Dynamic Programming and Viscosity Solutions *

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Abstract

In a celebrated 1984 paper, Crandall and Lions provided an elegant complete weak theory for all first order nonlinear partial differential equations, which they called the *viscsoity solutions*. In this introductory paper, we discuss this theory in the context of deterministic optimal control. After deriving the viscosity property of the value function, we will discuss several important technical contributions of the theory; formulation of the state constraints, discsontinuous viscosity solutions and weak formulation of boundary data.

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1 Introduction

The concept of control can be described as the process of influencing the behavior of a dynamical system to achieve a desired goal. If the goal is to optimize some pay-off function (or cost function) which depends on the control inputs to the system, then the problem is one of *optimal control.*

In this introduction, we are concerned with deterministic optimal control models in which the dynamics of the system being controlled are governed by a set of ordinary differential equations. In these models the system operates for times s in some interval I. The state at time $s \in I$ is a vector in n-dimensional Euclidean $\mathcal{R}^n$. At each time s, a control $u(s)$ is chosen from some given set U (called the *control space.*) It often happens that a system is being controlled only for $x(s) \in \bar{O}$ where $\bar{O}$ is the closure of some given open set $O \subset \mathcal{R}^n$. Two versions will be formulated in this paper. In one version, control occurs only until the time of exit from the closed cyclinderical region $\bar{Q} := \bar{I} \times \bar{O}$. In the other version, only controls which keep $x(s) \in \bar{O}$ for all $s \in I$ are allowed (this is called a *state constraint control problem.*)

The method of dynamic programming is the one which will be used in these notes. Although only deterministic problems are analyzed in this lecture, this methodology applies to stochastic models as well. In dynamic programming, a *value function* V is introduced which is the optimum value of the payoff considered as a function of the initial data. This value function V for a deterministic optimal control problem satisfies, at least formally, a first-order nonlinear partial differential equation which we call the *dynamic programming equation.* Often, V does not have the smoothness properties needed to interpret it as a solution to the dynamic programming partial differential equation in the usual ("classical") sense. Indeed the lack of smoothness of the value function is more of a rule than exception. Therefore a weak formulation of solutions to these equations is necssary in order to pursue the method of dynamic programming.

In their celebrated 1984 paper Crandall and Lions [CL84] provided such a weak formulation which they call *viscosity solutions.* Although the term "viscosity' refers to a certain relaxation scheme, the definition of a viscosity solution is an intrinsic one. Indeed, viscosity solutions remain stable under any reasonable relaxation or approximation of the equation. A uniqueness result was another very important contribution of 1984 paper. Later, elegant equivalent reformulations of viscosity solutions were obtained by Crandall, Evans, and Lions [CEL84]. Survey article of Crandall, Ishii, and Lions [CIL92], book by Fleming and the author [FS93], and more recent books Barles [B94], Bardi and Capuzzo-Dolcetta [BC97] contain most of the references to a very large literature that developed since the first paper of Crandall and Lions. Here, we very closely follow [FS93].

The theory of viscosity solutions is not limited to dynamic programming equations. Indeed, the chief property that is required is maximum principle. This property is enjoyed by all second-order parabolic or elliptic equations. In this paper, we restrict ourselves to first order equations or more specificaly to determinitic optimal control. But we discuss in detail problems with state space constraint and dicontinuous viscosity solutions. We also outline the theory of weak bondary data.

2 Deterministic Optimal Control

We start our discsussion by giving some examples. We shall analyze these examples in detail after we discuss dynamic programing principle and viscosity solutions.

2.1 Examples

Example 2.1. Consider a simple harmonic oscillator, in which a forcing term $u(s)$ is taken as the control. Let $x_1(s)$, $x_2(s)$ denote, respectively, the positiion and velocity at time s. Then,

$$\begin{aligned} \frac{d}{ds}x_1(s) &= x_2(s), \qquad (2.1) \\ \frac{d}{ds}x_2(s) &= -x_1(s) + u(s) \ . \end{aligned}$$

We may require that $u(s) \in U$, where U is a closed interval. For instance, if $U = [-a, a]$, then the upper bound $|u(s)| \leq a$ is imposed on the forcing term.

Let us consider the problem of controling the harmonic oscilator on a finite time interval $t \leq s \leq t_1$. After specifying an initial position and velocity $x := (x_1, x_2)$, we seek to minimize a quadratic criterion of the form

$$J(x, u(\cdot)) := \int_t^{t_1} \left[\alpha|x(s)|^2 + |u(s)|^2\right] \, ds + \beta|x(t_1)|^2 \ ,$$

where α and β are positive constants. In minimizing the above functional, we try to bring the state $x(\cdot)$ to rest by using minimal forcing term. If there is no constraint on the forcing term ($U = \mathcal{R}^n$), this is a particular case of the *linear quadratic regulator problem.* If $U = [-a, a]$, it is an example of a linear quadratic conrol problem with *saturation constraints.* (See [FS93;Example 2.3] for the linear quadratic regulator problem.)

Example 2.2. The simplest kind of problem in classical calculus of variations is to determine a function $x(\cdot)$ which minimizes a functional

$$\int_t^{t_1} \ L(s, x(s), \frac{dx}{ds}(s)) \ ds + \phi(x(t_1)) \ ,$$

subject to given conditions on $x(t)$ and $x(t_1)$. Let us fix the left end point, by requiring $x(t) = x$ where $x \in \mathcal{R}^n$ is given. For the right end point, let us fix t_1 and require that $x(t_1) \in \mathcal{M}$, where $\mathcal{M}$ is a given closed subset of $\mathcal{R}^n$. If $\mathcal{M} = \{x_1\}$ consists of a single point, then the right end point (t_1, x_1) is fixed. At the opposite extreme, we may put no constraints on $x(t_1)$ by taking $\mathcal{M} = \mathcal{R}^n$.

Example 2.3. Consider a finite number of smooth vector fields

$$f^i : \mathcal{R}^n \rightarrow \mathcal{R}^n , \qquad i = 1, \ldots, M .$$

Given a closed set $\mathcal{M}$ as in the previous example, let $\mathcal{A}$ be the set of all Lipscihtz curves $x(\cdot)$ satisfying $x(t) = x$,

$$\frac{dx}{ds}(s) \in \{f^i(x(s)) \ : \ i = 1, \ldots, M \ \}$$

for almost every $s > 0$, and

$$x(t_1) \in \mathcal{M}$$

at some time $t_1 \geq t$. $\mathcal{M}$ is called the *target*. Then, the *minimum time problem* is to find the curve $x(\cdot) \in \mathcal{A}$ which minimizes $t_1 > t$ so that $x(t_1) \in \mathcal{M}$.

Minimum time problem can be viewed as a special case of the previous example by taking $L \equiv 1$, fixing the left endpoint $x(t) = x$, and leaving t_1 free. Most important difference is in the minimum time problem the velocity of the state, $\frac{dx}{ds}$ is constrained to be equal to one of the given vector fields f^i.

In Section 4.3, we will provide solutions to the following simple minimum time problems. All of them are two dimensional problems, with two vector fields, and the target set $\mathcal{M} = \{(0,0)\}$.

(a). As a very simple example in $\mathcal{R}^2$, take

$$f^1(x_1, x_2) = (x_2, -x_1), \qquad f^2(x_1, x_2) = (-1, 0) .$$

In polar coordinates f^1 corresponds to $\dot{r} = 0$, $\dot{\theta} = -1$. We shall see that it is optimal to use the first vector field f^1 until the state $x(\cdot)$ is on the positive x-axis, then we switch to the second vector field.

We may consider a *state constraint* problem by requiring that

$$x(s) \notin C := \{ \ (r, \theta) \mid 0 < r < 1, \ \frac{\pi}{2} < \theta < \pi \ \} .$$

If the optimal arc for the unconstraint problem hits the contraint region C, then we have modify that arc appropriately.

(b). As in the previous example, we take $f^2(x_1, x_2) = (-1, 0)$ but

$$f^1(x_1, x_2) = \frac{1}{r} \ (x_2, -x_1) .$$

Under f^1, $\dot{r} = 0$, $\dot{\theta} = -1/r$. The optimnal control is described in §4.3 below.

3 Formulation

A terminal time t_1 is fixed throughout ($t_1 = \infty$ is possible). Let $t_0 < t_1$ and consider initial times $t \in [t_0, t_1)$. The objective is to minimize some payoff functional J, which depends on states $x(s)$ and control $u(s)$ for $t \le s \le t_1$. Let us first formulate the state dynamics for the control problem. Let $Q_0 := [t_0, t_1) \times \mathcal{R}^n$ and $\bar{Q}_0$ be the closure of Q_0. Let U be a closed subset of $\mathcal{R}^m$; we call U the *control space*. The state dynamics are given by a function

$$f : \bar{Q}_0 \times U \to \mathcal{R}^n \ .$$

It is assumed that f is continuous and for a suitable K_ρ,

$$|f(t,x,u) - f(t,y,u)| \le K_\rho |x - y| \ , \tag{3.1}$$

for all $t \in [t_0, t_1]$, $x, y \in \mathcal{R}^n$ and $|v| \le \rho$.

A *control* is a bounded, lebesgue measurable function $u(\cdot)$ on $[t, t_1]$ with values in U. Assumption (3.1) implies that, given any control $u(\cdot)$, the differential equation

$$\frac{dx}{ds}(s) = f(s, x(s), u(s)) \ , \qquad s \in [t, t_1] \ , \tag{3.2}$$

with initial condition

$$x(t) = x \tag{3.3}$$

has a unique solution. The solution $x(s)$ of (3.2) and (3.3) is called the *state* of the system at time s. Clearly the state depends on the control $u(\cdot)$ and the initial condition x, but this dependence is supressed in our notation.

In order to complete the formulation of an optimal control problem, we must specify for each initial data (t, x) a subset $\mathcal{U}(t, x)$ of admissible controls and a payoff functional $J(t, x; u(\cdot))$ to be minimized. Let $O \subset \mathcal{R}^n$ be open and $Q := [t_0, t_1) \times O$ and let

$$J(t, x; u(\cdot)) := \int_t^\tau \ L(s, x(s), u(s))ds + \Psi(\tau, x(\tau)) \ ,$$

where L is the *running cost*, Ψ is the *boundary cost*, and τ is the exit time of $(s, x(s))$ from $\bar{Q}$. Thus,

$$\tau := \begin{cases} \inf\{ \ s \in [t, t_1) \mid x(s) \notin \bar{O} \ \} \\ \\ t_1 \qquad \text{if } x(s) \in \bar{O} \quad \forall s \in [t, t_1) \end{cases} .$$

We admit controls $u(\cdot) \in \mathcal{U}(t, x)$. We assume that $\mathcal{U}(t, x)$ is nonempty and it satisfies a "switching" condition. Roughly speaking, this condition states that if we replace an admissible control by another admissible control after a certain time, then the resulting control is still admissible. More precisely, let $u(\cdot) \in \mathcal{U}(t,x)$ and $\tilde{u}(\cdot) \in \mathcal{U}(r, x(r))$ for some $r > t$. Define a new control $\bar{u}$ by

$$\bar{u}(s) := u(s)\chi_{\{s \le r\}} + \tilde{u}(s)\chi_{\{s>r\}} \ .$$

Let $\bar{x}(s)$ be the state corresponding to the control $\bar{u}(\cdot)$ and initial data $\bar{x}(t) = x$. Then, we assume that the restriction, $\bar{u}_s$, of the control $\bar{u}$ to $[s, t_1]$ satisfies,

$$\bar{u}_s(\cdot) \in \mathcal{U}(s, \bar{x}(s)) \ .$$

The *optimal control problem* is as follows: Given initial data $(t, x) \in \bar{Q}$, find $u^*(\cdot) \in \mathcal{U}(t,x)$ such that

$$J(t, x; u^*(\cdot)) \le J(t, x; u(\cdot)) \qquad \forall \ u(\cdot) \in \mathcal{U}(t,x) \ .$$

3.1 Dynamic Programming

It is convinient to consider a family of optimization problems with different initial conditions (t, x). Consider the minimum value of the payoff function as a function of this initial point. Thus define a *value function* by

$$v(t,x) := \inf\{ \ \ J(t,x;u(\cdot)) \ \ | \ \ u(\cdot) \in \mathcal{U}(t,x) \ \ \} \ ,$$

for all $(t, x) \in \bar{Q}$. We shall assume that $v(t, x) > -\infty$.

The method of dynamic programming uses the value function as a tool in the analysis of the optimal control problem. In this subsection and the following, we study some basic properties of the value function. Then, we illustrate the use of these properties in several simple problems that can be solved explicitely and introduce the idea of *feedback control policy.*

Lemma 3.1 *For any $r > t$ and $u(\cdot) \in \mathcal{U}(t,x)$,*

$$\begin{aligned} v(t,x) = \inf\{\int_t^{r\wedge\tau} L(s,x(s),u(s)) \ ds \ &+ \ \Psi(\tau, x(\tau))\chi_{\{\tau<r\}} \\ &+ \ v(r, x(r))\chi_{\{\tau \ge r\}} \ \ | \ u(\cdot) \in \mathcal{U}(t,x) \ \} \ . \end{aligned} \tag{3.4}$$

Proof of this lemma can be found in Section 1.4 of [FS93].

3.2 Dynamic Programming Equation

In this section, we derive a first order differential equation satisfied by the value function. We will first assume that the the value function is continuously differentiable and proceed formally to obtain a nonlinear partial differential equation. In general, the value function may not be differentiable. In that case, the differential equation holds in the viscosity sense that will be defined later.

Fix an initial condition (t, x) and let $r = t+h$ in dynamic programming. Subtract $v(t, x)$ both sides of (3.4) and divide by h. The result is:

$$0 = \inf_{u(\cdot)} \{ \frac{1}{h} \int_t^{r \wedge \tau} L(s, x(s), u(s))\, ds \quad + \quad \frac{1}{h} \Psi(\tau, x(\tau)) \chi_{\{\tau < r\}} \\ \quad + \quad \frac{1}{h} [v(r, x(r)) - v(t, x)] \chi_{\{\tau \geq r\}} \} \ .$$

When v is differentiable

$$v(r, x(r)) - v(t, x) = \int_t^r \ [v_t(s, x(s)) + \nabla v(s, x(s)) \cdot f(s, x(s), u(s)) \ ]\ ds \ ,$$

where $v_t = \frac{\partial v}{\partial t}$. Subsitute this into the previous identity and formally let h go to zero. The result is

$$0 = v_t(t, x) + \inf\{ \ \nabla v(t, s) \cdot f(t, x, u(t)) + L(s, t, u(t)) \mid u(\cdot) \in \mathcal{U}(t, x) \ \} \ .$$

Here we assumed that $u(\cdot) \in \mathcal{U}(t, x)$ has a limit $u(t)$ at time t. To complete this formal derivation, we further assume that for all $w \in U$ there exists $u(\cdot) \in \mathcal{U}(t, x)$ so that

$$w = \lim_{s \downarrow t} u(s) \ . \tag{3.5}$$

Then, we may write the above equation as

$$-v_t(t, x) + H(t, x, \nabla v(t, x)) = 0 \ , \tag{3.6}$$

where

$$H(t, x, p) := \sup \{ \ \ -f(t, x, w) \cdot p - L(t, x, w) \ \mid \ w \in U \ \} \ .$$

3.3 Infinite Horizon

For a large class of important control problems, such as the minimum time problem, the final time t_1 is infinity. If in addition, f is independent of t and

$$L(t, x, u) = e^{-\ \beta t} \hat{L}(x, u) \ , \quad \Psi(t, x) = e^{-\ \beta t} \hat{\Psi}(x)$$

for some discount factor β, then the value function has the following form:

$$v(t, x) = e^{-\ \beta t} \hat{v}(x) \ .$$

Substituting this into the dynamic programming equation (3.6), we derive an equation:

$$\beta\hat{v}(x) + \hat{H}(x, \nabla\hat{v}(x)) = 0 \ ,$$

where

$$\hat{H}(x,p) := \sup\{\ -f(x,u)\cdot p - \hat{L}(x,u) \ |\ \ u \in U\ \}\ .$$

3.4 Examples; continued

We continue by analyzing the dynamic programing equations for the examples considered in §2.1.

Harmonic Oscilator.

In that case, $U = [-a, a]$ with some $a \leq \infty$,

$$L(t,x,u) = \alpha|x|^2 + u^2, \qquad \Psi(t,x) = \beta|x|^2\ .$$

Hence,

$$H(x,p) = \sup_{u\in[-a,a]}\{\ \ -u^2 - up_2\ \} - \alpha|x|^2 - x_2p_1 + x_1p_2\ .$$

If $a = \infty$

$$H(x,p) = \frac{1}{4}p_2^2 - \alpha|x|^2 - x_2p_1 + x_1p_2\ ,$$

and in this case, we seek for a solution of the type

$$v(t,x) = a(t)x_1^2 + b(t)x_2^2 + c(t)x_1x_2\ .$$

Substitute this into (3.6). The result is

$$0 = [-a' - \alpha + 2c - \frac{1}{4}c^2]x_1^2 + [-c' - 2a + 2b - bc]x_1x_2 + [-b' - \alpha - 2c - b^2]x_2^2\ .$$

Hence

$$\begin{aligned} a' &= -\alpha + 2c + \frac{1}{4}c^2 \\ b' &= -\alpha - 2c + b^2 \\ c' &= -2a + 2b + bc\ . \end{aligned}$$

This is a special *Ricatti* equation, which is to be solved on (t_0, t_1) together with the boundary conditions,

$$a(t_1) = \beta, \qquad b(t_1) = \beta, \qquad c(t_1) = 0\ .$$

Minimal Time Problem.

This is an infinite horizon problem with $L \equiv 1$ and the equation is

$$\max_i \{ \ f^i(x) \cdot \nabla v(x)\} = 1 \qquad \forall x \notin \mathcal{M} \ ,$$

and $v(x) = 0$ on $\mathcal{M}$.

3.5 Verification and Feedback Control

In some cases, it may be possible to obtain a smooth solution of the dynamic programming equation together with the boundary condition. An example is the linear regulator problem. In such cases, this smooth solution is the value function and a feedback control is obtained from the derivatives of the value function. However, in addition to the smoothness, technical conditions on the growth of the solutions are also required. These conditions and the exact statement of the verification theorems can be found in [FS93;§1]. Here we will illustrate the method in the linear quadratic control problem with no sturation constraint (i.e., $a = \infty$.) Recall that in that case, the Dynamic Programming Equation has a smooth solution of the form

$$v(x) = a(t)x_1^2 + b(t)x_2^2 + c(t)x_1x_2 \ .$$

To obtain the feedback control we observe that

$$0 = -v_t + H(x, v_{x_1}, v_{x_2}) \ ,$$

where

$$H(x, v_{x_1}, v_{x_2}) = \sup_u \{ \ -u^2 - uv_{x_2} \ \} - \alpha |x|^2 - x_2 v_{x_1} + x_1 v_{x_2} \ .$$

The maximizer u^* in the above equation

$$u^* = u^*(x) = -\frac{1}{2} \ v_{x_2}(x) = -b(t)x_2 - \frac{1}{2}c(t)x_1 \ .$$

It can be proved that, [FS93;§1], this feedback policy is optimal. Notice that the optimal feedback policy is linear state; therefore, the optimal state process is also a solution of a linear ordinary differential equation. This very important property is enjoyed by the general linear regulator problem as well.

4 Viscosity Solutions

In this section, we rederive the dynamic programming equation rigourously. To simplify the presentation, we assume that the value function is continuous.

Fix an initial condition $(t,x) \in (t_0,t_1) \times O$, and suppose that φ be a smooth function satisfying

$$0 = (v-\varphi)(t,x) = \max(v-\varphi) \ . \tag{4.1}$$

In our analysis we consider all such smooth test functions. However, at a given point (t,x), there may be no such smooth function. In dynamic programming, let $r = t+h$. Subtract $v(t,x)$ both sides of (3.4) and divide by h. This yields

$$\begin{aligned} 0 &= \inf_{u(\cdot)} \{ \frac{1}{h} \int_t^{r\wedge\tau} L(s,x(s),u(s))\ ds + \frac{1}{h} \Psi(\tau, x(\tau)) \chi_{\{\tau<r\}} \\ &\qquad + \frac{1}{h}[v(r,x(r)) - v(t,x)] \chi_{\{\tau \geq r\}} \ \} \\ &\leq \inf_{u(\cdot)} \{ \frac{1}{h} \int_t^{r\wedge\tau} L(s,x(s),u(s))\ ds + \frac{1}{h} \Psi(\tau, x(\tau)) \chi_{\{\tau<r\}} \\ &\qquad + \chi_{\{\tau \geq r\}} \frac{1}{h} \int_t^r [\ \varphi_t(s,x(s)) + f(s,x(s),u(s)) \cdot \nabla\varphi(s,x(s))\ ]ds\ \} \ . \end{aligned} \tag{4.2}$$

Fix $w \in U$ let $u(\cdot) \in \mathcal{U}(t,x)$ be a control satifying (3.5). We use this control in (4.2) and send h to zero. The result is

$$0 \leq L(t,x,w) + f(t,x,w) \cdot \nabla\varphi(t,x) + \varphi_t(t,x), \qquad \forall w \in U \ .$$

Since this holds for all w, we conclude that

$$-\varphi_t(t,x) + H(t,x,\nabla\varphi(t,x)) \leq 0 \ . \tag{4.3}$$

We have proved the following:

Subsolution property.

At any $(t,x) \in (t_0,t_1) \times O$ and a smooth function φ satisfying (4.1), *we have* (4.3).

To obtain the opposite inequality, we consider a smooth test function φ satisfying

$$0 = (v-\varphi)(t,x) = \min(v-\varphi) \ . \tag{4.4}$$

Set $h = 1/m$ in $r_m := t + 1/m$ (4.2), and choose $u^m(\cdot) \in \mathcal{U}(t,x)$ so that

$$\begin{aligned} 0 &\geq \frac{1}{m} + m \int_t^{r_m \wedge \tau_m} L(s,x^m(s),u^m(s))\ ds + m\Psi(\tau_m, x^m(\tau_m)) \chi_{\{\tau_m < r_m\}} \\ &\qquad + m[v(r_m, x^m(r_m)) - v(t,x)] \chi_{\{\tau_m \geq r_m\}} \end{aligned}$$

$$\geq \frac{1}{m} + m\int_t^{r_m \wedge \tau_m} L(s,x(s),u(s))\ ds + \frac{1}{h}\Psi(\tau_m, x(\tau_m))\chi_{\{\tau_m < r_m\}}$$

$$+m\chi_{\{\tau_m \geq r_m\}} \int_t^{r_m} [\ \varphi_t(s,x(s)) + f(s,x(s),u(s)) \cdot \nabla\varphi(s,x(s))\ ]ds\ .$$

Hence,

$$0 = l_m + f_m \cdot \nabla\varphi_t(t,x) + \varphi(t,x) + e_m\ ,$$

where

$$l_m \ := \ m\int_t^{t+1/m} L(s,x,u^m(s))\ ds$$

$$f_m \ := \ m\int_t^{t+1/m} f(s,x,u^m(s))\ ds\ ,$$

and e_m is the error term. A careful analysis, show that the error term tends to zero (see [FS93; §2].) Let $\bar{co}FL(t,x)$ be the closed convex hull of

$$FL(t,x) := \{\ \ (f,l) = (f(t,x,u), L(t,x,u)) \quad u \in U\ \ \}\ .$$

Since $(f_m, l_m) \in \bar{co}FL(t,x)$,

$$0 = -l_m - f_m \cdot \nabla\varphi(t,x) - \varphi_t(t,x) - e_m \leq -\varphi_t(t,x) + \hat{H}(t,x,\nabla\varphi(t,x)) - e_m\ ,$$

where

$$\hat{H}(t,x,p) := \sup\ \{\ \ -f\cdot p - l\ \ |\ \ (f,l) \in \bar{co}FL(t,x)\ \}\ .$$

Notice that

$$H(t,x,p) := \sup\ \{\ \ -f\cdot p - l\ \ |\ \ (f,l) \in FL(t,x)\ \}\ .$$

Hence, $\hat{H}(t,x,p) = H(t,x,p)$. Therefore, we conclude that

$$-\varphi_t(t,x) + H(t,x,\nabla\varphi(t,x)) \geq 0\ . \tag{4.5}$$

So we have proved the following

Supersolution property.

At any (t,x) and a smooth function φ satisfying (4.4), *we have* (4.5).

Definition 4.1 *We say that any continuous function v satisfying the subsolution property is a viscosity subsolution of the dynamic programming equation (3.6).*

Any continuous function v satisfying the supersolution property is called a viscosity supersolution of the dynamic programming equation (3.6).

Finally a viscosity solution is a function which is both a sub and a supersolution.

Note that we have shown that the value function is a viscosity solution if it is continuous. Here we require continuity in order to simplify the presentation. Indeed the notion of a viscosity solution is defined for only locally bounded functions; see the next subsection.

In their seminal paper, Crandall and Lions proved that *there exists at most one continuous viscosity solution of the dynamic programing equation satisfying the boundary data*

$$v(t,x) = \Psi(t,x), \qquad \forall \ (t_0,t_1) \times O \ . \tag{4.6}$$

Indeed, they have proved more general comparison result for a more general class of first order differential equations. Of course technical assumptions on the data are needed and we refer the reader to [CL84, CEL 84] for a precise statement. A brief discussion of boundary condition is given in §5 below.

4.1 Discontinuous Solutions

Let w be a locally bounded funtion defined on a subset A of a Euclidean space. Let w^* be the *upper semicontinuous envelope* of w, i.e., w^* is the smallest upper semicontinuous function greater than or equal to w. There is costructive definition of w^* as well:

$$w^*(z) := \lim_{r\downarrow 0} \ \sup\{ \ w(y) \mid y \in A, \ |y-z| \le r \ \} \ , \quad z \in A.$$

Let w_* be the *lower semicontinuous envelope* of w, i.e., w_* is the largest lower semicontinuous function less than or equal to w:

$$w_*(z) := \lim_{r\downarrow 0} \ \inf\{ \ w(y) \mid y \in A, \ |y-z| \le r \ \} \ , \quad z \in A.$$

Next we consider a general first order partial differential equation

$$F(z, w(z), \nabla w(z)) = 0, \qquad z \in A. \tag{4.7}$$

Definition 4.2 a. *A locally bounded function w is a viscosity subsolution of (4.7) if for every smooth test fuction ϕ*

$$F(z_0, w^*(z_0), \nabla\phi(z_0)) \le 0 \ \ ,$$

at the local maximum z_0 of the difference $w^* - \phi$.

b. *A locally bounded function w is a viscosity supersolution of (4.7) if for every smooth test fuction ϕ*

$$F(z_0, w_*(z_0), \nabla\phi(z_0)) \ge 0 \ \ ,$$

at the local minimum z_0 of the difference $w_* - \phi$.

c. *A viscosity solution is both a sub and a supersolution.*

Notice that w^* is used to define a subsolution and w_* is for a supersolution. We expect that a subsolution to be less than a supersolution, but by definition, $w^* \geq w_*$. This observation is extremely usefull in the proof of convergence results; see [FS93;§VII.3].

4.2 State Constraints

In this section, we consider a class of problems in which the state process is constrained to stay within a cloased set $\bar{O}$. Hence, $\mathcal{U}(t,x)$ is the set of controls such that the corresponding state process stays within $\bar{O}$. Clearly, $\mathcal{U}(t,x)$ satisfies the condition (3.5) for all $(t,x) \in (t_0,t_1) \times O$. Hence the value function is a viscosity solution of the dynamic programing equation. To uniquely solve the equation we need boundary conditions. Since the state process never leaves the domian $\bar{O}$, the boundary condition (4.6) is not necessarily satisfied by the value of the state constrained optimal control problem. Following [S86], we fix a boundary point $(t,x) \in (t_0,t_1) \times \partial O$ and assume that a smooth test function, φ satisfies

$$0 = (v - \varphi)(t,x) = \min\{ \ (v-\varphi)(s,y) \mid (s,y) \in (t_0,t_1) \times \bar{O} \ \} \ .$$

We then follow the calculations that resulted in (4.5). This shows that φ still satisfies (4.5).

The fact that the maximizer (t,x) is a boundary point, implies more than a pointwise equation. Indeed this forces a boundary condition on v, which is strong enough to characterize it uniquely; see [S84], [FS93; §II.12]. To emphasize this, we consider a very simple one dimensional problem. Let $O = (-1,1)$, $t_1 = 1$, $U = \mathcal{R}^1$, $f \equiv u$, $\Psi \equiv 0$, and

$$L(t,x,u) = 1 + \frac{1}{4}u^2 \ .$$

The corresponding dynamic programming equation is

$$-v_t(t,x) + [v_x(t,x)]^2 = 1, \qquad (t,x) \in (0,1) \times (-1,1) \ . \tag{4.8}$$

It is easy to show that the value function corresponding to the state constraint problem is

$$v(t,x) = 1 - t \ ,$$

while

$$\tilde{v}(t,x) = \min\{ \ 1-t \ , \ 1 - |x| \ \}$$

is the value of the exit time problem with $\Psi \equiv 0$.

Now suppose that φ is a smooth function satisfying

$$0 = (v-\varphi)(t,1) = \max(v - \varphi) \ ,$$

at some $t < 1$. Since $v(t,x) = 1-t$, this implies that $\varphi_t(t,1) = -1$, and therefore, (4.5) is satisfied. More interesting, $v = 1-t$ is the unique viscosity solution of (4.8) which also satisfies the additional supersolution property at the boundary; [S86].

4.3 Examples; continued

We conclude by this section solving three specific minimal time problems considered in Example 2.3. For part a, the dynamic programming equation is

$$\max\{ \ -x_2 v_{x_1} + x_1 v_{x_2} \ ; \ v_{x_1} \ \} = 1 \ .$$

An obvious candidate for the optimal solution is to follow the first vector field until the state hits the positive x-axis and then to switch over to the second vector field. This strategy takes $\theta + r$ amount of time, where $\theta \in [0, 2\pi), r = |x|$ is the polar coordinates of x. So we set

$$v(x) = \theta + r \ ,$$

so that

$$v_{x_1} = -\frac{x_2}{r^2}, \qquad v_{x_2} = \frac{x_1}{r^2} \ .$$

Hence v satisfies the equation whenever it is differentiable, or quivalently everywhere away from the positive x-axis. On the x-axis we may show that it is still a viscosity solution. Note that on the positive x-axis, the optimal direction is orthogonal to the direction of the singularity.

Let us consider the state constraint next. State dynamics are the same but the state is not allowed to enter into the region

$$C := \{ \ \ (r, \theta) \ \ | \ \ r \in (0,1), \ \theta \in (\frac{\pi}{2}, \pi) \ \ \} \ .$$

If the optimal trajectory without the state constraint does not enter into C, then it is still optimal. Hence,

$$v(x) = \theta + r \ ,$$

for all $r \geq 1$ or $r < 1$ but $\theta \in [0, \frac{\pi}{2}]$. On the region, $r < 1$ and $\theta \in [\pi, 2\pi)$, we expect that the optimal strategy is to use f^1 until the state hits the constraint set C, then to use f^2 until the radis is equal to one, and switch to f^1 until state reaches the positive x-axis, and finally switch over to f^2. This yields a value of

$$v(x) = \theta + 2 - r \ .$$

It is straightforward to check that v satsifies the dynamic programming equation. On the boundary of C, suppose that

$$(v - \varphi)(x_1, 0) = \min(v - \varphi) \ ,$$

for some $x_1 \in (-1, 0)$. Then,

$$\varphi_{x_2}(x_1, 0) \geq v_{x_2}(x_1, 0) \ , \qquad \varphi_{x_1}(x_1, 0) = v_{x_1}(x_1, 0) = 1 \ .$$

Hence, at $(x_1, 0)$,

$$\max\{ \ -x_2\varphi_{x_1} + x_1\varphi_{x_2} \ ; \ \varphi_{x_1} \ \} \geq \varphi_{x_1} = 1 \ ,$$

and the modified boundary conditions are also satisfied by v. Therefore, v is the value function, and the described startegy is optimal.

For part b of Example 2.3, we try the following strategy. If $\theta \geq 3\pi/2$, then we use f^1 to reach the positive x-axis and then switch over to f^2. If, however, we start in the fourth quadrant, then we use f^2 until the state enters into the third quadrant and then follow the outlined strategy. The resulting time as a function of the initial data is

$$v(x) = \begin{cases} (\theta+1)r \ , & \theta \in [0, 3\pi/2] \ , \\ x - y(1 + 3\pi/2) & \theta \in (3\pi/2, 2\pi) \ . \end{cases}$$

This function satisfies the dynamic programing equation,

$$\max\{ \ -\frac{x_2}{r}v_{x_1} + \frac{x_1}{r}v_{x_2} \ ; \ v_{x_1} \ \} = 1 \ .$$

Hence it is equal to the value function.

5 Boundary Conditions

It is well known that solutions to first order differential equations may not satisfy the boundary conditions pointwise. As an example consider a one dimensional, infinite horizon problem with

$$L(t,x,u) \equiv 1, \quad f(t,x,u) \equiv 1, \quad O = (0,1) \ ,$$

and a general boundary condition

$$\Psi(t,x) = e^{-\beta t}\hat{\Psi}(x) \ .$$

Then, the value function is equal to $v(t,x) = e^{-\ \beta t}\hat{v}(x)$, and since there is no control effect

$$\hat{v}(x) = \int_0^\tau \ e^{-\ \beta s} \ ds + e^{-\ \beta\tau}\hat{\Psi}(x(\tau)) \ ,$$

where $x(s) = x+s$ is the state and τ is the exit time of $x(\cdot)$ from $O = (0,1)$. Hence, $x(\tau) = 1$ and $\tau = 1 - x$, and

$$\hat{v}(x) = \int_0^{(1-x)} \ e^{-\ \beta s} \ ds + e^{-\ \beta(1-x)}\hat{\Psi}(1) \ .$$

In particular,

$$\hat{v}(0) = [1 - e^{-\ \beta}]/\beta + \hat{\Psi}(1)/e \ ,$$

which is not equal to $\hat{\Psi}(0)$. In this example, the boundary condition at $x = 0$ is irrelevant since the state flows away from that boundary point.

In summary we do not expect the boundary conditions to be satisfied by the value function, and therefore, a weak formulation of the boundary condition is needed. Such a formulation is obtained after observing that when it is optimal not to exit from a certain boundary point, then essentially a state constarint is satisfied at that point. This observation leads us to the following weak formulation of the boundary conditions.

Consider the differential equation (4.7) together with the boundary condition

$$w(z) = g(z) \ , \qquad z \in \partial A \ , \tag{5.1}$$

where A is the closure of an open subset O of a Euclidean space.

Definition 5.3 a. *A viscosity subsolution w of (4.7) on O, is a viscosity subsolution of (4.7) together with the boundary condition (5.1) if for smooth test function ϕ*

$$\min\{ \ w^*(z_0) - g(z_0) \ ; \ F(z_0, w^*(z_0), \nabla\phi(z_0)) \ \} \leq 0 \ ,$$

at every maximizer $z_0 \in \partial A$ satisfying

$$w^*(z_0) - \phi(z_0) = \max\{ \ w^*(z) - \phi(z) \ : \ z \in A \ \} \ .$$

b. *A viscosity supersolution w of (4.7) on O, is a viscosity supersolution of (4.7) together with the boundary condition (5.1) if for smooth test function ϕ*

$$\max\{ \ w_*(z_0) - g(z_0) \ ; \ F(z_0, w_*(z_0), \nabla\phi(z_0)) \ \} \leq 0 \ ,$$

at every minimizer $z_0 \in \partial A$ satisfying

$$w_*(z_0) - \phi(z_0) = \min\{ \ w_*(z) - \phi(z) \ : \ z \in A \ \} \ .$$

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