CONDITIONAL DAVIS PRICING

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Abstract. We study the set of marginal utility-based prices of a financial derivative in the case where the investor has a non-replicable random endowment. We provide an example showing that even in the simplest of settings - such as Samuelson's geometric Brownian motion model - the interval of marginal utility-based prices can be a non-trivial strict subinterval of the set of all no-arbitrage prices. This is in stark contrast to the case with a replicable endowment where non- uniqueness is exceptional. We provide formulas for the end points for these prices and illustrate the theory with several examples.

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1. Introduction

We consider an investor in a frictionless but incomplete financial market. The price dynamics are modeled by a locally bounded semimartingale S. The investor will receive an endowment B at a future time $T > 0$ and would like to price a derivative with payoff φ at time T. To obtain the set of possible prices for the payoff φ, we use the marginal utility-based pricing approach of Mark Davis in [8]. With the investor's utility function $U : (0, \infty) \to \mathbb{R}$ given, we start by defining the primal value function $v(\cdot; B)$ "conditioned" on the presence of the endowment $B \in \mathbb{L}^\infty_{++} := \cup_{a>0}(a + \mathbb{L}^\infty_+)$. Its domain is a set of random variables φ interpreted as future derivative payments, and

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its value is

$$v(\varphi; B) := \sup_H \mathbb{E}[U(\varphi + B + \int_0^T H_u dS_u)], \quad \varphi \in \mathbb{L}^\infty, \tag{1.1}$$

where H ranges over a set of admissible integrands which is defined in Section 2 below. Then, given a fixed derivative with payoff φ, we call a constant $p \in \mathbb{R}$ a *conditional Davis price of* φ (conditional on the endowment B), if p satisfies the following inequality

$$v\big(\varepsilon(\varphi - p); B\big) \leq v(0; B), \quad \forall \varepsilon \in \mathbb{R}. \tag{1.2}$$

The mathematical details are given in Definition 3.4 below.

The conditional Davis pricing concept above can also be seen as a variation of the classical case where the utility function is no longer deterministic. We could consider the random endowment B a part of the preference structure of the agent, i.e., think of $x \mapsto U(x + B(\omega))$ as a stochastic utility function and view $\mathbb{E}[U(\varphi + B)]$ as the expected utility of the position φ.

The set of spanned endowments B provides an interesting special case. Indeed, if $B \in \mathbb{L}^\infty_{++}$ satisfies $B = x + \int_0^T H_u dS_u$ for some initial value x and an admissible H, then x is unique and the primal utility function satisfies $v(\varphi; B) = v(\varphi; x)$. In this case, the conditional Davis price is exactly the marginal utility-based price of the payoff φ given the (constant) initial wealth $x > 0$ defined in Definition 3.1 in [16]. This paper's main goal is to extend the theory developed for spanned endowments to the case of unspanned endowments, which is one of the main problems one faces in many incomplete-market-equilibrium frameworks.

An alternate approach is to consider the endowment B as the payoff of a financial derivative as well. This perspective requires us to price multiple derivatives simultaneously and was considered in [15, 23]. Namely, let ξ be the d-dimensional payoff of d derivatives and fix an amount $q_0 \in \mathbb{R}^d$. Then, in Remark 1 of [15], $p_0 \in \mathbb{R}^d$ is called the *marginal utility-based price of* ξ *given* q_0 if p_0 satisfies,

$$v(q \cdot (\xi - p_0); q_0 \cdot \xi) \leq v(0; q_0 \cdot \xi), \quad \forall q \in \mathbb{R}^d.$$

So alternatively, one may define conditional Davis prices of φ as the second component of the two-dimensional marginal utility-based prices of the pair $\xi := (B, \varphi)$ with $q_0 := (1, 0)$. Indeed, we give the precise definition in Section 4 and in Section 4.1 we show that these two notions agree.

Apart from using the same definition, previous studies do not cover the case of conditional Davis prices with an unspanned endowment B. In particular, [16] study the one dimensional problem with a constant (spanned) endowment $B := x > 0$, while [15] provide the multi-dimensional definition, but only investigate the utility maximization problem. [23] study the multi-dimensional pricing problem only for small values of q_0 under a decay assumption on ξ which we discuss later. Since we need to fix q_0 to be

$(1, 0)$, the asymptotic results of [23] cannot be applied to the general conditional Davis prices. To highlight one non-trivial difference between our setting and [23], let us recall that it has been known since [16] that even when $B := x > 0$ is constant, marginal utility-based prices of the payoff φ can form a non-trivial interval. Such an occurrence is, however, treated as a rare pathology and explicitly assumed away via a decay condition in [23]. In our setting, where B is unspanned, we provide an example showing that even in Samuelson's geometric Brownian motion model with constant coefficients, there exists a whole spectrum of explicit bounded payoffs φ with a non-trivial and explicitly computable interval of marginal utility-based prices.

While the notion of marginal utility-based prices has been around for more than two decades (we discuss related literature below), a characterization of all constants p satisfying (1.2) when both B and φ are unspanned is currently not available. The main contribution of this paper provides a set of conditions imposed on (B, φ) under which the two endpoints of the interval of marginal utility-based prices for the payoff φ can be explicitly computed. Our main results are:

(1) For an arbitrary endowment $B \in \mathbb{L}^\infty_{++}$, we show that the interval of marginal utility-based prices for the payoff $\varphi \in \mathbb{L}^\infty(\mathbb{P})$ is given by the set of all $\langle \varphi, \hat{\mathbb{Q}} \rangle \subset \mathbb{R}$, when $\hat{\mathbb{Q}} \in$ba$(\mathbb{P})$ ranges through the set of finitely-additive minimizers of the associated dual utility problem (as introduced in [6]). Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $\mathbb{L}^\infty(\mathbb{P})$ and its dual ba$(\mathbb{P}) := (\mathbb{L}^\infty(\mathbb{P}))'$.

(2) It is well-known that marginal utility-based prices are linked to derivatives of the primal utility value function (1.1). We show that under a mild growth condition on the utility function $U(\xi)$ at $\xi = 0$, the directional derivative of $v(\varphi; B)$ in the direction of φ can be characterized as the value of a certain linear stochastic control problem. Furthermore, we show by means of an example that even in the *log*-utility case, the mapping

$$\mathbb{R} \ni x \to v(x; B);$$

that is, the restriction of $v(\cdot; B)$ to constant payoffs $\varphi = x \in \mathbb{R}$ can fail to be differentiable on the interior of its domain[1]. See also the discussion in Erratum [7].

(3) Under the additional assumption of unique super-replicability from [27] placed on (B, φ), we solve the linear stochastic control problem mentioned in (2) explicitly. This gives us formulas for the two endpoints describing the interval of marginal utility-based prices. As

[1]When $B := x > 0$ is constant, Theorem 2.2 in [22] ensures smoothness of the primal value function (1.1). However, when $B \in \mathbb{L}^\infty_{++}$ is unspanned, Example 6.1 below illustrates that smoothness can fail.

an offshoot, we show additionally that the mapping $\varphi \to v(\varphi; B)$ is smooth whenever B is uniquely super-replicable.

We wish to stress that while some known results related to marginal utility-based prices extend verbatim from the constant endowment case $B := x > 0$ to the general, unspanned case of $B \in \mathbb{L}^\infty_+$, not all results do. Indeed, the example mentioned in (2) above illustrates a non-trivial difference.

Because of market incompleteness, the interval of arbitrage-free prices for φ often takes on the extreme form with endpoints given by the essential suprema and infima of φ. We refer the reader to monographs [3] and [14] for further general information and thorough historical overviews of various ways to price unspanned payoffs. Several authors have used ad hoc methods to reduce the width of the interval of arbitrage-free prices (see, e.g., [5] and its extension [2] where so-called good deal bounds, based on the Hansen-Jagannathan bound for the Sharpe ratio, are used). In this respect, our results show how marginal utility-based prices can be used to narrow down the interval of arbitrage-free prices.

We continue by elaborating on the lack of uniqueness of marginal utility-based prices mentioned above. For utility functions defined on the positive axis (such as power and log), and with $B := x > 0$ constant, [16] show that Davis prices are unique for all $\varphi \in \mathbb{L}^\infty$ if and only if the dual utility optimizer $\hat{\mathbb{Q}}$ is a martingale measure (in which case, the unique marginal utility-based price of the payoff φ is given by $\mathbb{E}^{\hat{\mathbb{Q}}}[\varphi]$). However, it has been known since [17] that there exist arbitrage-free models where the dual utility optimizers fail the martingale property; see also [22] for further examples of models satisfying NFLVR (no free lunch with vanishing risk) where the dual utility optimizers fail the martingale property. As a consequence, there are models for which there exists a bounded payoff φ with non-unique marginal utility-based prices (see [16] for an abstract construction of such a payoff φ). When uniqueness is considered indispensable, one could restrict attention to financial models and utility functions which produce martingale dual utility optimizers (see, e.g., the BMO-type condition used in [24]), or consider only payoffs φ with unique Davis prices (as done in [23]). We do not impose such restrictions and our intervals of marginal utility-based prices are generally nontrivial.

We finish this introduction with a brief summary of the sizable literature on marginal utility-based prices in the case when $B := x > 0$ is constant (more generally, when $B \in \mathbb{L}^\infty_{++}$ is spanned). The strand of literature that comes closest to this paper where U is finite only on the positive axis includes [6], [15], [23], and [16].[2] Let us comment on their similarities and differences with our paper:

[2]When U's domain is $\mathbb{R}$ (such as exponential utility) the corresponding dual optimizer is always a martingale; see [1]. Consequently, for such utility functions, marginal utility-based prices are always unique and our analysis offers nothing new in that case.

- [6] focus on the utility-maximization problem itself and do not consider pricing.
- The notion of marginal utility-based prices is defined in Remark 1 on page 849 of [15], and is not studied beyond a standard super-differential characterization.
- The authors of [16] perform an in-depth study of marginal utility-based prices in the constant endowment case, i.e., $B := x > 0$ for some constant x (or more generally for B spanned). [16] created the first abstract example exhibiting non- unique marginal utility-based prices.
- In [23], the authors perform an asymptotic analysis. As discussed earlier and further detailed in Remark 4.4 below, these results do not apply to our setting. Moreover, [23] work under assumptions guaranteeing that φ has a unique marginal utility-based price. These assumptions come in the form of a decay condition on φ and can be found already in [16]. This decay condition, in particular, is not satisfied for a generic bounded payoff. We do not require such a decay property and consequently, the set of claims we consider and the set considered in [23] do not nest in either direction. Indeed, based on the famous counterexample in [11], we construct an explicit example of a family of payoffs with constant $B := x > 0$ (as in [23]) which has a non-trivial interval of marginal utility-based prices. This example illustrates that even for a constant endowment $B := x > 0$, our setting allows for non-unique prices whereas the setting of [23] always produces a unique marginal-utility based price.

The paper is organized as follows. The model is described, the terminology and notation set, standing assumptions imposed, and preliminary analysis of our central utility-maximization problem is performed in Section 2. In Section 3 we define conditional Davis prices. In Section 4 we recall the definition of marginal utility-based prices from [15] and we show that conditional Davis prices can be seen as a projection of marginal-utility based prices. Section 5 characterizes the Davis prices from the dual point of view and lays out some of the first consequences of this characterization. Directional derivatives of the primal utility- maximization problem are studied in Section 6 which also gives the explicit example of non-smoothness mentioned in (2) above. Section 6 also gives a characterization of the directional derivative in terms of a linear stochastic control problem. Section 7 recalls the definition of unique super-replicability from [27] and provides a family of examples of uniquely super-replicable claims. The main result of Section 7 gives an explicit expression for the directional derivative of the utility-maximization value function under the unique super-replicability condition. This result is subsequently used in Section 8 to give explicit formulas for the interval of marginal utility-based prices in a general setting. These formulas are then used in two examples where one example is set in a Samuelson-Black- Scholes-Merton type model and illustrates the fact that non-uniqueness can arise even in the simplest of settings (this supports our claim in the abstract). These examples also allow us to give explicit expressions for the first-order approximation of the hedging portfolios associated to the end-points of the interval of marginal utility-based prices.

2. The setup and assumptions

2.1. **The market model.** Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space which satisfies the usual conditions, and let $\{S_t\}_{t\in[0,T]}$ be a locally bounded semimartingale. $L(S)$ denotes the set of all predictable S-integrable processes and $\mathcal{M}$ denotes the set of all $\mathbb{P}$-equivalent countably-additive probability measures $\mathbb{Q}$ on $\mathcal{F}$ for which S is a $\mathbb{Q}$-local martingale.

Standing Assumption 2.1 (NFLVR)**.** $\mathcal{M} \neq \emptyset$. □

Remark 2.2. We assume that the asset-price process S is locally bounded and postulate the existence of a local martingale measure. While it is possible to relax our setting to the non-locally-bounded case (as used in, e.g., [6]), it is not be possible to relax Assumption 2.1 so as to imply the existence of a supermartingale deflator only. Indeed, the presence of a non-replicable endowment B makes the admissibility class which produces only nonnegative wealth processes too small to host an optimizer. This delicate issue is discussed and illustrated on pages 240, 241 in [26]. To keep the focus of the current paper on the issues directly related to conditional Davis pricing, we have opted for a set of assumptions which is slightly stronger than absolutely necessary.

2.2. **Gains and admissibility.** The investor's gains process has the following dynamics

$$(\pi \cdot S)_t := \int_0^t \pi_u \, dS_u, \quad t \in [0,T], \tag{2.1}$$

for some $\pi \in L(S)$. We call $\pi \in L(S)$ admissible if the gains process is uniformly lower bounded by a constant in which case we write $\pi \in \mathcal{A}$. The set of terminal outcomes is denoted by $\mathcal{K}$, i.e., we define

$$\mathcal{K} := \{(\pi \cdot S)_T \, : \, \pi \in \mathcal{A}\}.$$

2.3. **The primal problem.** Let U be a utility function on $(0, \infty)$, i.e., U is strictly concave, strictly increasing, and continuously differentiable with $U'(0+) = +\infty$ and $U'(+\infty) = 0$. When necessary, we extend the domain of U to $\mathbb{R}$ by setting $U(x) = -\infty$ for $x < 0$ and $U(0) = \inf_{x>0} U(x)$. Finally, U is said to be reasonably elastic (as defined in [22]) if

$$\limsup_{x\to\infty} \tfrac{xU'(x)}{U(x)} < 1.$$

Even though we need it for some of our results, we do not impose the condition of reasonable elasticity from the start.

Let v be defined as in (1.1) with the above notion of admissibility. Then, for any $B \in \mathbb{L}^\infty_{++} := \cup_{x>0}(x + \mathbb{L}^\infty_+)$ we define

$$\mathfrak{U}(B) := v(0; B) = \sup_{X \in \mathcal{K}} \mathbb{E}\Big[U\Big(B + X\Big)\Big] \tag{2.2}$$

with the convention that $\mathbb{E}[U(B + X)] = -\infty$ if $\mathbb{E}[U(B + X)^-] = +\infty$. Because $\mathfrak{U}(B) \geq U(\text{essinf } B) > -\infty$, $\mathfrak{U}$ is $(-\infty, \infty]$-valued on $\mathbb{L}^\infty_{++}$. In (2.3) below, we impose a dual properness assumption which among other things ensures that $\mathfrak{U}$ is finitely valued on $\mathbb{L}^\infty_{++}$.

2.4. **The dual utility maximization problem.** The set of equivalent local martingale measures $\mathcal{M}$ can be identified - via Radon-Nikodym derivatives with respect to $\mathbb{P}$ - with a subset of $\mathbb{L}^1_+(\mathbb{P})$ and embedded, naturally, into $\text{ba}(\mathbb{P}) := \mathbb{L}^\infty(\mathbb{P})^* \supseteq \mathbb{L}^1(\mathbb{P})$. We define $\overline{\mathcal{M}}^*$ as the weak*-closure of $\mathcal{M}$ and we define $\mathcal{D} \subset \text{ba}_+(\mathbb{P})$ as the family of all $y\mathbb{Q}$ where $y \in [0, \infty)$ and $\mathbb{Q} \in \overline{\mathcal{M}}^*$. We can then define the dual utility functional by

$$\mathbb{V}_B(\mu) := \sup_{X \in \mathbb{L}^\infty} \Big(\mathbb{E}[U(B + X)] - \langle \mu, X\rangle\Big), \quad \mu \in \text{ba}(\mathbb{P}).$$

In particular, $\mathbb{V}_B$ is convex, lower weak*-semicontinuous on $\text{ba}(\mathbb{P})$ and bounded from below by $\mathbb{E}[U(B)] \in \mathbb{R}$. For the reminder of the paper we impose a properness assumption. While not the weakest possible in our setting, this assumption allows us to deal swiftly, and yet with a minimal loss of generality, with several technical points that are not central to the message of the paper:

Standing Assumption 2.3 (Properness)**.** There exist $y_0 \in (0, \infty)$ and $\mathbb{Q}_0 \in \mathcal{M}$ such that $\mu_0 := y_0\mathbb{Q}_0$ satisfies

$$\mathbb{V}_B(\mu_0) < \infty. \tag{2.3}$$

□

Thanks to a minimal modification of Lemma 2.1 on p. 138 in [30] and the discussion before it, $\mathbb{V}_B$ admits the following representation

$$\mathbb{V}_B(\mu) = \mathbb{E}\left[V\left(\tfrac{d\mu^r}{d\mathbb{P}}\right)\right] + \langle \mu, B\rangle, \quad \mu \in \mathcal{D}, \tag{2.4}$$

where V is the dual utility function (strictly convex) defined by

$$V(y) := \sup_{x>0} \Big(U(x) - xy\Big), \quad y > 0.$$

Consequently, Fenchel's inequality and (2.3) guarantee that the primal value function $\mathfrak{U}$ satisfies $\mathfrak{U}(B) < \infty$ for all $B \in \mathbb{L}^\infty_{++}$. Furthermore, (2.3) also ensures that the corresponding dual value function defined by

$$\mathfrak{V}(B) := \inf_{\mu \in \mathcal{D}} \mathbb{V}_B(\mu), \quad B \in \mathbb{L}^\infty_{++}, \tag{2.5}$$

is finitely valued. For $B \in \mathbb{L}^\infty_{++}$ we let $\hat{\mathcal{D}}(B)$ denote the set of all minimizers, i.e., all those $\mu \in \mathcal{D}$ such that $\mathfrak{V}(B) = \mathbb{V}_B(\mu)$.

The next result collects some basic facts we will need in the following sections:

Lemma 2.4. *For each $B \in \mathbb{L}^\infty_{++}$, the set $\hat{\mathcal{D}}(B)$ is a nonempty weak*-compact subset of* $\mathrm{ba}(\mathbb{P})$ *and there exists a nonnegative random variable $\hat{Y} = \hat{Y}(B)$ such that $\mathbb{P}[\hat{Y} > 0] > 0$ and*

$$\hat{Y} = \tfrac{d\mu^r}{d\mathbb{P}} \quad \text{for all } \mu \in \hat{\mathcal{D}}(B).$$

Furthermore, the strong duality $\mathfrak{U}(B) = \mathfrak{V}(B)$ holds for all $B \in \mathbb{L}^\infty_{++}$.

Proof. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for $\mathbb{V}_B$ of the form

$$\mu_n = y_n \mathbb{Q}_n \text{ where } \mathbb{Q}_n \in \overline{\mathcal{M}}^* \text{ and } \{y_n\}_{n\in\mathbb{N}} \subseteq [0, \infty).$$

To see that $\{y_n\}_{n\in\mathbb{N}}$ is bounded, we note that (2.3) produces the finite upper bound:

$$\begin{aligned}\mathbb{V}_B(\mu_0) &\geq \limsup_n \mathbb{E}[V(y_n \tfrac{d\mathbb{Q}_n^r}{d\mathbb{P}})] + y_n \langle \mathbb{Q}_n, B\rangle \\ &\geq \limsup_n V(y_n) + y_n \operatorname{essinf} B.\end{aligned}$$

The first inequality follows from (2.4) and the minimizing property of the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ whereas the last inequality follows from the non-increasing property of V and Jensen's inequality. Because $V'(\infty) = 0$ and $\operatorname{essinf} B > 0$ the boundedness property of $\{y_n\}_{n\in\mathbb{N}}$ follows.

Because the finitely-additive probabilities $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ belong to the weak*-compact set $\overline{\mathcal{M}}^*$, we can conclude that $\{\mu_n\}_{n\in\mathbb{N}}$ admits a weak*-convergent subnet μ_α such that $\mu_\alpha \to \mu$, for some $\mu \in \mathcal{D}$. The functional $\mathbb{V}_B$ is lower semicontinuous and we get

$$\mathfrak{V}(B) = \lim_\alpha \mathbb{V}_B(\mu_\alpha) \geq \mathbb{V}_B(\mu).$$

Therefore, μ is a minimizer over $\mathcal{D}$ and we have $\hat{\mathcal{D}}(B) \neq \emptyset$.

Next, we show that all $\mu \in \hat{\mathcal{D}}(B)$ have the same regular part. For that, suppose that $\mu_1^r \neq \mu_2^r$. Then, $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 \in \mathcal{D}$ and by (2.4) we have

$$\tfrac{1}{2}\mathbb{V}_B(\mu_1) + \tfrac{1}{2}\mathbb{V}_B(\mu_2) = \tfrac{1}{2}\mathbb{E}\left[V(\tfrac{d\mu_1^r}{d\mathbb{P}})\right] + \tfrac{1}{2}\mathbb{E}\left[V(\tfrac{d\mu_2^r}{d\mathbb{P}})\right] + \langle \mu, B\rangle > \mathbb{V}_B(\mu),$$

by the strict convexity of V. However, this is in contradiction with the minimality of μ_1 and μ_2.

To see that $\hat{Y} \neq 0$ we argue by contradiction and suppose that $\mathbb{P}[\hat{Y} = 0] = 1$. In that case $V(0) < \infty$ and, so, thanks to Jensen's inequality, we have $\mathbb{V}_B(\mu) < \infty$ for all $\mu \in \mathcal{D}$. In particular, we have for some $\mathbb{Q} \in \mathcal{M}$ and $\hat{\mu} \in \hat{\mathcal{D}}(B)$

$$\mathbb{V}_B(\mu^\varepsilon) < \infty, \quad \text{where} \quad \mu_\varepsilon := \varepsilon\mathbb{Q} + (1-\varepsilon)\hat{\mu}, \quad \varepsilon \in [0,1].$$

Because the regular-part functional is additive we have

$$\mu_\varepsilon^r = \varepsilon\mathbb{Q} + (1-\varepsilon)\hat{\mu}^r = \varepsilon\mathbb{Q}.$$

Therefore, $\hat{\mu}$'s minimality produces

$$\mathbb{E}[V(\varepsilon\tfrac{d\mathbb{Q}}{d\mathbb{P}})] + \langle \mu_\varepsilon, B\rangle = \mathbb{V}_B(\mu_\varepsilon) \geq \mathbb{V}_B(\hat{\mu}) = V(0) + \langle \hat{\mu}, B\rangle.$$

Fatou's lemma then implies

$$\langle \mathbb{Q} - \hat{\mu}, B\rangle \geq \liminf_{\varepsilon\searrow 0} \tfrac{1}{\epsilon}\Big(V(0) - \mathbb{E}\left[V\left(\varepsilon\tfrac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\Big) = -V'(0) = +\infty.$$

This is a contradiction because $B \in \mathbb{L}^\infty(\mathbb{P})$ ensures that the left-hand-side is finite.

From the above, we know that $\hat{\mu}(\Omega)$ is uniformly bounded over $\hat{\mathcal{D}}(B)$. Therefore, the weak*-closed set $\hat{\mathcal{D}}(B)$ is norm bounded and the compactness property of $\hat{\mathcal{D}}(B)$ follows from the Banach-Alaoglu theorem.

Finally, to establish the strong duality property, we define the nested sequence of weak*-compact dual sets

$$\mathcal{D}_n := \{\mu \in \mathcal{D} : ||\mu|| \leq n\}, \quad n \in \mathbb{N},$$

as well as the primal set

$$\mathcal{C} := (\mathcal{K} - \mathbb{L}^0_+) \cap \mathbb{L}^\infty = \{X \in \mathbb{L}^\infty \,:\, \langle \mathbb{Q}, X\rangle \leq 0 \text{ for all } \mathbb{Q} \in \mathcal{M}\}.$$

For a proof of the last identity see, e.g., Corollary 3.4(1) in [28]. As a consequence, we have the following identity for $X \in \mathbb{L}^\infty(\mathbb{P})$

$$\lim_{n\to\infty} \sup_{\mu\in\mathcal{D}_n} \langle \mu, X\rangle = \begin{cases} 0, & X \in \mathcal{C}, \\ +\infty, & X \notin \mathcal{C}. \end{cases}$$

The minimax theorem (see, e.g., Theorem 2.10.2, p. 144 in [31]) can then be used to produce

$$\begin{aligned}\mathfrak{V}(B) &= \lim_{n\to\infty} \inf_{\mu\in\mathcal{D}_n} \sup_{X\in\mathbb{L}^\infty(\mathbb{P})} \Big(\mathbb{E}[U(X+B)] - \langle \mu, X\rangle\Big) \\ &= \sup_{X\in\mathcal{C}} \mathbb{E}[U(X+B)].\end{aligned}$$

The monotone convergence theorem ensures that this expression equals the primal value function $\mathfrak{U}(B)$. □

3. Conditional Davis prices

Definition 3.1. For $B \in \mathbb{L}^\infty_{++}$, a random variable $R \in \mathbb{L}^\infty$ is said to be B-**irrelevant**, denoted by $R \in \mathcal{I}(B)$, if

$$\mathfrak{U}(B + \varepsilon R) \leq \mathfrak{U}(B), \quad \forall \varepsilon \in \mathbb{R}. \tag{3.1}$$

□

Remark 3.2. The function $\mathfrak{U}$ finite-valued at B, as well as in a $\mathbb{L}^\infty$-open ball around B. Therefore, both sides of (3.1) are real-valued for small enough ε. Thanks to the concavity of $\mathfrak{U}$, the right-hand-side of (3.1) may only take the negative infinite value for large values of ε. Therefore, for $R \in \mathcal{I}(B)$, it is enough to check (3.1) only for ε in a neighborhood of 0.

Lemma 3.3. *$\mathcal{I}(B)$ is a nonempty, weak* closed linear subspace in $\mathbb{L}^\infty$.*

Proof. The function $\mathfrak{U}$ is concave at B, so $\mathcal{I}(B)$ is the set of those directions R with the property that the directional derivative of $\mathfrak{U}$ in directions R and $-R$ are nonpositive. In other words, we have

$$\sup_{\mu\in\partial\mathfrak{U}(B)} \langle R,\mu\rangle \leq 0 \text{ and } \sup_{\mu\in\partial\mathfrak{U}(B)} -\langle R,\mu\rangle \leq 0,$$

where $\partial\mathfrak{U}(B) \subseteq \mathrm{ba}(\mathbb{P})$ is the super-differential of $\mathfrak{U}$. Therefore, $\mathcal{I}(B)$ is the annihilator of $\partial\mathfrak{U}(B)$, i.e.,

$$\mathcal{I}(B) = \{R \in \mathbb{L}^\infty \,:\, \langle\mu, R\rangle = 0 \text{ for all } \mu \in \partial\mathfrak{U}(B)\},$$

which implies the statement. □

The following definition is due to Mark Davis and originates in [8]:

Definition 3.4. A number $p \in \mathbb{R}$ is said to be a B-**conditional Davis price** (or a B-**marginal utility-based price**) and simply a **conditional Davis price** if B is clear from the context, for a payoff $\varphi \in \mathbb{L}^\infty$ if

$$\varphi - p \text{ is } B\text{-irrelevant.}$$

The set of all B-conditional Davis prices of φ is denoted by $P(\varphi|B)$. □

Consequently, $p \in P(\varphi|B)$ if and only if

$$\mathfrak{U}(B + \varepsilon(\varphi - p)) \leq \mathfrak{U}(B), \quad \forall\varepsilon \in \mathbb{R}. \tag{3.2}$$

4. MARGINAL UTILITY-BASED PRICES

In this section we recall the definition of marginal utility-based prices and make the connection to the conditional Davis prices defined in the previous section. We start with a definition given in [15]. Let the derivative payoff ξ be a $\mathbb{R}^d$-valued, bounded, $\mathcal{F}_T$ measurable random variable. Recall that the function v is defined in (1.1).

Definition 4.1 (Remark 1., p. 849 in [15])**.** A vector $p_0 \in \mathbb{R}^d$ is said to be a *marginal utility-based price of ξ at $(x_0, q_0) \in \mathbb{R}^{d+1}$* if

$$v(q\cdot(\xi - p_0); x_0 + q_0\cdot\xi) \leq v(0; x_0 + q_0\cdot\xi), \quad \forall q \in \mathbb{R}^d.$$

□

In our context, $B := x_0 + q_0\cdot\xi$ and the investor prices units of ξ in addition to q_0. As observed in [15], to study these prices it is convenient to introduce a finite-dimensional value function. Then, marginal utility prices can be expressed as sub-differentials of this concave function. Indeed, consider the value function $u(q, x)$ defined on $\mathbb{R}^{d+1}$ by

$$u(q,x) := \mathfrak{U}(x + q\cdot\xi) = \sup_{X\in\mathcal{K}} \mathbb{E}\Big[U\Big(x + q\cdot\xi + X\Big)\Big]. \tag{4.1}$$

Under our standing assumptions, u is a proper concave function on $\mathbb{R}^{d+1}$. Moreover, if there is no gains process $X \in \mathcal{K}$ such that $x + q \cdot \xi + X \geq 0$, then the value function u is by definition equal to minus infinity.

The elementary connection between marginal-utility based prices and the sub-differential of u in the sense of convex analysis is given in Remark 1 in [15], Equation (3.11) in [16], and Equation (24) in [23]. We re-state it here for future reference. First, we note that at any (q_0, x_0) in the interior of the domain of u, the set of sub-differentials is non-empty and compact. Moreover, the second component of any $(z_q, z_x) \in \partial u(q_0, x_0)$ satisfies $z_x > 0$.

Lemma 4.2 ([15], [16], and [23])**.** *Let $y_0 := (q_0, x_0) \in \mathbb{R}^{d+1}$ be in the interior of the domain of u. Then, $p_0 \in \mathbb{R}^d$ is a marginal utility based price of ξ at y_0 if and only if*

$$p_0 = \frac{z_q}{z_x},$$

for some $(z_q, z_x) \in \partial u(y_0)$.

Proof. By definition $p_0 \in \mathbb{R}^d$ is a utility based price of ξ at y_0 if and only if

$$u(q_0 + q, x_0 - q \cdot p_0) \leq u(q_0, x_0), \quad \forall q \in \mathbb{R}^d.$$

We define

$$f(q) := u(q_0 + q, x_0 - q \cdot p_0), \quad q \in \mathbb{R}^d.$$

Then, f is a concave function and $p_0 \in \mathbb{R}^d$ is a marginal utility based price of ξ at y_0 if and only if $0 \in \partial f(0)$. Moreover, the sub-differential of f is connected to the sub-differential of u by,

$$\partial f(0) = \left\{ -z_x p_0 + z_q \in \mathbb{R}^d \ : \ z = (z_q, z_x) \in \partial u(y_0) \right\}.$$

Therefore, $0 \in \partial f(0)$ if and only if there exists $(z_q, z_x) \in \partial u(y_0)$ such that $-z_x p_0 + z_q = 0$. □

4.1. **Conditional Davis prices and marginal utility-based prices.** In this subsection, we show that conditional Davis prices defined above can be seen as the projection of marginal utility-based prices from the previous section at an appropriately chosen point. For given $B \in \mathbb{L}^\infty_{++}$ and $\varphi \in \mathbb{L}^\infty$, we let $\mathcal{Z}(\varphi|B)$ be the set of all marginal utility-based prices $p_0 \in \mathbb{R}^2$ for the random variable $\xi := (B, \varphi)$ at the point $x_0 := 0$, $q_0 := (1, 0)$. With these parameter choices $p_0 \in \mathcal{Z}(\varphi|B)$ provided that we have

$$\mathfrak{U}(B + q \cdot ((B, \varphi) - p_0)) \leq \mathfrak{U}(B), \quad \forall q \in \mathbb{R}^2. \tag{4.2}$$

Next, we show that the projection of $\mathcal{Z}(\varphi|B)$ onto its second component is the set of conditional Davis prices $P(\varphi|B)$ from Definition 3.4 above. Also, because $B \in \mathbb{L}^\infty_{++}$ and $\varphi \in \mathbb{L}^\infty$, the point $(x_0, q_0) = (0, (1, 0))$ is in the interior of the domain of u defined in (4.1).

Lemma 4.3. *For $B \in \mathbb{L}^\infty_{++}$ and $\varphi \in \mathbb{L}^\infty$ we have*

$$P(\varphi|B) = \{ p \in \mathbb{R} \ : \ \exists\ p_0 \in \mathcal{Z}(\varphi|B) \ \text{ such that } \ p = p_0 \cdot (0,1) \}.$$

Proof. Let $(p_B, p) \in \mathcal{Z}(\varphi|B)$. We use (4.2) with $q := (0, \varepsilon)$. The result is,

$$\mathfrak{U}(B + \varepsilon(\varphi - p)) \leq \mathfrak{U}(B),$$

for all $\varepsilon \in \mathbb{R}$. In view of (3.2), $p \in P(\varphi|B)$.

To prove the converse, fix $p \in P(\varphi|B)$. Then, by (3.2)

$$u((1, \varepsilon), -\varepsilon p) = \mathfrak{U}(B + \varepsilon(\varphi - p)) \leq \mathfrak{U}(B) = u((1, 0), 0), \quad \forall \varepsilon \in \mathbb{R}.$$

Set

$$g(\varepsilon) := u((1, \varepsilon), -\varepsilon p).$$

Then, $0 \in \partial g(0)$. Also, as in the proof of Lemma 4.2, we have

$$\partial g(0) = \{-z_x p + z_q \cdot (0, 1) \in \mathbb{R} : \quad z = (z_q, z_x) \in \partial u((1, 0), 0)\}.$$

Hence, there exists $z \in \partial u((1, 0), 0)$ such that

$$0 = -z_x p + z_q \cdot (0, 1).$$

We define $p_0 := z_q / z_x \in \mathbb{R}^2$ and use Lemma 4.2 to arrive at $p_0 \in \mathcal{Z}(\varphi|B)$. It is also clear that $p = p_0 \cdot (0, 1)$. □

Remark 4.4. In our context, we are given an *endowment* B and a derivative with payoff φ (both B and φ pay off at time T). Our goal is to study marginal-utility based prices of φ conditioned on the fact that an endowment B is given. Clearly, one does not price the endowment B. Hence, the appropriate price is a projection of the set of marginal-utility based prices onto its second component with $\xi := (B, \varphi)$ at the points $x_0 := 0$ and $q_0 := (1, 0)$. In Lemma 4.3, we proved that these two approaches are equivalent.

We can also use the above notation to summarize the related literature as follows:

(1) Definition 4.1 and Lemma 4.2 are from [15]. [15] study only the utility maximization problem and do not study marginal utility-based prices beyond their Remark 1.
(2) Lemma 4.2 can also be found in [16]. Furthermore, when $B := x > 0$ is constant, [16] provide a growth condition on the claim's payoff φ which ensures uniqueness of marginal utility-based prices and exemplify that such prices can fail to be unique. In the case when $B := x > 0$ is constant, Theorem 8.2 below supplements the results in [16] with formulas for the two endpoints describing the non-trivial interval of marginal utility-based prices. We stress that when B is unspanned, the results in [16] do not apply. Example 6.1 below illustrates that there can be major differences between the two cases: (i) $B := x > 0$ is constant and (ii) B is unspanned.
(3) [23] use the growth condition from [16] mentioned in (2) above which ensures uniqueness of the marginal-utility based prices. [23] linearly expand the marginal-utility based price from the base case $B := x > 0$ constant. Our analysis differs in three crucial ways from [23]: (i) we allow for non-uniqueness even when $B := x > 0$ constant, (ii) we allow for B being unspanned, and (iii) we do not

perform an asymptotic expansion in small quantities q of the claim's payoff φ but we instead provide closed-form expressions for the interval of marginal utility-based prices in Theorem 8.2 below. These non-trivial interval end-points are explicitly calculated in the two examples in Section 8.1.

While the notion of pricing in Definition 4.1 is consistent with the existing literature, no prior results cover the case where the investor's endowment B is unspanned.

5. Characterization of Conditional Davis Prices

5.1. **A dual characterization.** The dual characterization of the set of conditional Davis prices in Theorem 5.2 below rests on the following, simple, lemma:

Lemma 5.1. *A random variable $R \in \mathbb{L}^\infty$ is B-irrelevant if and only if*

$$\inf_{\mu \in \mathcal{D}} \Big(\mathbb{V}_B(\mu) + |\langle \mu, R \rangle| \Big) = \inf_{\mu \in \mathcal{D}} \mathbb{V}_B(\mu). \tag{5.1}$$

Proof. Because $\mathcal{I}(B)$ is a vector space, we can scale R so that, without loss generality, we can assume that $B \pm R \in \mathbb{L}^\infty_{++}$. Then, by the minimax theorem (see Theorem 2.10.2, p. 144 in [31]), we have

$$\begin{aligned} \inf_{\mu \in \mathcal{D}} \Big(\mathbb{V}_B(\mu) + |\langle \mu, R \rangle| \Big) &= \inf_{\mu \in \mathcal{D}} \sup_{|\varepsilon| \leq 1} \Big(\mathbb{V}_B(\mu) + \varepsilon \langle \mu, R \rangle \Big) \\ &= \sup_{|\varepsilon| \leq 1} \inf_{\mu \in \mathcal{D}} \Big(\mathbb{V}_B(\mu) + \varepsilon \langle \mu, R \rangle \Big) = \sup_{|\varepsilon| \leq 1} \mathfrak{U}(B + \varepsilon R). \end{aligned}$$

The same equality with $R = 0$, implies that (5.1) is equivalent to

$$\mathfrak{U}(B) = \sup_{|\varepsilon| \leq 1} \mathfrak{U}(B + \varepsilon R)$$

which is, in turn, equivalent to $R \in \mathcal{I}(B)$. $\square$

By Lemma 2.4, we have $\mu(\Omega) > 0$ for each $\mu \in \hat{\mathcal{D}}(B)$. Therefore, the family

$$\hat{\mathcal{D}}_0(B) := \{ \tfrac{1}{\mu(\Omega)} \mu \, : \, \mu \in \hat{\mathcal{D}}(B) \} \tag{5.2}$$

is a well-defined nonempty family of finitely-additive probabilities. We now have everything set up for our main characterization of conditional Davis prices:

Theorem 5.2. *For $\varphi \in \mathbb{L}^\infty(\mathbb{P})$ the following two statements are equivalent*

1. *$p \in P(\varphi | B)$, i.e., p is a B-conditional Davis price of φ.*
2. *$p = \langle \mathbb{Q}, \varphi \rangle$, for some $\mathbb{Q} \in \hat{\mathcal{D}}_0(B)$.*

In particular, $P(\varphi | B)$ is a nonempty compact subinterval of $\mathbb{R}$.

Proof. (1) $\Rightarrow$ (2): The first part of the proof of Lemma 2.4 applies to the functional $\mu \mapsto \mathbb{V}_B(\mu) + |\langle \mu, \varphi - p\rangle|$, and we can conclude that it admits a minimizer $\hat{\mu}$. By Lemma 5.1, the same $\hat{\mu}$ must minimize the functional $\mu \mapsto \mathbb{V}_B(\mu)$, as well, and, so, $\hat{\mu} \in \hat{\mathcal{D}}(B)$ and $\langle \hat{\mu}, \varphi - p\rangle = 0$.

(2) $\Rightarrow$ (1)**:** Suppose that p is such that $\langle \mu^*, \varphi - p\rangle = 0$, for some $\mu^* \in \hat{\mathcal{D}}_0(B)$. Then, for any μ, we have

$$\mathbb{V}_B(\mu^*) + |\langle \mu^*, \varphi - p\rangle| = \mathbb{V}_B(\mu^*) \leq \mathbb{V}_B(\mu) \leq \mathbb{V}_B(\mu) + |\langle \mu, \varphi - p\rangle|,$$

and Lemma 5.1 can be used.

Finally, Lemma 2.4 ensures that $\hat{\mathcal{D}}(B)$ is weak*-compact and the last claim follows. $\square$

5.2. **First consequences.** A reinterpretation in the setting of portfolios with convex constraints leads to the following dual characterization:

Corollary 5.3. *Suppose that U is reasonably elastic. Then, for each constant $c \geq 0$ and each $R \in \mathbb{L}^\infty$ we have*

$$\inf_{\mu \in \mathcal{D}} \Big(\mathbb{V}_B(\mu) + c|\langle \mu, R\rangle| \Big) = \inf_{y \geq 0, \mathbb{Q} \in \mathcal{M}} \Big(\mathbb{V}_B(y\mathbb{Q}) + c|\langle y\mathbb{Q}, R\rangle| \Big).$$

Proof. Let $\mathcal{C} := (\mathcal{K} - \mathbb{L}^0_+) \cap \mathbb{L}^\infty$ and let $\mathcal{C}'$ be the family of all random variables $X' \in \mathbb{L}^\infty$ of the form

$$X' = B + X + \varepsilon R, \text{ where } X \in \mathcal{C}, \quad \varepsilon \in [-c, c].$$

The support function $\alpha_{\mathcal{C}'}$ for the set $\mathcal{C}'$ is then given by

$$\begin{aligned} \alpha_{\mathcal{C}'}(\mu) &= \sup_{X' \in \mathcal{C}'} \langle \mu, X\rangle \\ &= \langle \mu, B\rangle + \sup_{X \in \mathcal{C},\ \varepsilon \in [-c,c]} \Big(\langle \mu, X\rangle + \varepsilon \langle \mu, R\rangle \Big) \\ &= \langle \mu, B\rangle + c|\langle \mu, R\rangle| + \begin{cases} 0, & \mu \in \mathcal{D}, \\ +\infty, & \mu \notin \mathcal{D}. \end{cases} \end{aligned}$$

It follows that

$$\inf_{\mu \in \mathcal{D}} \Big(\mathbb{V}_B(\mu) + c|\langle \mu, R\rangle| \Big) = \inf_{\mu \in \mathrm{ba}(\mathbb{P})} \Big(\mathbb{V}_0(\mu) + \alpha_{\mathcal{C}'}(\mu) \Big). \tag{5.3}$$

Moreover, the set $\mathcal{C}$ is weak*-closed by Theorem 4.2 in [9]; hence, so is $\mathcal{C}'$. Hence, the assumptions of Proposition 3.14, p. 686 of [28] are satisfied (via Corollary 3.4, p. 679 in [28]) and, so, the infimum on the right-hand side of (5.3) can be replaced by an infimum over σ-additive measures. $\square$

Our next two consequences of Theorem 5.2 provide a partial generalization and an alternative method of proof for Theorem 3.1, p. 206 in [16].

Proposition 5.4. *Suppose that U is reasonably elastic and that the dual problem* (2.5) *admits a non-σ-additive optimizer. Then there exists $A \in \mathcal{F}$ such that $\varphi = \mathbf{1}_A$ has multiple B-conditional Davis prices.*

Proof. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for the problem $\inf_{\mu\in\mathcal{D}} \mathbb{V}_B(\mu)$. By Corollary 5.3 we can assume that each μ is countably additive. Moreover, the argument of Lemma 2.4 guarantees that the sequence $\{\mu(\Omega)_n\}_{n\in\mathbb{N}}$ is bounded. Therefore, $\{\mu_n\}_{n\in\mathbb{N}}$ belongs to a weak*-compact subset of $\mathrm{ba}(\mathbb{P})$. By extracting a further subsequence, we may assume that the sequence of total masses $\mu_n(\Omega)$ converges towards a positive constant $y > 0$ (Lemma 2.4 ensures that $y \neq 0$).

We suppose first that $\{\mu_n\}_{n\in\mathbb{N}}$ is not weak*-convergent. Then, two of its convergent subnets will have different limits, and both of these will be elements of $\hat{\mathcal{D}}(B)$ with the same total mass $y > 0$. Hence, the set $\hat{\mathcal{D}}_0(B)$ of (5.2) is not a singleton, and, by Corollary 5.2, there exists $\varphi = \mathbf{1}_A$, with $A \in \mathcal{F}$, with two different conditional Davis prices.

On the other hand, suppose that $\{\mu_n\}_{n\in\mathbb{N}}$ converges to $\hat{\mu}$ in the weak*-sense. Then we have $\hat{\mu} \in \hat{\mathcal{D}}(B)$. Furthermore, by the Vitali-Hahn-Sachs theorem (see [12], Corollary 8 on p.159) the limit $\hat{\mu}$ is countably additive. Hence, the set $\hat{\mathcal{D}}(B)$ will have at least two different elements - one countably additive and one not. Then a random variable $\varphi = \mathbf{1}_A$ with two different conditional Davis prices can be constructed as above. □

The next consequence of Theorem 5.2 gives a sufficient condition (analogous to that of Theorem 3.1 on p. 206 of [16]) for the uniqueness of conditional Davis prices. Before we state it, we recall that, under the condition of reasonable elasticity, [6] show there exists a process $\hat{\pi} \in \mathcal{A}$ such that $\hat{X} := (\hat{\pi} \cdot S)_T + B$ satisfies

$$\mathbb{E}[U(\hat{X})] = \mathfrak{U}(B) \text{ and } U'(\hat{X}) = \frac{d\hat{\mu}^r}{d\mathbb{P}}, \tag{5.4}$$

where $\hat{\mu} \in \hat{D}(B)$. The random variable $\hat{X}$ is $\mathbb{P}$-a.s. unique with this property.

Corollary 5.5. *Suppose U is reasonably elastic and that $|\varphi| \leq c\hat{X}$, for some constant $c \geq 0$, where $\hat{X}$ is as in (5.4). Then the set $P(\varphi|B)$ of B-conditional Davis prices for $\varphi \in \mathbb{L}^\infty$ is a singleton.*

Proof. In view of Lemma 2.4 and Theorem 5.2, it will be enough to show that $\langle \hat{\mu}^s, \hat{X} \rangle = 0$, for each $\hat{\mu} \in \hat{\mathcal{D}}(B)$. This, in turn, follows directly from the first part of Equation (4.7) in [6]. □

6. Directional derivatives of the primal value function

Our next task is to study directional differentiability of the primal utility-maximization value function $\mathfrak{U}$ defined by (2.2). Its relevance in the context of Davis pricing has been noted by several authors (including Davis in [8]), and we use the obtained results in the later sections to give a workable characterization of the interval of conditional Davis prices. First we show, by means of an example, that smoothness - even in the most "benign" directions

- cannot be expected in general. Then we give a characterization of the directional derivative in terms of a linear control problem. We hope that both our counterexample and the later characterization hold some independent interest outside of the context of Davis pricing.

6.1. **An example of nonsmoothness.** Our next example shows that the set $\hat{\mathcal{D}}(B)$ of dual minimizers may contain measures with different total masses. In other words, $\hat{\mu}(\Omega)$ may not be constant over $\hat{\mu} \in \hat{\mathcal{D}}(B)$. Consequently, $\mathfrak{U}$ may fail to be differentiable even in "constant directions" in the sense that $\varepsilon \to \mathfrak{U}(B+\varepsilon)$ may fail to be differentiable at $\varepsilon := 0$. Once we introduce the concept of unique superreplicability in the next section, we will see how it can be used to regain differentiability in certain cases of interest.

For simplicity and concreteness, we base the example on Example 5.1' in [22], and use the following notation and conventions: All random variables X will be defined on the sample space $\Omega := \mathbb{N}_0$, and we write X_n for $X(\{n\})$. Countably-additive measures are identified with sequences in ℓ^1_+ and for $\mathbb{Q} = (q_n) \in \ell^1_+$ we write $\langle \mathbb{Q}, X\rangle$ for $\sum q_n X_n$ whenever $X = (X_n) \in \ell^\infty$.

Example 6.1. We start by recalling the elements of (a special case of) the one period Example 5.1' in [22] where $\Omega := \mathbb{N}_0$ and $\mathbb{P} = (p_n)$ with

$$p_0 := \tfrac{3}{4}, \quad p_n := \tfrac{2^{-n}}{4} \text{ for } n \in \mathbb{N}.$$

The one-period stock-price increment $\Delta S = (\Delta S_n)$ is defined as follows

$$\Delta S_0 := 1 \text{ and } \Delta S_n := \tfrac{1-n}{n} \text{ for } n \in \mathbb{N}.$$

With $U := \log$, the primal problem is defined by

$$u(x) := \sup_{\pi \in [-x,x]} \mathbb{E}[U(x + \pi \Delta S)], \quad x > 0.$$

Let $\mathcal{Q}$ denote the set of all finite martingale measures, i.e.,

$$\mathcal{Q} := \{\mathbb{Q} \in \ell^1_+ \,:\, \langle \mathbb{Q}, \Delta S\rangle = 0\},$$

and let $\mathcal{M} := \{\mathbb{Q} \in \mathcal{Q} \,:\, \langle \mathbb{Q}, 1\rangle = 1\}$. Because $V(y) = -1 - \log(y)$, the dual problem is given by

$$v(y) := \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}[V(y\tfrac{d\mathbb{Q}}{d\mathbb{P}})] = V(y) + v^*, \text{ where } v^* := \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}[-\log(\tfrac{d\mathbb{Q}}{d\mathbb{P}})].$$

We will start by showing that no minimizing sequence $(\mathbb{Q}^N)_N \subset \mathcal{M}$ for v^* (equivalently, for $v(y)$) can be weakly convergent in the sense that $\langle \mathbb{Q}^N, f\rangle$ cannot converge for all test functions $f \in \ell^\infty$. It is a consequence of the Vitali-Hahn-Sachs Theorem (see, e.g., Corollary 8, p.159 in [12]) that ℓ^1 is weakly sequentially complete, so any weakly convergent sequence is necessarily weakly convergent in ℓ^1. Therefore, any weak limit of any minimizing sequence $(\mathbb{Q}^N)_N$ must also belong to $\mathcal{M}$ and is, therefore, a minimizer for v^*. However, this would contradict the strict supermartingale property of the dual log-optimizer shown in Example 5.1' in [22].

As a consequence of the above, for a given minimizing sequence $(\mathbb{Q}^N)_N$, there exists a random variable $H \in \ell^\infty$ such that

$$\langle \mathbb{Q}^N, H\rangle \text{ does not converge in } \mathbb{R} \text{ as } N \to \infty. \tag{6.1}$$

Because $\langle \mathbb{Q}^N, 1\rangle = 1$, for each N, we can assume that $H \geq 1$. Moreover, there exist two subsequences $(\mathbb{Q}^{1,N})_N$ and $(\mathbb{Q}^{2,N})_N$ of $(\mathbb{Q}^N)_N$ such that the limits

$$y_1 = \lim_N \langle \mathbb{Q}^{1,N}, H\rangle \text{ and } y_2 = \lim_N \langle \mathbb{Q}^{2,N}, H\rangle \text{ exist with } y_1 \neq y_2. \tag{6.2}$$

With H as above, we define $B := 1/H$ and a new stock price process with increments

$$\Delta\tilde{S} := B\,\Delta S,$$

and then consider the log-utility maximization problem with the random endowment B and the stock-price increments $\Delta\tilde{S}$. The associated dual problem[3] is given by

$$\begin{aligned}
\tilde{v}(y) &:= \inf_{\tilde{\mathbb{Q}}\in\tilde{\mathcal{M}}} \mathbb{E}[V(y\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}})] + y\langle\tilde{\mathbb{Q}}, B\rangle \\
&= -1 + \inf_{\tilde{\mathbb{Q}}\in\tilde{\mathcal{M}}} \Big(\mathbb{E}[-\log(y\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}})] + y\langle\tilde{\mathbb{Q}}, B\rangle\Big) \\
&= -1 + \mathbb{E}[\log(B)] + \inf_{\tilde{\mathbb{Q}}\in\tilde{\mathcal{M}}} \Big(\mathbb{E}[-\log(y\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}B)] + y\langle\tilde{\mathbb{Q}}B, 1\rangle\Big) \\
&= \mathbb{E}[\log(B)] + \inf_{\tilde{\mathbb{Q}}\in\tilde{\mathcal{M}}} \Big(\mathbb{E}[V(y\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}B)] + y\langle\tilde{\mathbb{Q}}B, 1\rangle\Big), \quad y > 0,
\end{aligned}$$

where

$$\tilde{\mathcal{Q}} := \{\tilde{\mathbb{Q}} \in \ell^1_+ \,:\, \langle\tilde{\mathbb{Q}}, \Delta\tilde{S}\rangle = 0\} \text{ and } \tilde{\mathcal{M}} := \{\tilde{\mathbb{Q}} \in \tilde{\mathcal{Q}} \,:\, \langle\tilde{\mathbb{Q}}, 1\rangle = 1\}.$$

Because $\tilde{\mathbb{Q}} \in \tilde{\mathcal{Q}}$ if and only if $\mathbb{Q} = \tilde{\mathbb{Q}}B \in \mathcal{Q}$, we have

$$\begin{aligned}
\inf_{y>0} \tilde{v}(y) &= \mathbb{E}[\log(B)] + \inf_{y>0}\inf_{\mathbb{Q}\in\mathcal{M}} \Big(\mathbb{E}[V(y\frac{d\mathbb{Q}}{d\mathbb{P}})] + y\Big) \\
&= \mathbb{E}[\log(B)] + \inf_{y>0} \Big(v(y) + y\Big) \\
&= \mathbb{E}[\log(B)] + \inf_{y>0} \Big(V(y) + y + v^*\Big) \\
&= \mathbb{E}[\log(B)] + v^*.
\end{aligned}$$

[3] It has been shown in [28, Lemma 3.12] that under the reasonable asymptotic elasticity condition, infimization over the set of countably-additive martingale measures - as opposed to its finitely-additive enlargement as in [6] - leads to the same value function.

By using the minimizing sequences $(\mathbb{Q}^{1,N})_N$, and $(\mathbb{Q}^{2,N})_N$ constructed above we define the sequence of probability measures

$$\tilde{\mathbb{Q}}^{i,N} := \frac{\mathbb{Q}^{i,N}H}{\langle \mathbb{Q}^{i,N}H, 1\rangle} \in \tilde{\mathcal{M}} \text{ for } i = 1, 2.$$

We can use (6.2) and the fact that $(\mathbb{Q}^{i,N})_N$, $i = 1, 2$, are minimizing sequences for v^* to see

$$\begin{aligned}\mathbb{E}[V(y_i \tfrac{d\tilde{\mathbb{Q}}^{i,N}}{d\mathbb{P}})] + y_i\langle \tilde{\mathbb{Q}}^{i,N}, B\rangle &= \mathbb{E}[V(\tfrac{y_i H}{\langle \mathbb{Q}^{i,N}H,1\rangle}\tfrac{d\mathbb{Q}^{i,N}}{d\mathbb{P}})] + \tfrac{y_i}{\langle \mathbb{Q}^{i,N}H,1\rangle} \\ &= \mathbb{E}[V(\tfrac{y_i H}{\langle \mathbb{Q}^{i,N}H,1\rangle})] + \tfrac{y_i}{\langle q^{i,N}H,1\rangle} - \mathbb{E}[\log(\tfrac{d\mathbb{Q}^{i,N}}{d\mathbb{P}})] \\ &\to \mathbb{E}[\log(B)] + v^* \\ &= \inf_{y>0} \tilde{v}(y).\end{aligned}$$

Clearly, $\tilde{v}(y_i) \geq \inf_{y>0} \tilde{v}(y)$, for $i = 1, 2$, which implies that $\tilde{\mathbb{Q}}^{i,N}$ is a minimizing sequence for $\tilde{v}(y_i)$. Therefore, $\tilde{v}(y_1) = \tilde{v}(y_2) = \inf_{y>0} \tilde{v}(y)$ which implies that $\tilde{v}$ is constant on $[y_1, y_2]$. This, in turn implies, that the conjugate function to $\tilde{v}$ fails to be differentiable at 0 (indeed, the entire segment $[y_1, y_2]$ belongs to its superdifferential at zero). □

Remark 6.2.

(1) The construction of the random endowment B in Example 6.1 above rests on the weak sequential completeness property of ℓ^1 which, in fact, holds for any $\mathbb{L}^1$-space. Example 6.1 above is therefore generic in the sense that it can be applied to any model which produces non-trivial singular components in the dual optimizer for the log-investor (with constant endowment). This implies that there also exist random endowments in the Brownian setting of Example 5.1 in [22] which produce a non-differentiable primal utility function.

(2) Example 6.1 seems to contradict the claimed smoothness of the primal value function stated in Theorem 3.1(i) in [6][4]: With the notation from Example 6.1 we can define the primal utility function

$$\tilde{u}(x) := \sup_{\pi\in\mathbb{R}} \mathbb{E}[U(x + \pi\Delta\tilde{S} + B)], \quad x \in \mathbb{R}, \tag{6.3}$$

where we use the convention $\mathbb{E}[U(x + \pi\Delta\tilde{S} + B)] = -\infty$ if $\mathbb{E}[U(x + \pi\Delta\tilde{S} + B)^-] = +\infty$. Then $\tilde{u}$ is not differentiable at $x = 0$ which is an interior point in $\tilde{u}$'s domain.

6.2. **A characterization via a linear stochastic control problem.** Even though the superdifferential of $\mathfrak{U}$ at B consists of finitely-additive measures related to the solution of the dual problem, it is possible to give a characterization of directional derivatives without any recourse to finite additivity. This is the most attractive feature of our linear characterization in

[4]The authors first learned from Pietro Siorpaes about the potential lack of correctness of Remark 4.2 in [6]. We also refer the reader to Erratum [7] for further discussions.

Proposition 6.5 below; however, as we shall see later, it also leads to explicit computations in many cases. The price we pay is the increased complexity of the linearized problem's domain.

Throughout the reminder of the paper we impose the following assumption, where $\hat{X}$ is the primal optimizer characterized by (5.4), and whose existence is guaranteed by the assumption of reasonable elasticity:

Assumption 6.3. U is reasonably elastic and there exists a constant $b > 0$ such that

$$\hat{X}\, U'\Big((1-b)\hat{X}\Big) \in \mathbb{L}^1(\mathbb{P}). \tag{6.4}$$

□

Remark 6.4. Assumption 6.3 holds automatically if, for example, U belongs to the class of CRRA (power) utilities

$$U(x) = \tfrac{x^p}{p}, \text{ for } p \in (-\infty, 1) \setminus \{0\} \text{ or } U(x) = \log(x).$$

Given the optimizer $\hat{\pi} \in \mathcal{A}$ and the random variable $\hat{X}$, we let $\Delta(\varphi) := \cup_{\varepsilon>0}\Delta^{\varepsilon}(\varphi)$ where $\Delta^{\varepsilon}(\varphi)$ denotes the class of all $\delta \in L(S)$, such that

$$\hat{\pi} + \varepsilon\delta \in \mathcal{A} \text{ and } \hat{X} + \varepsilon(\varphi + (\delta \cdot S)_T) \geq 0. \tag{6.5}$$

Because $\mathcal{A}$ is a convex cone and $\hat{X} \geq 0$, the family $\Delta^{\varepsilon}(\varphi)$ is nonincreasing in $\varepsilon \geq 0$ in the sense

$$\varepsilon_1 \leq \varepsilon_2 \Rightarrow \Delta^{\varepsilon_2}(\varphi) \subseteq \Delta^{\varepsilon_1}(\varphi). \tag{6.6}$$

Similarly, the family $\Delta^{\varepsilon}(\varphi)$ is nondecreasing in $\varphi \in \mathbb{L}^{\infty}$ in the sense

$$\varphi_1 \leq \varphi_2 \Rightarrow \Delta^{\varepsilon}(\varphi_1) \subseteq \Delta^{\varepsilon}(\varphi_2). \tag{6.7}$$

Proposition 6.5. *Under Assumption 6.3 we have for* $\varphi \in \mathbb{L}^{\infty}(\mathbb{P})$

$$\lim_{\varepsilon \searrow 0} \tfrac{1}{\varepsilon}\left(\mathfrak{U}(B+\varepsilon\varphi) - \mathfrak{U}(B)\right) = \sup_{\delta \in \Delta(\varphi)} \mathbb{E}[\hat{Y}\big((\delta \cdot S)_T + \varphi\big)], \tag{6.8}$$

where $\hat{Y} := \frac{d\hat{\mu}^r}{d\mathbb{P}}$.

Proof. For small enough $\varepsilon > 0$ we can find $\pi^{\varepsilon} \in \mathcal{A}$ such that $X^{\varepsilon} = (\pi^{\varepsilon} \cdot S)_T + B + \varepsilon\varphi$ has the property that

$$\mathbb{E}[U(X^{\varepsilon}_T)] \geq \mathfrak{U}(B + \varepsilon\varphi) - \varepsilon^2.$$

For such an $\varepsilon > 0$ we define

$$\delta^{\varepsilon} = \tfrac{1}{\varepsilon}\Big(\pi^{\varepsilon} - \hat{\pi}\Big).$$

Since $\hat{\pi} + \varepsilon\delta^{\varepsilon} = \pi^{\varepsilon} \in \mathcal{A}$, the first part of (6.5) above holds. To see that the second part of (6.5) holds, we note that $\hat{X} + \varepsilon\big((\delta^{\varepsilon} \cdot S)_T + \varphi\big) = X^{\varepsilon}$ and

$\mathbb{E}[U(X^\varepsilon)] > -\infty$ which implies $\hat{X} + \varepsilon\big((\delta^\varepsilon \cdot S)_T + \varphi\big) \geq 0$. Therefore, we have $\delta^\varepsilon \in \Delta^\varepsilon(\varphi)$. The concavity of $\mathfrak{U}$ then implies that

$$\begin{aligned}\mathfrak{U}(B + \varepsilon\varphi) &\leq \mathbb{E}[U(X^\varepsilon)] + \varepsilon^2 \\ &\leq \mathbb{E}[U(\hat{X})] + \varepsilon\mathbb{E}[U'(\hat{X})(\varphi + (\delta^\varepsilon \cdot S)_T)] + \varepsilon^2 \\ &\leq \mathfrak{U}(B) + \varepsilon \sup_{\delta\in\Delta^\varepsilon(\varphi)} \mathbb{E}[\hat{Y}((\delta \cdot S)_T + \varphi)] + \varepsilon^2 \\ &\leq \mathfrak{U}(B) + \varepsilon \sup_{\delta\in\Delta(\varphi)} \mathbb{E}[\hat{Y}((\delta \cdot S)_T + \varphi)] + \varepsilon^2.\end{aligned}$$

This produces the upper bound inequality

$$\limsup_{\varepsilon\searrow 0} \tfrac{1}{\varepsilon}\Big(\mathfrak{U}(B + \varepsilon\varphi) - \mathfrak{U}(B)\Big) \leq \sup_{\delta\in\Delta(\varphi)} \mathbb{E}[\hat{Y}((\delta \cdot S)_T + \varphi)].$$

To prove the opposite inequality, we pick $\varepsilon_0 > 0$ and $\delta \in \Delta^{\varepsilon_0}(\varphi)$, so that $\hat{\pi} + \varepsilon_0\delta \in \mathcal{A}$ and $\hat{X} + \varepsilon_0 D \geq 0$, where

$$D = (\delta \cdot S)_T + \varphi.$$

Because $b > 0$, we also have

$$\hat{X} + b\varepsilon_0 D \geq (1 - b)\hat{X}.$$

Therefore, for $\varepsilon \in (0, \varepsilon_1)$ with $\varepsilon_1 := b\varepsilon_0$ we have

$$\hat{X} + \varepsilon D \geq (1 - b)\hat{X} > 0. \tag{6.9}$$

The concavity of U implies that for $\varepsilon \in (0, \varepsilon_1)$ we have

$$U(\hat{X} + \varepsilon D) \geq U(\hat{X}) + \varepsilon Y^\varepsilon D \text{ where } Y^\varepsilon = U'\Big(\hat{X} + \varepsilon D\Big).$$

Therefore, for $\varepsilon \in (0, \varepsilon_1)$ we obtain

$$\mathfrak{U}(B + \varepsilon\varphi) \geq \mathbb{E}[U(\hat{X} + \varepsilon D)] \geq \mathfrak{U}(B) + \varepsilon\mathbb{E}[Y^\varepsilon D].$$

In order to pass ε to zero we note that (6.9) gives us

$$\Big(Y^\varepsilon D\Big)^- \leq U'\big((1 - b)\hat{X}\big)D^- \leq U'\big((1 - b)\hat{X}\big)\tfrac{1}{\varepsilon_0}\hat{X}, \tag{6.10}$$

which is integrable by assumption. The uniform bound in (6.10) allows us to use Fatou's lemma together with $Y^\varepsilon \to U'(\hat{X}) =: \hat{Y}$, $\mathbb{P}$-a.s., to conclude that

$$\liminf_{\varepsilon\searrow 0} \tfrac{1}{\varepsilon}\Big(\mathfrak{U}(B + \varepsilon\varphi) - \mathfrak{U}(\varphi)\Big) \geq \liminf_{\varepsilon\searrow 0} \mathbb{E}[Y^\varepsilon D] \geq \mathbb{E}[\hat{Y}D]. \qquad \square$$

The following example highlights the role strict local martingales play in the linear optimization problem appearing in (6.8). The next section identifies the key components which make this toy example work.

Example 6.6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting two independent Brownian motions (Z, W) and we let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be their augmented filtration up to some maturity $T > 0$. We define the stock price dynamics to be

$$dS_t := S_t\big(\lambda_t dt + dZ_t\big), \quad S_0 > 0, \tag{6.11}$$

where the process λ is as in [11] so that the *minimal martingale density*

$$\mathcal{E}(-\lambda \cdot Z)_t := e^{-\int_0^t \lambda_u dZ_u - \frac{1}{2}\int_0^t \lambda_u^2 du}, \quad t \in [0, T],$$

fails the martingale property even though the set $\mathcal{M}$ of equivalent local martingale measures is non-empty. As a consequence, Example 5.1 in [22] shows that the log-investor's dual utility optimizer $\hat{Y}_t := \hat{Y}_0 \mathcal{E}(-\lambda \cdot Z)_t$ is a strict local martingale.

We consider the simple case where $\hat{X}_0 := 1$ and the payoff φ is constant. The fact that we are working with the log-utility implies that $\hat{Y}_0 = 1$, and Remark 3.2 in [16] states that the unique Davis price of φ is φ itself, a quantity different from $\mathbb{E}[\hat{Y}_T \varphi]$.

For $\delta \in \Delta^\varepsilon(\varphi)$ we have

$$(\delta \cdot S)_t \geq -\varphi - \tfrac{1}{\varepsilon}\hat{X}_t, \quad t \in [0, T].$$

Thanks to the fact that $\hat{Y}\hat{X} = 1$, which is the standard myopic property of optimizers in logarithmic utility maximization, the local martingale $\hat{Y}_t\big((\delta \cdot S)_t + \varphi\big)$ is a lower bounded by $-\frac{1}{\varepsilon}$; hence, it is a supermartingale. Therefore, the limit on the left-hand side of (6.8) is bounded from above by

$$\sup_{\delta\in\Delta(\varphi)} \mathbb{E}[\hat{Y}_T\big((\delta \cdot S)_T + \varphi\big)] \leq \varphi. \tag{6.12}$$

Because $\hat{Y}$ is a strict local martingale, we see that for any $\delta \in \Delta(\varphi)$ for which the local martingale $\hat{Y}_t(\delta \cdot S)_t$ is a martingale the expression $\mathbb{E}[\hat{Y}_T\big((\delta \cdot S)_T + \varphi\big)]$ stays bounded away from the upper bound in (6.12). On the other hand, that upper bound is attained at any $\delta \in \Delta(\varphi)$ which satisfies the requirement

$$\hat{Y}_T(\delta \cdot S)_T = \varphi(\hat{Y}\lambda \cdot Z)_T. \qquad \square$$

7. Uniquely superreplicable random variables

While the linear control problem of Proposition 6.5 provides a useful characterization of $\mathfrak{U}$'s directional derivatives, the linear problem seems to be difficult to solve explicitly in full generality. The present section outlines a relevant class of payoffs φ for which such a tractable solution is, indeed, available. It involves the notion of unique superreplicability similar to Condition (B1) in [27].

Definition 7.1. A random variable $\psi \in \mathbb{L}^\infty(\mathbb{P})$ is said to be

(1) **replicable** if there exists a constant $\psi_0 \in \mathbb{R}$ and $\pi_\psi \in \mathcal{A} \cap (-\mathcal{A})$ such that

$$\psi = \psi_0 + (\pi_\psi \cdot S)_T.$$

(2) **uniquely superreplicable (by Ψ)** if $\Psi \in \mathbb{L}^\infty(\mathbb{P})$ is replicable, $\Psi \geq \psi$, and

$$x + (\pi \cdot S)_T \geq \psi \Rightarrow x + (\pi \cdot S)_T \geq \Psi$$

for all $x \in \mathbb{R}$ and $\pi \in \mathcal{A}$.

Remark 7.2.

(1) The need to use uniformly bounded gains processes for replication purposes such as in Definition 7.1(1) has long been recognized; see, e.g., Definition 1.15 in [4] and the first part of Remark 3.2 in [16].

(2) The representation in Definition 7.1(1) of a replicable claim ψ in terms of (ψ_0, π_ψ) is unique. Moreover, the process $(\pi_\psi \cdot S)_t$ is a bounded $\mathbb{Q}$-martingale for each $\mathbb{Q} \in \mathcal{M}$. Consequently, because each $\mu \in \mathcal{D}$ is the weak* limit of a net $y_\alpha \mathbb{Q}_\alpha$ with $y_\alpha \in [0, \infty)$ and $\mathbb{Q}_\alpha \in \mathcal{M}$, we have

$$\langle \mu, (\pi_\psi \cdot S)_t \rangle = \lim_\alpha y_\alpha \langle \mathbb{Q}_\alpha, (\pi_\psi \cdot S)_t \rangle = 0.$$

(3) Provided it exists, the random variable Ψ in Definition 7.1(2) is unique. If $\Psi = \Psi_0 + (\pi_\Psi \cdot S)_T$ uniquely superreplicates ψ, we have the representation

$$\Psi_0 = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\psi].$$

(4) Unique superreplicability is scale invariant: If ψ is uniquely superreplicable by Ψ, then $\alpha\psi$ is uniquely superreplicable by $\alpha\Psi$ for $\alpha \geq 0$. It is also invariant under translation by replicable random variables. In particular, replicable random variables are uniquely superreplicable.

Example 7.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting two independent Brownian motions (β, W) and we let $\{\mathcal{F}_t\}_{t \in [0,T]}$ be their augmented filtration up to some maturity $T > 0$. With the set of all pathwise p-integrable predictable processes denoted by $\mathcal{L}^p$, we let S be the Itô process

$$dS_t := S_t \sigma_t \big(\lambda_t dt + d\beta_t\big), \quad S_0 > 0, \tag{7.1}$$

where $\sigma, \lambda \in \mathcal{L}^2$ are such that NFLVR holds.

We focus on payoffs of the form $\varphi = \varphi(W_T)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a bounded Lipschitz function. To show that such $\varphi(W_T)$ is uniquely superreplicable by the constant $\sup_a \varphi(a)$, we start by assuming that

$$x + (\pi \cdot S)_T \geq \varphi(W_T) \text{ a.s.},$$

for some $x \in \mathbb{R}$ and some $\pi \in \mathcal{A}$. Then, for each $t \in [0, T)$ we have

$$x + (\pi \cdot S)_t \geq \operatorname*{esssup}_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[x + (\pi \cdot S)_T | \mathcal{F}_t] \geq \operatorname*{esssup}_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\varphi(W_T) | \mathcal{F}_t]. \tag{7.2}$$

Lemma 7.4 below gives conditions under which the limit as $t \uparrow T$ of the right-hand side of (7.2) equals $\sup_a \varphi(a)$. When these conditions are met, the continuity of the paths of the stochastic integral with respect to S implies that $x+(\pi\cdot S)_T \geq \sup_a \varphi(a)$. This, in turn, confirms that $\varphi(W_T)$ is uniquely superreplicable by the constant $\sup_a \varphi(a)$. $\square$

Lemma 7.4. *In the setting of Example 7.3 above with $\varphi:\mathbb{R}\to\mathbb{R}$ bounded and Lipschitz, assume that there exists a nonnegative (deterministic) function $f \in \mathbb{L}^1([0,T])$ and a predictable process $\nu^{(0)} \in \mathcal{L}^2$ such that*

(1) *$|\nu_u^{(0)}| \leq f(u)$, for Lebesgue-almost all $u \in [0,T]$, $\mathbb{P}$-a.s., and*
(2) *the stochastic exponential $Z_T^{(0)} := \mathcal{E}(-\lambda\cdot\beta - \nu^{(0)}\cdot W)_T$ is the Radon-Nikodym density of some $\mathbb{Q}^{(0)} \in \mathcal{M}$ with respect to $\mathbb{P}$.*

Then

$$\lim_{t\uparrow T} \operatorname*{esssup}_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\varphi(W_T)|\mathcal{F}_t] = \sup_a \varphi(a). \tag{7.3}$$

Proof. For a bounded and predictable process δ we define the process $Z^{(\delta)}$ by

$$dZ_t^{(\delta)} := -Z_t^{(\delta)}\big(\lambda_t\, d\beta_t + (\nu_t^{(0)} + \delta_t)\, dW_t\big) \quad Z_0^{(\delta)} := 1.$$

A simple calculation yields the following expression

$$Z_T^{(\delta)} = Z_T^{(0)}\mathcal{E}(-\delta\cdot W^{(0)})_T,$$

where $W_t^{(0)} := W_t + \int_0^t \nu_u^{(0)}\, du$ is a $\mathbb{Q}^{(0)}$-Brownian motion. With $\mathbb{E}^{(0)}$ denoting the expectation with respect to $\mathbb{Q}^{(0)}$, we have

$$\mathbb{E}[Z_T^{(\delta)}] = \mathbb{E}^{(0)}[\mathcal{E}(-\delta\cdot W^{(0)})] = 1,$$

where the last equality follows from the boundedness of δ. Hence, $Z^{(\delta)}$ is a (true) martingale and can be used as a density of a probability measure $\mathbb{Q}^{(\delta)} \in \mathcal{M}$.

To proceed, we fix $t_0 \in (0,T)$ and $a \in \mathbb{R}$ and define

$$\delta_t^{(a)} := \tfrac{1}{T-t_0}(W_{t_0}\mathbf{1}_{\{|W_{t_0}|\leq 1/(T-t_0)\}} - a)\mathbf{1}_{\{t\geq t_0\}}, \quad t\in[t_0,T],$$
$$W_t^{(a)} := W_t + \int_0^t (\nu_u^{(0)} + \delta_u^{(a)})\, du, \quad t\in[0,T].$$

Then we have

$$W_T - a = W_T^{(a)} - W_{t_0}^{(a)} - \int_{t_0}^T \nu_u^{(0)}\, du + W_{t_0}\mathbf{1}_{\{|W_{t_0}|>1/(T-t_0)\}}.$$

The process $W^{(a)}$ is a $\mathbb{Q}^{(a)}$-Brownian motion, where $\mathbb{Q}^{(a)}$ is a short for $\mathbb{Q}^{(\delta^{(a)})}$. Therefore, the bound $|\nu^{(0)}| \leq f$ implies that

$$\mathbb{E}^{\mathbb{Q}^{(a)}}\big[|W_T - a|\big|\mathcal{F}_{t_0}\big] \leq C(t_0)$$

where

$$C(t_0) := \sqrt{\tfrac{2(T-t_0)}{\pi}} + \int_{t_0}^T f_u\, du + |W_{t_0}|\mathbf{1}_{\{|W_{t_0}|>1/(T-t_0)\}}.$$

With L_φ denoting the Lipschitz constant of φ, we have

$$|\mathbb{E}^{\mathbb{Q}^{(a)}}[\varphi(W_T)|\mathcal{F}_{t_0}] - \varphi(a)| \leq L_\varphi \, \mathbb{E}^{\mathbb{Q}^{(a)}}[|W_T - a||\mathcal{F}_{t_0}] \leq L_\varphi C(t_0).$$

Therefore,

$$\begin{aligned}\limsup_{t_0 \nearrow T} \operatorname*{esssup}_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\varphi(W_T)|\mathcal{F}_{t_0}] &\geq \limsup_{t_0 \nearrow T} \mathbb{E}^{\mathbb{Q}^{(a)}}[\varphi(W_T)|\mathcal{F}_{t_0}] \\ &\geq \limsup_{t_0 \nearrow T} \Big(\varphi(a) - L_\varphi C(t_0)\Big) = \varphi(a).\end{aligned}$$

It remains to note that the left-hand side above does not depend on a and that $\sup_a \varphi(a)$ is a trivial upper bound in (7.3). □

Example 7.5. (Continuation of Example 7.3) We continue Example 7.3 by examining two cases in which Lemma 7.4 applies. In the first one we simply take $f := 0$. That can be done if and only if the minimal martingale density $\mathcal{E}(-\lambda \cdot B)$ defines a martingale which is the case in many popular models including the incomplete models developed in [25] and [20].

In the second case, $\mathcal{E}(-\lambda \cdot B)$ is a strict local martingale but NFLVR nevertheless still holds. A famous example of a model where this occurs is given in [11]. We present here a time-changed version (using the standard logarithmic time transform $t \mapsto -\log(1-t)$), as the original version in [11] is defined on an infinite horizon. In the notation of Example 7.3, and with $T = 1$, we define the local martingales $(\beta'_t)_{t\in[0,1)}$ and $(W'_t)_{t\in[0,1)}$ by

$$\beta'_t := \int_0^t \tfrac{1}{\sqrt{1-u}}\, d\beta_u \text{ and } W'_t := \int_0^t \tfrac{1}{\sqrt{1-u}}\, dW_u, \quad t \in [0,1),$$

as well as the stopping times

$$\tau := \inf\{t > 0 : \mathcal{E}(\beta') = 1/2\} \text{ and } \sigma := \inf\{t > 0 : \mathcal{E}(W') = 2\}.$$

With the processes $(\lambda_t)_{t\in[0,1]}$ and $(\nu_t^{(0)})_{t\in[0,1]}$ given by

$$\lambda_t := -\frac{1_{\sigma\wedge\tau}(t)}{\sqrt{1-t}}, \quad \nu_t^{(0)} := -\frac{1_{\sigma\wedge\tau}(t)}{\sqrt{1-t}},$$

it remains to apply the results of [11] to conclude that the NFLVR condition is satisfied, but that the minimal martingale density $\mathcal{E}(-\lambda\cdot\beta)$ is a strict local martingale. Our Lemma 7.4 applies because $|\nu_t^{(0)}| \leq \frac{1}{\sqrt{1-t}} \in \mathbb{L}^1([0,1])$.

We conclude the by mentioning that Examples 7.3 and 7.5, as well as Lemma 7.4 will be used again in the examples in Section 8. □

The next example shows that it is quite easy to construct bounded payoffs ψ which fail to be uniquely superreplicable.

Example 7.6. We consider the following one period model with three states:

$$\Delta S := (1, 0, -1)', \quad \psi := (-1, 0, -1)'.$$

The set of pairs (x, π) for which $x + \pi\Delta S \geq \psi$ is given by $x \geq 0$ and $\pi \in [-1-x, 1+x]$. However, the corresponding set of gain outcomes $(x+\pi, x, x-\pi)'$

with $x \geq 0$ and $\pi \in [-1-x, 1+x]$ does not contain a smallest element. Indeed, if (a,b,c) is a smallest element, we would have $a \leq -1, b \leq 0$, and $c \leq -1$ but such an element (a,b,c) is not the outcome of any gains process $x + \pi\Delta S$ with $x \geq 0$ and $\pi \in [-1-x, 1+x]$. □

The main technical result of this section is the following proposition:

Proposition 7.7. *Under Assumption 6.3, suppose that $-B$ and $-(B+\varepsilon\varphi)$ are uniquely superreplicable by $-\underline{B}$ and $-(\underline{B+\varepsilon\varphi})$, respectively, for all $\varepsilon > 0$ in some neighborhood of 0. Then, for each $\hat{\mu} \in \hat{\mathcal{D}}(B)$, we have*

$$\lim_{\varepsilon \searrow 0} \tfrac{1}{\varepsilon}\Big(\mathfrak{U}(B+\varepsilon\varphi) - \mathfrak{U}(B)\Big) = \mathbb{E}[\hat{Y}\varphi] + \lim_{\varepsilon \searrow 0} \tfrac{1}{\varepsilon}\Big\langle \hat{\mu}^s, \underline{B+\varepsilon\varphi} - \underline{B}\Big\rangle, \tag{7.4}$$

where $\hat{Y} = \frac{d\hat{\mu}^r}{d\mathbb{P}}$.

Proof. For $\varepsilon > 0$ we let x_ε, $x_0 \in \mathbb{R}$ and $\pi_\varepsilon, \pi_0 \in \mathcal{A} \cap (-\mathcal{A})$ be such that

$$\underline{B+\varepsilon\varphi} = \varepsilon x_\varepsilon + \varepsilon\,(\pi_\varepsilon \cdot S)_T \text{ and } \underline{B} = x_0 + (\pi_0 \cdot S)_T\,.$$

Because B is bounded away from zero and $\varphi \in \mathbb{L}^\infty(\mathbb{P})$ we can consider $\varepsilon > 0$ so small that $x_\varepsilon, x_0 > 0$. For $\delta \in \Delta^\varepsilon(\varphi)$ we have $\varepsilon\delta + \hat{\pi} \in \mathcal{A}$ and

$$\varepsilon(\delta \cdot S)_T + (\hat{\pi} \cdot S)_T \geq -\varepsilon\varphi - B.$$

Therefore, by the unique superreplicability of $\underline{B+\varepsilon\varphi}$, we have

$$0 \leq x_\varepsilon + (\delta \cdot S)_T + \tfrac{1}{\varepsilon}(\hat{\pi} \cdot S)_T + (\pi_\varepsilon \cdot S)_T\,. \tag{7.5}$$

Since $\underline{B}$ uniquely superreplicates B we have $\underline{B} + (\hat{\pi} \cdot S)_T \geq 0$. Therefore, for any $\hat{\mu} \in \hat{\mathcal{D}}(B)$, the first part of Equation (4.7) in [6] produces

$$0 \leq \langle \hat{\mu}^s, \underline{B} + (\hat{\pi} \cdot S)_T\rangle \leq \langle \hat{\mu}^s, B + (\hat{\pi} \cdot S)_T\rangle = 0. \tag{7.6}$$

The second part of Equation (4.7) in [6] ensures $\langle \hat{\mu}, (\hat{\pi} \cdot S)_T\rangle = 0$ and combining this with $\langle \hat{\mu}^s, \underline{B} + (\hat{\pi} \cdot S)_T\rangle = 0$ we see

$$\begin{aligned}\langle \hat{\mu}^r, \underline{B} + (\hat{\pi} \cdot S)_T\rangle &= \langle \hat{\mu}, \underline{B} + (\hat{\pi} \cdot S)_T\rangle \\ &= \langle \hat{\mu}, \underline{B}\rangle.\end{aligned}$$

Because $\hat{Y} = \frac{d\hat{\mu}^r}{d\mathbb{P}}$ we obtain the representation

$$\mathbb{E}[\hat{Y}(\hat{\pi} \cdot S)_T] = \langle \hat{\mu}^s, \underline{B}\rangle. \tag{7.7}$$

The property $\varepsilon\delta + \hat{\pi} \in \mathcal{A}$ produces $\langle \hat{\mu}, \varepsilon(\delta \cdot S)_T + (\hat{\pi} \cdot S)_T\rangle \leq 0$ and $\langle \hat{\mu}, (\pi_\varepsilon \cdot S)_T\rangle = 0$, for each $\varepsilon > 0$. Therefore, by (7.5), we find

$$\begin{aligned}&\mathbb{E}[\hat{Y}(x_\varepsilon + (\pi_\varepsilon \cdot S)_T + (\delta \cdot S)_T + \tfrac{1}{\varepsilon}(\hat{\pi} \cdot S)_T)] \\ &\leq \langle \hat{\mu}, x_\varepsilon + (\pi_\varepsilon \cdot S)_T + (\delta \cdot S)_T + \tfrac{1}{\varepsilon}(\hat{\pi} \cdot S)_T\rangle \\ &\leq \langle \hat{\mu}, x_\varepsilon\rangle.\end{aligned} \tag{7.8}$$

To show that the upper bound in (7.8) above is attained we pick

$$\delta_\varepsilon = (\tfrac{x_\varepsilon}{x_0} - \tfrac{1}{\varepsilon})\hat{\pi} + \tfrac{x_\varepsilon}{x_0}\pi_0 - \pi_\varepsilon. \tag{7.9}$$

Because $x_\varepsilon > 0$ and $x_0 > 0$ one can check that $\delta_\varepsilon \in \Delta^\varepsilon(\varphi)$. Then we have

$$\begin{aligned}
&\mathbb{E}[\hat{Y}(x_\varepsilon + (\pi_\varepsilon \cdot S)_T + (\delta_\varepsilon \cdot S)_T + \tfrac{1}{\varepsilon}(\hat{\pi} \cdot S)_T)] \\
&\quad = \mathbb{E}[\hat{Y}(x_\varepsilon + \tfrac{x_\varepsilon}{x_0}((\hat{\pi} + \pi_0) \cdot S)_T)] \\
&\quad = \tfrac{x_\varepsilon}{x_0}\mathbb{E}[\hat{Y}(\underline{B} + (\hat{\pi} \cdot S)_T)] \\
&\quad = \tfrac{x_\varepsilon}{x_0}\langle \hat{\mu}, \underline{B} + (\hat{\pi} \cdot S)_T \rangle \\
&\quad = \langle \hat{\mu}, x_\varepsilon \rangle,
\end{aligned}$$

where the last equality follows from $\langle \hat{\mu}, \underline{B} \rangle = \langle \hat{\mu}, x_0 \rangle$ and $\langle \hat{\mu}, (\hat{\pi} \cdot S)_T \rangle = 0$. Therefore, δ_ε indeed attains the upper bound of (7.8), and, so,

$$\begin{aligned}
&\sup_{\delta \in \Delta^\varepsilon(\varphi)} \mathbb{E}[\hat{Y}((\delta \cdot S)_T + \varphi)] = \\
&\quad = \langle \hat{\mu}, x_\varepsilon \rangle - \tfrac{1}{\varepsilon}\mathbb{E}[\hat{Y}(\underline{B + \varepsilon\varphi})] - \tfrac{1}{\varepsilon}\mathbb{E}[\hat{Y}(\hat{\pi} \cdot S)_T] + \mathbb{E}[\hat{Y}\varphi] \\
&\quad = \langle \hat{\mu}^r, \varphi \rangle + \tfrac{1}{\varepsilon}\langle \hat{\mu}^s, \underline{B + \varepsilon\varphi} - \underline{B} \rangle.
\end{aligned}$$

The sets $\Delta^\varepsilon(\varphi)$ monotonically increase to $\Delta(\varphi)$ as $\epsilon \searrow 0$. This implies

$$\sup_{\delta \in \Delta^\varepsilon(\varphi)} \mathbb{E}[\hat{Y}((\delta \cdot S)_T + \varphi)] \nearrow \sup_{\delta \in \Delta(\varphi)} \mathbb{E}[\hat{Y}((\delta \cdot S)_T + \varphi)] \tag{7.10}$$

as $\varepsilon \searrow 0$. Because the left-hand-side of (7.10) equals $\langle \hat{\mu}^r, \varphi \rangle + \frac{1}{\varepsilon}\langle \hat{\mu}^s, \underline{B + \varepsilon\varphi} - \underline{B} \rangle$, we see that (7.4) holds by Proposition 6.5. □

A first consequence of Proposition 7.7 is that the situation encountered in Example 6.1 cannot happen if $-B$ if uniquely superreplicable. Indeed, the primal value function $\mathfrak{U}$ is then smooth in all replicable directions:

Corollary 7.8. *Suppose that Assumption 6.3 holds, that $-B$ is uniquely superreplicable and that φ is replicable. Then the following two-sided limit exists*

$$\lim_{\varepsilon \to 0} \tfrac{1}{\varepsilon}\Big(\mathfrak{U}(B + \varepsilon\varphi) - \mathfrak{U}(B)\Big) = \langle \hat{\mu}, \varphi \rangle \text{ for each } \hat{\mu} \in \hat{\mathcal{D}}(B). \tag{7.11}$$

In particular, there exists a constant $y_B > 0$ such that

$$y_B = \hat{\mu}(\Omega) \text{ for each } \hat{\mu} \in \hat{\mathcal{D}}(B).$$

Proof. First observe that for replicable φ we have $\underline{B + \varepsilon\varphi} = \underline{B} + \varepsilon\varphi$. Then we can apply Proposition 7.7 to both φ and $-\varphi$ to see (7.11). The last claim follows by setting $\varphi = 1$. □

Now that we know that all dual minimizers $\hat{\mu} \in \hat{\mathcal{D}}(B)$ have the same total mass, the following result follows directly from Corollary 5.2 above.

Corollary 7.9. *Suppose that Assumption 6.3 holds and that $-B$ is uniquely superreplicable. Then each replicable $\varphi \in \mathbb{L}^\infty(\mathbb{P})$ has the unique B-conditionally Davis price $\langle \hat{\mu}, \varphi \rangle / \hat{\mu}(\Omega)$.*

When B is a constant (and more generally, when B is replicable), it is known that the product of the primal and dual optimizers is a martingale (see, e.g., the discussion on p.911-2 in [22]). When the dual optimizer is only a finitely-additive measure, the following corollary may serve as a surrogate. The result relies on [19] where a positive supermartingale deflator $\{\hat{Y}_t\}_{t\in[0,T]}$ is constructed from $\hat{\mu} \in \hat{\mathcal{D}}(B)$ (see Equation 2.5 in [19]).

Corollary 7.10. *Suppose that Assumption 6.3 holds, that $-B$ is uniquely superreplicable by $-\underline{B}$, and write*

$$\underline{B} = x_0 + (\pi_0 \cdot S)_T.$$

Then the process

$$\hat{Y}_t\Big(x_0 + \big((\pi_0 + \hat{\pi}) \cdot S\big)_t\Big), \quad t \in [0,T],$$

is a nonnegative martingale where $\{\hat{Y}_t\}_{t\in[0,T]}$ is the supermartingale deflator corresponding to $\hat{\mu} \in \hat{\mathcal{D}}(B)$.

Proof. From Theorem 2.10 in [19] we know that the process in question is a nonnegative supermartingale. Furthermore, also from [19], we have $\hat{Y}_T = \frac{d\mu^r}{d\mathbb{P}}$ and $\hat{Y}_0 \le \hat{\mu}(\Omega)$. To obtain the constant expectation property we use (7.6) to get

$$\begin{aligned}\langle \hat{\mu}, x_0\rangle &= \langle \hat{\mu}, (\hat{\pi}\cdot S)_T + \underline{B}\rangle = \langle \hat{\mu}^r, (\hat{\pi}\cdot S)_T + \underline{B}\rangle \\ &= \mathbb{E}\Big[\hat{Y}_T\big((\hat{\pi}\cdot S)_T + \underline{B}\big)\Big] \le \hat{Y}_0 x_0 \le \hat{\mu}(\Omega) x_0,\end{aligned}$$

and the claimed martingale property follows. □

8. The interval of Conditional Davis prices

This section closes the loop and gives an explicit expression for the interval of conditional Davis prices under the assumption of unique superreplicability. We start with a standard characterization of conditional Davis prices in terms of perturbed value functions. Given $B \in \mathbb{L}^\infty_{++}$ and $\varphi \in \mathbb{L}^\infty$ (we do not impose any unique superreplicability assumption on either, yet). We let the function $u : \mathbb{R}^2 \to [-\infty, \infty)$ be defined by

$$u(\varepsilon, x) := \mathfrak{U}(B + x + \varepsilon\varphi),$$

and let its supergradient at $(0,0)$ be denoted by $\partial u(0,0)$.

Lemma 8.1. *Let $\varphi \in \mathbb{L}^\infty$. If $(0,0) \in \partial u(0,0)$, then $P(\varphi|B) = \mathbb{R}$. Otherwise,*

$$P(\varphi|B) = \{\delta/y \,:\, y \ne 0, (\delta, y) \in \partial u(0,0)\}.$$

Proof. Thanks to the assumption that $B \in \mathbb{L}^\infty_{++}$, u is concave and finite-valued in some neighborhood of $(0,0)$. Moreover Definition 3.4 translates into the following statement:

$$p \in P(\varphi|B) \text{ if and only if } u(0,0) \ge u(\varepsilon, -\varepsilon p) \text{ for all } \varepsilon.$$

By concavity, this is equivalent to the nonpositivity of the directional derivative of u, at $(0,0)$ in the directions $(-1,p)$ and $(1,-p)$, i.e.

$$\inf_{(\delta,y)\in\partial u(0,0)} -\delta + py \le 0 \quad\text{and}\quad \inf_{(\delta,y)\in\partial u(0,0)} \delta - py \le 0.$$

By the convexity of the supergradient, this is equivalent to existence of a pair $(\delta,y)\in\partial u(0,0)$ such that $py=\delta$. □

Theorem 8.2. *Suppose that Assumption 6.3 holds and that $-B$ and $-(B+\varepsilon\varphi)$ are uniquely superreplicable by $-\underline{B}$ and $-(\underline{B+\varepsilon\varphi})$, respectively, for all ε in some neighborhood of 0. The interval of B-Davis prices of φ is given by*

$$\frac{1}{y_B}\mathbb{E}[\hat{Y}\varphi] + \frac{1}{y_B}\Big[\lim_{\varepsilon\searrow 0}\tfrac{1}{\varepsilon}\Big\langle\hat{\mu}^s, \underline{B+\varepsilon\varphi}-\underline{B}\Big\rangle, \lim_{\varepsilon\nearrow 0}\tfrac{1}{\varepsilon}\Big\langle\hat{\mu}^s, \underline{B+\varepsilon\varphi}-\underline{B}\Big\rangle\Big], \tag{8.1}$$

where y_B is the common value of $\hat{\mu}(\Omega)$ for all $\hat{\mu}\in\hat{\mathcal{D}}(B)$.

Proof. By Corollary 7.8, the function u is differentiable in x at $x=0$, with derivative y_B. The interval of B-conditional Davis prices, according to Lemma 8.1, is given by

$$\tfrac{1}{y_B}[\partial_{\varepsilon+}u(0,0), \partial_{\varepsilon-}u(0,0)].$$

This, in turn, coincides with the expression in (8.1) thanks to Proposition 7.7. □

8.1. **Two illustrative examples.** We conclude by giving two illustrative examples, both in an incomplete Brownian setting, of situations where our results can be applied directly and lead to explicit formulas for the nontrivial interval of conditional Davis prices.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space supporting two independent Brownian motions (Z,W) and we let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be their augmented filtration up to some maturity $T>0$. The set of all pathwise p-integrable predictable processes is denoted by $\mathcal{L}^p$ and the space finitely-additive measures which are $\mathbb{P}$-absolutely continuous is denoted ba$(\mathbb{P})$ so that ba$(\mathbb{P})$ can be identified with $\mathbb{L}^\infty(\mathbb{P})'$.

In both examples, the stock-price dynamics are given by a one-dimensional Itô process

$$dS_t := S_t\sigma_t\big(\lambda_t dt + dZ_t\big), \quad S_0>0, \tag{8.2}$$

with processes $\sigma,\lambda\in\mathcal{L}^2$. With more driving Brownian motions than assets, this leads to a incomplete financial market. Both examples will feature (an unspanned) contingent claim paying out $\varphi(W_T)$ at time T, where $\varphi:\mathbb{R}\to\mathbb{R}$ is a non-constant, bounded, and continuous function.

The major difference between the examples is that in the first example the illiquid random endowment degenerates ($B:=x$ for a constant $x>0$), while in the second example the random endowment B is non-replicable. The first example illustrates that even when $B:=x>0$ is constant, our setting differs from that of [23] because the corresponding Davis prices are non-unique whereas the growth condition placed on the claim's payoff in [23] always produces unique Davis prices (the growth condition used in [23] originates from [16]). In other words, the payoffs considered in the first example are not included in [23]. The second example backs up the claim we made in both the abstract and in the introduction: When the endowment B is non-replicable, the generic case is that Davis prices are non-unique.

8.1.1. *Example 1.* We adopt the setting used in Example 7.3 above which is based on [11]. The endowment is taken to be $B := x > 0$ constant. It follows from Example 7.3 that the interval of arbitrage-free prices for $\varphi(W_T)$ is given by $(\underline{\varphi}, \overline{\varphi})$ where

$$\underline{\varphi} := \inf_{a\in\mathbb{R}} \varphi(a), \quad \overline{\varphi} := \sup_{a\in\mathbb{R}} \varphi(a). \tag{8.3}$$

Our Theorem 8.2 with $B := x > 0$ constant shows that the interval of log-investor's Davis' prices for $\varphi(W_T)$ is given by $[\underline{p}, \overline{p}]$ where

$$\underline{p} := \tfrac{1}{\hat{Y}_0}\mathbb{E}[\hat{Y}_T(\varphi - \underline{\varphi})] + \underline{\varphi}, \quad \overline{p} := \overline{\varphi} - \tfrac{1}{\hat{Y}_0}\mathbb{E}[\hat{Y}_T(\overline{\varphi} - \varphi)].$$

Therefore, since the function φ is not constant, we have

$$\overline{p} - \underline{p} = (\overline{\varphi} - \underline{\varphi})(1 - \mathbb{E}[\hat{Y}_T]/\hat{Y}_0) > 0.$$

8.1.2. *Example 2.* In this example, we consider the Samuelson-model setting used in Section 2 in [27] where the stock price dynamics are given by (8.2) with both $\sigma_t := \sigma > 0$ and $\lambda_t := \lambda > 0$ being constants. Let $U(\xi) := \frac{\xi^\gamma}{\gamma}$, $\xi > 0$, $\gamma < 1$, be a utility function in the "power" family, with constant relative risk-aversion parameter (as usual $\gamma := 0$ is interpreted as the *log* investor).

The investor receives the random endowment of the form $B(W_T)$ at time $T > 0$, where B is a non-constant, bounded and continuous function. The payoff φ whose B-conditional Davis prices we are computing, as well as the quantities $\underline{\varphi}$ and $\overline{\varphi}$ are defined exactly as in Example 1 above. We also define the following quantities

$$\underline{B}(\varepsilon) := \inf_{a\in\mathbb{R}} \Big(B(a) + \varepsilon\varphi(a)\Big), \quad \overline{B}(\varepsilon) := \sup_{a\in\mathbb{R}} \Big(\varepsilon\varphi(a) - B(a)\Big), \quad \varepsilon \geq 0.$$

Proposition 2.4 in [27] states that the dual optimizer $\hat{\mathbb{Q}} \in \mathrm{ba}(\mathbb{P})$ for the utility-maximization problem with the random endowment of the form $B(W_T)$ has a non-trivial singular part in the Yosida-Hewitt decomposition $\hat{\mathbb{Q}} = \hat{\mathbb{Q}}^r + \hat{\mathbb{Q}}^s$ after a possible shift of the function B by a constant. Moreover, such a shift can always be arranged so as to keep the values of B positive and bounded away from 0. Therefore, we assume, without loss of generality that such a shift has already been performed, so that, in particular, we have $\underline{B}(0) > 0$. This loss-of-mass property for $\hat{\mathbb{Q}}^r$ can be partially quantified as follows: Theorem 3.7 in [13] and Proposition 3.2 in [29] allow us to write $\frac{d\hat{\mathbb{Q}}^r}{d\mathbb{P}} = \hat{Y}_T$ where

$$d\hat{Y}_t = -\hat{Y}_t\big(\tfrac{\mu}{\sigma}dZ_t + \hat{\nu}_t dW_t\big), \quad \hat{Y}_0 > 0,$$

for some process $\hat{\nu} \in \mathcal{L}^2$. The presence of the non-trivial singular part $\hat{\mathbb{Q}}^s$ implies that $\hat{Y}$ is a strict local martingale, i.e., $\mathbb{E}[\hat{Y}_T] < \hat{Y}_0$.

Example 7.3 takes care of the conditions of Theorem 8.2 dealing with unique superreplicability. Indeed, both $-B$ and $-(B + \varepsilon\varphi)$ are of the form treated there, and are, therefore, uniquely superreplicable by $-\underline{B}$ and $-\underline{B}(\varepsilon)$, respectively, for $\varepsilon \geq 0$.

The last step before Theorem 8.2 is applied is to simplify the two ε-limits appearing in (8.1). That is an easy task thanks to the fact that the random variable $\frac{1}{\varepsilon}\big(\underline{B + \varepsilon\varphi} - \underline{B}\big)$ is constant and equal to $\frac{1}{\varepsilon}\big(\underline{B}(\varepsilon) - \underline{B}(0)\big)$. Theorem 8.2 guarantees that this quotient admits a left and a right limit at $\varepsilon = 0$ and we introduce the following notation

$$\underline{B}'(0+) := \lim_{\varepsilon \searrow 0} \tfrac{1}{\varepsilon}\big(\underline{B}(\varepsilon) - \underline{B}(0)\big) \text{ and } \underline{B}'(0-) := \lim_{\varepsilon \nearrow 0} \tfrac{1}{\varepsilon}\big(\underline{B}(\varepsilon) - \underline{B}(0)\big).$$

The total mass in $\hat{\mathbb{Q}}^s$ is given by $\hat{Y}_0 - \mathbb{E}[\hat{Y}_T]$, and, so the interval of $B(W_T)$-conditional Davis prices for the payoff $\varphi(W_T)$ is given by $[\underline{p}, \overline{p}]$ where

$$\underline{p} := \tfrac{1}{\hat{Y}_0}\mathbb{E}\left[\hat{Y}_T\big(\varphi(W_T) - \underline{B}'(0+)\big)\right] + \underline{B}'(0+),$$
$$\overline{p} := \overline{B}'(0+) - \tfrac{1}{\hat{Y}_0}\mathbb{E}\left[\hat{Y}_T\big(\overline{B}'(0+) - \varphi(W_T)\big)\right].$$

8.1.3. *Linear Approximation.* We close the paper with a result which complements the pricing formula of the previous two examples with some asymptotic hedging information. More precisely, we provide two first-order approximations to the primal utility maximizer in the Brownian setting used above. We focus on the right limit ($\varepsilon \searrow 0$), as one gets the left-limit corrector by applying the result to $-\varphi$. As a preparation, we note that the function $\underline{B}$ is concave and that its right derivative $\underline{B}'(0+)$ at 0 satisfies

$$|\underline{B}'(0+)| \leq \sup_{a \in \mathbb{R}} |\varphi(a)|,$$

and remind the reader that both B and φ are normalized so that $\underline{B}(0) > 0$. As always, $\hat{\pi}$ denotes the primal optimizer for the utility-maximization problem with the random endowment $B = B(W_T)$ and $\hat{Y}$ is the common regular part of all dual optimizers.

Proposition 8.3. *In the setting described in the beginning of subsection 8.1, and under Assumption 6.3, the process*

$$(1 + \varepsilon\tfrac{\underline{B}'(0+)}{\underline{B}(0)})\hat{\pi}$$

is first-order optimal in the sense that

$$\mathfrak{U}\big(B + \varepsilon\varphi(W_T)\big) - \mathbb{E}\left[U\Big(\big(1 + \varepsilon\tfrac{\underline{B}'(0+)}{\underline{B}(0)}\big)(\hat{\pi}\cdot S)_T + B + \varepsilon\varphi(W_T)\Big)\right] = o(\varepsilon)$$

as $\varepsilon \searrow 0$.

Proof. We base the proof on the abstract expression (7.4) for the right derivative of the value function of Proposition 7.7.

We let δ^ε be defined by (7.9) in the proof of Proposition 7.7, which thanks to Example 7.3, takes the simple form

$$\delta_t^\varepsilon := \frac{1}{\underline{B}(0)} \frac{\underline{B}(\varepsilon) - \underline{B}(0)}{\varepsilon} \hat{\pi}_t.$$

Then we have

$$\begin{aligned}\sup_{\delta \in \Delta(\varphi)} \mathbb{E}[\hat{Y}(\delta \cdot S)_T] &= \lim_{\varepsilon \searrow 0} \mathbb{E}\Big[\hat{Y}(\delta^\varepsilon \cdot S)_T\Big] \\ &= \tfrac{\underline{B}'(0+)}{\underline{B}(0)} \mathbb{E}\Big[\hat{Y}(\hat{\pi} \cdot S)_T\Big] = \underline{B}'(0+)\hat{\mu}^s(\Omega),\end{aligned}$$

where the last equality uses (7.7). We can then re-purpose the proof of Proposition 6.5 to see that

$$\mathbb{E}\left[U\Big(\big(1 + \varepsilon \tfrac{\underline{B}'(0+)}{\underline{B}(0)}\big)(\hat{\pi} \cdot S)_T + B + \varepsilon\varphi\big(W_T\big)\Big)\right] - \mathfrak{U}\big(B\big) - \varepsilon\Delta = o(\varepsilon),$$

as $\varepsilon \searrow 0$ where we have defined

$$\Delta : = \mathbb{E}[\hat{Y}\varphi] + \underline{B}'(0+)\hat{\mu}^s(\Omega).$$

It remains to apply the triangle inequality and Proposition 7.7. □

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