Flow by mean curvature of surfaces of any codimension

Luigi Ambrosio
Istituto di Matematiche Applicate
Via Bonanno 25/B
56126 Pisa, Italy

Halil Mete Soner
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213, USA

Introduction

In this paper we summarize the main results of [1], where we develop a a level set approach for the description of mean curvature flow of surfaces Γ_t of codimension k in $\mathbf{R}^n$, generalizing previous works by Evans & Spruck and Chen, Giga & Goto (see [5], [9]) devoted to the evolution of hypersurfaces.

The main idea, suggested by De Giorgi in [8], is to surround the surface Γ_t by a family of hypersurfaces evolving with normal velocity equal to the sum of the smallest $(n-k)$ principal curvatures. If the hypersurfaces are the level sets of a time depending function u, it turns out that u must satisfy a second order nonlinear, degenerate, parabolic PDE. The theory of viscosity solutions developed in [5], [6], [7], [10] can be applied, yielding existence of a weak solution to the co–dimension k mean curvature flow.

We show the consistency of our weak solutions with classical solutions. The proof is based on the analysis of the properties of the squared distance function η from a smooth manifold Γ and on the relation between the second fundamental form of Γ and the third order derivatives of η on Γ and near to Γ.

Moreover, we compare our level set solution with other solutions already proposed in the literature: the measure theoretic subsolutions of Brakke (see [4]) and Ilmanen (see [12]), the distance solutions of Soner (see [14]) and the minimal barriers of De Giorgi (see [8]).

1. Smooth flow by mean curvature

Let $(\Gamma_t)_{t\in[0,T]}$ be $(n-k)$-dimensional smooth compact manifolds in $\mathbf{R}^n$ without boundary.

Definition. *$(\Gamma_t)_{t\in[0,T]}$ is a smooth flow if there exists a smooth, one to one deformation map $\phi(\cdot,t):\Gamma_0\to\Gamma_t$ such that the Jacobian $J\phi(x,t)$ is not singular, $\phi(x,0)=x$ and $\phi_t(x,t)$ is perpendicular to Γ_t at $\phi(x,t)$ or any $x\in\Gamma_0$, $t\in[0,T]$.*
A smooth flow is a mean curvature flow if

$$\phi_t(x,t)=\mathbf{H}\big(\phi(x,t),t\big) \qquad\qquad \forall x\in\Gamma_0,\ t\in[0,T]$$

where $\mathbf{H}(y,t)$ is the mean curvature vector of Γ_t at y.

In the above definition the time derivative $\phi_t(x,t)$ represents the velocity at time t of the point $\phi(x,t)\in\Gamma_t$. The mean curvature vector of a manifold Γ is locally defined by the formula

$$\mathbf{H}(y):=-\sum_{j=1}^{k}\mathrm{div}^{\Gamma}\big(\nu^j\big)\nu^j(y) \tag{1.1}$$

where $\nu^1,\dots,\nu^k$ is a smooth orthonormal vector field generating the normal space to Γ near y and

$$\mathrm{div}^{\Gamma} g:=\sum_{i=1}^{n}\delta_i g_i$$

is the tangential divergence. In the co–dimension 1 case, (1.1) reduces to

$$\mathbf{H}(y):=-\mathrm{div}^{\Gamma}\big(\nu\big)\nu(y)=-\mathrm{div}\big(\nu\big)\nu(y) \tag{1.2}$$

provided ν is a unit vector field defined in a full neighbourhood U of y, perpendicular to Γ at any point in $U\cap\Gamma$. We see from (1.1) that the mean curvature vector $\mathbf{H}$ does not depend on the orientation and is normal to Γ.

Geometrically, $\mathbf{H}$ points in the direction where the $(n-k)$ dimensional area $\mathcal{H}^{n-k}$ of Γ decreases most. This can be seen choosing a vector field $g\in C^1_0\big(\mathbf{R}^n,\mathbf{R}^n\big)$, defining

$$\Phi_\tau(x):=x+\tau g(x), \qquad\qquad \Gamma_\tau:=\Phi_\tau(\Gamma)$$

for $|\tau|<<1$ and looking at the derivative of $\tau\mapsto\mathcal{H}^{n-k}(\Gamma_\tau)$ at $\tau=0$. Using the divergence theorem on Γ one finds

$$\frac{d}{d\tau}\Big(\mathcal{H}^{n-k}\big(\Gamma_\tau\big)\Big)\Big|_{\tau=0}=\int_\Gamma \mathrm{div}^{\Gamma} g\,d\mathcal{H}^{n-k}=-\int_\Gamma\langle g,\mathbf{H}\rangle\,d\mathcal{H}^{n-k}.$$

In the case of smooth flow by mean curvature, $\Gamma_{t+\tau} \sim \big(Id + \tau \mathbf{H}_t\big)\Gamma_t$ as $\tau \sim 0$, hence

$$\frac{d}{dt}\Big(\mathcal{H}^{n-k}\big(\Gamma_t\big)\Big) = -\int_{\Gamma_t} |\mathbf{H}_t|^2 \, d\mathcal{H}^{n-k}. \tag{1.3}$$

It it well known that the flow by mean curvature may develop singularities in finite time even if the initial surface is smooth, making meaningless (with the exception of simple plane curves, boundaries of convex sets and graphs) the parametric approach. This is the main motivation for the research of a weak definition of flow by mean curvature, consistent with the classical one and defined for all times, even after the appearance of singularities.

2. The level set approach

The basic idea of the level set approach (see [13], [9], [5]) is to find a parabolic PDE such that *all* level sets of any solution u flow by mean curvature. The crucial property is that

$$t \mapsto \mathrm{dist}\big(\Gamma_t, \Gamma'_t\big)$$

is non decreasing for hypersurfaces Γ_t, Γ'_t flowing by mean curvature, so that initially disjoint hypersurfaces remain disjoint, and we can view them as level sets of a time depending function u. We introduce the following notations

$$\nu := \frac{\nabla u}{|\nabla u|} \qquad\qquad P_w := I - \frac{w \otimes w}{|w|^2} \quad w \neq 0.$$

Hence, ν is a unit normal vector to the level sets of u and P_w is the orthogonal projection on the hyperplane normal to w.

Using (1.2) and the fact that the velocity (in the direction ν) of a point $x \in \mathbf{R}^n$ is given by $-u_t(x,t)/|\nabla u|(x,t)$, we find that the PDE is

$$\begin{aligned} u_t &= |\nabla u| \mathrm{div}\Big(\frac{\nabla u}{|\nabla u|}\Big) = \big(\delta_{ij} - \nu_i \nu_j\big)\frac{\partial^2 u}{\partial x_i \partial x_j} \\ &= F\big(\nabla u, \nabla^2 u\big) \end{aligned} \tag{2.1}$$

with $F(w,X) := \mathrm{trace}\big(P_w X\big)$ (here and in the following we use the summation convention on repeated indices). Generalized solutions of the mean curvature flow are defined in the following steps:

1) Find u_0 such that $\Gamma_0 = \big\{x : u_0(x) = 0\big\}$.
2) Solve the PDE with the initial condition $u(x,0) = u_0(x)$.
3) Define $\Gamma_t := \big\{x : u(x,t) = 0\big\}$.

4) Show that Γ_t depends only on Γ_0, i.e., it does not depend on the choice of u_0.

However, there are some difficulties: the equation $u_t = F\big(\nabla u, \nabla^2 u\big)$ is nonlinear, degenerate and the 0-homogeneous map $w \mapsto F(w, X)$ has no continuous extension at 0. All these difficulties can be removed using the theory of viscosity solutions (see [6], [7]). Let us recall the definition of viscosity solution of a second order parabolic PDE:

Definition. *Let $u : \mathbf{R}^n \times (0,T) \to \mathbf{R}$ be a continuous function, let A be a dense subset of $\mathbf{R}^n \times \mathbf{S}^{n\times n}$ and $F : A \to \mathbf{R}$.*

$$\begin{array}{ccc} u \text{ subsolution of} & & u - \phi \text{ relative maximum at } (x_0, t_0), \\ & \Longleftrightarrow & \phi \text{ of class } C^2, \text{ implies} \\ u_t = F\big(\nabla u, \nabla^2 u\big) & & \phi_t \le F^*\big(\nabla \phi, \nabla^2 \phi\big) \text{ at } (x_0, t_0) \end{array}$$

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$$u \text{ viscosity solution} \quad \Longleftrightarrow \quad u \text{ subsolution and supersolution.}$$

In the above definition $\mathbf{S}^{n\times n}$ denotes the set of symme tric $n \times n$ matrices and F^*, F_* denote the upper and lower semicontinuous extensions of F

$$F^*(w, X) := \sup\left\{ \limsup_{h\to+\infty} F(w_h, X_h) : (w_h, X_h) \in A, \quad (w_h, X_h) \to (w, X) \right\}$$

$$F_*(w, X) := \inf\left\{ \liminf_{h\to+\infty} F(w_h, X_h) : (w_h, X_h) \in A, \quad (w_h, X_h) \to (w, X) \right\}$$

defined for all pairs $w \in \mathbf{R}^n$ and $X \in \mathbf{S}^{n\times n}$.

The basic assumption of the theory of viscosity solutions is the *degenerate ellipticity* of F_* and F^*:

$$X \ge Y \qquad \Longrightarrow \qquad F_*\big(w, X\big) \ge F_*\big(w, Y\big), \quad F^*\big(w, X\big) \ge F^*\big(w, Y\big).$$

The degenerate ellipticity implies that classical solutions of the equation are viscosity solutions. Indeed, if $u \in C^2\big(\mathbf{R}^n \times (0,T)\big)$ is a classical solution of the equation and $u-\phi$ has a relative maximum at (x_0,t_0), then $\nabla u = \nabla\phi$, $u_t = \phi_t$, $\nabla^2 u \le \nabla^2\phi$ at (x_0,t_0), hence

$$\begin{aligned} F^*\big(\nabla\phi(x_0,t_0),\nabla^2\phi(x_0,t_0)\big) &= F^*\big(\nabla u(x_0,t_0),\nabla^2\phi(x_0,t_0)\big) \\ &\ge F^*\big(\nabla u(x_0,t_0),\nabla^2 u(x_0,t_0)\big) \\ &= u_t(x_0,t_0) = \phi_t(x_0,t_0) \end{aligned}$$

and u is a viscosity subsolution. A similar argument works for supersolutios. The main advantages of viscosity solutions are:

1) General existence results, by Perron's method (see [6], [7], [5]).
2) Comparison theorems (see [10]):

$$\left.\begin{array}{l} u \text{ subsolution in } \mathbf{R}^n\times(0,T) \\ v \text{ supersolution in } \mathbf{R}^n\times(0,T) \\ \\ u,\ v \text{ uniformly continuous} \\ u(x,0)\le v(x,0) \text{ for any } x\in\mathbf{R}^n \end{array}\right. \quad\Longrightarrow\quad u\le v \text{ in } \mathbf{R}^n\times(0,T).$$

3) Strong stability properties with respect to uniform convergence (even Γ-convergence) of F and/or u (see [5]).

Evans & Spruck in [9], and Chen, Giga & Goto in [5] independently proved existence and uniqueness of viscosity solutions of

$$\begin{cases} u_t = F\big(\nabla u,\nabla^2 u\big) \\ \\ u(x,0) = u_0(x). \end{cases}$$

Moreover, they proved that

$$\Gamma_t := \big\{x : u(x,t)=0\big\}$$

depends only on Γ_0 and that the smoothness of Γ_0 implies the coincidence of the level sets Γ_t with the *classical* evolution of Γ_0, as long as the latter is defined. Moreover, short time existence results for the classical evolution starting from Γ_0 have been obtained by De Mottoni & Schatzman, Huisken, Evans & Spruck.

3. Higher codimension level set flow

The extension of the level set approach to the description of higher codimension flows is not straightforward. The main difficulty is due to the fact that a co–dimension k manifold can be represented by the intersection of k level sets of scalar functions. Hence, the level set approach leads to a *system* of PDE, without maximum principle. Geometrically, the absence of a maximum principle is related to the fact that initially disjoint manifolds could intersect during their flow by mean curvature. This happens, for instance, if Γ_0 consists of two chained rings. An additional difficulty is related to the computation of $\mathbf{H}$, requiring by (1.1) the local knowledge of an orthonormal basis of the manifold.

The basic idea, due to De Giorgi, is to look at the evolution of all positive level sets of a *scalar* function $u \geq 0$ under a new geometric law. Specifically, the level sets are required to flow with velocity equal to the sum of the smallest $(n-k)$ principal curvatures. Quite surprisingly, it turns out that the 0 level set of u is forced to flow by mean curvature!
The necessity of removing the highest $(k-1)$ eigenvalues to get a nonconstant motion can be understood by looking at the evolution of a simple smooth curve Γ in $\mathbf{R}^3$. In this case, if we look at a small tubular surface S around Γ, we find that one of the principal curvatures is very high, its order of magnitude being the inverse of $\mathrm{dist}(S,\Gamma)$, and the other one is related to the geometry of Γ. In particular, if we let S flow by mean curvature, we find that S becomes empty in a very short time.

Let u be a nonnegative function, assume that $u \in C^2\big(\{u > 0\}\big)$ and $|\nabla u| > 0$ for $u > 0$. For $\tau > 0$, let

$$E^\tau := \big\{x : u(x) = \tau\big\}$$

oriented da ν. The principal curvatures of E^τ (with respect to the orientation induced by ν) are given by the eigenvalues of the symmetric bilinear form

$$B(\xi,\eta) := \langle \xi, d_\eta \nu^{E^\tau} \rangle \qquad\qquad \xi,\, \eta \in \nu^\perp.$$

With this sign convention, if u is a convex function, the sets $\big\{x : u(x) < \tau\big\}$ have nonnegative principal curvatures. A simple computation shows that

$$B = \frac{P_\nu \nabla^2 u P_\nu}{|\nabla u|} \qquad\qquad \text{on } \nu^\perp. \tag{3.1}$$

Hence, we define

$$\begin{aligned} G(w,X) :&= \text{sum of the } (n-k) \text{ smallest eigenvalues} \\ &\qquad \text{of } Y := P_w X P_w \text{ on the space } w^\perp \\ &= \lambda_1(Y) + \ldots + \lambda_{n-k}(Y) \end{aligned} \tag{3.2}$$

removing the highest $(k-1)$ eigenvalues of Y.
By (3.1), (3.2), the positive level sets of a function $u(x,t)$ flow (in the direction $-\nu$) with velocity equal to the sum of their smallest $(n-k)$ principal curvatures if and only if

$$u_t = G\big(\nabla u, \nabla^2 u\big), \qquad u(x,0) = u_0(x) \geq 0. \tag{3.3}$$

To our knowledge, there are no results in the literature concerning short time existence of $C^{1,1}$ solutions of (3.3), assuming that $u_0 \in C^2\big(\{u_0 > 0\}\big)$ and $|\nabla u_0| > 0$ for $u_0 > 0$. Even if u_0 is smooth, we cannot expect smooth solutions, even for a short time, because $X \mapsto G(w,X)$ is not a C^1 function (only a Lipschitz function) in $\mathbf{S}^{n\times n}$.
However, the theory of viscosity solutions can be used to get weak solutions of (3.3). In order to apply this theory, we need only to check the degenerate ellipticity of $X \mapsto G(w,X)$. The problem is the following: given a $(n-1)$–dimensional vector space E, a symmetric bilinear form Y on E, and given the ordered list of the eigenvalues of Y

$$\lambda_1(Y) \leq \lambda_2(Y) \leq \ldots \leq \lambda_{n-1}(Y)$$

can we say that

$$Y \leq Y' \qquad \Longrightarrow \qquad \lambda_i(Y) \leq \lambda_i(Y') \quad i = 1, \ldots, (n-1) \quad ?$$

The answer is positive because the numbers $\lambda_i(Y)$ solve a max–min problem depending monotonically on Y:

$$\lambda_i(Y) = \max\bigg\{ \min_{p\in F\setminus\{0\}} \frac{\langle Yp, p\rangle}{|p|^2} : F \subset E,\ \ \mathrm{codim}\, F \leq (i-1) \bigg\}. \tag{3.4}$$

The proof of (3.4) is elementary. Indeed, the inequality $\leq$ follows by choosing as F the vector space spanned by the eigenvectors corresponding to $\lambda_i(Y), \lambda_{i+1}(Y), \ldots, \lambda_n(Y)$. The opposite inequality follows by the fact that each subspace F with codimension not greater than $(i-1)$ has at least a nonzero vector in common with the vector space F_0 generated by the eigenvectors corresponding to $\lambda_1(Y) \ldots, \lambda_i(Y)$. Since

$$Y \leq \lambda_i(Y) I \quad \text{on } F_0 \qquad \text{and} \qquad Y \geq \min_{p\in F\setminus\{0\}} \frac{\langle Yp, p\rangle}{|p|^2} I \quad \text{on } F$$

we obtain

$$\lambda_i(Y) \geq \min_{p\in F\setminus\{0\}} \frac{\langle Yp, p\rangle}{|p|^2}.$$

Hence, $G(w,X) = \lambda_1(Y) + \ldots + \lambda_{n-k}(Y)$ is degenerate elliptic. Geometrically, the ellipticity of G is related to the validity of an inclusion principle for open sets whose boundaries flow by the sum of the smallest $(n-k)$ principal curvatures, i.e.,

$$\Omega(t_0) \subset \Omega'(t_0) \qquad \Longrightarrow \qquad \Omega(t) \subset \Omega'(t) \quad \forall t \geq t_0.$$

Therefore, the level set approach can be used to describe this motion. The surprising property is that the flow according to this law of a family of hypersurfaces Γ_t^τ filling $\mathbf{R}^n \setminus \Gamma_t$ forces Γ_t to flow by mean curvature.

Applying a general existence theorem of Chen, Giga & Goto for "geometric" parabolic equations (see [5]) we proved that

$$\begin{cases} u_t = G\big(\nabla u, \nabla^2 u\big) \\ u(x,0) = u_0(x) \end{cases} \tag{3.5}$$

has a unique viscosity solution for any uniformly continuous function u_0.

Definition. *Given a closed set Γ_0, we define generalized co–dimension k flow of Γ_0 the sets*

$$\Gamma_t := \big\{x : u(x,t) = 0\big\}$$

where $u(x,t)$ is the solution of the problem (3.5) with a nonnegative, uniformly continuous function u_0 such that

$$\Gamma_0 = \big\{x : u_0(x) = 0\big\}.$$

The definition is well posed: using comparison theorems we proved that

Γ_t does not depend on the choice of u_0. In the case $k = 1$ it is easy to check that G is equal to the function F considered by Evans & Spruck, Chen, Giga & Goto. Since the map $t \mapsto u(t,\cdot)$ has the semigroup property, we have the

Semigroup property: Given s, $t \geq 0$, the set Γ_{t+s} coincides with the evolution at time t starting from Γ_s.

4. Main properties of co–dimension k flow

The co–dimension k level set flow Γ_t defined in §3 can be compared with the classical solutions and other weak solutions of the flow by mean curvature proposed by Brakke, Ilmanen, De Giorgi. It turns out that, if Γ_0 is smooth, Γ_t coincides with the classical solution of flow by mean curvature as long as the latter is defined. We will explain in the next section the ideas involved in the proof of this consistency result.

Let us very briefly recall Brakke's definition of motion by mean curvature. A family of Radon measures $(\mu_t)_{t\geq 0}$ in $\mathbf{R}^n$ is said to be a *m-dimensional Brakke motion* if

$$\frac{d^+}{dt}\left(\int_{\mathbf{R}^n}\phi\,d\mu_t\right)\leq \mathcal{B}(\mu_t,\phi) \qquad \forall\phi\in C^2_0\big((\mathbf{R}^n,[0,+\infty)\big) \tag{4.1}$$

where d^+/dt denotes the upper derivative and

$$\mathcal{B}(\mu_t,\phi)=\int_{\mathbf{R}^n}\big(-\phi|\mathbf{H}_t|^2+\langle\nabla\phi,\mathbf{H}_t\rangle\big)\,d\mu_t \tag{4.2}$$

if μ_t is equal to a m-varifold with integer density and absolutely continuous mean curvature $H_t\mu_t$ in $\{\phi>0\}$ (see [4], [12]) and $\mathcal{B}(\mu_t,\phi)=-\infty$ otherwise. For a smooth mean curvature flow, (4.2) holds and the equality in (4.1) can be proved by a localization of (1.3).

As Ilmanen proved in [12], we can think to the level set flow as a minimal set theoretic supersolution and to the Brakke's flow as a measure theorethic subsolution, because only the inequality $\leq$ is required in (4.1). Hence, the best result we can hope for is an inclusion property of $(n-k)$–dimensional Brakke's motions in co–dimension k level set motions. Indeed for any Brakke motion (see [1], Theorem 5.4) the following implication holds:

$$\operatorname{supp}\mu_0\subset\Gamma_0 \qquad\Longrightarrow\qquad \operatorname{supp}\mu_t\subset\Gamma_t \quad \forall t\geq 0.$$

Similar comparison properties can be stated and proved for distance solutions (see [14] and §4 of [1]) and for the barrier solutions introduced by De Giorgi (see [2], [3], [8] and §6 of [1]).

5. Consistency with smooth flows

The typical idea to prove the consistency result (already exploited in codimension 1) is to describe the flow by mean curvature using a PDE satisfied by the distance function, and use this to compare classical and viscosity solutions. In the case $k = 1$ it is well known that

$$\Gamma_t = \partial\Omega_t \text{ is a smooth mean curvature flow}$$
$$\Updownarrow$$
$$r_t = \Delta r \quad \text{on} \quad \big\{r = 0\big\}$$

where $r(x,t)$ is the signed distance function from Γ_t:

$$r(x,t) = \begin{cases} -\mathrm{dist}\big(x, \Gamma_t\big) & \text{if } x \in \Omega_t; \\ \\ \ \ \mathrm{dist}\big(x, \Gamma_t\big) & \text{if } x \notin \Omega_t. \end{cases}$$

In higher codimension there is no possibility to define a signed distance function, so that the first idea could be to work with the distance function

$$\delta(x) := \mathrm{dist}\big(x, \Gamma\big).$$

However, δ is a Lipschitz function in $\mathbf{R}^n$ but its first order derivatives (defined in $\{0 < \delta < \tau\}$ for small τ) are discontinuous on Γ. Moreover $(k-1)$ eigenvalues of the hessian matrix of δ are unbounded near Γ and even if we remove them there is still lack of continuity. Indeed, we proved that

$$\nexists \lim_{y \to x \in \Gamma} G\big(\nabla\delta(y), \nabla^2\delta(y)\big)$$

because the limit above exists only on lines normal to Γ, depends on the direction $p \in \mathbf{S}^{n-1}$ of the line and is equal to $-\langle \mathbf{H}(x), p\rangle$.

As suggested by De Giorgi, to transfer informations from Γ to a neighbourhood of Γ and to characterize the flow by mean curvature we will use the *squared distance function*

$$\eta(x) := \frac{1}{2}\delta^2(x) = \frac{1}{2}\mathrm{dist}^2\big(x, \Gamma\big).$$

The first result we proved (see [1], Theorem 3.5 and Lemma 3.7) is the following:

Theorem 1. *Let* $\big\{\Gamma_t\big\}_{t\in[0,T]}$ *be a smooth flow of codimension* k *and let*

$$\eta(x,t) := \mathrm{dist}^2\big(x, \Gamma_t\big)/2.$$

Then, Γ_t is a mean curvature flow if and only if

$$\big(\nabla\eta\big)_t = \Delta\big(\nabla\eta\big) \qquad \text{on } \big\{\eta = 0\big\}.$$

The proof of Theorem 1 is based on the fact that $-\big(\nabla\eta\big)_t(x,t)$ is the normal velocity of the point $x \in \Gamma_t$ and $-\Delta\big(\nabla\eta(x,t)\big)$ is the mean curvature vector $\mathbf{H}_t(x)$ of Γ_t at x. Hence, one of the advantages of the function η is that it provides a simple method for the computation of the mean curvature vector.

The second result (see [1], Theorem 3.8) shows that the system in η is equivalent to a differential inequality in δ, where

$$\delta(x,t) := \sqrt{2\eta} = \operatorname{dist}\big(x, \Gamma_t\big).$$

Theorem 2. *Let $\big\{\Gamma_t\big\}_{t\in[0,T]}$ be a smooth flow of codimension k, and let $\Omega \subset \mathbf{R}^n \times (0,T)$ be the maximal open set where δ is smooth. Then, Γ_t flows by mean curvature if and only if*

$$\delta_t \geq G\big(\nabla\delta, \nabla^2\delta\big) \qquad \text{on } \Omega.$$

The proof of Theorem 2 also shows that the inequality

$$\delta_t \geq G\big(\nabla\delta, \nabla^2\delta\big) \tag{5.1}$$

holds in $\mathbf{R}^n \times (0,T)$, in the viscosity sense. Moreover,

$$\delta_t \leq G\big(\nabla\delta, \nabla^2\delta\big) + C\delta \qquad \text{on } \big\{0 < \delta < \sigma\big\} \tag{5.2}$$

for suitable constants C, $\sigma > 0$. From (5.2) we get that $W := e^{-Ct}\big(\delta \wedge \sigma/2\big)$ satisfies

$$W_t \leq G\big(\nabla W, \nabla^2 W\big) \tag{5.3}$$

in $\mathbf{R}^n \times (0,T)$, in the viscosity sense. Hence, using (5.1) and (5.3) we can compare δ with the viscosity solution of

$$\begin{cases} u_t = G\big(\nabla u, \nabla^2 u\big) \\ u(x,0) = \delta(x) \end{cases}$$

obtaining $W \leq u \leq \delta$ in $\mathbf{R}^n \times (0,T)$. In particula r, since W and δ have the same 0 level set,

$$\Gamma_t = \big\{x : \delta(x,t) = 0\big\} = \big\{x : u(x,t) = 0\big\} \qquad \forall t \in (0,T)$$

and this shows the consistency with smooth flows.

The proof of Theorem 2 is based on the propagation properties of the eigenvalues of $\nabla^2\eta$ on normal lines to Γ. Given a smooth, compact, codimension k surface Γ without boundary, let $\tau > 0$ such that $\eta \in C^\infty\big(\{\eta \le \tau^2/2\}\big)$ and let $x \notin \Gamma$ such that $\delta(x) = \tau$. Denoting by $y \in \Gamma$ the point in Γ of least distance from x, set

$$p := \frac{x-y}{\tau}, \qquad x_s := y + sp, \qquad B(s) := \nabla^2\eta(x_s)$$

for $s \in [0,\tau]$. Then, the following theorem holds (see [1], Theorem 3.2):

Theorem 3. *(1) For $s \in [0,\tau]$ the eigenvectors of $B(s)$ do not depend on s, $B(s)$ has k eigenvalues equal to 1, all others $\lambda_i(s)$ are strictly less than 1 and satisfy*

$$\big|\lambda_i(s)\big| \le Cs \qquad i = 1,\ldots,n-k$$

with C depending only on Γ, τ.
(2) For $s \in (0,\tau]$ the matrix $\nabla^2\delta(x_s)$ has $(k-1)$ eigenvalues equal to $1/s$, one (corresponding a p) equal to 0, all others $\beta_i(s)$ are strictly less than $1/s$ and satisfy

$$\big|\beta_i(s)\big| \le C \qquad i = 1,\ldots,n-k.$$

(3) The maps

$$s \mapsto \beta_i(s), \qquad s \mapsto G\big(\nabla\delta(x_s), \nabla^2\delta(x_s)\big)$$

are nonincreasing in $(0,\tau]$.

The geometric interpretation of Theorem 3 is the following: consider, for instance, the level set $E_s := \{\delta = s\}$ of the distance function δ from a smooth, simple curve Γ in $\mathbf{R}^n$. Then, if E_s lies in the region where δ^2 is smooth, we know that $(n-2)$ principal curvatures of E_s are *exactly* equal to the $1/s$.

The proof of Theorem 3 is based on the continuity of the map $s \mapsto B(s)$ in $[0,\tau]$ and on the fact that $B(s)$ solves the ODE

$$B'(s) = \frac{B(s) - B^2(s)}{s} \qquad s \in (0,\tau]. \tag{5.4}$$

The ODE is obtained by differentiating twice $|\nabla\eta|^2 = 2\eta$ (this identity directly follows from the equality $|\nabla\delta|^2 = 1$). Indeed, the first differentiation gives $\eta_{ij}\eta_j = \eta_i$ and the second one gives

$$\eta_{ijk}\eta_j + \eta_{ij}\eta_{jk} = \eta_{ik}. \tag{5.5}$$

It is not hard to see that $\eta_{ijk}(x_s)\eta_j(x_s)$ is equal to $B'_{ij}(s)s$, hence (5.4) follows by (5.5). The ODE implies that the matrices $B(s)$ are diagonal in a common basis and their eigenvalues satisfy

$$\lambda_i'(s) = \frac{\lambda_i(s) - \lambda_i^2(s)}{s}. \tag{5.6}$$

On the other hand, $B(0)$ is the orthogonal projection on the normal space to Γ at y, hence $B(0)$ has k eigenvalues equal to 1. Studying the solutions of (5.6) backward in time we obtain that $B(s)$ must have exactly k eigenvalues equal to 1 and the other ones must be infinitesimal (with order s) as $s \to 0^+$. The connection between the eigenvalues of $\nabla^2\delta$ and the eigenvalues of $\nabla^2\eta$ comes from the fact that $\nabla\delta$ is an eigenvalue of $\nabla^2\eta$ and from the identity

$$\delta\delta_{ij} = \eta_{ij} - \delta_i\delta_j$$

that can be obtained by differentiation of $\eta_i = \delta\delta_i$.

Now we can explain the connection, stated in Theorem 2, between the differential inequality

$$\delta_t \geq G\big(\nabla\delta, \nabla^2\delta\big) \qquad \text{on} \qquad \big\{0 < \delta \leq \tau\big\}$$

and the system

$$\big(\nabla\eta\big)_t = \Delta\big(\nabla\eta\big) \qquad \text{on} \qquad \big\{\eta = 0\big\}.$$

Let us set

$$\alpha_i := (\eta_i)_t - \Delta(\eta_i), \qquad \beta_1 \leq \beta_2 \leq \ldots \leq \beta_n \quad \text{eigenvalues of } \nabla^2\delta.$$

By Theorem 3, $\beta_{n-k+2} = \cdots = \beta_n = \delta^{-1}$. Using again identities obtained by differentiation of $|\nabla\delta|^2 = 1$ we get the following identities

$$\begin{aligned}
\alpha_i\delta_i &= \delta_t - \Delta\delta + \delta\|\nabla^2\delta\|^2 \\
&= \delta_t - \sum_{i=1}^{n}\beta_i + \delta\sum_{i=1}^{n}\beta_i^2 && \text{(by the symmetry of } \nabla^2\delta) \\
&= \delta_t - \sum_{i=1}^{n-k+1}\beta_i + \delta\sum_{i=1}^{n-k+1}\beta_i^2 && \text{(by Theorem 3)} \\
&= \delta_t - G\big(\nabla\delta, \nabla^2\delta\big) + O(\delta) && \text{(by Theorem 3).}
\end{aligned}$$

in the set

$$\big\{(x,t) : 0 < \delta(x,t) \leq \tau\big\}.$$

Assume that $\alpha_i = 0$ for $\eta = 0$. Passing to the limit we get

$$\lim_{x \to y \in \Gamma} \delta_t - G\big(\nabla \delta, \nabla^2 \delta\big) = 0.$$

Since $\delta_t(x,t)$ is constant and $G\big(\nabla\delta, \nabla^2\delta\big)$ is nonincreasing on normal lines to Γ (moving away from Γ) we infer

$$\delta_t \geq G\big(\nabla\delta, \nabla^2\delta\big) \qquad \text{on} \quad \big\{0 < \delta \leq \tau\big\}.$$

Conversely, assume that the differential inequality holds. Approaching to Γ on a line parallel to $p \in \mathbf{S}^{n-1}$, since $\delta_i = p_i$ on the line, we get

$$\alpha_i p_i \geq 0 \qquad \text{on} \quad \big\{\eta = 0\big\}.$$

Since $\alpha = \mathbf{H} - V$ is normal to Γ and p is arbitrary, $\alpha = 0$.

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