dy) \right] \mu(dx) = \int_Y \left[\int_X \zeta(x,y)\mu(y,dx) \right] \nu(dy),$$

for every $\zeta \in \mathcal{B}_b(\Omega)$. Since η is a probability measure, we may define $\eta_\xi \in \mathcal{C}_b'$ by,

$$\eta_\xi(f) := \int_{\mathcal{N}_\xi} f\, d\eta, \quad \forall f \in \mathcal{C}_b.$$

Then, one can directly show that for every Borel set $A \subset X$,

$$\left(\eta_\xi\right)_x (A) = \int_A \left[\int_Y \chi_{\mathcal{N}_\xi}(x,y)\nu(x,dy) \right] \mu(dx).$$

Since $\left(\eta_\xi\right)_x$ is absolutely continuous with respect to μ, the construction of h_ξ implies that

$$(\mathfrak{b} \oplus \mathbf{0})(\eta_\xi) = \mathfrak{b}\left(\left(\eta_\xi\right)_x\right) = \left(\eta_\xi\right)_x (h_\xi) = \int_\Omega h_\xi(x)\chi_{\mathcal{N}_\xi}(x,y)\eta(dx,dy).$$

Similarly, one can show that,

$$(\mathbf{0} \oplus \mathfrak{c})(\eta_\xi) = \mathfrak{b}\left(\left(\eta_\xi\right)_y\right) = \left(\eta_\xi\right)_y (g_\xi) = \int_\Omega g_\xi(y)\chi_{\mathcal{N}_\xi}(x,y)\eta(dx,dy).$$

These imply that

$$\begin{aligned} 0 \leqq (\mathfrak{b} \oplus \mathfrak{c} - \mathfrak{I}(\xi))(\eta_\xi) &= \int_\Omega [(h_\xi \oplus g_\xi) - \xi]\, \chi_{\mathcal{N}_\xi} d\eta \\ &= \int -\zeta_\xi \chi_{\{\zeta_\xi > 0\}} d\eta = -\eta((\zeta_\xi)^+) \leqq 0. \end{aligned}$$

Therefore, $\eta((\zeta_\xi)^+) = 0$ for every $\eta \in \mathcal{Q}_{ot}$ and $(\zeta_\xi)^+ \in \mathcal{Z}_{ot}$.
Step 3. The definition of ζ_ξ implies that

$$\xi = [\xi - h_\xi \oplus g_\xi] + h_\xi \oplus g_\xi \leqq (\zeta_\xi)^+ + h_\xi \oplus g_\xi.$$

Since $\mu(h_\xi) = \nu(g_\xi) = 0$, this proves that $\xi \in \hat{\mathcal{H}}_{ot}^\infty$. □

Remark 5.6. The above dual problem with the hedging set $\hat{\mathcal{H}}_{ot}^\infty$ can also be seen as quasi-sure super-replication in the sense defined in [17,31,32]. Indeed, we say two functions ℓ, ξ satisfy $\ell \geqq \xi$, $\mathcal{Q}_{ot}$ quasi-surely and write $\ell \geqq \xi$, $\mathcal{Q}_{ot} - q.s.$, if $\eta(\{\ell < \xi\}) = 0$ for every $\eta \in \mathcal{Q}_{ot}$ or equivalently, if the set $\{\ell < \xi\}$ is a $\mathcal{Q}_{ot}$ polar set. Then, we have the following immediate representation of the dual problem:

$$\begin{aligned} &\mathbf{D}\left(\xi; \hat{\mathcal{H}}_{ot}^\infty + \mathcal{B}_b(\Omega)_-\right) \\ &\quad = \inf\left\{c \in \mathbb{R} \ :\ \exists h \oplus g \in \mathcal{H}_{ot}^\infty, \ \text{ s.t. } \ c\mathbf{1} + h \oplus g \geqq \xi, \ \mathcal{Q}_{ot} - q.s.\right\}. \end{aligned}$$

In the classical paper of Kellerer [25], the duality is proved with the hedging set $\mathcal{H}_{ot}^\infty$ without augmenting it with the $\mathcal{Q}_{ot}$ polar sets. In particular, Kellerer's duality result shows that every $\mathcal{Q}_{ot}$ polar set N is dominated from above by the sum of a μ null set A and a ν null set B, i.e.,

$$\chi_N(x, y) \leqq \chi_A(x) + \chi_B(y), \quad \text{and} \quad \mu(A) = \nu(B) = 0. \tag{5.1}$$

Kellerer proved the above result by using the classical Choquet capacity theory. Indeed, the result would follow if the primal functional is shown to have certain regularity properties as assumed in the Choquet Theorem. Then one shows these properties using the Lusin theorem and other approximations.

On the other hand, our results imply that there are $\mathfrak{b} \in \mathcal{C}_x''$, $\mathfrak{c} \in \mathcal{C}_y''$ satisfying,

$$\mathfrak{I}(\chi_N) \leqq \mathfrak{b} \oplus \mathfrak{c}, \quad \text{and} \quad \mathfrak{b}(\mu) = \mathfrak{c}(\nu) = 0.$$

To prove (5.1) from the above statement is an interesting analytical question. In particular, it is not clear if the Kellerer approach via the Choquet capacity theory is the only possible way. We leave these questions to further research and do not pursue them here.

6. Constrained Optimal Transport

In this section, we investigate an extension of the classical optimal transport. In this extension, we are given a finite subset $\{f_1, \ldots, f_N\} \subset \mathcal{C}_b$. We set $\mathcal{Q}_0 := \mathcal{Q}_{ot}$ and for $k = 1, \ldots, N$, define,

$$\mathcal{Q}_k := \mathcal{H}_k^{\perp} \cap \partial B'_+, \quad \text{where} \quad \mathcal{H}_k := \left\{ f + \sum_{i=1}^{k} a_i f_i \ : \ f \in \mathcal{H}_{ot}, \ \ a_i \in \mathbb{R} \right\}.$$

We make the following structural assumption:

Assumption 6.1. For $k = 1, \ldots, N$, we assume that

$$\inf_{\eta \in \mathcal{Q}_{k-1}} \eta(f_k) < 0 < \sup_{\eta \in \mathcal{Q}_{k-1}} \eta(f_k).$$

Remark 6.2. The above assumption is equivalent to the following:

$$-\mathbf{P}(-f_k; \mathcal{Q}_{k-1}) < 0 < \mathbf{P}(f_k; \mathcal{Q}_{k-1}),$$

for every $k = 1, 2, \ldots, N$. In this assumption, the value zero is not important. Indeed, if $\bar{f}_k$ satisfies

$$-\mathbf{P}(-\bar{f}_k; \mathcal{Q}_{k-1}) < \mathbf{P}(\bar{f}_k; \mathcal{Q}_{k-1})$$

then there exists a constant b_k so that $f_k := \bar{f}_k - b_k$ satisfies the Assumption 6.1.

Moreover, Assumption 6.1 implies that $\mathcal{Q}_k$ is non-empty and satisfies the Assumption 3.1. Conversely, the inequality

$$-\mathbf{P}(-f_k; \mathcal{Q}_{k-1}) \leqq 0 \leqq \mathbf{P}(f_k; \mathcal{Q}_{k-1}),$$

is necessary for $\mathcal{Q}_k$ to be non-empty.

Set $\mathfrak{H}_0 := \mathfrak{H}_{ot}$, $\mathcal{Q}_{cot} := \mathcal{Q}_N$, $\mathcal{H}_{cot} := \mathcal{H}_N$, $\mathfrak{H}_{cot} := (\mathcal{H}_{cot}^{\perp})^{\perp}$,

$$\mathfrak{K}_{cot} := \left\{ \mathfrak{z} \in \mathcal{C}_b'' \;:\; \mathfrak{z}(\eta) \leqq 0, \;\forall \eta \in \mathcal{Q}_{cot} \right\},$$

and let $\mathfrak{F}_k$ be the subspace spanned by $\mathfrak{I}(f_k)$.

Proposition 6.3. *Suppose that Assumption 6.1 holds. Then,*

$$\mathfrak{H}_{cot} = \mathfrak{H}_{ot} + \sum_{i=1}^{N} \mathfrak{F}_i, \qquad \mathfrak{K}_{cot} = \mathfrak{H}_{cot} + \left(\mathcal{C}_b''\right)_- .$$

In particular,

$$\begin{aligned} \mathbf{P}(\mathfrak{a}; \mathcal{Q}_{cot}) &= \mathbf{D}\left(\mathfrak{a}; \mathfrak{H}_{cot} + \left(\mathcal{C}_b''\right)_-\right) \\ &= \min \Big\{ c \in \mathbb{R} \;:\; \exists \mathfrak{h} \in \mathcal{C}_x'', \;\; \mathfrak{g} \in \mathcal{C}_y'', \;\; a_1, \ldots, a_n \in \mathbb{R} \\ &\qquad\qquad \text{such that } \; \mathfrak{h} \oplus \mathfrak{g} + \sum_{k=1}^{N} a_k \mathfrak{I}(f_k) \geqq \mathfrak{a} \Big\}. \end{aligned}$$

Proof. Set

$$\mathfrak{H}_k := \left(\mathcal{H}_k^{\perp}\right)^{\perp}, \quad \mathfrak{K}_k := \left\{ \mathfrak{z} \in \mathcal{C}_b'' \;:\; \mathfrak{z}(\eta) \leqq 0, \;\forall \eta \in \mathcal{Q}_k \right\}, \qquad k = 1, \ldots, N.$$

It is clear that $\mathfrak{H}_{cot} = \mathfrak{H}_N$ and $\mathfrak{K}_{cot} = \mathfrak{K}_N$.

We shall prove by induction that

$$\mathfrak{H}_k = \mathfrak{H}_{ot} + \sum_{i=1}^{k} \mathfrak{F}_i, \qquad \mathfrak{K}_k = \mathfrak{H}_k + \left(\mathcal{C}_b''\right)_-, \qquad k = 1, \ldots, N.$$

We know that the above statements hold for $k = 0$. Indeed $\mathfrak{H}_0 = \mathfrak{H}_{ot}$ by definition and by Proposition 5.4, $\mathfrak{K}_0 = \mathfrak{H}_{ot} + \left(\mathcal{C}_b''\right)_-$. Suppose now that the claim holds for $k-1$ with some $k \geqq 1$. Set $\mathfrak{H} := \mathfrak{H}_{k-1} + \mathfrak{F}_k$ and $\mathfrak{K} := \mathfrak{K}_{k-1} + \mathfrak{F}_k$. We claim that both $\mathfrak{H}$ and $\mathfrak{K}$ are weak* closed. Since both proofs are similar, we prove only the second statement by an application of Lemma 10.1.

Fix an arbitrary $\mathfrak{a} \in \mathfrak{K} \cap B_1''$. Then, there are $\mathfrak{z}_{k-1} \in \mathfrak{K}_{k-1}$ and $a_k \in \mathbb{R}$ satisfying,

$$\mathfrak{a} = \mathfrak{z}_{k-1} + a_k \mathfrak{I}(f_k).$$

Since $\mathfrak{K}_{k-1} = \mathfrak{H}_{k-1} + \left(\mathcal{C}_b''\right)_-$, there are $\mathfrak{h}_{k-1} \in \mathfrak{H}_{k-1}$ and $\mathfrak{n}_{k-1} \leqq 0$ such that $\mathfrak{z}_{k-1} = \mathfrak{h}_{k-1} + \mathfrak{n}_{k-1}$. Also, Assumption 6.1 states that

$$\underline{p} := \inf_{\eta \in \mathcal{Q}_{k-1}} \eta(f_k) < 0 < \overline{p} := \sup_{\eta \in \mathcal{Q}_{k-1}} \eta(f_k).$$

We analyse two cases separately. First, suppose that $a_k \geqq 0$. Then,

$$a_k \, \underline{p} \geqq \inf_{\eta \in \mathcal{Q}_{k-1}} \left(\mathfrak{z}_{k-1} + a_k \mathfrak{I}(f_k)\right)(\eta) = \inf_{\eta \in \mathcal{Q}_{k-1}} \mathfrak{a}(\eta) \geqq -1.$$

Since $\underline{p} < 0$, $a_k \leqq 1/(-\underline{p})$.

Next, suppose that $a_k \leqq 0$. Then,

$$a_k \, \overline{p} \geqq \sup_{\eta \in \mathcal{Q}_{k-1}} \left(\mathfrak{z}_{k-1} + a_k \mathfrak{I}(f_k) \right)(\eta) = \sup_{\eta \in \mathcal{Q}_{k-1}} \mathfrak{a}(\eta) \geqq -1.$$

Hence $a_k \geqq -1/(\overline{p})$.

Combining both cases, we conclude that

$$|a_k| \leqq c_k^* := \max \left\{ \frac{1}{\overline{p}} \, , \; \frac{1}{-\underline{p}} \right\}.$$

Therefore,

$$\|\mathfrak{z}_{k-1}\|_\infty \leqq 1 + c_k^* \| f_k \|_\infty.$$

We now apply Lemma 10.1 of the appendix to conclude that $\mathfrak{K}$ is regularly convex. Hence, $\mathfrak{K} = \mathfrak{K}_k$.

Since $\mathfrak{K}$ is defined to be $\mathfrak{K}_{k-1} + \mathfrak{F}_k$, by the induction hypothesis,

$$\mathfrak{K}_{k-1} = \mathfrak{H}_{k-1} + \left(\mathcal{C}_b''\right)_- \quad \Rightarrow \quad \mathfrak{K}_k = \mathfrak{K} = \mathfrak{H}_{k-1} + \mathfrak{F}_k + \left(\mathcal{C}_b''\right)_- .$$

A similar induction argument shows that

$$\mathfrak{H}_k = \mathfrak{H}_{k-1} + \mathfrak{F}_k.$$

Hence, we conclude that

$$\mathfrak{K}_k = \mathfrak{H}_k + \left(\mathcal{C}_b''\right)_- \quad \text{and} \quad \mathfrak{H}_k = \mathfrak{H}_{ot} + \sum_{i=1}^{k} \mathfrak{F}_i.$$

The duality statement follows from the above characterizations, Theorem 4.1 and the fact that $\mathcal{Q}_{cot}$ satisfies the Assumption 3.1. □

We can prove the duality in $\mathcal{B}_b(\Omega)$ as in Proposition 5.5. We state this result without a proof for completeness. Let $\mathcal{Z}_{cot}$ be the set of all bounded functions ζ such that $\eta(\{\zeta \neq 0\}) = 0$ for every $\eta \in \mathcal{Q}_{cot}$. Set

$$\hat{\mathcal{H}}_{cot}^\infty := \left\{ \zeta + h \oplus g + \sum_{i=1}^{N} a_i \, f_i \; : \; \zeta \in \mathcal{Z}_{cot}, \; h \oplus g \in \mathcal{H}_{ot}^\infty, \;\; a_i \in \mathbb{R} \right\}.$$

Corollary 6.4. *For any $\xi \in \mathcal{B}_b(\Omega)$,*

$$\mathbf{P}_{cot}(\xi) = \mathbf{D}\left(\xi; \hat{\mathcal{H}}_{cot}^\infty + \mathcal{B}_b(\Omega)_-\right).$$

7. Martingale Measures and Super-Martingales

Suppose that $X = Y \subset \mathbb{R}^d$ be convex and closed sets. Recall that

$$\mathcal{C}_\ell = C_\ell(\Omega) := \{ f \in C(\Omega) \; : \; \|f\|_\ell < \infty \},$$

with the weighted norm defined in (2.4).

7.1. Definitions

Define a linear functional $T : C_b(X;\mathbb{R}^d) \mapsto \mathcal{C}_\ell$ by

$$T(\gamma)(x,y) := \gamma(x)\cdot(x-y), \;\; \forall\,(x,y)\in\Omega \;\; \gamma\in C_b(X;\mathbb{R}^d).$$

It is clear that $T(\gamma)\in\mathcal{C}_\ell$ and the adjoint $T' : \mathcal{C}'_\ell \mapsto C'_b(X;\mathbb{R}^d)$ satisfies,

$$T'(\eta)(\gamma) = \eta(T(\gamma)), \quad \forall\,\omega=(x,y)\in\Omega, \quad \gamma\in C_b(X;\mathbb{R}^d), \quad \eta\in\mathcal{C}'_\ell.$$

Let $T'' : C''_b(X;\mathbb{R}^d)\mapsto \mathcal{C}''_\ell$ to be the adjoint of T'. Then,

$$T''(\mathfrak{I}(\gamma)) = \mathfrak{I}(T(\gamma)), \quad \forall\,\gamma\in C_b(X;\mathbb{R}^d),$$

where $\mathfrak{I}$ is the canonical map of $C_b(X;\mathbb{R}^d)$ into $C''_b(X;\mathbb{R}^d)$. We then define a subset $\mathfrak{D}\subset\mathcal{C}''_\ell$ by,

$$\mathfrak{D} := \left\{ \mathfrak{f}\in\mathcal{C}''_\ell \;:\; \exists\mathfrak{g}\in C''_b(X;\mathbb{R}^d) \text{ such that } \mathfrak{f} = T''(\mathfrak{g}) \right\}.$$

Equivalently, $\mathfrak{D}$ is the range of the adjoint operator T''. Finally, set

$$\mathfrak{M} := \mathcal{M}^\perp, \quad \text{where} \quad \mathcal{M} := \left\{ \eta\in\mathcal{C}'_l \;:\; \eta(T(\gamma))=0, \;\; \forall\,\gamma\in C_b(X;\mathbb{R}^d) \right\},$$
$$\mathfrak{S} := \left\{ \mathfrak{h}\in\mathcal{C}''_l \;:\; \mathfrak{h}(\eta)\leqq 0, \;\; \forall\,\eta\in\mathcal{M}\cap(\mathcal{C}'_\ell)_+ \right\}.$$

It is clear that $\mathfrak{D}\subset\mathfrak{M}$ and $\mathfrak{D}+(\mathcal{C}''_\ell)_-\subset\mathfrak{S}$.

Definition 7.1. Any element η of $\mathcal{M}$ is called a martingale measure, any $\mathfrak{m}\in\mathfrak{M}$ a martingale, and any $\mathfrak{h}\in\mathfrak{S}$ a super-martingale.

7.2. The case $X = Y = \mathbb{R}^d$

The following result characterizes the sets defined above and also motivates the terminology used in that definition.

Proposition 7.2. *Let $X = Y = \mathbb{R}^d$. Then, $\mathfrak{D}$ and $\mathfrak{D}+(\mathcal{C}''_l)_-$ are regularly convex. In particular,*

$$\mathfrak{D} = \mathfrak{M} \quad \textit{and} \quad \mathfrak{D}+(\mathcal{C}''_\ell)_- = \mathfrak{S}.$$

Hence, in $\Omega = \mathbb{R}^d\times\mathbb{R}^d$, any martingale has the form $T''(\mathfrak{g})$ and $\mathfrak{h}\in\mathcal{C}''_\ell$ is a super-martingale if and only if it is dominated by a martingale.

Proof. We again use the closed range theorem to prove this result.

Step 1. (*Range of T is closed*).

Let γ_n be a sequence in $C_b(X;\mathbb{R}^d)$ so that $\xi_n = T(\gamma_n)$ is strongly convergent to $\xi\in\mathcal{C}_\ell$. For each $x\in X$, and positive integers n, m, set

$$y_{n,m}(x) := x - \frac{\gamma_n(x)-\gamma_m(x)}{|\gamma_n(x)-\gamma_m(x)|}.$$

Then,

$$|\gamma_n(x) - \gamma_m(x)| = \left|\xi_n(x, y_{n,m}(x)) - \xi_m(x, y_{n,m}(x))\right|$$
$$\leqq \|\xi_n - \xi_m\|_{\mathcal{C}_\ell} \,[2 + |x| + |y_{n,m}(x)|] \leqq 3[1 + |x|]\, \|\xi_n - \xi_m\|_{\mathcal{C}_\ell}.$$

Therefore, $\{\gamma_n\}$ is locally Cauchy in $C_b(X, \mathbb{R}^d)$. Consequently, as n tends to infinity, γ_n converges locally uniformly to a continuous function $\gamma \in C(X; \mathbb{R}^d)$. Moreover, it is clear that $\xi = T(\gamma)$. We claim that γ is bounded. Indeed, set

$$y(x, \lambda) := x - \lambda\, \frac{\gamma(x)}{|\gamma(x)|}, \qquad \lambda > 0,\ x \in X,\ \gamma(x) \neq 0,$$

and set $y(x, \lambda) = 0$ when $\gamma(x) = 0$. Then, we directly estimate that,

$$|\gamma(x)| = \frac{1}{\lambda}\, \xi(x, y(x, \lambda)) \leqq \frac{1}{\lambda}\, \|\xi\|_{\mathcal{C}_\ell}\, [2 + |x| + |y(x, \lambda)|]$$
$$\leqq \frac{1}{\lambda}\, \|\xi\|_{\mathcal{C}_\ell}\, [2 + 2|x| + \lambda].$$

We let λ to infinity to arrive at the following estimate:

$$\|\gamma\|_{C_b(X;\mathbb{R}^d)} \leqq \|\xi\|_{\mathcal{C}_\ell} = \|T(\gamma)\|_{\mathcal{C}_\ell}, \quad \forall\, \gamma \in C_b(X; \mathbb{R}^d). \tag{7.1}$$

Step 2. ($\mathfrak{D} = \mathfrak{M}$).

The previous step shows that the range of T is closed. Then, by the closed range Theorem [30][Theorem 4.14], the range of T' is weak* and also norm closed. We now apply the same theorem to T' to conclude that the range of T'' is weak* closed. Since $\mathfrak{D}$ is defined as the range of T'' and since it is linear, we conclude that it is regularly convex. Hence, $\mathfrak{M} = \mathfrak{D}$.

Step 3. (*A map*).

Define a linear map $L : \mathcal{C}_\ell \mapsto \mathcal{C}_\ell$ by

$$L(f)(x, y) := f(x, 2x - y), \quad \forall\, (x, y) \in \Omega, \quad f \in \mathcal{C}_\ell.$$

Then, one can directly verify that for any $(x, y) \in \Omega$ and $\gamma \in C_b(X; \mathbb{R}^d)$ the following identity holds:

$$L(T(\gamma))(x, y) = T(\gamma)(x, 2x - y) = \gamma(x) \cdot (x - (2x - y)) = -T(\gamma)(x, y).$$

We use this with $\eta \in \mathcal{C}_\ell'$ and $\gamma \in C_b(X; \mathbb{R}^d)$ to arrive at

$$T''(\mathfrak{I}(\gamma))(\eta) = \eta(T(\gamma)) = -\eta(L(T(\gamma))) = -L'(\eta)(T(\gamma))$$
$$= -T''(\mathfrak{I}(\gamma))(L'(\eta)).$$

By the weak* density of $\mathfrak{I}(C_b(X; \mathbb{R}^d))$ in $C_b''(X; \mathbb{R}^d)$, we conclude that the above holds for any element of $C_b''(X; \mathbb{R}^d)$, i.e.,

$$T''(\mathfrak{g})(\eta) = -T''(\mathfrak{g})(L'(\eta)), \quad \forall\, \mathfrak{g} \in C_b''(X; \mathbb{R}^d),\ \eta \in \mathcal{C}_\ell'.$$

Moreover, for any $f \in \mathcal{C}_\ell$ and $(x, y) \in \Omega$,

$$|L(f)(x, y)| = |f(x, 2x - y)| \leqq \|f\|_{\mathcal{C}_\ell} \ell(x, 2x - y) \leqq 3\|f\|_{\mathcal{C}_\ell} \ell(x, y).$$

Hence, for any $\eta \geqq 0$,

$$\|L'(\eta)\|_{\mathcal{C}'_\ell} \leqq 3\|\eta\|_{\mathcal{C}'_\ell}.$$

Also, $L'(\eta) \geqq 0$ whenever $\eta \geqq 0$.

Step 4. (*An estimate*).

Let $\mathfrak{a}$ be an arbitrary element of $\mathfrak{D} + (\mathcal{C}''_\ell)_-$. Then, there exists $\mathfrak{g} \in C''_b(X; \mathbb{R}^d)$ such that $T''(\mathfrak{g})(\eta) \geqq \mathfrak{a}(\eta)$. For any $\eta \in (\mathcal{C}'_\ell)_+$, the previous step implies the following:

$$T''(\mathfrak{g})(\eta) = -T''(\mathfrak{g})(L'(\eta)) \leqq -\mathfrak{a}(L'(\eta)) \leqq 3\|\mathfrak{a}\|_{\mathcal{C}''}\|\eta\|_{\mathcal{C}'_\ell}.$$

We also have,

$$T''(\mathfrak{g})(\eta) \geqq \mathfrak{a}(\eta) \geqq -\|\mathfrak{a}\|_{\mathcal{C}''}\|\eta\|_{\mathcal{C}'_\ell}.$$

Hence,

$$\left\|T''(\mathfrak{g})\right\|_{\mathcal{C}''_\ell} \leqq 3\|\mathfrak{a}\|_{\mathcal{C}''_\ell}, \quad \text{whenever} \quad T''(\mathfrak{g}) \geqq \mathfrak{a}.$$

We summarize the above estimate into the following

$$\left(\mathfrak{D} + (\mathcal{C}''_\ell)_-\right) \cap B''_{\ell,R} \subset \mathfrak{D} \cap B''_{\ell,3R} + (\mathcal{C}''_\ell)_- \cap B''_{\ell,4R}, \quad \forall\, R > 0,$$

where

$$B''_{\ell,R} = \left\{ \mathfrak{a} \in C''_\ell \;:\; \|\mathfrak{a}\|_{\mathcal{C}''_\ell} \leqq 1 \right\}.$$

Step 5. ($\mathfrak{D} + (\mathcal{C}''_\ell)_-$ *is regularly convex*).

Since $\mathfrak{D}$ is proved to be regularly convex, by the previous step and Lemma 10.1 of the Appendix, we conclude that $\mathfrak{D} + (\mathcal{C}''_\ell)_-$ is also regularly convex. Therefore, $\mathfrak{D} + (\mathcal{C}''_\ell)_-$ is equal to its regularly convex envelope $\mathfrak{S}$. □

We now give without proof the following corollary which is a direct consequence of the regular convexity of $\mathfrak{D} + (\mathcal{C}''_\ell)_-$.

Corollary 7.3. *For* $\mathfrak{a} \in \mathcal{C}''_\ell$,

$$\begin{aligned}\mathbf{P}(\mathfrak{a}; \mathcal{M} \cap \partial B'_+) &= \mathbf{D}(\mathfrak{a}; \mathfrak{D} + (\mathcal{C}''_\ell)_-) \\ &= \min\left\{ c \in \mathbb{R} : \exists \mathfrak{g} \in C''_b(X; \mathbb{R}^d),\ \textit{such that}\ c\,\mathbf{1} + T''(\mathfrak{g}) \geqq \mathfrak{a}\right\}.\end{aligned}$$

8. Martingale Optimal Transport

In this example, we take $X = Y = \mathbb{R}^d$ and set $\mathcal{X} = \mathcal{C}_\ell$ with the unit element

$$\mathbf{e}(x, y) = \ell(x, y) := \ell_X(x) + \ell_Y(y) = (1 + |x|) + (1 + |y|).$$

We also use the notation

$$\mathcal{C}_X := \mathcal{C}_{\ell_X}(X), \quad \mathcal{C}_Y := \mathcal{C}_{\ell_Y}(Y).$$

8.1. Set-Up

As in Section 5, we fix μ, ν. We assume that they are in convex order. For any $\alpha \in \mathcal{C}_X, \beta \in \mathcal{C}_Y$, set

$$m^* := m^*_X + m^*_Y, \quad m^*_X := \mu(\ell_X), \quad m^*_Y := \nu(\ell_Y).$$

Recall that the set of martingale measures $\mathcal{M}$ is defined in Section 7 as the annihilators of the range of the map T again introduced in that section. Set,

$$\mathcal{H}_{mot} := \left\{ h \oplus g + T(\gamma) \ : h \in \mathcal{C}_X, \ g \in \mathcal{C}_Y, \ \gamma \in C_b(X; \mathbb{R}^d) \right\},$$
$$\hat{\mathcal{Q}}_{mot} := \mathcal{H}^{\perp}_{mot} \cap \partial B'_+.$$

Then, any $\eta \in \hat{\mathcal{Q}}_{mot}$ satisfies $\eta \in ca^+_r(\Omega)$ and $\eta_x = \eta(\Omega)\mu$, $\eta_y = \eta(\Omega)\nu$. In particular,

$$\eta(\mathbf{e}) = \eta(\ell) = \int_\Omega \ell(x, y)\eta(dx, dy) = m^*\eta(\Omega) = 1.$$

Set,

$$\mathcal{Q}_{mot} := \left\{ \eta \in \mathcal{C}'_\ell \ : \ \eta/m^* \in \hat{\mathcal{Q}}_{mot} \right\}.$$

Then, $\mathcal{Q}_{mot} = \mathcal{Q}_{ot} \cap \mathcal{M}$. We also note that, by Strassen's Theorem [33], $\mathcal{Q}_{mot}$ is non-empty if and only if μ and ν are in convex order which we always assume.

For any $\eta \in \mathcal{Q}_{mot}$, $\eta(\ell) = m^*$. Hence,

$$\eta \in \hat{\mathcal{Q}}_{mot} \quad \Leftrightarrow \quad m^*\eta \in \mathcal{Q}_{mot}.$$

In particular,

$$\mathbf{P}_{mot}(\cdot) = m^* \, \mathbf{P}(\cdot; \hat{\mathcal{Q}}_{mot}), \quad \text{where} \quad \mathbf{P}_{mot}(\mathfrak{a}) := \sup_{\eta \in \mathcal{Q}_{mot}} \mathfrak{a}(\eta).$$

Set

$$\mathfrak{H}_{mot} = \left(\mathcal{H}^{\perp}_{mot}\right)^{\perp}, \quad \mathfrak{K}_{mot} := \mathfrak{K}_{\mathcal{Q}_{mot}} = \mathfrak{K}_{\hat{\mathcal{Q}}_{mot}}.$$

By Theorem 4.1,

$$\mathbf{P}(\mathfrak{a}; \hat{\mathcal{Q}}_{mot}) = \mathbf{P}(\mathfrak{a}; \mathfrak{K}_{mot}), \qquad \forall \, \mathfrak{a} \in \mathcal{C}''_\ell.$$

One may directly verify that $\mathfrak{I}(\ell - m^*) \in \mathfrak{H}_{mot}$. Therefore, for any $\mathfrak{a} \in \mathcal{C}''_\ell$,

$$\begin{aligned}
\mathbf{P}_{mot}(\mathfrak{a}) &= m^* \, \mathbf{P}(\mathfrak{a}; \hat{\mathcal{Q}}_{mot}) = m^* \, \mathbf{D}(\mathfrak{a}; \mathfrak{K}_{mot}), \\
&= m^* \ \min \{c \in \mathbb{R} \ : \exists \, \mathfrak{z} \in \mathfrak{K}_{mot} \quad \text{such that} \quad c\mathfrak{I}(\ell) + \mathfrak{z} = \mathfrak{a}\} \\
&= \min \big\{ c \, m^* \in \mathbb{R} \ : \exists \, \mathfrak{z} \in \mathfrak{K}_{mot} \quad \text{such that} \\
&\quad c \, m^* \mathbf{1} + [c \, \mathfrak{I}(\ell - m^*) + \mathfrak{z}] = \mathfrak{a} \big\}.
\end{aligned}$$

For any $\eta \in \mathcal{Q}_{mot}$ and for any $\mathfrak{z} \in \mathcal{X}''$, $c \in \mathbb{R}$, we have,

$$\eta(c\,\mathfrak{I}(\ell - m^*) + \mathfrak{z}) = c[\eta(\ell) - m^*] + \eta(\mathfrak{z}) = \eta(\mathfrak{z}).$$

Hence, $\tilde{\mathfrak{z}} := c\,\mathfrak{I}(\ell - m^*) + \mathfrak{z} \in \mathfrak{K}_{mot}$ if and only if $\mathfrak{z} \in \mathfrak{K}_{mot}$. This implies that

$$\begin{aligned} \mathbf{P}_{mot}(\mathfrak{a}) &= \min\left\{\tilde{c} \in \mathbb{R} \ : \exists\, \tilde{\mathfrak{z}} \in \mathfrak{K}_{mot} \quad \text{such that} \quad \tilde{c}\mathbf{1} + \tilde{\mathfrak{z}} = \mathfrak{a}\right\} \\ &=: \mathbf{D}_{mot}(\mathfrak{a}). \end{aligned} \tag{8.1}$$

We summarize the above result in the following.

Theorem 8.1. *Assume that μ and ν are in convex order. Then, for every $\mathfrak{a} \in \mathcal{C}''_\ell$,*

$$\mathbf{P}_{mot}(\mathfrak{a}) = \sup_{\eta \in \mathcal{Q}_{mot}} \mathfrak{a}(\eta) = \mathbf{D}_{mot}(\mathfrak{a}).$$

8.2. Map T

In this subsection, we write the map T and its adjoints in the coordinate form to better explain the constructions that will be given in the next subsection. For $i = 1, \ldots, d, \gamma^{(i)} \in C_b(X), \omega = (x, y), x = (x^{(1)}, \ldots, x^{(d)})$ and $y = (y^{(1)}, \ldots, y^{(d)})$, set

$$T^{(i)}(\gamma^{(i)})(\omega) := \gamma^{(i)}(x)(x^{(i)} - y^{(i)}).$$

It is clear that for $\gamma = (\gamma^{(1)}, \ldots, \gamma^{(d)}) \in C_b(X; \mathbb{R}^d)$,

$$T(\gamma)(\omega) = \sum_{i=1}^{d} T^{(i)}(\gamma^{(i)})(\omega).$$

Then, $T'(\eta) = \left(T'^{(1)}(\eta), \ldots, T'^{(d)}(\eta)\right)$ and

$$T'^{(i)}(\eta)(\gamma) = T'^{(i)}(\eta)(\gamma^{(i)}) = \eta\left(T^{(i)}(\gamma^{(i)})\right) = \int_\Omega \gamma^{(i)}(x)(x^{(i)} - y^{(i)})\,\eta(dx, dy).$$

For any $\mathfrak{g} = (\mathfrak{g}^{(1)}, \ldots, \mathfrak{g}^{(d)}) \in C''_b(\Omega; \mathbb{R}^d)$, we have $\mathfrak{g}^{(i)} \in C''_b(X)$ for each $i = 1, \ldots, d$ and for $\rho = (\rho^{(1)}, \ldots, \rho^{(d)})$ with $\rho^{(i)} \in C'_b(X)$,

$$\mathfrak{g}(\rho) = \sum_{i=1}^{d} \mathfrak{g}^{(i)}(\rho^{(i)}).$$

Then, for any $\eta \in \mathcal{C}''_\ell$,

$$T''(\mathfrak{g})(\eta) = \mathfrak{g}(T'(\eta)) = \sum_{i=1}^{d} \mathfrak{g}^{(i)}(T'^{(i)}(\eta)).$$

8.3. Dual Elements

The following result characterizes the dual elements $\mathfrak{H}_{mot}$. We introduce the sets

$$\mathfrak{B}_\ell = \mathfrak{B}_{\ell,\mu} := \left\{\mathfrak{a} \in \mathcal{C}''_\ell \ : \ \exists \mathfrak{b} \in \mathcal{C}''_X, \ \text{ such that } \ \mathfrak{a} = \mathfrak{b} \oplus \mathbf{0} \ \text{ and } \ \mathfrak{b}(\mu) = 0\right\}$$
$$\mathfrak{C}_\ell = \mathfrak{C}_{\ell,\nu} := \left\{\mathfrak{a} \in \mathcal{C}''_\ell \ : \ \exists \mathfrak{c} \in \mathcal{C}''_Y, \ \text{ such that } \ \mathfrak{a} = \mathbf{0} \oplus \mathfrak{c} \ \text{ and } \ \mathfrak{c}(\nu) = 0\right\}.$$

Theorem 8.2. *Let $X = Y = \mathbb{R}^d$. Suppose that μ and ν are in convex order. Then,*

$$\mathfrak{H}_{mot} = \left(\mathcal{H}^\perp_{mot}\right)^\perp = \mathfrak{B}_\ell + \mathfrak{C}_\ell + \mathfrak{M}. \tag{8.2}$$

In particular, the dual set $\mathfrak{K}_{mot}$ is the weak closure of $\mathfrak{B}_\ell + \mathfrak{C}_\ell + \mathfrak{M} + \mathcal{X}''_-$.*

Proof. It is clear from their definitions that $\mathfrak{B}_\ell$, $\mathfrak{C}_\ell$ and $\mathfrak{M}$ are all regularly convex.

Step 1: We first show that $\mathfrak{H} := \mathfrak{B}_\ell + \mathfrak{C}_\ell + \mathfrak{M}$ is regularly convex. In view of Lemma 10.1, regular convexity of this sum would follow from the following estimate:

$$\mathfrak{H} \cap B''_1 \subset \left(\mathfrak{B}_\ell \cap B''_{c^*}\right) + \left(\mathfrak{C}_\ell \cap B''_{c^*}\right) + \mathfrak{M}, \tag{8.3}$$

for some constant c^*.

We continue by proving this estimate. Fix $\mathfrak{a}_0 \in \mathfrak{H} \cap B''_1$. Then, there are $\mathfrak{b}_0 \in \mathfrak{B}_\ell$, $\mathfrak{c}_0 \in \mathfrak{C}_\ell$ and $\mathfrak{g}_0 \in \mathcal{C}''_b(X; \mathbb{R}^d)$ so that

$$\mathfrak{a}_0 = \mathfrak{b}_0 \oplus \mathfrak{c}_0 + T''(\mathfrak{g}_0).$$

Define $\mathfrak{g}_1 \in \mathcal{C}''_b(X; \mathbb{R}^d)$ by,

$$\mathfrak{g}_1(\rho) := \mathfrak{g}_0(\rho) - \mathfrak{g}_0(\rho(X)\delta_0), \quad \rho \in \mathcal{C}'_b(X; \mathbb{R}^d).$$

In the above, note that for $\rho \in \mathcal{C}'_b(X; \mathbb{R}^d)$, $\rho(A) \in \mathbb{R}^d$ for any Borel subset A of X. In the coordinate form,

$$\mathfrak{g}_0\left(\rho(X)\delta_0\right) = \sum_{i=1}^d \rho^{(i)}(X)\, \mathfrak{g}_0^{(i)}\left(\delta_0\right).$$

Then, for any $\eta \in \mathcal{C}'_\ell$,

$$\mathfrak{g}_0\left(T'(\eta)(X)\delta_0\right) = \sum_{i=1}^d T'^{(i)}(\eta)(X)\, \mathfrak{g}_0^{(i)}\left(\delta_0\right) = \mathfrak{g}_0(\delta_0) \cdot \int_\Omega (x-y)\eta(dx, dy).$$

Observe that

$$\begin{aligned} T''(\mathfrak{g}_1)(\eta) &= \mathfrak{g}_1\left(T'(\eta)\right) = \mathfrak{g}_0\left(T'(\eta)\right) - \mathfrak{g}_0\left(T'(\eta)(X)\delta_0\right) \\ &= T''(\mathfrak{g}_0)(\eta) - \mathfrak{g}_0(\delta_0) \cdot \int_\Omega (x-y)\eta(dx, dy) \\ &= T''(\mathfrak{g}_0)(\eta) + \tilde{\mathfrak{b}}_1(\eta_x) + \tilde{\mathfrak{c}}_1(\eta_y), \end{aligned}$$

where

$$\tilde{\mathfrak{b}}_1(\alpha) := -\mathfrak{g}_0(\delta_0) \cdot \int_X x\, \alpha(dx), \quad \alpha \in \mathcal{C}'_X,$$
$$\tilde{\mathfrak{c}}_1(\rho) := \mathfrak{g}_0(\delta_0) \cdot \int_Y y\, \rho(dy), \quad \rho \in \mathcal{C}'_Y.$$

Set $\mathfrak{b}_1 := \mathfrak{b}_0 - \tilde{\mathfrak{b}}_1$, $\mathfrak{c}_1 := \mathfrak{c}_0 - \tilde{\mathfrak{c}}_1$. Then, $\mathfrak{a} = \mathfrak{b}_1 \oplus \mathfrak{c}_1 + T''(\mathfrak{g}_1)$ and $\mathfrak{g}_1(c\, \delta_0) = 0$ for any constant $c \in \mathbb{R}^d$. Since μ and ν are in convex order,

$$\int_X x\, \mu(dx) = \int_Y y\, \nu(dy).$$

Therefore, $(\mathfrak{b}_1 \oplus \mathfrak{c}_1)(\mu \times \nu) = (\mathfrak{b}_0 \oplus \mathfrak{c}_0)(\mu \times \nu) = 0$.

For any $\beta \in \mathcal{C}'_Y$, set $\eta_\beta := \delta_0 \times \beta$. We directly calculate that for $\gamma \in \mathcal{C}_b(X; \mathbb{R}^d)$,

$$T'(\eta_\beta)(\gamma) = -\gamma(0) \cdot \int_Y y\, \beta(dy) =: \delta_0(\gamma) \cdot c_\beta \quad \Rightarrow$$
$$T'^{(i)}(\eta_\beta) = c_\beta^{(i)} \delta_0, \quad i = 1, \ldots, d.$$

Hence,

$$T''(\mathfrak{g}_1)(\eta_\beta) = \mathfrak{g}_1\left(T'(\eta_\beta)\right) = \mathfrak{g}_1(c_\beta \delta_0) = 0.$$

Set

$$\mathfrak{b}_2(\alpha) := \mathfrak{b}_1(\alpha) - \mathfrak{b}_1(\delta_0)\alpha(X) \quad \mathfrak{c}_2(\beta) := \mathfrak{c}_1(\beta) + \mathfrak{b}_1(\delta_0)\beta(Y).$$

Then, since $\eta_x(X) = \eta_y(Y) = \eta(X \times Y)$, for $\eta \in \mathcal{C}'_\ell$,

$$\begin{aligned}(\mathfrak{b}_2 \oplus \mathfrak{c}_2)(\eta) &= \mathfrak{b}_2(\eta_x) + \mathfrak{c}_2(\eta_y) = \mathfrak{b}_1(\eta_x) + \mathfrak{c}_1(\eta_y) - \mathfrak{b}_1(\delta_0)[\eta_x(X) - \eta_y(Y)] \\ &= \mathfrak{b}_1(\eta_x) + \mathfrak{c}_1(\eta_y).\end{aligned}$$

Therefore, the triplet $(\mathfrak{b}_2, \mathfrak{c}_2, \mathfrak{g}_1)$ satisfies,

$$\mathfrak{b}_2 \oplus \mathfrak{c}_2 + T''(\mathfrak{g}_1) = \mathfrak{b}_1 \oplus \mathfrak{c}_1 + T''(\mathfrak{g}_1) = \mathfrak{a}$$

and $\mathfrak{b}_2(\delta_0) = \mathfrak{g}_1(c\delta_0) = 0$ for any $c \in \mathbb{R}^d$. In particular,

$$\mathfrak{c}_2(\beta) = \mathfrak{a}(\eta_\beta), \quad \forall \beta \in \mathcal{C}'_Y.$$

Hence, $\|\mathfrak{c}_2\|_{\mathcal{C}''_Y} \leqq \|\mathfrak{a}\|_{\mathcal{X}''} = 1$. Let $\mathcal{Q}(\alpha)$ be the set of all martingale measures η with $\eta_x = \alpha$. Then, for any $\eta \in \mathcal{Q}(\alpha)$, $\mathfrak{b}_2(\alpha) = \mathfrak{a}(\eta) - \mathfrak{c}_2(\eta_y)$. Hence, $\|\mathfrak{b}_2\|_{\mathcal{C}''_Y} \leqq 2$. Moreover,

$$(\mathfrak{b}_2 \oplus \mathfrak{c}_2)(\mu \times \nu) = (\mathfrak{b}_1 \oplus \mathfrak{c}_1)(\mu \times \nu) = 0.$$

Set $\mathfrak{b}_3(\alpha) := \mathfrak{b}_2(\alpha) - \mathfrak{b}_2(\mu)\alpha(X)$, $\mathfrak{c}_3(\beta) := \mathfrak{c}_2(\beta) - \mathfrak{c}_2(\nu)\beta(Y)$. By their definitions, $\mathfrak{b}_3(\mu) = \mathfrak{c}_3(\nu) = 0$. We use these to conclude that for any $\eta \in \mathcal{C}'_\ell$,

$$(\mathfrak{b}_3 \oplus \mathfrak{c}_3)(\eta) = (\mathfrak{b}_2 \oplus \mathfrak{c}_2)(\eta) - (\mathfrak{b}_2 \oplus \mathfrak{c}_2)(\mu \times \nu)\, \eta(\Omega) = (\mathfrak{b}_2 \oplus \mathfrak{c}_2)(\eta).$$

The above equations imply that

$$\mathfrak{a} = \mathfrak{b}_3 \oplus \mathfrak{c}_3 + T''(\mathfrak{g}_1),$$

and $\mathfrak{b}_3 \in \mathfrak{B}_{\ell,\mu}$, $\mathfrak{c}_3 \in \mathfrak{C}_{\ell,\nu}$. Moreover,

$$\|\mathfrak{b}_3\|_{\mathcal{C}''_X} \leqq 4, \quad \|\mathfrak{c}_3\|_{\mathcal{C}''_Y} \leqq 2.$$

Therefore (8.3) holds with $c^* = 4$ and $\mathfrak{B}_\ell + \mathfrak{C}_\ell + \mathfrak{M}$ is regularly convex.

Step 2: In this step, we show that the regular convexity of $\mathfrak{B}_\ell + \mathfrak{C}_\ell + \mathfrak{M}$ implies that it is equal to $\mathfrak{H}_{mot}$. It is clear that

$$\mathfrak{B}_\ell + \mathfrak{C}_\ell + \mathfrak{M} \subset \mathfrak{H}_{mot}.$$

In working towards a contradiction assume that the above inclusion is strict. By regular convexity, there exist $\mathfrak{a}_0 \in \mathfrak{H}_{mot}$ and $\eta_0 \in \mathcal{C}'_\ell$ such that

$$\sup_{\mathfrak{a} \in \mathfrak{B}_\ell + \mathfrak{C}_\ell + \mathfrak{M}} \mathfrak{a}(\eta_0) < \mathfrak{a}_0(\eta_0). \tag{8.4}$$

By linearity, the left hand side of the above inequality is equal to zero and therefore, $\eta_0 \in (\mathfrak{B}_\ell + \mathfrak{C}_\ell + \mathfrak{M})_\perp$. Since $(\mathfrak{B}_\ell + \mathfrak{C}_\ell + \mathfrak{M})_\perp \subset \mathcal{H}^\perp_{mot}$, we conclude that $\eta_0 \in \mathcal{H}^\perp_{mot}$. However, $\mathfrak{a}_0 \in \mathfrak{H}_{mot}$ and consequently, $\mathfrak{a}_0(\eta_0) = 0$. This is a contradiction with (8.4). Consequently, (8.2) holds and the proof of this Theorem is complete. □

8.4. Polar Sets

Let $\mathcal{N}_{mot}$ be the set of all Borel subsets Z of Ω such that $\eta(Z) = 0$ for every $\eta \in \mathcal{Q}_{mot}$. It is immediate that $\mathfrak{I}(\chi_Z) \in \mathfrak{K}_{mot}$ for every $Z \in \mathcal{N}_{mot}$. However, it is not clear whether $\mathfrak{I}(\chi_Z)$ belongs to the set $\mathfrak{B}_\ell + \mathfrak{C}_\ell + \mathfrak{M}$. Hence, these sets must be used in the hedging set as observed in [8].

On the other hand, functions of the type χ_A are not included in the original dual set $\mathcal{H}_{mot}$. This observation suggests that the duality with the set $\mathfrak{I}(\mathcal{H}_{mot}) + \mathcal{X}''$ is not expected. Indeed, Example 8.1 of [8], a similar counter-example in $\mathcal{B}_b(\Omega)$ is constructed. This example shows a duality gap in $\mathcal{B}_b(\Omega)$, when the dual elements do not contain the functions of the form χ_A with $A \in \mathcal{N}_{mot}$.

So one needs to augment the set of dual elements by adding at least the polar sets, $\mathcal{N}_{mot}$, of the set of probability measures $\mathcal{Q}_{mot}$. Equivalently, one needs to consider all equalities and inequalities $\mathcal{Q}_{mot}$- *quasi-surely*; c.f. [31,32].

We close this section by providing constructions of some polar sets discussed above in two separate examples. In these two examples, we restrict ourselves to the one-dimensional case $X = Y = \mathbb{R}$.

Example 8.3. Let μ, ν be absolutely continuous with respect to the Lebesgue measure. Consider their potential functions defined by

$$u_\mu(x) = \int_{\mathbb{R}} |x - t| \mu(dt), \qquad u_\nu(x) = \int_{\mathbb{R}} |x - t| \nu(dt).$$

Assume that there exists $x_0 \in \mathbb{R}$ so that $u_\mu(x) < u_\nu(x)$ for all $x \neq x_0$ and $u_\mu(x_0) = u_\nu(x_0)$. Then, the set $A = (-\infty, x_0) \times (x_0, \infty)$ is in $\mathcal{N}_{mot}$. Set

$$a(x, y) = |y - x_0| - |x - x_0| - \frac{x - x_0}{|x - x_0|}(y - x),$$

when $x \neq x_0$ and we set

$$a(x, y) = |y - x_0|, \text{ if } x = x_0.$$

Also let $\mathfrak{a} := \mathfrak{I}(a)$. Note that by the convexity of the absolute value function, $\mathfrak{a} \geqq 0$ and $\mathfrak{a} > 0$ on A. Thus, for all $n \geqq 0$ there exists $\mathfrak{n}_n \leqq 0$ such that $n\mathfrak{a} + \mathfrak{n}_n \in \mathfrak{H}_{mot} + \mathcal{X}''_- \subset \mathfrak{K}_{mot}$ converges to $\mathfrak{I}(\chi_A)$ increasingly. Therefore, by the monotone convergence theorem, this convergence is also in the weak* topology. Consequently, $\mathfrak{I}(\chi_A) \in \overline{\mathfrak{H}_{mot} + \mathcal{X}''_-}^* = \mathfrak{K}_{mot}$. $\square$

Example 8.4. We now explain the duality gap in the Example 8.1 of [8] through the polar sets. The existence of the duality gap can be proved using the set of all elements in $ba(\Omega)$ that are martingales and have marginals μ and ν.

In this example, we let $\mu = \nu = \lambda$ where λ is the Lebesgue measure on [0, 1]. Then, the only martingale coupling $\mathbb{Q}^*$ is the uniform probability distribution on the diagonal

$$D := \{(x, x) : x \in [0, 1]\}.$$

Thus D^c, the complement of D, is a polar set of the set of measures $\mathcal{Q} = \{\mathbb{Q}^*\}$.

For $k \geqq 2$, let η^k be the uniform probability measure on the set

$$D_k = \left\{\left(x, x + \frac{1}{k}\right) : x \in \left[0, 1 - \frac{1}{k}\right]\right\}.$$

Since D^c is a polar set, we have $\mathfrak{I}(\chi_{D^c}) \in \mathfrak{K}_{mot}$. On the other hand, we claim that

$$\mathfrak{I}(\chi_{D^c}) \notin \overline{\mathfrak{B}_{\ell,\lambda} + \mathfrak{C}_{\ell,\lambda} + \mathfrak{M} + \mathcal{X}''_-}. \tag{8.5}$$

In view Theorem 4.1, this would prove the existence of the duality gap when one uses $\mathfrak{B}_{\ell,\lambda} + \mathfrak{C}_{\ell,\lambda} + \mathfrak{M} + \mathcal{X}''_-$ as the hedging strategies.

We prove (8.5) by showing that for all $(\mathfrak{b}, \mathfrak{c}, \mathfrak{g}, \mathfrak{n}) \in \mathfrak{B}_{\ell,\lambda} \times \mathfrak{C}_{\ell,\lambda} \times C''_b(\mathbb{R}; \mathbb{R}) \times \mathcal{X}''_-$ one has the following estimate,

$$\limsup_{k\to\infty} \mathfrak{b}\left(\eta^k_x\right) + \mathfrak{c}\left(\eta^k_y\right) + T''(\mathfrak{g})(\eta^k) + \mathfrak{n}(\eta^k) - \eta^k(D^c) \leqq -1.$$

Note that η^k_x and η^k_y converges to λ in total variation. Also,

$$\sup_{|g|\leqq 1} T'(\eta^k)(g) \leqq \sup_{|g|\leqq 1} \left|\int g(x)(y - x)\eta^k(dx, dy)\right| \leqq \int |y - x|\eta^k(dx, dy) \leqq \frac{c_k}{k}$$

where c_k is a sequence of positive bounded constants. Therefore,

$$\lim_{k\to\infty} T''(\mathfrak{g})(\eta^k) = \lim_{k\to\infty} \mathfrak{g}\left(T'(\eta^k)\right) = 0.$$

These imply that

$$\limsup_{k\to\infty} \mathfrak{b}(\eta_x^k)+\mathfrak{c}\left(\eta_y^k\right)+T''(\mathfrak{g})(\eta^k)+\mathfrak{n}(\eta^k)-\eta^k(D^c) \leqq \limsup_{k\to\infty} -\eta^k(D^c) = -1.$$

Hence, (8.5) holds and imply that the strong and the weak* closures of the set

$$\mathfrak{B}_{\ell,\lambda}+\mathfrak{C}_{\ell,\lambda}+\mathfrak{M}+\mathcal{X}''_-$$

are distinct. In view of the main duality theorem and the characterization of $\mathfrak{K}_{mot}$, we conclude that there would be a duality gap if one uses only the above set in the definition of the dual problem. □

9. Convex Envelopes

Assume that $X = Y$ are closed convex subsets of $\mathbb{R}^d$.

Motivated by the results of the previous sections, we define the notion convexity in the bidual $\mathcal{C}''$. Recall that $ca_{r,c}(\Omega)$ is the set of all countably additive Borel measures that are compactly supported and $ca^+_{r,c}(\Omega)$ are its positive elements. Set

$$\mathcal{M}_+ := \mathcal{M}\cap\left(\mathcal{C}'_\ell\right)_+, \quad \text{and} \quad \mathcal{M}_{c,+} := \mathcal{M}\cap\mathcal{C}'_+ = \mathcal{M}\cap ca^+_{r,c}(\Omega).$$

Definition 9.1. We call $\mathfrak{c}\in\mathcal{C}''(X)$ convex if

$$(\mathfrak{c}\oplus(-\mathfrak{c}))(\eta)\leqq 0, \quad \forall\, \eta\in\mathcal{M}_{c,+}.$$

To obtain equivalent characterizations of convexity, we recall that $\alpha\in\mathcal{M}_1(X)$ is in *convex order* with $\beta\in\mathcal{M}_1(X)$ and write $\alpha\leqq_c\beta$ if and only if

$$\int_X \phi d\alpha \;\leqq\; \int_X \phi d\beta,$$

for every $\phi : X\mapsto\mathbb{R}$ convex. The following follows directly from the results of Strassen [33]:

Proposition 9.2. $\mathfrak{b}\in\mathcal{C}''_\ell(X)$ *is convex if and only if* $\mathfrak{b}(\alpha)\leqq\mathfrak{b}(\beta)$ *for all measures* $\alpha,\beta\in\mathcal{M}_1(X)$ *that are in convex order. Moreover, if* $X=\mathbb{R}^d$ *and if* $\mathfrak{b}\in\mathcal{C}''_\ell(\mathbb{R}^d)$ *is convex, then there exists* $\mathfrak{g}\in C''_b(\mathbb{R}^d;\mathbb{R}^d)$ *such that*

$$\mathfrak{b}(\eta_x)\leqq\mathfrak{b}(\eta_y)+T''(\mathfrak{g})(\eta), \quad \forall\,\eta\in\left(\mathcal{C}'_\ell(\Omega)\right)_+.$$

Proof. The first statement follows directly from Strassen [33]. Indeed, if a measure $\eta\in\mathcal{M}_{c,+}$ then $\eta_x\in ca^+_{c,r}(X)$, $\eta_y\in ca^+_{c,r}(Y)$ and they are in convex order. Conversely, if $\alpha,\beta\in\mathcal{M}_1(X)\cap ca^+_{c,r}(X)$ are in convex order, then there exists $\eta\in\mathcal{M}_{c,+}$ such that $\eta_x=\alpha$ and $\eta_y=\beta$. Then, the general statement follows from the density of $ca_{c,r}(X)$ in $\mathcal{C}'_\ell(X)$.

Now suppose that $X=Y=\mathbb{R}^d$. Given $\mathfrak{b}\in\mathcal{C}''_\ell(\mathbb{R}^d)$, define

$$\mathfrak{a} := \mathfrak{b}\oplus(-\mathfrak{b}).$$

Then, by the definition of convexity and Proposition 7.2, $\mathfrak{b}$ is convex if and only if $\mathfrak{a} \in \mathfrak{D} + (\mathcal{C}''_\ell)_-$. Hence, there exists $\mathfrak{g} \in C''_b(\mathbb{R}^d; \mathbb{R}^d)$ such that for any $\eta \in \mathcal{M}_+$,

$$\mathfrak{b}(\eta_x) - \mathfrak{b}(\eta_y) = \mathfrak{a}(\eta) \leqq T''(\mathfrak{g})(\eta).$$

□

The following is a natural extension of the classical definition. Although a definition in the larger class $\mathcal{C}''$ can be given, we restrict ourselves to $\mathcal{C}_\ell(X)$ to simplify the presentation.

Definition 9.3. For any $\mathfrak{b} \in \mathcal{C}''_\ell(X)$ its convex envelope is defined by,

$$\mathfrak{b}^c(\alpha) := \inf\left\{\mathfrak{b}(\eta_y) \;:\; \eta \in \mathcal{M}_+,\;\; \eta_x = \alpha\right\}, \quad \alpha \in \left(\mathcal{C}'_\ell(X)\right)_+,$$

and for general $\alpha \in \mathcal{C}'_\ell(X)$, $\mathfrak{b}^c(\alpha) := \mathfrak{b}^c(\alpha^+) - \mathfrak{b}^c(\alpha^-)$.

Lemma 9.4. *For any* $\mathfrak{b} \in \mathcal{C}_\ell(X)$*,* $\mathfrak{b}^c \in \mathcal{C}_\ell(X)$*. Moreover,* $\|\mathfrak{b}\|_{\mathcal{C}''_\ell(X)} = \|\mathfrak{b}^c\|_{\mathcal{C}''_\ell(X)}$ *and for every* $\alpha \geqq 0$

$$\mathfrak{b}^c(\alpha) = \sup\left\{\mathfrak{c}(\alpha) \;:\; \mathfrak{c} \text{ is convex and } \mathfrak{c} \leqq \mathfrak{b}\right\}.$$

Proof. For $\alpha \in \left(\mathcal{C}'_\ell(X)\right)_+$ set

$$\mathcal{Q}(\alpha) := \{\eta \in \mathcal{M}_+ \;:\; \eta_x = \alpha\}.$$

We claim that for any $\alpha, \beta \in \left(\mathcal{C}'_\ell(X)\right)_+$,

$$\mathcal{Q}(\alpha) + \mathcal{Q}(\beta) = \mathcal{Q}(\delta), \quad \text{where} \quad \delta := \alpha + \beta. \tag{9.1}$$

Indeed, the inclusion $\mathcal{Q}(\alpha) + \mathcal{Q}(\beta) \subset \mathcal{Q}(\delta)$ follows directly from the definitions. For any Borel subset $A \subset X$, set

$$z^\alpha := \frac{d\alpha}{d\delta}, \quad z^\beta := \frac{d\beta}{d\delta}, \quad \delta = \alpha + \beta.$$

Then, both z^α, and z^β are Borel functions of the first variable x. Moreover, $z^\alpha(x)$, $z^\beta(x) \in [0, 1]$ and $z^\alpha + z^\beta \equiv 1$. For any $\eta \in \mathcal{Q}(\delta)$, set

$$\eta^\alpha(A) := \int_A z^\alpha(x)\eta(dx, dy), \quad \eta^\beta(A) := \int_A z^\beta(x)\eta(dx, dy).$$

It is clear that $\eta^\alpha, \eta^\beta \in (\mathcal{C}'_\ell)_+$. For any $h \in \mathcal{C}_\ell(X)$ and $\eta \in \mathcal{Q}(\delta)$, we calculate that

$$\begin{aligned}\int_X h(x)\eta^\alpha_x(dx) &= \int_\Omega h(x)\eta^\alpha(dx, dy) = \int_\Omega h(x)z^\alpha(x)\eta(dx, dy)\\ &= \int_X h(x)z^\alpha(x)\eta_x(dx) = \int_X h(x)z^\alpha(x)\delta(dx)\\ &= \int_X h(x)\alpha(dx).\end{aligned}$$

So, we conclude that $\eta^\alpha_x = \alpha$. Also for any $\gamma \in \mathcal{C}_b(X;\mathbb{R}^d)$,

$$\int_\Omega T(\gamma)d\eta^\alpha = \int_X z^\alpha(x)\gamma(x)\cdot\left(\int_Y (x-y)\eta(dx,dy)\right) = 0.$$

Hence, $\eta^\alpha \in \mathcal{Q}(\alpha)$. Similarly one can show that $\eta^\beta \in \mathcal{Q}(\beta)$. Since $\eta^\alpha + \eta^\beta = \eta$, this proves the claim (9.1).

The linearity of the map $\alpha \in \mathcal{C}'_\ell(X)$ to $\mathfrak{b}^c(\alpha)$ now follows directly from the definitions and (9.1). Similarly the norm statement is immediately apparent from the definitions. □

10. Appendix: Regularly Convex Sets

In this Appendix we state a slight extension of a condition for regular convexity that is proved in [26]. Let E be any Banach space. First note that a set $\mathfrak{K} \in E'$ is regularly convex if and only if it is closed in the weak* topology and is convex. The following result is an immediate corollary of Theorem 7 of [26] and is used repeatedly in our arguments:

Lemma 10.1. *Let $\mathfrak{K} \subset E'$. Suppose that for each $R > 0$ there exists a regularly convex set $\mathfrak{L}_R$ so that*

$$\mathfrak{K} \cap B_R \subset \mathfrak{L}_R \subset \mathfrak{K},$$

where B_R is the closed ball in E' centered around the origin with radius R. Then, $\mathfrak{K}$ is regularly convex.

Proof. Let $\mathfrak{U}$ be bounded, regularly convex and B_R be a ball that contains $\mathfrak{U}$. Then, we have the set inclusions

$$\mathfrak{K} \cap \mathfrak{U} \subset \mathfrak{K} \cap B_R \subset \mathfrak{L}_R \subset \mathfrak{K}.$$

Therefore,

$$\mathfrak{K} \cap \mathfrak{U} = \mathfrak{L}_R \cap \mathfrak{U}.$$

Since both $\mathfrak{L}_R$ and $\mathfrak{U}$ are regularly convex, so is their intersection. Therefore, for every regularly convex, bounded $\mathfrak{U}$, $\mathfrak{K} \cap \mathfrak{U}$ is regularly convex. By Theorem 7 [26], this proves the regular convexity of $\mathfrak{K}$. □

Acknowledgements. The authors would like to thank MATTI KIISKI for many comments and fruitful discussions and would like to thank the anonymous reviewer for valuable comments.

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(*Received October 12, 2016 / Accepted September 26, 2017*)