Front Propogation

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Foreword

During the six lectures I have given at the CRM 1995 Summer School in Banff, I outlined recent weak theories for front propogation and approximation by Ginzburg-Landau reaction-diffusion equations. The level set type, weak-viscosity theories were initiated by the papers of Evans and Spruck [ES] and Chen, Giga and Goto [CGG] who were motivated by earlier papers of Osher and Sethin [Se, OS] and Ohta, Jasnow and Kawasaki [OJK]. In my lectures, I followed the more intrinsic approach developed in my paper [S1].

This new weak-viscosity theory was later used to prove the global in-time convergence of the scalar Ginzburg-Landau (or Cahn-Allen) equation to mean curvature flow; thus verifying the formal results of Rubinstein, Sternberg and Keller [RSK] and extending a short time result of Chen [Ch]. First global result was proved by Evans, Soner and Souganidis [ESS] and later extended by Barles, Soner and Souganidis [BSS], Ilmanen [Il] and by myself [S2]. A more complete list of references is given in [S2]. In these notes, I followed the more recent approach of Jerrard and myself [JS] which also applies to the systems of Ginzburg-Landau as well. Also, this new method shows that, for the scalar equations, a convergence result to smooth flows is sufficient to prove the convergence to weak flows. To make this statement rigorous, I used the properties of the intrinsic solutions instead of the level set solutions.

In these lecture notes, I have not covered the higher codimension flows such as the evolution of vortex lines in three dimensional space. Recently, together with Ambrosio, I have developed a weak-viscosity theory for the mean curvature flow in any codimension, and the corresponding approximation of the higher co-dimension flows by the Ginzburg-Landau systems is proved in a paper by

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Jerrard and myself [JS]. Also, I have not covered the analysis of the phase field equations of solidification [S3].

1 Codimension-one Geometric Flows

1.1 Introduction

The problem is an initial value problem of finding a one parameter family of compact sets $\{\Gamma(t)\}_{t\geq 0} \subset \mathcal{R}^d$ that are the boundaries of open sets $\Omega(t)$ and satisfy:

$$v = V(x, t, \vec{n}, D\vec{n}), \text{ on } t > 0, x \in \Gamma(t), \tag{1.1}$$

$$\Gamma(0) = \Gamma_0, \tag{1.2}$$

where $\Gamma_0 = \partial\Omega_0$ is a given initial set, v is the outward normal velocity of $\Gamma(t)$, $\vec{n}$ is the outward, unit normal vector at $x \in \Gamma(t)$ and V is a given nonlinear function. We will always assume that $\vec{n}$ is extended off $\Gamma(t)$, as a unit vector, and D is differentiation in $\mathcal{R}^d$.

Typical examples are the following.

Example 1.1 (Mean Curvature flow) In this example, the normal velocity is equal to the sum of the principal curvatures of the hypersurface $\Gamma(t)$ and, therefore, the nonlinear function V in (1.1) is given by,

$$V = -\nabla \cdot \vec{n}.$$

In the vector form, we may rewrite this equation as

$$\vec{v} := V\vec{n} = -\vec{H} := -(\nabla \cdot \vec{n})\vec{n},$$

where $\vec{v}$ and $\vec{H}$ are, respectively, the normal velocity vector and the mean curvature vector. Note that both $\vec{v}$ and $\vec{H}$ are orientation independent. Hence, in this example, $\{\Gamma(t)\}_{t\geq 0}$ need not be a boundary to be defined as a solution.

Example 1.2 In this example, V is independent of $\vec{n}$ and $V = \alpha(x, t)$, where α is a given function. In contrast to mean curvature flow, this flow is orientation dependent, and therefore, we have to take $\Gamma(t)$ to be a boundary.

Example 1.3 (Gurtin's anisotropic flow) This type of flows arise in several models for supercooled solidification: see Gurtin [Gu]. For a given convex, positively homogenous of degree one function H,

$$V = -\nabla \cdot \big(DH(\vec{n})\big) + c,$$

where c is a constant. In two dimensions, we may rewrite this equation as:

$$V = -\big(f(\theta) + f''(\theta)\big)k + c,$$

where k is the curvature of the curve $\Gamma(t)$, θ is such that $\vec{n} = (\cos\theta, \sin\theta)$, and

$$f(\theta) = H(\cos\theta, \sin\theta).$$

It is well known that even if $\Gamma(0)$ is smooth, in finite time solutions develop singularities: see for instance, Barles-Soner-Souganidis [BSS], or consider the equation $v = 1$, with initial data

$$\Omega(0) = \big\{x \in \mathcal{R}^2 : |x| < 1\big\} \cup \big\{x \in \mathcal{R}^2 : |x - (4,0)| < 1\big\}.$$

Clearly at $t = 1$, the solution develops a geometric singularity. Hence a notion of a weak solution is necessary.

1.2 Level set formulation

Suppose that $\big\{\Gamma(t) = \partial\Omega(t)\big\}_{t\le 0}$ is a solution of (1.1). We assume that there is a smooth auxiliary function $\varphi(x,t)$ satisfying:

$$\Omega(t) = \big\{x : \varphi(x,t) > 0\big\}, \quad \Gamma(t) = \big\{x : \varphi(x,t) = 0\big\}, \quad \big|\nabla\varphi(x,t)\big| > 0, \quad \forall x \in \Gamma(t).$$

Then, a direct computation shows that, on $\Gamma(t)$,

$$v = \frac{\varphi_t}{|\nabla\varphi|}, \quad \vec{n} = -\frac{\nabla\varphi}{|\nabla\varphi|},$$

and therefore

$$v = \frac{\varphi_t}{|\nabla\varphi|} = V\left(x, t, -\frac{\nabla\varphi}{|\nabla\varphi|}, -\nabla\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right)\right), \text{ on } \Gamma(t).$$

We rewrite this equation as

$$\varphi_t = F(x, t, \nabla\varphi, \nabla^2\varphi), \tag{1.3}$$

where for $p \in \mathcal{R}^d \setminus \{0\}$ and a symmetric $d \times d$ matrix A,

$$F(x,t,p,A) = |p| V\left(x, t, -\frac{p}{|p|}, -\left[I - \frac{p \otimes p}{|p|^2}\right]\frac{A}{|p|}\right).$$

Note that F, defined as above, has the following property, which we call *geometric*,

$$F(x, t, \lambda p, \lambda A + \mu p \otimes p) = \lambda F(x,t,p,A), \quad \forall \lambda > 0, \mu \in \mathcal{R}^1. \tag{1.4}$$

We also assume that V is such that, F is (degenerate) em elliptic, i.e.,

$$F(x,t,p,A+B) \ge F(x,t,p,A), \quad \forall B \ge 0. \tag{1.5}$$

The idea of Osher and Sethian [Se], [OS] and Ohta, Jasnow and Kawasaki [OJK] is to solve (1.3) on the whole $\mathcal{R}^d \times (0,\infty)$ instead of solving (1.1) on the unknown hypersurface $\Gamma(t)$. Then, a weak solution of (1.1) and (1.2) is defined as follows: given $\Gamma_0 = \partial\Omega_0$, choose a continuous function $\varphi_0(x)$ so that

$$\Omega_0 = \big\{x : \varphi_0(x) > 0\big\}, \quad \Gamma_0 = \partial\Omega_0 = \big\{x : \varphi_0(x) = 0\big\}.$$

There are such functions, for instance the signed distance to Γ_0:

$$\varphi_0(x) = -dist(x, \overline{\Omega}_0) + dist(x, \mathcal{R}^d \setminus \Omega_0).$$

Let $\varphi(x,t)$ be a viscosity solution (a notion that will be defined later) of (1.3) in $\mathcal{R}^d \times (0,\infty)$ satisfying:

$$\varphi(x,0) = \varphi_0(x), \quad x \in \mathcal{R}^d. \tag{1.6}$$

Then, we define a weak solution of (1.1) and (1.2) as the zero level set of φ:

$$\Gamma(t) = \big\{x : \varphi(x,t) = 0\big\}.$$

This definition makes sense provided that there is a unique solution φ of (1.3)-(1.6) and that, for all $t > 0$, the zero level set of $\varphi(\cdot,t)$ depends *only* on the zero level set of $\varphi_0(\cdot)$ but not φ_0 itself. The latter is required because we are given only $\Gamma_0 = \partial\Omega_0$ but not φ_0 itself, and therefore, the arbitrary choice we make for φ_0 should not alter $\Gamma(t)$. Indeed, in §1.4, we will show that if $\varphi(x,t)$ and $\widetilde{\varphi}(x,t)$ are uniformly continuous viscosity solutions of (1.3) and if

$$\big\{\varphi(x,0) = 0\big\} = \big\{\widetilde{\varphi}(x,0) = 0\big\} \quad \text{and} \quad \big\{\varphi(x,0) < 0\big\} = \big\{\widetilde{\varphi}(x,0) < 0\big\},$$

then, for all $t \geq 0$,

$$\big\{x : \varphi(x,t) = 0\big\} = \big\{x : \widetilde{\varphi}(x,t) = 0\big\}, \big\{x : \varphi(x,t) < 0\big\} = \big\{x : \widetilde{\varphi}(x,t) < 0\big\}.$$

We close this section by computing the function F for the examples considered in §1.2.

Example 1.4 For the mean curvature flow

$$V = -\nabla \cdot \vec{n} = -trace(D\vec{n}).$$

Since $\vec{n} = -\nabla\varphi / |\nabla\varphi|$,

$$V = \nabla \cdot \left(\frac{\nabla\varphi}{|\nabla\varphi|}\right) = \frac{1}{|\nabla\varphi|}\left(\nabla\varphi - \frac{D^2\varphi\nabla\varphi\cdot\nabla\varphi}{|\nabla\varphi|^2}\right),$$

and therefore, for a symmetric matrix A and $p \in \mathcal{R}^d \setminus \{0\}$,

$$F(p,A) = trace\left(I - \frac{p\otimes p}{|p|^2}\right)A.$$

The level set equation (E) takes the form

$$\varphi_t = \Delta\varphi - \frac{D^2\varphi\nabla\varphi\cdot\nabla\varphi}{|\nabla\varphi|^2}. \tag{1.7}$$

Note that F is not well defined when $p = 0$!

Example 1.5 In this example F is simpler:

$$F(x,t,p) = \alpha(x,t)|p|.$$

Example 1.6 For the Gurtin's equation a similar computation yields

$$F(p,A) = trace \left[D^2 H\left(\frac{p}{|p|}\right)\left(I - \frac{p \otimes p}{|p|^2}\right) A \right] + c|p|,$$

and again F is not defined at $p = 0$.

All these examples indicate that F is degenerate in the p direction, a property that every geometric F has in view of (1.4), and also F may not be well defined at $p = 0$. Therefore the appropriate notion of a solution of (1.3) is the viscosity solutions of Crandall and Lions [CL]: also see Crandall-Ishii-Lions [CIL] and Fleming-Soner [FS].

1.3 Viscosity solutions

In this section we will give a brief introduction to viscosity solutions. For more information we refer to the User's Guide [CIL] and [FS]. Let $\mathcal{O}$ be a subset of Euclidian space and w be a scalar valued function on $\mathcal{O}$. On the closure $\bar{\mathcal{O}}$ of $\mathcal{O}$ we define two functions, the upper semicontinuous envelope $w^\star$ and the lower semicontinuous envelope $w_\star$ by

$$\begin{aligned} w^\star(x) &:= \lim_{\varepsilon \downarrow 0} \sup_{\substack{|y-x| \le \varepsilon \\ y \in \mathcal{O}}} w(y), \qquad x \in \bar{\mathcal{O}}, \\ w_\star(x) &= -(-w)^\star(x) \\ &= \lim_{\varepsilon \downarrow 0} \inf_{\substack{|y-x| \le \varepsilon \\ y \in \mathcal{O}}} w(y), \qquad x \in \bar{\mathcal{O}}. \end{aligned}$$

Definition 1.7 Let u be a locally bounded function on $\mathcal{R}^d \times [0,T]$ with $T \le \infty$.

(a) We say that u is a *viscosity subsolution* of (1.3) in $\mathcal{R}^d \times (0,T)$, if for any $\varphi \in C^\infty\big(\mathcal{R}^d \times [0,T]\big)$

$$\varphi_t(x_0,t_0) \le F^\star\big(x_0,t_0,D\varphi(x_0,t_0),D^2\varphi(x_0,t_0)\big), \tag{1.8}$$

at every local, strict maximizer $(x_0,t_0) \in \mathcal{R}^d \times (0,T)$ of the difference $u^\star - \varphi$.

(b) We say that u is a *viscosity supersolution* of (1.3) in $\mathcal{R}^d \times (0,T)$, if for any $\varphi \in C^\infty\big([0,T] \times \mathcal{R}^d\big)$

$$\varphi_t(x_0,t_0) \ge F_\star\big(x_0,t_0,D\varphi(x_0,t_0),D^2\varphi(x_0,t_0)\big), \tag{1.9}$$

at every local, strict maximizer $(x_0,t_0) \in \mathcal{R}^d \times (0,T)$ of the difference $u_\star - \varphi$.

(c) A *viscosity solution* is both a sub and a supersolution.

Since F is degenerate elliptic, (1.5), if $u \in C^{2,1}\big(\mathcal{R}^d \times (0,T)\big)$ is a classical subsolution (or a supersolution) then, by calculus, it is a viscosity subsolution (or a supersolution, resp.).

Inequalities (1.8) (or (1.9)) also hold at any maximizer (or minimizer, resp.) which is not necessarily strict.

To clarify the definition consider the mean curvature flow. Then,

$$F(p,A) = trace\left(I - \frac{p \otimes p}{|p|^2}\right)A, \quad \forall p \neq 0.$$

Hence, for $p \neq 0$, $F^\star(p,A) = F_\star(p,A) = F(p,A)$ and for $p = 0$:

$$F^\star(0,A) = \sup_{|\nu|=1} trace(I - \nu \otimes \nu)A, \quad F_\star(0,A) = \inf_{|\nu|=1} trace(I - \nu \otimes \nu)A.$$

Exercise 1.8 Given $R_0 > 0$, let

$$R(t) = \sqrt{R_0^2 - 2(d-1)t}, \quad \forall t \in \left[0, \frac{R_0^2}{2(d-1)}\right].$$

Then, $R(t)$ solves:

$$\frac{d}{dt}R(t) = \frac{d-1}{R(t)}, \quad t \in \left(0, \frac{R_0^2}{2(d-1)}\right).$$

Hence $\Omega(t) = \big\{x \in \mathcal{R}^d : |x| < R(t)\big\}$ is a classical solution of the mean curvature flow with initial data $\Omega_0 = \big\{|x| < R_0\big\}$. Show that

$$u(x,t) = \left\{ \begin{array}{ll} 0, & |x| < R(t) \text{ and } t < R_0^2/2(d-1), \\ 1, & |x| \geq R(t) \text{ or } t \geq R_0^2/2(d-1), \end{array} \right.$$

is a viscosity solution of (1.7) in $\mathcal{R}^d \times (0,\infty)$.

Exercise 1.9 This is a generalization of the previous exercise. Assume that

$$F^\star(x,t,0,\mathcal{O}) \geq 0 \geq F_\star(x,t,0,\mathcal{O}).$$

Suppose that $\big\{\Gamma(t) = \partial\Omega(t)\big\}_{t\in[0,T]}$ is a classical solution of (1.1). Show that the indicator of $\Omega(t)$, $X_{\Omega(t)}(x)$ is a viscosity solution of (1.3) in $\mathcal{R}^d \times (0,T)$.

The following property of viscosity solutions is very powerful. Note that in the following statement (due to Barles and Perthame), we make no assumptions on the convergence of the derivatives of the sequence u_n and we do not use the fact that (1.3) is a geometric equation.

Theorem 1.10 (Stability) *Let $\{u_n\}$ be a sequence of viscosity subsolutions of*

$$u_{n,t} \le F_n(x,t,Du_n,D^2u_n) \text{ in } \mathcal{R}^d \times (0,T).$$

Suppose that for any compact set $K \subset \mathcal{R}^d \times [0,T]$,

$$\sup_n \sup_K |u_n| < \infty,$$

and

$$\limsup_{\substack{(x_n,t_n,p_n,A_n) \to (x,t,p,A) \\ n \to \infty}} F_n(x_n,t_n,p_n,A_n) \le F^\star(x,t,p,A).$$

Then,

$$\bar{u}(x,t) := \limsup_{\substack{(x_n,t_n) \to (x,t) \\ n \to \infty}} u_n^\star(x_n,t_n), \quad (x,t) \in \mathcal{R}^d \times (0,T),$$

is a viscosity subsolution of (1.3) in $\mathcal{R}^d \times (0,T)$. An analogous statement holds for supersolutions.

PROOF. Observe that $\bar{u}^\star = \bar{u}$. Let $\varphi \in C^\infty(\mathcal{R}^d \times [0,T])$ and $(x_0,t_0) \in \mathcal{R}^d \times (0,T)$ be a strict, local maximizer of $\bar{u} - \varphi$. Then, there is a subsequence n_k and a sequence $(x_k,t_k) \to (x_0,t_0)$ such that (x_k,t_k) is a local maximizer of $u_{n_k}^\star - \varphi$. Then, by the viscosity property of u_{n_k},

$$\varphi_t(x_k,t_k) \le F_{n_k}\big(x_k,t_k,D\varphi(x_k,t_k),D^2\varphi(x_k,t_k)\big).$$

Now send $k \to \infty$ and use the assumption on F_n. $\square$

The following property is a consequence of the geometric property (1.4).

Theorem 1.11 (Relabelling) *Let θ be a continuous, non decreasing scalar function of the real line and u be a viscosity subsolution (or a supersolution) of (1.3) in $\mathcal{R}^d \times (0,T)$. Then $\theta(u)$ is a viscosity subsolution (or a supersolution, resp.) of (1.3) in $\mathcal{R}^d \times (0,T)$.*

PROOF. (1). First suppose that θ is twice continuously differentiable and $\theta' > 0$. Let $\varphi \in C^\infty\big(\mathcal{R}^d \times [0,T]\big)$ and $(x_0,t_0) \in \mathcal{R}^d \times (0,T)$ be a strict, local maximizer of $\big(\theta(u)\big)^\star - \varphi$. Note that, by the monotonicity of θ, $\big(\theta(u)\big)^\star = \theta(u^\star)$ and $u^\star - \psi$ has a strict, local maximum at (x_0,t_0), where

$$\psi = \theta^{-1}(\varphi),$$

and θ^{-1} is the inverse of θ. Clearly ψ is smooth and by the viscosity property of u,

$$\psi_t(x_0,t_0) \le F^\star\big(x_0,t_0,D\psi(x_0,t_0),D^2\psi(x_0,t_0)\big).$$

(2). Set $G = \theta^{-1}$ and compute

$$D\psi = G'(\varphi)D\varphi, \quad D^2\psi = G''(\varphi)D\varphi \otimes D\varphi + G'(\varphi)D^2\varphi.$$

Then, by (1.4),

$$F^\star(x_0, t_0, D\psi, D^2\psi) = G'(\varphi)F(x_0, t_0, D\varphi, D^2\varphi).$$

Therefore, (1.8) holds at (x_0, t_0).

(3). Now suppose that θ is continuous and nondecreasing. Let $\xi(r) \in [0, 1]$ be a smooth function with compact support and $\int \xi(r)dr = 1$. For a positive integer n, set

$$\theta_n(r) = \int \theta\left(r - \frac{\rho}{n}\right)\xi(\rho)d\rho + \frac{1}{n}r$$

so that θ_n is smooth and $\theta_n' \geq 1/n$ and, by Step 1, $w_n = \theta_n(u)$ is a viscosity subsolution of (1.3) in $\mathcal{R}^d \times (0, T)$. Since $\theta_n \to \theta$ uniformly on compact sets, by Theorem 1.10, $w = \theta(u)$ is a viscosity subsolution of (1.3) in $\mathcal{R}^d \times (0, T)$.

(4). The supersolution property is proved exactly the same way. $\square$

For two real numbers a, b, set $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

Proposition 1.12 *Let u, v be a viscosity subsolution of (1.3) in $\mathcal{R}^d \times (0, T)$ and $k > 0$ be a constant. Then $u \vee v$, $u \wedge k$ and $u \vee k$ are viscosity subsolutions of (1.3) in $\mathcal{R}^d \times (0, T)$. An analogous statement holds for supersolutions.*

PROOF. Let $\varphi \in C^\infty(\mathcal{R}^d \times [0, T])$ and $(x_0, t_0) \in \mathcal{R}^d \times (0, T)$ be a strict, local maximizer of $(u \vee v)^\star - \varphi$. Suppose that $u^\star(x_0, t_0) \geq v^\star(x_0, t_0)$. Since $(u \vee v)^\star = u^\star \vee v^\star$, (x_0, t_0) is a local maximizer of $u^\star - \varphi$ and the viscosity property of u yields (1.8). If $u^\star(x_0, t_0) \leq v^\star(x_0, t_0)$, then (x_0, t_0) is a local maximizer of $v^\star - \varphi$ and the viscosity property of v yields (1.8).

Since the function $\theta(r) = r \vee k$ and $\widehat{\theta}(r) = r \vee k$ are non decreasing, by Theorem 1.11, $u \wedge k = \theta(u)$ and $u \vee k = \widehat{\theta}(u)$ are viscosity subsolutions of (1.3) in $\mathcal{R}^d \times (0, T)$. $\square$

1.4 Level set solutions: Definition and consistency

In this section we prove that the level set solutions are well defined and agree with classical solutions whenever the latter exist. All of these results crucially depend on a comparison result between the viscosity sub and supersolutions of (1.3). Under some technical assumptions on F, these results were proved by Chen, Giga and Goto [CGG] and by Evans and Spruck [ES] for the mean curvature flow. (Also see Giga, Goto, Ishii and Sato [GGIS].) Here we shall assume that there is a comparison-principle. More precisely, set

$$Q = \mathcal{R}^d \times (0, \infty).$$

Let $UC(\mathcal{R}^d)$ be the set of all uniformly continuous functions on $\mathcal{R}^d$ and $\mathcal{A}$ be the collection of all function satisfying

$$K_w(T) := \sup_{x \in \mathcal{R}^d,\ t \leq T} \frac{|w(x,t)|}{|x| + 1} < \infty, \quad \forall T < \infty.$$

Definition 1.13 We say that *the equation (1.3) has comparison in* Q if for any viscosity subsolution u of (1.3), and a viscosity supersolution v of (1.3) satisfy

$$\sup_{\bar{Q}} (u^\star - v_\star) \leq \sup_{\mathcal{R}^d} \big(u^\star(\cdot, 0) - v_\star(\cdot, 0)\big),$$

provided that u or v is in $\mathcal{A}$.

All the examples of Section 1.1 including the mean curvature flow has comparison in Q: see Giga, Goto, Ishii and Sato [GGIS].

The additional hypothesis that u or v is in $\mathcal{A}$ is necessary due to a counterexample of Ilmanen [Il].

In what follows, we shall assume that

(1) Given $u_0 \in UC(\mathcal{R}^d)$, there exists a unique viscosity solution $u \in \mathcal{A}$ of (1.3) in Q, satisfying the initial condition $u(\cdot, 0) = u_0(\cdot)$.

(2) Equation (1.3) has comparison in Q.

(3) $F(t, x, p, A)$ is smooth on $p \neq 0$.

The following result implies that the level set definition is independent of the choice of the initial data and therefore has the semigroup property. Moreover, by the uniqueness of solutions of (1.3), (1.6), the level set definition is unique.

Theorem 1.14 *Let* $\Gamma_0 = \partial\Omega_0$ *be a closed subset of* $\mathcal{R}^d$, Ω_0 *be an open set and* u_0 *be a uniformly continuous function satisfying*

$$\Gamma_0 = \big\{x : u_0(x) = 0\big\}, \quad \Omega_0 = \big\{x : u_0(x) > 0\big\}.$$

Let $u \in \mathcal{A}$ *be the unique viscosity solution of (1.3) in* Q *satisfying (1.6). Under our standard assumptions, the zero level sets*

$$\Gamma(t) := \big\{x : u(x,t) = 0\big\},$$

and the set

$$\Omega(t) := \big\{x : u(x,t) > 0\big\},$$

are independent of the choice of u_0, *and therefore* $\Gamma(t)$ *is a well defined weak solution of (1.1).*

PROOF. (1). Let $\widetilde{u}_0(x) \in UC(\mathcal{R}^d)$ be an initial data satisfying

$$\Gamma(0) = \big\{x \colon \widetilde{u}_0(x) = 0\big\}, \quad \Omega(0) = \big\{x \colon \widetilde{u}_0(x) > 0\big\}.$$

Let $\widetilde{u} \in \mathcal{A}$ be the viscosity solution of (1.3) in Q with initial data $\widetilde{u}_0$. Set

$$\widetilde{\Gamma}(t) := \big\{x \colon \widetilde{u}(x,t) = 0\big\}, \quad \widetilde{\Omega}(t) := \big\{x \colon \widetilde{u}(x,t) > 0\big\}.$$

We will show that

$$\widetilde{\Gamma}(t) = \Gamma(t), \quad \widetilde{\Omega}(t) = \Omega(t), \quad \forall t \geq 0.$$

(2). Set

$$w(x,t) := \begin{cases} 0, & \text{if } u(x,t) \leq 0 \\ 1, & \text{if } u(x,t) > 0. \end{cases}$$

Then, $w_\star = w$ and

$$w(x,t) = \liminf_{\substack{(x_n,t_n) \to (x,t) \\ n \to \infty}} \theta_n\big(u(x_n,t_n)\big),$$

where $\theta_n(r) = 0$ on $r \leq 0$, $\theta_n(r) = 1$ on $r \geq 1/n$ and on $[0,1/n]$, $\theta_n(r) = nr$. By Theorem 1.10, w is a viscosity supersolution of (1.3) in Q and by Theorem 1.11, $\widetilde{u}(x,t) \wedge 1$ is a viscosity solution of (1.3) in Q. Hence, by comparison,

$$\widetilde{u}(x,t) \wedge 1 - w(x,t) \leq \sup\{\widetilde{u}_0 \wedge 1 - w(\cdot,0)\} = 0,$$

and therefore,

$$\big\{x \colon \widetilde{u}(x,t) > 0\big\} \subset \big\{x \colon w(x,t) = 1\big\} = \big\{x \colon u(x,t) > 0\big\}, \quad \forall t \geq 0.$$

Since the argument is symmetric,

$$\big\{x \colon \widetilde{u}(x,t) > 0\big\} = \big\{x \colon u(x,t) > 0\big\}, \quad \forall t \geq 0.$$

(3). Set

$$z(x,t) := \begin{cases} -1 & \text{if } u(x,t) < 0 \\ 0 & \text{if } u(x,t) \geq 0. \end{cases}$$

Then, $z^\star = z$ and

$$z(x,t) = \limsup_{\substack{(x_n,t_n) \to (x,t) \\ n \to \infty}} \left\{\theta_n\left(u(x,t) + \frac{1}{n}\right) - 1\right\},$$

where θ_n is as in Step 2. Arguing as in the previous step, we conclude that

$$z(x,t) \leq \widetilde{u}(x,t) \vee (-1).$$

Hence

$$\big\{x \colon u(x,t) \geq 0\big\} = \big\{x \colon z(x,t) = 0\big\} \subset \big\{x : \widetilde{u}(x,t) \geq 0\big\}.$$

and, by symmetry,

$$\big\{x \colon u(x,t) \geq 0\big\} = \big\{x \colon \widetilde{u}(x,t) \geq 0\big\}, \quad \forall t \geq 0.$$

$\square$

Our next result shows that the level set definition agrees with the classical solutions whenever the latter exist.

Theorem 1.15 (Consistency) *Let $\Gamma(t) = \partial\Omega(t)$ be the level set solution of (1.1) with (1.2) and $\big\{\widetilde{\Gamma}(t) = \partial\widetilde{\Omega}(t)\big\}_{t\in[0,T]}$ be a family of smooth, compact sets solving (1.1), (1.2). Then*

$$\Gamma(t) = \widetilde{\Gamma}(t), \quad \Omega(t) = \widetilde{\Omega}(t), \quad \forall t \in [0,T].$$

PROOF. Set

$$\begin{aligned} d_0(x) &:= -dist(x, \overline{\Omega}_0) + dist(x, \mathcal{R}^d \setminus \Omega_0), \\ d(x,t) &:= -dist\big(x, \widetilde{\Omega}(t)\big) + dist\big(x, \mathcal{R}^d \setminus \widetilde{\Omega}(t)\big), \quad t \in [0,T], \end{aligned}$$

and, for $\delta > 0$, set

$$I_\delta := \big\{(x,t) \in \mathcal{R}^d \times [0,T] \colon \big|d(x,t)\big| < \delta\big\}.$$

Let $u(x,t)$ be the unique solution of (1.3) in Q with initial data d_0.

(1). Choose $\delta > 0$ so that d is smooth on $\overline{I}_\delta$ and

$$c^\star := \|d\|_{C^2(\bar{I}_g)} < \infty.$$

Since on $\widetilde{\Gamma}(t)$,

$$v = d_t, \quad \vec{n} = -\nabla d,$$

d satisfies

$$d_t = F(x,t,Dd,D^2d) \text{ on } \widetilde{\Gamma}(t), \quad t \in (0,T).$$

Therefore, there is a constant K depending on F and $c^\star$, such that

$$\big|d_t - F(x,t,Dd,D^2d)\big| \le K|d|, \mathrm{on I}_\delta.$$

(2). Set $w(x,t) := e^{Kt}\big((d(x,t) \vee 0) \wedge \delta\big)$. Then,

$$w_t - F(x,t,Dw,D^2w) \ge 0,$$

on $I_\delta \cap \{d > 0\}$ and, by Theorem 1.11, w is a supersolution of (1.3) in Q. Since $\widetilde{\Gamma}(t)$ is compact, $w \in \mathcal{A}$ and by comparison,

$$u(x,t) \wedge \delta \le w(x,t), \quad \forall x \in \mathcal{R}^d,\ t \in [0,T],$$

and therefore,

$$\big\{x : u(x,t) > 0\big\} \subset \big\{x \colon d(x,t) > 0\big\}, \quad \forall t \le T,$$

or equivalently,

$$\Omega(t) \subset \widetilde{\Omega}(t), \quad \forall t \le T.$$

(3). Set $z(x,t) := e^{-Kt}\big((d(x,t)\wedge 0)\vee(-\delta)\big)$ and argue as in Step 2. The result is:

$$z(x,t) \le u(x,t)\vee(-\delta), \quad \forall x\in\mathcal{R}^d, t\in[0,T],$$

and, by Step 2,

$$\widetilde{\Omega}(t) = \Omega(t), \quad \forall t \le T.$$

(4). There is constant $k>0$ such that for $|\varepsilon| \ll \delta$, the set

$$\Gamma_\varepsilon(t) := \big\{x\colon d(x,t) = \varepsilon + k\varepsilon t\big\}, \quad t\in[0,T],$$

is a classical supersolution of (1.1). Let $d_\varepsilon(x,t)$ be the signed distance to $\Gamma_\varepsilon(t)$. As in Step 2, set

$$w_\varepsilon(x,t) := e^{Kt}\big((d_\varepsilon(x,t)\vee 0)\wedge(\delta-\varepsilon)\big).$$

Then, w_ε is a viscosity supersolution of (1.3) in Q. Set

$$m(\varepsilon) := \inf_y \big(d_\varepsilon(y,0) - d_0(y)\big).$$

For $\varepsilon<0$, $m(\varepsilon)>0$. Since $u(x,t)+m(\varepsilon)$ is a viscosity solution of (1.3) on Q,

$$\big(u(x,t)+m(\varepsilon)\big)\wedge(\delta-\varepsilon) \le w_\varepsilon(x,t), \quad \forall x\in\mathcal{R}^d,\ t\le T,$$

and therefore,

$$\big\{x : u(x,t)\ge 0\big\} \subset \big\{x\colon u(x,t) > -m(\varepsilon)\big\} \subset \big\{x\colon w_\varepsilon(x,t)>0\big\} = \big\{x\colon d(x,t) > \varepsilon + k\varepsilon t\big\}.$$

By letting $\epsilon \uparrow 0$,

$$\big\{x\colon u(x,t)\ge 0\big\} \subset \big\{x\colon d(x,t)\ge 0\big\}.$$

(5). A similar argument shows that

$$\big\{x\colon u(x,t)\le 0\big\} \subset \big\{x\colon d(x,t)\le 0\big\}.$$

Combining the previous steps, $\Gamma(t) = \widetilde{\Gamma}(t)$, $\Omega(t) = \widetilde{\Omega}(t)$ for all $t\le T$. □

1.5 Distance solutions

The level set approach to geometric problems provide a unique weak solution with several useful properties. However, in some cases the zero level set may not be hypersurface. Indeed consider the mean curvature flow of the initial data

$$\Gamma_0 = \big\{(x_1,x_2) : |x_1| = |x_2|\big\} \subset \mathcal{R}^2.$$

Then, for all $t>0$, $\Gamma(t) = \big\{x\in\mathcal{R}^2\colon \varphi(x,t) = 0\big\}$ has nonempty interior.

In certain applications, it is convenient to have an intrinsic definition which restricts the solutions to be hypersurfaces. Brakke [Br] gave such a definition, by using the theory geometric measure theory. Here we follow [S1] to give an intrinsic weak solution again using the theory of viscosity solution. As a general rule, all of these intrinsic solutions are generally nonunique, but they are all included in the zero level set of φ.

Definition 1.16 Let $\{\Gamma(t) = \partial\Omega(t)\}_{t\geq 0}$ be a family of compact set. Set

$$d(x,t) := -dist\big(x, \overline{\Omega}(t)\big) + dist\big(x, \mathcal{R}^d \setminus \Omega(t)\big)$$

(1) We say that $\{\Gamma(t)\}_{t\geq 0}$ is a *distance subsolution* of (1.1) if $(d \wedge 0)$ is a viscosity subsolution of the level set equation (1.3).

(2) We say that $\{\Gamma(t)\}_{t\geq 0}$ is a *distance supersolution* of (1.1) if $(d \vee 0)$ is a viscosity supersolution of the level set equation (1.3).

(3) A *distance solution* is both a sub and a supersolution.

Let X_A be the indicator of the set $A \subset \mathcal{R}^d$.

Theorem 1.17 *$\{\Gamma(t) = \partial\Omega(t)\}_{t\in[0,t]}$ is a distance subsolution (or a supersolution) if and only if $X_{\Omega(t)}$ is a viscosity subsolution (or a supersolution, resp.) of (1.3).*

PROOF. (1). Let Γ be a distance subsolution of (1.1). For a positive integer n, let

$$\theta_n(r) := \begin{cases} 0, & r \leq -\frac{1}{n}, \\ 1, & r \geq 0, \\ 1 + nr, & r \in \big[-\frac{1}{n}, 0\big]. \end{cases}$$

Then,

$$w(x,t) := \limsup_{\substack{(x_n,t_n) \to (x,t) \\ n \to \infty}} \theta_n\big(d(x_n,t_n) \wedge 0\big),$$

is a viscosity subsolution of (1.3). Moreover, $w = (X_\Omega)^\star$.

(2). Suppose that (X_Ω) is a viscosity subsolution of (1.3). For $K > 0$, by Theorem 1.11, $K(X_\Omega - 1)$ is a viscosity subsolution of (1.3). Set

$$w_K(x,t) := \sup\big\{K\big(X_\Omega(y,t) - 1\big) - |x-y| \colon y \in \mathcal{R}^d\big\}.$$

Then, it is straightforward to show that w_K is a subsolution and therefore, by Theorem 1.10,

$$w := \lim_{K\uparrow\infty} w_K$$

is also a viscosity subsolution. Since

$$w_K(x,t) = \big(d(x,t) \wedge 0\big) \vee (-K), \quad w = d \wedge 0.$$

(3) Supersolution property is proved similarly. □

Level set solutions and the distance solutions are closely related as seen in the following result.

Theorem 1.18 *Let $u \in \mathcal{A}$ be a viscosity solution of (1.3). Then, the sets*

$$\Gamma_1(t) = \partial L(t), \quad L(t) := \big\{x \colon u(x,t) > 0\big\},$$

and

$$\Gamma_2(t) = \partial U(t), \quad U(t) := \big\{x \colon u(x,t) \geq 0\big\},$$

are distance solutions. Moreover,

$$L(t) \subset \Omega(t), \quad \widetilde{\Omega}(t) \subset U(t), \quad \forall t \geq 0, \tag{1.10}$$

for any distance supersolution $\Gamma(t) = \partial\Omega(t)$ and distance subsolution $\widetilde{\Gamma}(t) = \partial\widetilde{\Omega}(t)$ satisfying (1.10) at $t = 0$.

PROOF. (1). Let θ_n be as in Step 1 of Theorem 1.17. Then

$$(X_L)^\star(x,t) = \limsup_{\substack{(x_n,t_n) \to (x,t) \\ n \to \infty}} \theta_n\left(u(x_n,t_n) - \frac{1}{n}\right),$$

is a viscosity subsolution and therefore Γ_1 is a distance subsolution. Moreover,

$$(X_L)_\star(x,t) = \liminf_{\substack{(x_n,t_n) \to (x,t) \\ n \to \infty}} \theta_n\left(u(x_n,t_n) - \frac{1}{n}\right),$$

is a viscosity supersolution and therefore Γ_1 is a distance solution of (1.1).

(2). Also Γ_2 is a distance solution of (1.1) because:

$$\begin{aligned} (X_U)^\star(x,t) &= \limsup_{\substack{(x_n,t_n) \to (x,t) \\ n \to \infty}} \theta_n\big(u(x_n,t_n)\big), \\ (X_U)_\star(x,t) &= \liminf_{\substack{(x_n,t_n) \to (x,t) \\ n \to \infty}} \theta_n\big(u(x_n,t_n)\big). \end{aligned}$$

(3). Let $\bar{u}$ be the viscosity solution of (1.3) with initial data

$$\bar{u}(x,0) = -dist\big(x, \{u(\cdot,0) \geq 0\}\big) + dist\big(x, \{u(\cdot,0) < 0\}\big).$$

Then,

$$U(t) = \{\bar{u} \geq 0\}, \quad L(t) = \{\bar{u} > 0\}.$$

Let d be the signed distance to $\Omega(\cdot)$ and $\widetilde{d}$ be the signed distance to $\widetilde{\Omega}(\cdot)$. Since (1.10) satisfied at $t = 0$,

$$\widetilde{d}(x,0) \wedge 0 \leq \bar{u}(x,0) \leq d(x,0) \vee 0.$$

Then, by comparison, $\widetilde{d} \wedge 0 \leq \bar{u} \leq d \vee 0$ and (1.10) follows. □

Remark 1.19 If (1.4) holds for all $\lambda \in \mathcal{R}^1$, instead of $\lambda > 0$, then (1.1) is orientation independent. Mean curvature flow is an example of such equation. In this case, we can extend the notion of distance solution to set $\widehat{\Gamma}(t)$ that are not necessarily the boundary of an open set. We say that $\{\widehat{\Gamma}(t)\}_{t \geq 0}$ is an *unoriented distance solution* if $d(x,t) := dist\big(x, \widehat{\Gamma}(t)\big)$ a viscosity supersolution of (1.3).

Lemma 1.20 *Suppose that (1.4) holds for all $\lambda \in R$ and $\{\widehat{\Gamma}(t)\}$ is an unoriented distance solution. Let $u \in \mathcal{A}$ be the unique viscosity solution of (1.3) with initial data $u(x,0) = dist\big(x, \widehat{\Gamma}(0)\big)$. Then,*

$$\widehat{\Gamma}(t) \subset \{x \in \mathcal{R}^d : u(x,t) = 0\}, \quad \forall t \geq 0. \tag{1.11}$$

PROOF. Set $d(x,t) := dist\big(x, \widehat{\Gamma}(t)\big)$. By comparison, $d \geq u$ and (1.11) follows. □

2 Ginzburg-Landau Approximation

2.1 Introduction

For $0 < \varepsilon$, let

$$I^{\varepsilon}(\varphi) = \int_{\mathcal{R}^d} \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon^2}W(\varphi)\,dx$$

where $W(\varphi)$ is a bistable potential with zeroes at ± 1. The typical example is

$$W(\varphi) = \frac{1}{2}(1-\varphi^2)^2. \tag{2.1}$$

Then, it is known that, as $\varepsilon \downarrow 0$, the Γ-limit of I^{ε} is the surface area functional [M]. Since the mean curvature flow is the gradient flow of the area functional, formally we expect the gradient flow of I^{ε} to approximate the mean curvature flow. The gradient flow of I^{ε} is

$$u_t^{\varepsilon} - \Delta u^{\varepsilon} + \frac{1}{\varepsilon^2}W'(u^{\varepsilon}) = 0 \text{ in } \mathcal{R}^d \times (0,\infty). \tag{2.2}$$

(Allen and Cahn [AC] derived this equation from different considerations.)

In this section, we will prove that (2.2) approximates the mean curvature flow. Such approximations are also available for more general geometric flows: see for instance Barles, Soner, Souganidis [BSS].

First, we consider the case $d = 1$ and look for a stationary solution $u^{\varepsilon}(x,t) = q(x/\varepsilon)$. This yields the ordinary differential equation

$$q'' = W'(q),$$

with boundary data $q(\pm\infty) = \pm 1$, and $q(0) = 0$. The unique solution q satisfyies

$$q'(r) = \sqrt{2W(q)} := h(q) > 0.$$

When W is as in (2.1), $q(r) = \tanh(r)$. Note that

$$\begin{aligned}
I^{\varepsilon}\left(q\left(\frac{\cdot}{\varepsilon}\right)\right) &= \int_{-\infty}^{\infty} \frac{1}{\varepsilon^2}\left(\frac{1}{2}\left(q'\left(\frac{x}{\varepsilon}\right)\right)^2 + W\left(q\left(\frac{x}{\varepsilon}\right)\right)\right) dx \\
&= \frac{1}{\varepsilon}\int_{-\infty}^{\infty} 2W\big(q(y)\big)\,dy \\
&= \frac{1}{\varepsilon}\int_{-\infty}^{\infty} h\big(q(y)\big)q'(y)dy \\
&= \frac{H(1) - H(-1)}{\varepsilon},
\end{aligned}$$

where $H' = h$.

We now use the stationary wave q to analyze the asymptotic behavior of u^ε. Since $q' > 0$, its inverse q^{-1} exists and we define

$$Z^\varepsilon := \varepsilon q^{-1}(u^\varepsilon) \Rightarrow u^\varepsilon = q\left(\frac{Z^\varepsilon}{\varepsilon}\right).$$

Then,

$$Z^\varepsilon_t - \Delta Z^\varepsilon + \frac{2q''}{\varepsilon q'}\big(1 - |\nabla Z^\varepsilon|^2\big) = 0. \tag{2.3}$$

Formally, we conclude that, as $\varepsilon \downarrow 0$, $|\nabla Z^\varepsilon| \sim 1$. Therefore, Z^ε is approximately equal to the signed distance function, d, of a front $\Gamma(t)$. Moreover, when $Z^\varepsilon = 0$, $q'' = 0$ and $Z^\varepsilon_t = \Delta Z^\varepsilon$ and

$$d_t = \Delta d \quad \text{on } \{\ d = 0\ \}.$$

Hence, formally, the front $\Gamma(t)$ evolves according to its mean curvature.

For future reference we record that, by maximum principle,

$$\big|\nabla Z^\varepsilon(t,x)\big|^2 \le 1, \quad \forall (x,t), \tag{2.4}$$

provided (2.4) is satisfied by the initial data.

The main goal of the following three sections is to prove the following result. Set

$$d\mu^\varepsilon_t(x) := \varepsilon E^\varepsilon(t,x)\,dx, \quad E^\varepsilon(t,x) = \frac{1}{2}|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon^2}W(u^\varepsilon).$$

We assume that

$$C_0 := \sup_{\varepsilon>0} \mu^\varepsilon_0(\mathcal{R}^d) < \infty, \tag{2.5}$$

and, as $\varepsilon \downarrow 0$, μ^ε_0 converges to μ_0 in the weak* topology of Radon measures.

Theorem 2.1 *There are a subsequence $\varepsilon_n \downarrow 0$ and a Radon measure μ_t such that*

$$\mu^{\varepsilon_n}_t \overset{\star}{\rightharpoonup} \mu_t$$

in the weak* toplogy of Radon measures, and $\Gamma(t) = spt(\mu_t)$ is a distance solution of the mean curvature flow, i.e., $dist\big(x,\Gamma(t)\big)$ is a viscosity supersolution of (1.7). Off $\Gamma(t)$, u^{ε_n} converges to ± 1 locally uniformly.

Moreover, if $H^{d-1}\big(\Gamma(0)\big) < \infty$, then $H^{d-1}\big(\Gamma(t)\big) < \infty$ for all $t \ge 0$, where H^{d-1} is the $d-1$ dimensional Hausdorff measure.

Remark 2.2 Under the assumption $H^{d-1}\big(\Gamma(0)\big) < \infty$, $\Gamma(t)$ is a Brakke solution: see Ilmanen [I].

2.2 Energy estimates

Following the presentation in Jerrard and Soner [JS], we calculate that

$$E_t^\varepsilon = -|u_t^\varepsilon|^2 + div(\nabla u^\varepsilon u_t^\varepsilon), \tag{2.6}$$

$$\nabla E^\varepsilon = -\nabla u^\varepsilon u_t^\varepsilon + div(\nabla u^\varepsilon \otimes \nabla u^\varepsilon). \tag{2.7}$$

Let $\eta \geq 0$ be a smooth compactly supported function. Multiply (2.6) by η and (2.7) by $\nabla\eta$ and substract the two identities. Then use the resulting identity to compute the time derivative of the integral of ηE^ε. The result is:

$$\frac{d}{dt}\int \eta E^\varepsilon = \int (\eta_t - \Delta\eta)E^\varepsilon + D^2\eta\nabla u^\varepsilon \cdot \nabla u^\varepsilon - \int \eta |u_t^\varepsilon|^2. \tag{2.8}$$

If we add the two identities and then proceed similarly, we obtain the following identity:

$$\begin{aligned} \frac{d}{dt}\int \eta E^\varepsilon &= \int (\eta_t + \Delta\eta)E^\varepsilon - D^2\eta\nabla u^\varepsilon \cdot \nabla u^\varepsilon \\ &\quad + \frac{|\nabla\eta \cdot \nabla u^\varepsilon|^2}{\eta} - \int \eta \left| u_t^\varepsilon - \frac{\nabla\eta\cdot\nabla u^\varepsilon}{\eta}\right|^2. \end{aligned} \tag{2.9}$$

The special case of (2.8) with $\eta \equiv 1$ yields the classical energy estimate,

$$\int E^\varepsilon(x,t)\,dx + \int_0^t \int |u_t^\varepsilon|^2\,dx\,ds = \int E^\varepsilon(x,0)\,dx. \tag{2.10}$$

Suppose that $\{\Gamma(t)\}_{t\in[t_0,t_1]}$ is a classical solution of the mean curvature flow. Then, there are $\delta > 0$, $C > 0$ and a smooth function $\eta : \mathcal{R}^d \times [t_0,t_1] \to [0,\infty]$ satisfying

$$\begin{aligned} \eta(x,t) &= \big(dist(x,\Gamma(t)0)\big)^2/2, & &\forall dist\big(x,\Gamma(t)\big) < \delta, \\ \eta(x,t) &\geq \delta^2/2, & &\forall dist\big(x,\Gamma(t)\big) \geq \delta, \\ \eta(x,t) &= \delta^2, & &\forall dist\big(x,\Gamma(t)\big) \geq 2\delta, \\ \|\eta\|_{C^2} &\leq C. \end{aligned}$$

Set $\mathcal{O} = \{(x,t) \in \mathcal{R}^d \times [t_0,t_1] : dist\big(x,\Gamma(t)\big) < \delta\}$. Then, on $\mathcal{O}$,

$$D^2\eta(x,t)\xi\cdot\xi \leq |\xi|^2, \quad \forall \xi \in \mathcal{R}^d.$$

Let $d(x,t)$ be the signed distance of x to $\Gamma(t)$. Then, on $\mathcal{O}$,

$$\eta_t - \Delta\eta = d[d_t - \Delta d] - |\nabla d|^2.$$

Since $d_t - \Delta d = 0$ on $\Gamma(t)$, $|d_t - \Delta d| \leq C|d|$ on $\mathcal{O}$ and therefore, on $\mathcal{O}$,

$$|\eta_t - \Delta\eta + 1| = |d|\,|d_t - \Delta d| \leq C|d|^2 = 2C\eta.$$

Hence, on $\mathcal{O}$,

$$\begin{aligned}(\eta_t - \Delta\eta)E^\varepsilon + D^2\eta\nabla u^\varepsilon \cdot \nabla u^\varepsilon &\leq -E^\varepsilon + 2C\eta E^\varepsilon + |\nabla u^\varepsilon|^2 \\ &= 2C\eta E^\varepsilon + \frac{1}{2}|\nabla u^\varepsilon|^2 - \frac{1}{\varepsilon^2}W(u^\varepsilon).\end{aligned}$$

Since $u^\varepsilon = q(Z^\varepsilon/\varepsilon)$ and $|\nabla Z^\varepsilon|^2 \leq 1$,

$$\frac{1}{2}|\nabla u^\varepsilon|^2 - \frac{1}{\varepsilon^2}W(u^\varepsilon) = \frac{1}{2\varepsilon^2}\left(q'\left(\frac{Z^\varepsilon}{2}\right)\right)^2 \left(|\nabla Z^\varepsilon|^2 - 1\right) \leq 0.$$

Therefore,

$$(\eta_t - \Delta\eta)E^\varepsilon + D^2\eta\nabla u^\varepsilon \cdot \nabla u^\varepsilon \leq 2CE^\varepsilon\eta, \text{ on } \mathcal{O}. \tag{2.11}$$

On the complement of $\mathcal{O}$, η is smooth and positive. Hence, on $\mathcal{O}^c$,

$$(\eta_t - \Delta\eta)E^\varepsilon + D^2\eta\nabla u^\varepsilon \cdot \nabla u^\varepsilon \leq CE^\varepsilon \leq \frac{2C}{\delta^2}E^\varepsilon\eta,$$

and therefore, (2.11) holds, possibly with a larger C, on all of $\mathcal{R}^d \times [t_0, t_1]$. Substitute this into (2.8):

$$\frac{d}{dt}\int \eta E^\varepsilon \leq C\int \eta E^\varepsilon, \quad \forall t \in [t_0, t_1],$$

and integrate,

$$\int \eta(x,t)E^\varepsilon(x,t)\,dx \leq e^{C(t-t_0)}\int \eta(x,t_0)E^\varepsilon(x,t_0)\,dx, \quad \forall \in [t_0, t_1]. \tag{2.12}$$

The above estimate will be used in the next section.

Fix, $x_0 \in \mathcal{R}^d$, $t_0 > 0$ and use (2.9) with

$$\eta(x,t) = \rho(x,t) := \sqrt{4\pi(t_0 - t)}G(x - x_0, t_0 - t),$$

where G is the heat kernel, i.e.,

$$G(y,t) := (4\pi t)^{-d/2}\exp\left(-\frac{|y|^2}{4t}\right), \quad t > 0, \ y \in \mathcal{R}^d.$$

We compute:

$$\begin{aligned}\rho_t + \Delta\rho &= -\frac{1}{2(t_0 - t)}\rho, \\ D^2\rho\xi \cdot \xi &= -\frac{|\xi|^2}{2(t_0 - t)}\rho + \frac{(\xi - (x - x_0))^2}{4(t_0 - t)^2}\rho, \quad \forall \xi \in \mathcal{R}^d, \\ \frac{|\nabla\rho \cdot \xi|^2}{\rho} &= \frac{(\xi \cdot (x - x_0))^2}{4(t_0 - t)^2}\rho, \quad \forall \xi \in \mathcal{R}^d.\end{aligned}$$

Hence,

$$\frac{d}{dt}\int \rho E^\varepsilon \leq \frac{1}{2(t_0 - t)}\int \rho\left(|\nabla u^\varepsilon|^2 - E^\varepsilon\right) \leq 0, \quad \forall t < t_0. \tag{2.13}$$

This inequality is known as the *monotonicity formula*. For systems of equations, it was first used by Struwe [St] and later by Chen-Struwe [CS]. For (2.2), it was derived by Ilmanen [I] following the computation of Huisken. The following *clearing-out* Lemma is a powerful one in the analysis of (2.2) and the mean curvature flow. Set

$$\alpha^\varepsilon(t;x_0,t_0) = \int \rho(x,t)\, d\mu_t^\varepsilon(x), \quad t < t_0.$$

Lemma 2.3 (Clearing-out) *Suppose that* $\varepsilon\|\nabla u^\varepsilon\|_\infty \le k_1$. *Then there is a constant*
$C = C(k_1)$ *satisfying*

$$\begin{aligned} W\big(u^\varepsilon(x_0,t_0-\varepsilon^2)\big) &\le C(k_1)\big(\alpha^\varepsilon(t_0-\varepsilon^2;x_0,t_0)\big)^{1/d+1}, \\ &\le C(k_1)\big(\alpha^\varepsilon(t;x_0,t_0)\big)^{1/d+1}, \quad \forall t \le t_0-\varepsilon^2. \end{aligned}$$

PROOF. Set

$$w_0 := W\big(u^\varepsilon(x_0,t_0-\varepsilon^2)\big)$$

so that

$$W\big(u^\varepsilon(x,t_0-\varepsilon^2)\big) \ge w_0 - \|\nabla W\|_\infty\, \|\nabla u^\varepsilon\|_\infty\, |x-x_0| \ge \frac{w_0}{2},$$

for all $|x-x_0| \le \varepsilon K w_0$, with a constant $K \ge 1$, depending on k_1, and W. Since $\rho(x,t) = \widehat{\rho}\big(|x-x_0|, t_0-t\big)$ and $\widehat{\rho}$ is decreasing in $|x-x_0|$,

$$\begin{aligned} \alpha^\varepsilon(t_0-\varepsilon^2,x_0,t_0) &\ge \int_{B_{\varepsilon K w_0}(x_0)} \rho(x,t_0-\varepsilon^2)\, \frac{W(u^\varepsilon(x,t_0-\varepsilon^2))}{\varepsilon^2}\, dx, \\ &\ge (\varepsilon K w_0)^d \widehat{\rho}(\varepsilon K w_0, \varepsilon^2)\, \frac{w_0}{2\varepsilon^2}, \\ &= \widehat{C} w_0^{d+1} e^{-\frac{K^2 w_0^2}{4}} \ge C w_0^{d+1}. \end{aligned}$$

□

2.3 Convergence to a smooth flow

In this section, we prove Theorem 2.1 when there is a smooth solution of the mean curvature flow. Let μ_t^ε and Γ_0 be as in Theorem 2.1. Suppose that $\mu_0 = C^\star H^{d-1}\lfloor\,\Gamma(0)$ for some $C^\star > 0$ and a smooth, compact hypersurface $\Gamma(0) = \partial\Omega(0)$. (Here $H^{d-1}\lfloor\,\Gamma(0)$ is the surface measure of $\Gamma(0)$). Let $\big\{\Gamma(t) = \partial\Omega(t)\big\}_{t\in[0,T]}$ be the local, classical solution of the mean curvature flow with initial data $\Gamma(0)$.

Theorem 2.4 *As* $\varepsilon \downarrow 0$, u^ε *converges to* ± 1, *locally uniformly away from* $\Gamma(t)$.

PROOF. (1). By (2.12) with $t_0 = 0$ and $\{\Gamma(t)\}_{t\in[0,T]}$,

$$\int \eta(x,t)\, d\mu_t^\varepsilon(x) \leq C \int \eta(x,0)\, d\mu_0^\varepsilon(x), \quad \forall t \in [0,T].$$

Since $\mu_0^\varepsilon \stackrel{\star}{\rightharpoonup} H^{d-1} \lfloor \Gamma(0)$ and $\eta(x,0) = 0$ on $\Gamma(0)$,

$$\lim_{\varepsilon\downarrow 0} \int \eta(x,t)\, d\mu_t^\varepsilon(x) = 0, \quad \forall t \in [0,T].$$

Moreover, by the standard energy estimate and (2.5),

$$\mu_t^\varepsilon(\mathcal{R}^d) \leq \mu_0^\varepsilon(\mathcal{R}^d) \leq C_0, \quad \forall t \geq 0.$$

(2). Fix $t_0 \in (0,T]$, $x_0 \notin \Gamma(t_0)$ and set $R = (dist(x_0, \Gamma(t_0)))/2$. For $\varepsilon > 0$ and $t < t_0$,

$$\begin{aligned}
\alpha^\varepsilon(t; x_0, t_0) &= \int_{B_R(x_0)} \rho(x,t)\, d\mu_t^\varepsilon(x) + \int_{\mathcal{R}^d \setminus B_R(x_0)} \rho(x,t)\, d\mu_t^\varepsilon(x) \\
&\leq \sup_{y\in B_R(x_0)} \frac{\rho(y,t)}{\eta(y,t)} \int \eta(x,t)\, d\mu_t^\varepsilon(x) \\
&\quad + \sup_{y\notin B_R(x_0)} \rho(y,t) \mu_t^\varepsilon(\mathcal{R}^d) \\
&\leq \frac{C}{R^2(t_0-t)^{(d-1)/2}} \int \eta(x,t)\, d\mu_t^\varepsilon(x) + \widehat{\rho}(R, t_0 - t) C_0,
\end{aligned}$$

where, as before, $\rho(x,t) = \widehat{\rho}(|x - x_0|, t_0 - t)$. By Step 1 and the definition of $\widehat{\rho}$, for any $\gamma > 0$,

$$\lim_{\varepsilon\downarrow 0} \sup\{\alpha^\varepsilon(t_0 - \delta; x_0, t_0) \colon t \in [\delta, T], dist(x, \Gamma(t)) \geq \gamma\} \leq C_0 \widehat{\rho}(\gamma, \delta).$$

Hence, for any $\gamma > 0$,

$$\lim_{\delta\downarrow 0} \lim_{\varepsilon\downarrow 0} \sup\{\alpha^\varepsilon(t_0 - \delta; x_0, t_0) \colon t \in [\delta, T], dist(x, \Gamma(t)) \geq \gamma\} = 0.$$

By the clearing-out Lemma, for any $\gamma > 0$,

$$\limsup_{\varepsilon\downarrow 0} \sup\{1 - |u^\varepsilon(x,t)| \colon t \in [\gamma, T], dist(x, \Gamma(t)) \geq \gamma\} = 0.$$

Hence, $|u^\varepsilon| \to 1$ locally uniformly away from $\Gamma(t)$. Since u^ε is continuous, $u^\varepsilon \to \pm 1$ away from $\Gamma(t)$. □

The previous theorem and the comparison between solutions of (2.2) yield the following more general result.

Lemma 2.5 *Let* $\{\Gamma(t) = \partial\Omega(t)\}_{t\in[t_0,t_1]}$ *be a classical solution of the mean curvature flow. Suppose that*

$$\lim_{\varepsilon\downarrow 0}\int_{\mathcal{R}^d\setminus\Omega(t_0)} d\mu^\varepsilon(t_0,x) = 0.$$

Then,

$$\lim_{\varepsilon\downarrow 0}\int_{\mathcal{R}^d\setminus\Omega(t)} d\mu^\varepsilon(t,x) = 0, \quad \forall t\in[t_0,t_1].$$

PROOF. For $t\in[t_0,t_1]$, let $\eta(t,x)$ be as in the previous subsection. Set

$$\overline{\eta}(x,t) := \eta(x,t)\big(1 - X_{\Omega(t)}(x)\big),$$

$$I(\overline{\eta}) := (\overline{\eta}_t - \Delta\overline{\eta})E^\varepsilon + D^2\overline{\eta}\nabla u^\varepsilon\cdot\nabla u^\varepsilon,$$

so that $I(\overline{\eta}) = 0$ on $\Omega(t)$ and $I(\overline{\eta}) = I(\eta)$ on the complement of $\Omega(t)$. Therefore (2.12) holds with $\overline{\eta}$, i.e.,

$$\int \overline{\eta}(x,t)\, d\mu_t^\varepsilon(x) \le e^{C(t-t_0)}\int \overline{\eta}(x,t_0)\, d\mu_{t_0}^\varepsilon(x), \qquad t\in[t_0,t_1].$$

By the hypothesis of the lemma, for all $t\in[t_0,t_1]$,

$$\lim_{\varepsilon\downarrow 0}\int_{\mathcal{R}^d\setminus\Omega(t)} d\mu^\varepsilon(t,x) \le \lim\int \overline{\eta}(x,t_0)\, d\mu_{t_0}^\varepsilon(x) = 0.$$

□

An immediate corollary to Lemma 2.5 is the following.

Corrollary 2.6 *Suppose that*

$$\lim_{\varepsilon\downarrow 0}\mu_{t_0}^\varepsilon\big(\{|x|\le R_0\}\big) = 0.$$

Then,

$$\lim_{\varepsilon\downarrow 0}\mu_t^\varepsilon\big(\{|x|\le R(t)\}\big) = 0, \quad \forall t\ge 0,$$

where $R(t)$ *is the solution of* $R(t_0) = R_0$,

$$\frac{d}{dt}R(t) = -\frac{d-1}{R(t)}, \quad t > t_0.$$

2.4 Proof of Theorem 2.1

We start with a result first proved by Brakke.

Lemma 2.7 *There are $\varepsilon_n \downarrow 0$ and Radon measures μ_t such that $\mu_t^{\varepsilon_n}$ converges to μ_t, in the weak* topology, for all $t \geq 0$. Moreover, for any function $\xi(\cdot) \geq 0$, and $t_0 > 0$*

$$\lim_{t \uparrow t_0} \int \xi(x)\, d\mu_t(x) \geq \int \xi(x)\, d\mu_{t_0}(x) \geq \lim_{s \downarrow t_0} \int \xi(x)\, d\mu_s(x). \tag{2.14}$$

PROOF. (1). In view of (2.5), for each $t \geq 0$, there are subsequence along which μ_t^ε is convergent. Let $Q \subset [0, \infty)$ be dense set. By a Cantor diagonal argument, we construct $\varepsilon_n \downarrow 0$ and Radon measures μ_t so that

$$\mu_t^{\varepsilon_n} \stackrel{\star}{\rightharpoonup} \mu_t, \quad \forall t \in Q.$$

(2). Let $\big\{\varphi_m(x)\big\}$ be a dense subset of $\mathcal{D}(\mathcal{R}^d)$. Then, by (2.8) with $\eta = \varphi_m$,

$$\frac{d}{dt} \int \varphi_m(x)\, d\mu_t^\varepsilon(x) \leq \hat{k}_m \mu_t^\varepsilon(\mathcal{R}^d) \leq k_m,$$

where k_m is a constant depending on φ_m but not on ε. Hence, the function

$$f_{m,\varepsilon}(t) := \int \varphi_m(x)\, d\mu_t^\varepsilon(x) - k_m t, \quad t \geq 0,$$

is non increasing. Moreover,

$$f_m(t) := \lim_{n \to \infty} f_{m,\varepsilon_n}(t),$$

exists for all $t \in Q$ and $m = 1, 2, \ldots$. For $t \geq 0$, define

$$f_m(t) := \lim_{s \uparrow t} f_m(s).$$

Let $\widehat{Q} \subset [0, \infty)$ be the set of discontinuities of $\{f_m\}$. Clearly $\widehat{Q}$ is countable and

$$f_m(t) = \lim_{m \to \infty} f_{m,\varepsilon_n}(t), \quad \forall t \notin \widehat{Q} \setminus Q.$$

(3). By redefining ε_n, if necessary, we may assure that $Q \supset \widehat{Q}$. Then $f_{m,\varepsilon_n}(t)$ converges to $f_m(t)$ for all $t \geq 0$ and $m = 1, 2, \ldots$. Since $f_m(t)$ depends linearly on φ_m, there are Radon measures μ_t, such that

$$f_m(t) = \int \varphi_m\, d\mu_t - k_m t, \quad \forall t \geq 0, m = 1, 2, \ldots.$$

Since $\{\varphi_m\}$ is dense, $\mu_t^{\varepsilon_n} \stackrel{\star}{\rightharpoonup} \mu_t$ for all $t > 0$. □

By the monotonicity, proved in Step 2, $d\mu := d\mu_t\, dt$ is a Radon measure. Set

$$\Gamma(t) = spt\mu_t, \quad \Gamma = spt\mu.$$

Lemma 2.8

$$\Gamma = \overline{\bigcup_{t>0} \Gamma(t) \times \{t\}}.$$

Moreover, u^{ε_n} converges to ± 1, locally uniformly on the complement of Γ.

PROOF. (1). Let

$$C := \bigcup_{t>0} \Gamma(t) \times \{t\}.$$

The inclusion $\Gamma \subset \bar{C}$ is immediate. To prove the reverse inclusion, suppose that $t_0 > 0$ and $(x_0, t_0) \notin \Gamma$. Then, there are $\delta \in (0, t_0)$ and a smooth nonnegative function ξ, with compact support such that $\xi(x_0) > 0$ and

$$\int_{t_0-\delta}^{t_o+\delta} \int_{\mathcal{R}^d} \xi(x)\, d\mu_t(x)\, dt = 0.$$

Hence,

$$\int_{\mathcal{R}^d} \xi(x)\, d\mu_t(x) = 0,$$

for almost every $t \in (t_0 - \delta, t_0 + \delta)$. However, by (2.14), this holds for every $t \in (t_0 - \delta, t_0 + \delta)$. In particular, $x_0 \notin C$. Hence, $C \subset \Gamma \subset \bar{C}$. Since Γ is closed, $\Gamma = \bar{C}$.

(2). Let $(x_n, t_n) \to (x_0, t_0) \notin \Gamma$, as $n \to \infty$. Then,

$$\lim_{n\to\infty} \alpha^{\varepsilon_n}(t_n; x_n, t_n + \varepsilon_n^2) = 0,$$

and, by the clearing out lemma, $\left|u^{\varepsilon_n}(x_n, t_n)\right| \to 1$. Since u^{ε_n} is continuous, this proves the local uniform convergence of u^{ε_n} on the complement of Γ. □

We are now in a position to prove Theorem 2.1.

PROOF OF THEOREM 2.1. Set

$$\delta(x,t) := dist\big(x, \Gamma(t)\big), \quad d(x,t) := dist(x, \Gamma_t),$$

where Γ_t is the t-cross section of Γ. In view of Lemma 2.8, $\delta_\star = d$.

(1). We will first show that δ, or equivalently d, is a viscosity supersolution of (1.7) on $\{d > 0\}$. Suppose to the contrary. Then, there are $\varphi \in C^\infty(\bar{Q})$, a strict minimizer $(x_0, t_0) \in Q$ of $\delta_\star - \varphi = d - \varphi$ with $d(x_0, t_0) > 0$ such that

$$\beta := -\big[\varphi_t(x_0,t_0) - F_\star\big(D\varphi(x_0,t_0), D^2\varphi(x_0,t_0)\big)\big] > 0.$$

Since $d-\varphi$ has a minimum at (x_0,t_0), we may assume that $d(x_0,t_0)<\infty$.

In the next three steps, we will obtain a contradiction.

(2). Since any distance function is semiconcave on its positive set, and since $d-\varphi$ attains its minimum at (x_0,t_0) with $d(x_0,t_0)>0$, d is differentiable at (x_0,t_0), and therefore, $\big|\nabla\varphi(x_0,t_0)\big| = \big|\nabla d(x_0,t_0)\big| = 1$.

Choose $y_0\in\mathcal{R}^d$ so that

$$d(x_0,t_0) = |x_0-y_0|, \quad (y_0,t_0)\in\Gamma.$$

Set $\psi(x,t) := \varphi(x+x_0-y_0,t)$ and

$$\Omega(t) := \big\{x\in\mathcal{R}^d\colon \psi(x,t)<\varphi(x_0,t_0)\big\}.$$

Then, $y_0\in\partial\Omega(t_0)$.

For $0<r<t_0$, set $Q_r = B_r(y_0)\times[t_0-r,t_0+r]$ and choose $r^\star>0$ satisfying

$$\psi_t - F_\star(D\psi, D^2\psi) \le -\frac{\beta}{2},$$

$$\frac{1}{2}\le |D\psi|\le 2,$$

on $Q_{2r^\star}$. Then, $\partial\Omega(t)\cap B_{2r^\star}(x_0)\in C^\infty$ for all $|t-t_0|\le 2r^\star$.

(3). Note that

$$\Gamma_t\subset\overline{\Omega(t)}, \quad \forall t\ge 0,$$

and

$$\Gamma_t\cap\partial\Omega(t)=\phi, \quad \forall t\ne t_0, \qquad \Gamma_{t_0}\cap\partial\Omega(t_0)=\{y_0\}.$$

Set

$$\alpha_0 := \frac{\inf\{(\delta-\psi)(x,t)\colon (x,t)\notin Q_r\}}{2(1+\|\nabla\psi\|_\infty)}\wedge r^\star.$$

Then, a straightforward analysis yields

$$\big\{(x,t)\colon x\in\Gamma_t, dist\big(x,\partial\Omega(t)\big)\le\alpha_0\big\}\subset Q_{r^\star}.$$

(4). Let $\hat{d}(x,t)$ be the signed distance to $\partial\Omega(t)$, and let $H:\mathcal{R}^1\to\mathcal{R}^1$ be a smooth function satisfying :

$$H(r)=0,\ r\ge\alpha_0, \qquad H(r)=(r-\alpha_0)^2/2,\ r\in[-r^\star,\alpha_0], \qquad H'(r)=0,\ r\le -2r^\star.$$

Set $\widehat{\eta}(x,t) := H(\hat{d}(x,t))$ so that, by Step 3,

$$U_t := \Gamma_t\cap spt\big\{\widehat{\eta}_t + |D^2\widehat{\eta}|\big\}\subset B_{r^\star}(y_0)\cap\Omega(t), \quad \forall |t-t_0|\le 2r^\star,$$

and therefore, $\hat{d}$ is smooth on U_t. Then, for $|t-t_0|<2r^\star$,

$$\frac{d}{dt}\int\widehat{\eta}\,d\mu_t^\varepsilon \le \int(\widehat{\eta}_t-\Delta\widehat{\eta})\,d\mu_t^\varepsilon + \varepsilon\int D^2\widehat{\eta}\nabla u^\varepsilon\cdot\nabla u^\varepsilon := J_\varepsilon(t),$$

and

$$\liminf J_\varepsilon(t) = \liminf \int_{B_{r^\star}(y_0)\cap\Omega(t)} (\widehat{\eta}_t - \Delta\widehat{\eta})\, d\mu_t^\varepsilon + \varepsilon D^2\widehat{\eta}\nabla u^\varepsilon \cdot \nabla u^\varepsilon dx.$$

Fix $t \in [t_0 - r^\star, t_0 + r^\star]$ and $x \in U_t$. Then, $\hat{d}(x,t) \in [0, \alpha_0]$ and

$$\begin{aligned}\widehat{\eta}_t - \Delta\widehat{\eta} &= H'(\hat{d})[\hat{d}_t - \Delta\hat{d}] - H''(\hat{d})|\nabla\hat{d}|^2 \\ &= (\hat{d} - \alpha_0)[\hat{d}_t - \Delta\hat{d}] - 1.\end{aligned}$$

Let $y \in \partial\widehat{\Omega}(t)$ be such that $-|x-y| = \hat{d}(x,t)$. Then, $y \in B_{2r^\star}(y_0)$ and

$$\hat{d}_t(x,t) - \Delta\hat{d}(x,t) \le \hat{d}_t(y,t) - \Delta\hat{d}(y,t) = \frac{\psi_t(y,t) - F(D\psi, D^2\psi)}{|D\psi(y,t)|} < 0.$$

On U_t, $D^2\eta \le I$, and therefore,

$$(\widehat{\eta}_t - \Delta\widehat{\eta})d\mu_t^\varepsilon(x) + \varepsilon D^2\widehat{\eta}\nabla u^\varepsilon \cdot \nabla u^\varepsilon dx \le 0,$$

for all $|t - t_0| < r^\star$. Hence,

$$\frac{d}{dt}\int \eta\, d\mu_t \le \liminf J_\varepsilon(t) \le 0, \quad \forall |t - t_0| < r^\star.$$

Moreover, since $U_{t_0 - r^\star} = \emptyset$,

$$\int \eta(x, t_0 - r^\star)\, d\mu_{t_0 - r^\star}(x) = 0,$$

and consequently,

$$\int \eta(x, t_0)\, d\mu_{t_0}(x) = 0.$$

On the other hand, $y_0 \in \partial\Omega(t_0) \cap \Gamma(t_0)$ and $\widehat{\eta}(y_0, t_0) > 0$. This contradicts with the above statement. Hence, δ is a viscosity supersolution of (1.7) on $\{\delta > 0\}$.

In the next step, we will show that δ is a supersolution of (1.7) on the whole domain.

(5). For $\varepsilon > 0$, let $h_\varepsilon(r) = (r - \varepsilon)^+$. We claim that $h_\varepsilon(d)$ is a viscosity supersolution of (1.7) in Q. Let $\varphi \in C^\infty(\bar{Q})$ and $(x_0, t_0) \in Q$ be a strict minimizer of $h_\varepsilon(d) - \varphi$.

First suppose that $d(x_0, t_0) > 0$. Since d is a supersolution on $\{d > 0\}$, Theorem 1.11 yields

$$\varphi_t - F_\star(D\varphi, D^2\varphi) \ge 0 \text{ at } (x_0, t_0). \tag{2.15}$$

Now suppose that $d(x_0, t_0) = 0$. We claim that there are $x_n \to x_0$ and $t_n \uparrow t_0$ such that $d(x_n, t_n) = 0$. Indeed, if there is no such sequence, there exists $\delta > 0$ such that

$$\mu_t\big(B_\delta(x_0)\big) = 0, \quad \forall t \in [t_0 - \delta, t_0).$$

Then, by the clearing-out lemma,

$$\mu_t\big(B_{\delta/2}(x_0)\big) = 0,$$

for all t sufficiently close to t_0 and therefore $(x_0, t_0) \notin \Gamma$ which contradicts with $d(x_0, t_0) = 0$. Hence there is such a sequence and consequently $\varphi_t(x_0, t_0) \geq 0$. Moreover $h_\varepsilon\big(d(x, t_0)\big) = 0$ for all $|x - x_0| < \varepsilon$ and $D\varphi(x_0, t_0) = 0$, $D^2\varphi(x_0, t_0) \leq 0$. Hence (2.15) holds and $h_\varepsilon(d)$ is a viscosity supersolution of (1.7) in Q. Now let $\varepsilon \downarrow 0$ to conclude that d is also a viscosity supersolution of (1.7) in Q.

(6). The Hausdorff dimension estimates follow from the clearing-out lemma and a standard covering lemma . See Ilmanen [I] for the details. □

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