OPTIMAL CONTROL OF JUMP-MARKOV PROCESSES AND VISCOSITY SOLUTIONS*

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Abstract. We investigate the Bellman equation that arises in the optimal control of Markov processes. This is a fully nonlinear integro-differential equation. The notion of viscosity solutions is introduced and then existence and uniqueness results are obtained. Also, the connection between the optimal control problem and the Bellman equation is developed.

1. Introduction. In this paper we study the Bellman equation associated to the optimal control of jump-Markov processes. In the analysis of this equation one encounters the very same difficulties as in the deterministic case, such as non-differentiability of the solutions or non-uniqueness of almost-everywhere solutions. For these reasons, one is forced to use a notion of weak solutions. It is known that the viscosity solutions defined by M.G. Crandall and P.-L. Lions [2] provides a resolution to this problem. We refer to the paper of P.-L. Lions and P.E. Souganidis [6] in this proceedings and the references therein, for an introduction to the subject. The purpose of this article is to pursue the "viscosity approach" for the jump-Markov processes.

It is also possible to obtain uniqueness results in the class of semi-concave functions, when the equation is related to a control problem or equivalently when the Hamiltonian is convex. For jump-Markov processes, this approach was taken by H. Pragarauskas [7]. An earlier paper of R. Rishel [8], also verifies dynamic programming, when the jump-rate is finite. In addition to all these results, D. Vermes [12] showed that the optimal value function of controlled piecewise deterministic processes [3], is the maximal subsolution.

Under assumption H1-H6,we shall prove that there is a unique viscosity solution to (1.1), (see Theorem 4.3). Since, any semi-concave almost-everywhere solution of (1.1) is a viscosity solution, our result extends the ones obtained by H. Pragarauskas. Also, the "viscosity approach" is applicable to some other problems, such as switching and/or impulse control [5] and some large deviation problems [4].

More specifically, we shall consider a controlled jump-Markov process with infinitesimal generator $L^v\Psi(x)$ given by,

$$L^v\Psi(x) = \int_{R^n} [\Psi(x + c(x,v,z)) - \Psi(x) - c(x,v,x)\cdot\nabla\Psi(x)]f(x,v,z)m(dz) \tag{1.1}$$

for $\Psi \in C^2(R^n)$. Here, $m(dz) = dz|z|^{-n-1}m(x,v,A) = \int \chi_{\{z:c(x,v,z)\in A\}}f(x,v,z)m(dz)$ is the Lévy measure satisfying $\int |z|^2 m(x,v,dz) = \int |c(x,v,z)|^2 f(x,v,z)m(dz) < \infty$, where χ_B is the indicator function of the set $B \subset R^n$. Consider a finite horizon problem with running cost $\ell(x,v)$,

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discount rate $\lambda \geq 0$ and terminal cost $g(x)$. We shall define this problem more precisely in the last section. Then the Hamilton-Jacobi-Bellman (HJB) equation corresponding to this problem has the following form:

$$\lambda u(x,t) - \frac{\partial}{\partial t}u(x,t) + H(x, Du(x,t), u(\cdot,t)) = 0, \quad x \in R^n, t \in [0,T) \tag{1.2}$$

with terminal data $u(x,T) = g(x)$. Here, for $x, p \in R^n$ and $\Psi \in C^2(R^n)$ the Hamiltonian $H(x,p,\Psi)$ is a non-local operator defined by,

$$\begin{aligned} H(x,p,\Psi) = \sup_{v \in V}\{ & -b(x,v)\cdot p - \ell(x,v) - \int_{R^n} [\Psi(x + c(x,v,z)) \\ & - \Psi(x) - c(x,v,z)\cdot \nabla\Psi(x)] f(x,v,z) m(dz)\} \end{aligned} \tag{1.3}$$

The paper is organized as follows: assumptions and definitions are given in the next section. Section 3 is devoted to the uniqueness, and then an existence result, by analytical methods, is obtained in Section 4. Finally, a stochastic control problem is defined and the optimal value function is shown to be a viscosity solution, in the last section.

2. Assumptions and Definitions. Let R^n denote the n-dimensional Euclidean space and V be the control set, which is assumed to be a compact metric space. For $x, y, z \in R^n$, $v \in V$ the functions $b(x,v), c(x,v,z) \in R^n$, $\ell(x,v)$, $g(x) \in R$, $f(x,v,z) \geq 0$ and the positive measure $m(dz)$ satisfy,

$$|g(x)| + |\ell(x,v)| + |b(x,v)| + |f(x,v,z)| + |c(x,v,z)| \leq K \tag{H1}$$

$$\begin{aligned} &|g(x) - g(y)| + |\ell(x,v) - \ell(y,v)| + |b(x,v) - b(y,v)| + |c(x,v,z) - c(y,v,z)| \\ &\quad + |f(x,v,z) - f(y,v,z)| \leq K|x - y| \end{aligned} \tag{H2}$$

$$\int_{R^n} |c(x,v,z)|^2 f(x,v,z) m(dz) \leq K \tag{H3}$$

$$\int_{R^n} |c(x,v,z) - c(y,v,z)|^2 f(x,v,z) m(dz) \leq K|x-y|^2 \tag{H4}$$

$$\int_{R^n} |f(x,v,z) - f(y,v,z)|\, m(dz) \leq K|x-y| \tag{H5}$$

$$\lim_{\delta \downarrow 0} \sup_{x,v} \int_{|z| \leq \delta} |c(x,v,z)|^2 f(x,v,z) m(dz) = 0 \tag{H6}$$

A straightforward generalization of the notion of viscosity solutions defined by M.G. Crandall and P.-L. Lions is, DEFINITION 2.1. *Any $u \in BUC(R^n \times [0,T])$, the set of bounded, uniformly continuous functions, is a viscosity supersolution (subsolution) of (1.2) if for every $\Psi \in C^2(R^n \times [0,T])$*

$$\lambda u(x,t) - \frac{\partial}{\partial t}, \Psi(x,t) + H(x, \nabla\Psi(x,t), \Psi(\cdot,t)) \geq 0, \quad (\leq 0),$$

whenever $(u-\Psi)(x,t) = \min\{(u-\Psi)(y,s) : y,s \in R^n \times [0,T]\}$, (max), at some $x,t \in R^n \times [0,T)$. Any $u \in BUC(R^n \times [0,T])$ is a viscosity solution of (1.2) if it is both sub and supersolution.

For $\delta > 0$, define a Hamiltonian $H_\delta(x,p,\Psi,w)$, for $x,p \in R^n$, $\Psi \in C^2(R^n)$ and $w \in BUC(R^n)$ by,

(2.1)
$$\begin{aligned} H_\delta(x,p,\Psi,w) = \sup_{v\in V}\{&-b(x,v)\cdot p - \ell(x,v) - \\ &- \int_{|z|\geq\delta} [w(x+c(x,v,z)) - w(x) - p\cdot c(x,v,z)] f(x,v,z) m(dz) \\ &- \int_{|z|\leq\delta} [\Psi(x+c(x,v,z)) - \Psi(x) - c(x,v,z)\cdot\nabla\Psi(x)] f(x,v,z) m(dz)\} \end{aligned}$$

It is easy to verify the following (also, see Theorem 2.1 in [10]).

LEMMA 2.2. *Any $u \in BUC(R^n \times [0,T])$ is a viscosity supersolution (subsolution), of (1.2) if and only if, for every $\Psi \in C^2(R^n \times [0,T])$ and $\delta > 0$,*

$$\lambda u(x,t) - \frac{\partial}{\partial t}\Psi(x,t) + H_\delta(x, \nabla\Psi(x,t), \Psi(\cdot,t), w(\cdot,t)) \geq 0, \qquad (\leq 0),$$

whenever $(u-\Psi)(x,t) = \min(u-\Psi)$, (max), at some $x,t \in R^n \times [0,T)$.

3. Comparison Result. An immediate consequence of the following result is the uniqueness of viscosity solutions to (1.2). Also, in Section 4, we shall use it to obtain an *a priori* Lipschitz estimate of the solutions of an approximated equation.

LEMMA 3.1. *Suppose, u and w are viscosity sub and supersolutions of (1.2), respectively. Then, for all $x,t \in R^n \times [O,T]$,*

$$u(x,t) - w(x,t) \leq \sup\{u(y,T) - w(y,T), y \in R^n\}$$

Proof. For $\varepsilon > 0$, introduce the standard test function $\Phi^\epsilon(x,y,t,s)$ for $x,y \in R^n$ and $t,s \in [0,T]$;

$$\Phi^\epsilon(x,y,t,s) = u(x,t) - w(y,s) - \frac{1}{\varepsilon}(|x-y|^2 + |t-s|^2)$$

Assume that, there are $\bar{x},\bar{y} \in R^n, \bar{t},\bar{s} \in [0,T]$ such that $\Phi^\epsilon(\bar{x},\bar{y},\bar{t},\bar{s}) = \max\Phi^\epsilon$. One can always obtain such points by slightly perturbing u and w, see Theorem 4.1 in [1]. Then, for a suitable constant $k > 0$,

$$|\bar{x}-\bar{y}| + |\bar{t}-\bar{s}| \leq k\sqrt{\varepsilon}h(k\sqrt{\varepsilon}) \tag{3.1}$$

where $h(\cdot)$ is a modulus of continuity for u and w. First, suppose that $\bar{t} = T$. Then,

$$\begin{aligned} |u(\bar{x},\bar{t}) - w(\bar{y},\bar{s})| &\le u(\bar{x},T) - w(\bar{x},T) + h(|\bar{x}-\bar{y}| + |\bar{t}-\bar{s}|) \\ &\le \sup\{u(u,T) - w(y,T), y \in R^n\} + kh(k\sqrt{\epsilon}) \end{aligned} \tag{3.2}$$

Also, the same estimate holds, when $\bar{s} = \bar{T}$. Finally, assume that $\bar{t}, \bar{s} < T$. The viscosity property of u and w yields that, for every $\delta > 0$,

$$\lambda u(\bar{x},\bar{t}) - \frac{2}{\varepsilon}(\bar{t}-\bar{s}) + H_\delta(\bar{x}, \frac{2}{\varepsilon}(\bar{x}-\bar{y}), \frac{1}{\varepsilon}|\cdot - \bar{y}|^2, u(\cdot,\bar{t})) \le 0$$

$$\lambda w(\bar{y},\bar{s}) - \frac{2}{\varepsilon}(\bar{t}-\bar{s}) + H_\delta(\bar{y}, \frac{2}{\varepsilon}(\bar{x}-\bar{y}), -\frac{1}{\varepsilon}|\bar{x} - \cdot|^2, w(\cdot,\bar{s})) \ge 0$$

straightforward estimation, together with $H1 - H6$ yield,

$$\begin{aligned} &\lambda u(\bar{x},\bar{t}) + \lambda w(\bar{y},\bar{s}) \\ &\quad\le \frac{k}{\varepsilon}|\bar{x}-\bar{y}|^2 + k|\bar{x}-\bar{y}| \\ &\qquad + \sup_{v\in V}\{\frac{1}{\varepsilon}\int_{|z|\le\delta}[|\bar{x}+c(\bar{x},v,z)-\bar{y}|^2 - |\bar{x}-\bar{y}|^2 - 2(\bar{x}-\bar{y})\cdot c(\bar{x},v,z)]f(\bar{x},v,z)m(dz)\} \\ &\qquad + \sup_{v\in V}\{\frac{1}{\varepsilon}\int_{|z|\le\delta}[-|\bar{x}-\bar{y}-c(\bar{y},v,z)|^2 + |\bar{x}-\bar{y}|^2 - 2(\bar{x}-\bar{y})\cdot c(\bar{y},v,z)]f(\bar{y},v,z)m(dz)\} \\ &\qquad + \sup_{v\in V}\{\frac{1}{\varepsilon}\int_{|z|\le\delta}([u(\bar{x}+c(\bar{x},v,z),\bar{t}) - u(\bar{x},\bar{t}) - \frac{2}{\varepsilon}(\bar{x}-\bar{y})\cdot c(\bar{x},v,z)]f(\bar{x},v,z) \\ &\qquad - [w(\bar{y}+c(\bar{x},v,z),\bar{s}) - w(\bar{y},\bar{s}) - \frac{2}{\varepsilon}(\bar{x}-\bar{y})\cdot c(\bar{y},v,z)]f(\bar{y},v,z))m(dz)\} \\ &\quad\le \frac{k}{\varepsilon}|\bar{x}-\bar{y}| + k|\bar{x}-\bar{y}| + \frac{2}{\varepsilon}\sup_{v,x}\{\int_{|z|\le\delta}|c(x,v,z)|^2 f(x,v,z)m(dz)\} \\ &\qquad + (2\|w\|_\infty + \frac{2}{\varepsilon}|\bar{x}-\bar{y}|)\sup_v\{\int|f(\bar{x},v,z) - f(\bar{y},v,z)|m(dz)\} \\ &\qquad + \sup_v\{\int_{|z|\ge\delta}[u(\bar{x}+c(\bar{x},v,z),\bar{t}) - w(\bar{y}+c(\bar{y},v,z),\bar{s}) - u(\bar{x},\bar{t}) + w(\bar{y},\bar{s}) \\ &\qquad - \frac{2}{\varepsilon}(\bar{x}-\bar{y})\cdot(c(\bar{x},v,z) - c(\bar{y},v,z))]f(x,v,z)m(dz)\} \end{aligned} \tag{3.3}$$

Let I_δ denotes the last expression. Using the choice of $\bar{x}, \bar{y}, \bar{t}, \bar{s}$ to obtain,

$$\begin{aligned} I_\delta &\le \sup_{v\in V}\{\frac{1}{\varepsilon}\int_{|z|\ge\delta}[|\bar{x}+c(\bar{x},v,z)-\bar{y}-c(\bar{y},v,z)|^2 - |\bar{x}-\bar{y}|^2 \\ &\qquad - 2(\bar{x}-\bar{y})\cdot(c(\bar{x},v,z)-c(\bar{y},v,z))]f(\bar{x},v,z)m(dz)\} \\ &= \sup_{v\in V}\{\frac{1}{\varepsilon}\int_{|z|\ge\delta}|c(\bar{x},v,z)-c(\bar{y},v,z)|^2 f(\bar{x},v,z)m(dz)\} \\ &\le \frac{1}{\varepsilon}|\bar{x}-\bar{y}|^2 \end{aligned} \tag{3.4}$$

Therefore, (3.2) - (3.4) yields that

$$\begin{aligned} u(\bar{x},\bar{t}) - w(\bar{y},\bar{s}) &\le \frac{k}{\varepsilon}|\bar{x}-\bar{y}|^2 + k|\bar{x}-\bar{y}| + \sup\{u(y,T) - w(y,T), y \in R^n\} \\ &\quad + \frac{k}{\varepsilon}\sup_{v,x}\{\int_{|z|\le\delta}|c(x,v,z)|^2 f(x,v,z)m(dz)\}. \end{aligned}$$

Send δ to zero and invoke (3.1) to conclude that,

$$\Phi^{\epsilon}(\bar{x},\bar{y},\bar{t},\bar{s}) \leq k(h(k\sqrt{\varepsilon}))^2 + k\sqrt{\varepsilon} + \sup\{u(y,T) - w(y,T), y \in R^n\}$$

Now, one completes the proof of the lemma, after observing that $u(x,t) - w(x,t) = \Phi^{\epsilon}(x,x,t,t) \leq \Phi^{\epsilon}(\bar{x},\bar{y},\bar{t},\bar{s})$, for each $\varepsilon > 0$. □.

4. Existence. Define $\lambda(x,v) := \int \chi_{\{z:c(x,v,z)\neq 0\}} f(x,v,z)m(dz)$. Suppose that, $\lambda(x,v) \leq K$ at each $x \in R^n$, $v \in V$. Then, construct a controlled piecewise deterministic process with jump-rate $\lambda(x,v)$ and post-jump location distribution

$$\beta(x,v,A) = \lambda(x,v)^{-1} \int \chi_{\{z:c(x,v,z)+x\in A\}} \chi_{\{z:c(x,v,z)\neq 0\}} f(x,v,z)m(dz)$$

(see M. Davis [3] and D. Vermes [12], for this construction). Define $u(x,t)$ by,

$$u(x,t) = \inf_{v\in A} E_{x,t}\{\int_t^T e^{-\lambda(s,t)}\ell(x(s),v(s))ds + e^{-\lambda(T-t)}g(x(T))\} \tag{4.1}$$

where A is the set of non-anticipative control processes $v(t)$ and $x(s)$ is the piecewise deterministic process corresponding to $v(s)$. Then, $u(x,t)$ is a viscosity solution of (1.2) (see Theorem 1.1 in [10]). When $\lambda(x,v)$ is not finite, we approximate $c(x,v,z)$ by $c^{\epsilon}(x,v,z) = c(x,v,z)\chi_{\{z:|z|\geq\epsilon\}}$. The above argument implies that there is a viscosity solution $u^{\epsilon}(x,t)$ of the following equation.

$$\lambda u^{\epsilon}(x,t) - \frac{\partial}{\partial t}u^{\epsilon}(x,t) + H^{\epsilon}(x,\nabla u^{\epsilon}(x,t),u^{\epsilon}(\cdot,t)) = 0$$

with $u^{\epsilon}(x,T) = g(x)$, where H^{ϵ} is defined as in (1.3), by using c^{ϵ} instead of c. We need the following estimate to pass to the limit as ε tends to zero. Let $\|\cdot\|_{1,\infty}$ denote the norm of the Sobolev space of Lipschitz functions.

LEMMA 4.1. *Suppose that u^{ϵ}'s are Lipschitz continuous. Then, $\|u^{\epsilon}\|_{1,\infty} \leq K$ for every $\varepsilon > 0$.*

Proof. Since $H^{\epsilon}(x,\nabla g(x),g)$ is bounded in x, for sufficiently large K, $\Phi^{+}(x,t) = g(x) + K(T-t)$ and $\Phi^{-}(x,t) = g(x) - K(T-t)$ are super and subsolution of (4.2), respectively. Therefore, Lemma 3.1 yields,

$$|u^{\epsilon}(x,t) - g(x)| \leq K(T-t) \tag{4.3}$$

Similarly, one can prove that $\|u^{\epsilon}\|_{\infty} \leq K$. Consider, $w^{\epsilon}(x,t) = e^{-\rho(T-t)}u^{\epsilon}(x,t)$, where ρ to be chosen later. Then w^{ϵ} solves,

$$(\rho+\lambda)w^{\epsilon}(x,t) - \frac{\partial}{\partial t}w^{\epsilon}(x,t) + H^{\epsilon,\rho}(x,t),\nabla w^{\epsilon}(x,t),w^{\epsilon}(\cdot,t)) = 0 \tag{4.4}$$

For $x,p \in R^n, t \in [0,T]$ and $\Psi \in C^2(R^n)$, the Hamiltonian $H^{\epsilon,\rho}(x,t,p,\Psi)$ is given by,

$$\begin{aligned} H^{\epsilon,\rho}(x,t,p,\Psi) = \sup_{v\in V}\{&-b(x,v)\cdot p - \ell(x,v)e^{-\rho(T-t)} \\ &- \int_{|z|\geq\epsilon} [\Psi(x+c(x,v,z)) - \Psi(x) - \nabla\Psi(x)\cdot c(x,v,z)]f(x,v,t)m(dz)\} \end{aligned}$$

Since w^ε is Lipschitz, for each $\gamma > 0$ there are $t_0 < T$ and $x_0, \nu_0 \in R^n, |\nu_0| = 1$ such that $w^\varepsilon(x_0,t_0) - w^\varepsilon(x_0 + \alpha\nu_0, t_0) \geq (\|\nabla w^\varepsilon\|_\infty - \gamma)\alpha$, for all small $\alpha > 0$.

Now, define a test function $\Phi^\alpha(x,y,t,s)$ by,

$$\Phi^\alpha(x,y,t,s) = w^\varepsilon(x,t) - w^\varepsilon(y,s) - \frac{1}{2\alpha}|x-y|^2 - \frac{1}{2\alpha^2}|t-s|^2$$

In view of (4.3), $\Phi^\alpha(x,y,t,T) \leq \|\nabla g\|_\infty |x-y| + k|T-t| - \frac{1}{2\alpha}(x-y)^2 - \frac{1}{2\alpha^2}|T-t|^2 \leq \frac{1}{2}\|\nabla g\|_\infty^2 \alpha + \frac{1}{2}K^2\alpha^2$. On the other hand, $\Phi^\alpha(x_0, x_0 + \alpha(\|\nabla w^\varepsilon\|_\infty - \gamma)\nu_0, t_0, t_0) \geq \frac{1}{2}(\|\nabla w^\varepsilon\|_\infty - \gamma)^2\alpha$. Therefore, if $\|\nabla w^\varepsilon\|_\infty > \|\nabla g\|_\infty$, the maximum of Φ^α can not be achieved at $t = T$ or $s = T$, for small $\alpha > 0$. Again, as in the proof of Lemma 3.1, with no loss of generality, we assume that there are $\bar{x}, \bar{y} \in R^n$ and $\bar{t}, \bar{s} \in [0,T)$ at which Φ^α achieves its maximum. The following are easy to verify,

$$\Phi^\alpha(\bar{x},\bar{y},\bar{t},\bar{s}) \geq \frac{1}{2}(\|\nabla w^\varepsilon\|_\infty - \gamma)^2\alpha \tag{4.5}$$

$$|\bar{x}-\bar{y}| + |\bar{t}-\bar{s}| \leq \|\nabla w^\varepsilon\|_\infty \alpha \tag{4.6}$$

The inequality (4.5) holds for every $\gamma > 0$ and for all $\alpha \leq \alpha(\gamma)$. Invoke the viscosity property of w^ε at $\bar{x}, \bar{t}$ and $\bar{y}, \bar{s}$ to obtain the following for every $\delta > 0$.

$$(\rho+\lambda)w^\varepsilon(\bar{x},\bar{t}) - \frac{1}{\alpha^2}(\bar{t}-\bar{s}) + H_\delta^{\varepsilon,\rho}(\bar{x},\bar{t},\frac{1}{\alpha}(\bar{x}-\bar{y}), \frac{1}{2\alpha}|\cdot - \bar{y}|^2, w^\varepsilon(\cdot,\bar{t})) \leq 0$$

$$(\rho+\lambda)w^\varepsilon(\bar{y},\bar{s}) - \frac{1}{\alpha^2}(\bar{t}-\bar{s}) + H_\delta^{\varepsilon,\rho}(\bar{y},\bar{s},\frac{1}{\alpha}(\bar{x}-\bar{y}), -\frac{1}{2\alpha}|\bar{x} - \cdot|^2, w^\varepsilon(\cdot,\bar{s})) \geq 0$$

Proceed as in the proof of Lemma 3.1, to obtain

$$\begin{aligned}(\rho+\lambda)(w^\varepsilon(\bar{x},\bar{t}) - w^\varepsilon(\bar{y},\bar{s})) \leq \frac{K}{\alpha}|\bar{x}-\bar{y}|^2 + K|\bar{x}-\bar{y}| + K|\bar{t}-\bar{s}| \\ + \frac{1}{\alpha}\sup_{x,v}\{\int_{\varepsilon\leq|z|\leq\delta} |c(x,v,z)|^2 f(x,v,z) m(dz)\}\end{aligned}$$

The constant K in the above inequality, is independent of ε, α and ρ. Let $\rho = 2K$. Send δ to zero in the above inequality to obtain the following,

$$(\rho+\lambda)\Phi^\alpha(\bar{x},\bar{y},\bar{t},\bar{s}) \leq (\rho+\lambda)[w^\varepsilon(\bar{x},\bar{t}) - w^\varepsilon(\bar{y},\bar{s}) - \frac{1}{2\alpha}|\bar{x}-\bar{y}|^2] \leq K|\bar{x}-\bar{y}|$$

In view of (4.5) and (4.6), we have

$$\frac{1}{2}(\|\nabla w^\varepsilon\|_\infty - \gamma)^2\alpha \leq \Phi^\alpha(\bar{x},\bar{y},\bar{t},\bar{s}) \leq \frac{K}{\rho+\lambda}\|\nabla w^\varepsilon\|_\infty \alpha$$

Hence, $\|\nabla w^\varepsilon\|_\infty \leq \frac{K}{\rho+\lambda} \leq \frac{K}{K+\lambda} \leq 1$ and consequently, for $\rho = K$, $\|\nabla w^\varepsilon\|_\infty = \max\{\|\nabla g\|_\infty, 1\}$ □.

LEMMA 4.2. *For each $\varepsilon > 0$, u^ε is Lipschitz continuous.*

Proof. For $C, \alpha > 0$, define $\Phi^{\alpha,C}$ by,

$$\Phi^{\alpha,C}(x,y,t,s) = w^\varepsilon(x,t) - w^\varepsilon(y,s) - C|x-y| - \frac{1}{2\alpha}|t-s|^2$$

as in the proof of the previous lemma, $w^\varepsilon(x,t) = e^{-\rho(T-t)}u^\varepsilon(x,t)$ and $\rho > 0$ is to be chosen later. Once again, with no loss of generality, assume that there are $\bar{x}, \bar{y} \in R^n$ and $\bar{t}, \bar{s} \in [0,T]$ such that $\Phi^{\alpha,C}(\bar{x},\bar{y},\bar{t},\bar{s}) = \max \Phi^{\alpha,C}$.

Since $w^\varepsilon(x,t) - w^\varepsilon(y,t) - C|x-y| \leq \Phi^{\alpha,C}(\bar{x},\bar{y},\bar{t},\bar{s})$, to obtain the Lipschitz continuity of w^ε in the x-variable, it suffices to show that $\limsup_{\alpha\downarrow 0} \Phi^{\alpha,C}(\bar{x},\bar{y},\bar{t},\bar{s})$ is less than zero.

We shall analyse three different cases seperately.

(i) $\bar{t} = T$ or $\bar{s} = T$. In view of (4.3), we have $\Phi^{\alpha,C}(\bar{x},\bar{y},\bar{t},\bar{s}) \leq g(\bar{x}) - g(\bar{y}) + K|\bar{s}-\bar{t}| - C|)|\bar{x}-\bar{y}| - \frac{1}{2\alpha}|\bar{s}-\bar{t}|^2 \leq (\|\nabla g\|_\infty - C)|\bar{x}-\bar{y}| + \frac{1}{2}K^2\alpha$. Therefore, for $C \geq \|\nabla g\|_\infty$ $\limsup_{\alpha\downarrow 0} \Phi^{\alpha,C}(\bar{x},\bar{y},\bar{t},\bar{s}) \leq 0$.

(ii) $\bar{x} = \bar{y}$. $\Phi^{\alpha,C}(\bar{x},\bar{y},\bar{t},\bar{s}) \leq 0$.

(iii) $\bar{x} \neq \bar{y}$ and $\bar{t}, \bar{s} < T$. In this case, use the viscosity property of w^ε at $\bar{x}, \bar{t}$ and $\bar{y}, \bar{t}$.

$$(\rho+\lambda)w^\varepsilon(\bar{x},\bar{t}) - \frac{1}{\alpha}(\bar{t}-\bar{s}) + H^\varepsilon(\bar{x}, C(\bar{x}-\bar{y})|\bar{x}-\bar{y}|^{-1}, w^\varepsilon(\cdot,\bar{t})) \leq 0$$

$$(\rho+\lambda)w^\varepsilon(\bar{y},\bar{s}) - \frac{1}{\alpha}(\bar{t}-\bar{s}) + H^\varepsilon(\bar{y}, C(\bar{x}-\bar{y})|\bar{x}-\bar{y}|^{-1}, w^\varepsilon(\cdot,\bar{s})) \geq 0$$

Use the assumptions (H1)–(H6), to obtain:

$$\begin{aligned}
&(\rho+\lambda)(w^\varepsilon(\bar{x},\bar{t}) - w^\varepsilon(\bar{y},\bar{s})) \\
&\quad \leq \sup_{v\in V}\{C(\bar{x}-\bar{y})|\bar{x}-\bar{y}|^{-1} \cdot [b(\bar{x},v) - b(\bar{y},v) + \\
&\quad + \int_{|z|\geq\varepsilon} (c(\bar{x},v,z)f(\bar{x},v,z) - c(\bar{y},v,z)f(\bar{y},v,z))m(dz)] + \\
&\quad + \ell(\bar{x},v)e^{-\rho(T-\bar{t})} - \ell(\bar{y},v)e^{-\rho(T-\bar{s})} + \\
&\quad + \int_{|z|\geq\varepsilon} (w^\varepsilon(\bar{x}+c(\bar{x},v,z),\bar{t}) - w^\varepsilon(\bar{x},\bar{t}))f(\bar{x},v,z)m(dz) \\
&\quad + \int_{|z|\geq\varepsilon} (w^\varepsilon(\bar{y}+c(\bar{y},v,z),\bar{s}) - w^\varepsilon(\bar{y},\bar{s}))f(\bar{y},v,z)m(dz)\} \\
&\quad \leq C|\bar{x}-\bar{y}|(K + Km(\{z : |z| \geq \varepsilon\})) + K|\bar{t}-\bar{s}| \\
&\quad + \sup_{v\in V}\{\int_{|z|\geq\varepsilon} (w^\varepsilon(\bar{x}+c(\bar{x},v,z),\bar{t}) - w^\varepsilon(\bar{y}+c(\bar{y},v,z),\bar{s}) - w^\varepsilon(\bar{x},\bar{t}) \\
&\quad + w^\varepsilon(\bar{y},\bar{s}))f(\bar{x},v,z)m(dz)\}
\end{aligned}$$

Now, use the choice of $\bar{x}, \bar{y}, \bar{t}, \bar{s}$ and the fact that $|\bar{t}-\bar{s}| \leq \sqrt{\alpha}K$.

$$
\begin{aligned}
(\rho+\lambda)(w^\varepsilon(\bar{x},\bar{t}) - w^\varepsilon(\bar{y},\bar{s}))& \\
&\leq C|\bar{x}-\bar{y}|K(\varepsilon) + K\sqrt{\alpha}+ \\
&\quad + C\sup_{v\in V}\{\int_{|z|\geq\varepsilon}(|\bar{x}+c(\bar{x},v,z) \\
&\quad - \bar{y} - c(\bar{y},v,z)| - |\bar{x}-\bar{y}|)f(\bar{x},v,z)m(dz)\} \\
&\leq C|\bar{x}-\bar{y}|K(\varepsilon) + K\sqrt{\alpha}+ \\
&\quad + C\sup_{v\in V}\{\int_{|z|\geq\varepsilon}|c(\bar{x},v,z) - c(\bar{y},v,z)|m(dz)\} \\
&\leq C|\bar{x}-\bar{y}|K(\varepsilon) + K\sqrt{\alpha} + C|\bar{x}-\bar{y}|Km(\{z:|z|\geq\varepsilon\}) \\
&\leq K\sqrt{\alpha} + C|\bar{x}-\bar{y}|\bar{K}(\varepsilon)
\end{aligned}
$$

where $K(\varepsilon)$ and $\bar{K}(\varepsilon)$ are suitable constants that may only depend on $\varepsilon > 0$. Let $\rho = \bar{K}(\varepsilon)$. Then, $(\rho+\lambda)\Phi^{\alpha,C}(\bar{x},\bar{y},\bar{t},\bar{s}) \leq K\sqrt{\alpha}$. Hence, combining all three cases, we conclude that for $C = \|\nabla g\|_\infty$ and $\rho = \bar{K}(\varepsilon)$ the following holds,

$$
\limsup_{\alpha\downarrow 0}\Phi^{\alpha,C}(\bar{x},\bar{y},\bar{t},\bar{s}) \leq 0
$$

Therefore, $\|\nabla u^\varepsilon\|_\infty \leq e^{\rho T}\|\nabla w^\varepsilon\|_\infty \leq e^{\rho T}C$. One proves the time-derivative estimate after observing that $|H^\varepsilon(x,\nabla u^\varepsilon(x,t),u^\varepsilon(x,t),u^\varepsilon(\cdot,t))| \leq K(\varepsilon)\|\nabla u^\varepsilon\|_\infty$. (this inequality is to be understood in the viscosity sense.) □

Theory of viscosity solutions implies that any limit point of u^ε as ε approaches to zero is a viscosity solution of (1.2). The uniqueness result, Lemma 3.1, together with this observation yield that there is a Lipschitz continuous solution u of (1.2) and $u^\varepsilon \to u$ uniformly on bounded subsets of $R^n \times [0,T]$. We sum these into the following:

THEOREM 4.3. *There is a unique viscosity solution $u(x,t)$ of (1.2) with terminal data $u(x,T) = g(x)$. Moreover, $\|\nabla u(\cdot,t)\|_\infty \leq e^{\rho(T-t)}\|\nabla g\|_\infty$ for some large $\rho > 0$.*

5. The Stochastic Control Problem. In this section, we shall use a stronger version of (H5). That is there is $\nu_0 > 0$ such that

(H5′) $$f(x,v,z) = 1, \quad \text{for all} \quad |z| \leq \nu_0 \quad \text{and} \quad x \in R^n, v \in V$$

($H7$) $$\int_{R^n}|z|^2 m(dz) < \infty$$

For each $v \in V$, there is a solution of the martingale problem corresponding to the operator L^v given in (1.1), the reader may refer to D. Stroock [11]. But, instead of characterizing the controlled jump-Markov process by the martingale problem, we shall make use of the stronger assumptions on the coefficients b, f, c, to construct the stochastic process, pathwise.

First, we briefly sketch a stochastic integral equation driven by an independent increment process $\xi(t)$ with Lévy measure $M(dz) = m(dz)\chi_{|z|\leq\nu_0}$, i.e. the characteristic functional of ξ is given by,

$$
E(\exp\sqrt{-1}\eta\cdot(\xi(t)-\xi(s))) = \exp[(t-s)\int_{|z|\leq\nu_0}(e^{\sqrt{-1}\eta\cdot z} - 1 - \sqrt{-1}\eta\cdot z)m(dz)],
$$
$$
\eta \in R^n, t, s \geq 0
$$

From the jumps of $\xi(t)$, construct a random measure $p(dt \times dz)$ on $[0, \infty) \times R^n \setminus \{0\}$ by,

$$p([0,T] \times A) = \text{cardinality of } \{s \leq T : \xi(s) - \xi(s^-) \in A\}, \quad \text{for} \quad T > 0 \quad \text{and} \quad A \subset R^n \setminus \{0\}$$

For any Borel subset A of $R^n \setminus \{0\}$, the above set is almost-surely finite. Therefore, $p(dt \times dz)$ is an integer-valued random measure. Let $q(dt) = p(dz) - m(dz)dt$. Then, $q([0,t] \times A)$ is a martingale with respect to the filteration $\mathfrak{F}_t = \sigma(\xi(s) = 0 \leq \delta \leq t)$, for each Borel subset A of $R^n \setminus \{0\}$. For any $v(t)$, adapted to $\mathfrak{F}_t$, there is a unique solution $y(t)$ of the following stochastic integral equation, see V. Skorokhod [9] for details and the definition of the stochastic integrals with respect to $q(dt \times dz)$.

$$y(t) = x + \int_0^t b(y(s), v(s))ds + \int_0^t \int_{R^n} c(y(s^-), v(s^-), z)q(ds \times dz) \tag{5.1}$$

Moreover, $y(t)$ is a Markov process and when $v(t) \equiv v \in V$, the infinitesimal generator is given by,

$$\tilde{L}^v \Psi(x) = b(x,v) \cdot \nabla \Psi(x) + \int_{|z| \leq \nu_0} [\Psi(x + c(x,v,z)) - \Psi(x) - \nabla \Psi(x) \cdot c(x,v,z)] m(dz) \tag{5.2}$$

Observe that $\tilde{L}^v$ coincides with L^v for small jumps. To close the difference between $\tilde{L}^v$ and L^v, we proceed as follows:

Let $v(x,t)$ be adapted to $\mathfrak{F}_t$ for every $x \in R^n$ and $y(x,t)$ be the solution of (5.1). Define a random jump-time T_1 and a post-jump location Y_1 by,

$$P(T_1 \geq t | \mathfrak{F}_t) = \exp(-\int_0^t \lambda(y(x,s), v(x,s))ds)$$

$$P(Y_1 - y(x,T_1) \in A | \mathfrak{F}_t, T_1) = (\lambda(y(x,T_1)))^{-1} \cdot [\int_{|z| > \nu_0} \chi_{\{z: c(y(x,T_1), v(x,T_1), z) \in A \setminus \{0\}\}} f(y(x,T_1), v(x,T_1), z) m(dz)]$$

where

$$\lambda(x,v) = \int_{|z| > \nu_0} \chi_{\{z: c(x,v,z) \neq 0\}} f(x,v,z) m(dz)$$

Now, on $[0, T_1]$ define the process $x(t)$ and the control process $v(t)$ by,

$$x(t) = \begin{cases} y(x,t) & t \in [0, T_1) \\ Y_1 & t = T_1 \end{cases}$$

$$v(t) = \begin{cases} v(x,t) & t \in [0, T_1) \\ v(Y_1, 0) & t = T_1 \end{cases}$$

Then, repeat the same procedure by using $v(Y_1, t - T_1)$ instead of $v(x,t)$. Observe that this procedure is exactly the same one given for piecewise deterministic process [3], [12]. The only difference is that $y(x,t)$ is a random process rather than a deterministic flow. Since $y(x,t)$ is Markov, it is easy to check that $x(t)$ is also a Markov process. Moreover, when $v(t) \equiv v \in V$, the infinitesimal generator of $x(t)$ coincides with L^v on $C^2(R^n)$.

Now, we are ready to define the control problem: the pay-off functional $J(x,t,v(\cdot))$ is given by,

$$J(x,t,v(\cdot)) = E_{x,t}[\int_t^T e^{-\lambda(s,t)}\ell(x(s),v(s))ds + e^{-\lambda(T-t)}g(x(T))] \tag{5.3}$$

$$u(x,t) = \inf_{v(\cdot)\in\mathcal{A}} J(x,t,v(\cdot)) \tag{5.4}$$

where $x(s)$ is the jump-Markov process corresponding to $v(\cdot)\in\mathcal{A}$ and the initial condition $x(t)=x$. $\mathcal{A}$ is the set of all maps $v(x,t)\in V$, which are adapted to $\mathfrak{F}_t$, for each $x\in R^n$ and are continuous in both variables.

LEMMA 5.1. *The optimal value function $u(x,t)$ is uniformly continuous*

Proof. Let $u_0(x,t)$ be the optimal value of the following problem,

$$u_0(x,t) = \inf_{v(\cdot)\in\tilde{\mathcal{A}}} E_{x,t}[\int_t^T e^{\lambda(s-t)}\ell(y(s),v(s))ds + e^{-\lambda(T-t)}g(y(T))]$$

where $y(s)$ is the solution of (5.1) with $y(t)=x$ and $\tilde{\mathcal{A}}$ is the set of $\mathfrak{F}_t$ adapted process $v(s)$. Since $y(s)$ is smooth as a function of the initial condition x, (see Theorem 2 on page 86 in [9].), $u_0(x,t)$ is Lipschitz continuous. Now, define $u_n(x,t)$ for $n=1,2,\dots$ as follows:

$$u_n(x,t) = \inf_{v(\cdot)\in\tilde{\mathcal{A}}} E_{x,t}[\int_t^{T_1\wedge T} e^{-\lambda(s-t)}\ell(y(s),v(s))ds + e^{-\lambda(T_1\wedge T-t)}u_{n-1}(y(T_1\wedge T),T_1\wedge T)]$$

Inductively, one can prove that u_n's are Lipschitz continuous. Also, dynamic programming implies that,

$$u_n(x,t) = \inf_{v\in\mathcal{A}} E_{x,t}[\int_t^{T_n\wedge T} e^{-\lambda(s-t)}\ell(x(s),v(s))ds + e^{\lambda(T_n\wedge T-t)}u_0(x(T_n\wedge T),T_n\wedge T)]$$

Now, it is elementary to show that, as n approaches to infinity, u_n converges to the optimal value function u given in (5.4) (see Theorem 1.1 in [10]) □.

By using the continuity of the optimal value function, it is easy to verify that it is a viscosity solution to (1.2). Hence, we have the following result.

THEOREM 5.2. *The optimal value function $u(x,t)$ is the only viscosity solution of (1.2) with terminal data $u(x,T)=g(x)$. Moreover, it is Lipschitz continuous.*

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