Variational and Dynamic Problems for the Ginzburg-Landau Functional

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1 Introduction

These are the notes for the three lectures I have given in the CIM/CIME Euro-Summer School on evolving interfaces held in the Portugal island of Maderia, in July 2000. In these lectures, I surveyed several results on the Ginzburg-Landau functional

$$\hat{I}^\epsilon(u) := \int_U E^\epsilon(u)\ dx, \quad E^\epsilon(u) := \frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{4\epsilon}[1-|u|^2]^2,$$

where U is an open, bounded subset of $I\!R^n$ with smooth boundary, and $\epsilon > 0$ is a small parameter.

This functional is a simpler version of the Ginzburg-Landau functional for superconductivity. The model for superconductivity has two fields; the complex valued order parameter u and the vector valued magnetic potential A. The functional $\hat{I}^\epsilon$ is obtained by setting A to zero and by appropriate scaling. In superconductivity, the length of the order parameter $|u|$ is proportional to the density of superconducting electrons. Hence the zeroes of u, called *vortices*, are the places where superconductivity is lost. The parameter ϵ corresponds to the inverse of the Ginzburg-Landau parameter κ and a number of type II superconducting materials have large κ justifying the asymptotic regime considered in the notes. The lecture notes of Rubinstein [23] provides a good introduction.

The starting point of these lectures is the seminal work of Bethuel, Brezis and Helein [6] which gives a detailed asymptotic description of the minimizers u^ϵ of $\hat{I}^\epsilon$ as ϵ tends to zero when $U \subset I\!R^2$. To briefly describe this result, let u^ϵ be the minimizer of $\hat{I}^\epsilon$ among all functions u satisfying a given boundary data $u = g$. Since as ϵ tends to zero the second term in E^ϵ forces the solution to have length one, it is natural to assume that g takes values in the unit circle S^1. In the complex notation, g admits the representation $g = e^{i\phi}$ for some possibly multi-valued function ϕ. Then, the boundary condition is

$$u(x) = g(x) = e^{i\phi(x)}, \quad x \in \partial U. \tag{1.1}$$

The degree of g around the origin (or the winding number) is an important parameter. Using the local representation $g = e^{i\phi}$ and the fact that $U \subset I\!R^2$, the degree can be computed by the following formula

$$\deg(g; \partial U) = \frac{1}{2\pi} \int_{\partial U} \nabla\phi \cdot \vec{t}\ d\mathcal{H}^1(x), \tag{1.2}$$

where $\vec{t}$ is the unit tangent and $\int_{\partial U} \cdots d\mathcal{H}^1$ is the line integral around ∂U.

If the boundary data g has zero degree, then ϕ is single valued. It is then relatively straightforward to show that u^ϵ converges strongly to the smooth function $u = e^{i\varphi}$ where φ is unique harmonic function in U satisfying the boundary condition $\varphi = \phi$. Hence, the interesting the case is

$$d := \deg(g; \partial U) \neq 0.$$

Then, by topological considerations, any function satisfying the boundary and in particular the minimizer u^ϵ must have d zeroes. This fact makes the problem interesting as each zero carries an energy of at least $\pi \ln(1/\epsilon)$. To see this consider the problem with $U = B_R := \{|x| < R\}$, and $g(x) = e^{iN\theta}$ where $\theta(x)$ is the angle between x and the x-axis and N is an integer. Consider a test function

$$v^\epsilon(x) = f(|x|/\epsilon)\ e^{iN\theta},$$

with some smooth, positive function f satisfying $f(0) = 0$, $f(r) = 1$ for all $r \geq 1$. By calculus,

$$\hat{I}^\epsilon(v^\epsilon) = \ N^2\pi \ln(1/\epsilon) + O(1).$$

The N^2 term indicates that it is better to have N distinct zeroes of degree one, instead of less zeroes with higher degree. Hence the minimizer u^ϵ is expected to have d distinct zeroes $a^\epsilon := (a_1^\epsilon, \ldots, a_d^\epsilon)$, again called vortices, each having degree one. The minimum energy $\hat{I}^\epsilon(u^\epsilon)$ behaves like $d\pi \ln(1/\epsilon)$ and the asymptotic behavior of u^ϵ is determined by the location of the zeroes a^ϵ. Bethuel, Brezis and Helein, obtained the location of the zeroes by calculating the next term in the minimum energy, which they call the *renormalized energy.* The renormalized energy $W(a^\epsilon)$ is a function of the location of the zeroes, and it has a representation in terms of the solution of the Laplace equation with point sources at a^ϵ, or equivalently the canoniacl hramonic as defined in [6]. Then, the minimum energy has the form

$$\hat{I}^\epsilon(u^\epsilon) = d\ \pi\ \ln(1/\epsilon) + W(a^\epsilon) + o(1). \tag{1.3}$$

In view of this expansion of the minimum energy, it is clear that a^ϵ converges to a minimum of the renormalized energy W. Once the location of the zeroes is determined it is possible to calculate the limit of u^ϵ. We refer to [6] for more information.

The chief difference between the problem considered here and the model for superconductivity is the boundary condition. In superconductivity, Neumann condition is given and the vortices are formed by an exogenous forcing term which is the applied magnetic field. While in the above problem vortices created by the Dirichlet data. For this reason, local results which do not refer to a particular boundary condition are more useful for our understanding of the model for superconductivity (we refer to the recent paper of Sandier and Serfaty [27], and the references therein for infromation on mathematical results on the model for superconductivity.) The Gamma limit is such a result. For the scalar valued functions, the Gamma limit of the Ginzburg-Landau functional, with a different rescaling, is proved by Modica and Mortola [21, 21], and by Modica [20]. A brief definition of the Gamma limit is given in §4.

In view of the above calculations, we consider the Gamma limit of the rescaled Ginzburg-Landau functional

$$I^\epsilon(u) := \frac{\hat{I}^\epsilon(u)}{\ln(1/\epsilon)} \ = \frac{1}{\ln(1/\epsilon)} \int_U E^\epsilon(u)\ dx.$$

In §4 below we will show that the Gamma limit of I^ϵ is equal to

$$I(u): \; = \begin{cases} |Ju|(U), & \text{if } \; u \in B2V(U;S^1), \\ +\infty, & \text{otherwise,} \end{cases}$$

where Ju is the Jacobian of u and weakly it is given by (see §2.1)

$$Ju: \; = \frac{1}{2} \, \nabla \times j(u), \qquad\qquad j(u): \; = u \times Du = det(u; Du),$$

S^1 is the unit circle, and $B2V$ is the set of all functions whose weak Jacobian is a Radon measure. The weak definition of the Jacobian in higher dimensions is discussed in §5, and BnV with a general n is defined properly in §4.1. This class of functions and its properties are studied in the two papers of the author with Jerrard [14, 15]. The set BnV, called *functions of bounded n variations* is related to the classical BV space and to the *Cartesian currents* of Giaquinta, Modica and Soucek [10, 11].

It is shown in [14] and also in §4.1 below that for $u \in B2V(U;S^1)$, the Jacobian has a special structure:

$$Ju = \pi \sum_i \; k_i \; \delta_{a_i},$$

for some points $\{a_i\} \subset U$ and integers k_i. Here δ_{a_i} is the Dirac measure located at a_i. This is interpreted as encoding the location and the topological singularities (or zeroes) of u. Moreover, for $u \in B2V(U;S^1)$,

$$|Ju|(U) = \sum_i \; |k_i|.$$

Hence, the Gamma limit $I(u)$ counts the zeroes of u with multiplicity.

This Gamma limit is proved in several steps. The first step is a "local" energy lower bound of the form

$$\int_U E^\epsilon(u) \; dx \geq \pi \; \ln(1/\epsilon) \; \deg(u; \partial U) - C,$$

for some constant C. There are problems with this estimate as it is stated. The difficulty comes from the possible zeroes of u near or on ∂U. We will prove two such results, Theorem 2.1, and Theorem 2.6. They are local in nature, especially the second one. The proof technique is an elegant covering argument of Jerrard [12]. To explain this method clearly, we first prove it under slightly restrictive set of assumptions first to prove Theorem 2.1. We then modify this technique to obtain the sharper lower bound Theorem 2.6.

A corollary of this lower bound is a compactness result for the Jacobian. This estimate bounds the Jacobian by the Ginzburg-Landau energy, and yields a compactness result for the Jacobian. Indeed for any sequence u^ϵ satisfying

$$\sup_{\epsilon} \; I^\epsilon(u^\epsilon) \; < \; \infty,$$

the Jacobians Ju^ϵ are compact in dual $(C^{0,\alpha})^*$ norm for every $\alpha > 0$. Hence, on a subsequence Ju^ϵ converge to a distribution J not in a norm slightly weaker than the weak* topology of Radon measures. Although this convergence is not in the space of measures, we will show that the resulting distribution J is indeed a measure of a special form. For $B2V(U;\mathbb{R}^2)$ with $U \subset \mathbb{R}^m$ with $m = 2$, this is proved in §3.2, and for $m \geq 3$ it is stated in §5.

Then the Gamma limit is proved in §4 as a result of the lower bound and the compactness of the Jacobian.

The compactness of the Jacobian is also a useful tool in the analysis of the dynamic problems. To motivate the study of the evolution problems in this context, let us consider an experiment in superconductivity. In this experiment the vortices are formed by an external magnetic field. Then the magnetic field is turned off and the material turns back to superconducting state. To understand these transition from the vortex state to the superconducting state, both the parabolic

$$u_t - \Delta u = \frac{u}{\epsilon^2}\,[1 - |u|^2], \qquad t > 0,\; x \in \mathbb{R}^n, \tag{1.4}$$

and the Schrodinger

$$i\; u_t - \Delta u = \frac{u}{\epsilon^2}\,[1 - |u|^2], \qquad t > 0,\; x \in \mathbb{R}^n, \tag{1.5}$$

equations are proposed. The question then is to obtain evolution of the vortices that are forced into the system via the initial data. Mathematically this is achieved by studying the small ϵ asymptotics of the above equations for an initial data u_0^ϵ which contains several vortices with degree ± 1. Asymptotic expansion techniques used by Neu, Rubinstein and E to derive these equations; see the lecture notes of Rubinstein [23]. Since (1.4) is the gradient flow for $\hat{I}^\epsilon$, in view of the expansion (1.3), as ϵ tends to zero, the vortices should satisfy the gradient flow for the potential W. Indeed, let $a(t) = (a_1(t), \ldots, a_N(t))$ be the limit of vortices $a^\epsilon(t\; \ln(1/\epsilon))$. Then,

$$\frac{d}{dt}a(t) = -\nabla W(a(t)), \tag{1.6}$$

in the case of (1.4). Note that we need to speed up the dynamics by a factor of $\ln(1/\epsilon)$. For the Schroedinger equation, in the original time scale, we get the Hamiltonian dynamics. These results are rigorously proved in several papers. For the parabolic flow, in [18] Lin proved that the speed of vortices is $1/\ln(1/\epsilon)$ when the vortices all have same sign. The mixed vortex case, which is the relevant one in the experiment outline above, was proved in [16] by Jerrard and the author. First rigorous derivation of the vortex equation is also given in [16]. For $U = \mathbb{R}^2$, an explicit form of (1.6) is avaliable:

$$\frac{d}{dt}a_k(t) = 2\sum_{j=1}^{N} \; d_j d_k \; \frac{(a_k(t)-a_j(t))}{|a_k(t)-a_j(t)|^2},$$

where d_k is the degree of u^ϵ around a_k for small ϵ, and by hypothesis d_k is equal to ± 1. Note that the solutions of the above equation behaves like charged particles with a logarithmic potential; vortices with same degree expel each other while ones with opposite degree attract with a force proportional to the inverse of the distance between the vortices.

In $I\!R^n$, the set of zeroes of u^ϵ is a codimension two set, as ϵ tends to zero we obtain geometric equations for these sets; called vortex lines in $I\!R^3$. As expected from results on scalar version of (1.4), the limiting vortex line moves by mean curvature flow. We refer to [23, 24] and the references therein for the formal derivation of these equations. First rigorous results for the vector Ginzburg-Landau equation are [17] and [4].

In §6 we prove the convergence when there exists a smooth solution $\{\Gamma_t\}_{t\in[0,T]}$ of the codimension two mean curvature flow. The main idea set forward in [17] is to consider the limiting measures

$$\mu_t : \;\; = \text{weak}^* \text{ limit of } \mu_t^\epsilon,$$

where

$$\mu_t^\epsilon(V) : \;\; = \frac{1}{|\ln 1/\epsilon|}\int_V \; E^\epsilon(u^\epsilon(t,x))\; dx.$$

In Theorem 6.1, under appropriate assumptions on the initial data, we will show that

$$\text{support } \mu_t = \Gamma_t,$$

$$\mu_t \geq \pi \; \mathcal{H}^{n-2} \llcorner \Gamma_t,$$

where $\mathcal{H}^{n-2}\llcorner\Gamma_t$ is the Hausdorff measure restricted to Γ_t, i.e., it is the surface area measure on the surface Γ_t. Moreover, the limit J_t of the Jacobians $Ju^\epsilon(t,\cdot)$ satisfy

$$|J_t| = \pi \; \mathcal{H}^1 \llcorner \Gamma_t. \tag{1.7}$$

To prove the inclusion support $(\mu_t) \subset \Gamma_t$, we use the energy identities and a Pohazaev type inequality as in [17]. The idea is to estimate the time derivative of

$$\alpha(t) : \;\; = \int_{I\!R^n} \; \eta(t,x) \; \mu_t(dx),$$

when the test function η is the square distance function of Γ_t. Since $\{\Gamma_t\}_{t\in[0,T]}$ solves the mean curvature flow, the square distance function η satisfies

$$\nabla \eta_t = \nabla \Delta \eta, \qquad \text{on } \; \Gamma_t.$$

We use this and the other properties of the square distance function to prove that $\alpha(t) = 0$ for $t \in [0, T]$. This proves the inclusion support $(\mu_t) \subset \Gamma_t$.

The opposite inclusion is proved by studying the Jacobian. In view of the energy estimate

$$\int_{I\!R^n} E^\epsilon(u^\epsilon(T,x))\ dx + \int_0^T \int_{I\!R^n} |u_t^\epsilon(t,x)|^2\ dx\ dt = \int_{I\!R^n} E^\epsilon(u^\epsilon(0,x))\ dx,$$

our compactness result implies that $J_t^\epsilon := Ju^\epsilon(u^\epsilon(t,\cdot))$ is compact. Let J_t be a limit of J_t^ϵ. Then, by the previous inclusion we know that the support of J_t is in Γ_t. Moreover, by the weak formulation of the Jacobian (see §2.1), the Jacobian is divergence free. Since Γ_t is smooth manifold with no boundary, this implies that the density of the Jacobian on Γ_t is constant. We then show that this constant is equal to π for all $t \in [0, T]$, proving (1.7). Also, in view of the compactness result, Theorem 5.2, the energy measure dominates the Jacobian measure. Hence, the support of μ_t is equal to Γ_t.

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2 Energy Lower Bounds

In this section we prove an energy lower bound in terms of the topological degree. This bound is local in nature and is a key step in the proof of the Gamma limit as well as the dynamical properties of the vortices. Local energy lower bounds were proved by covering arguments by Jerrard [12] and Sandier [25]. Here, we follow the technique developed by Jerrard to prove these estimates. In the next subsection, we give a brief and a formal discussion of this technique. Then we will give the precise statement and the proof of the lower bound.

Let U is a bounded open subset of $\mathbb{R}^2$, and $u \in H^1(U; \mathbb{R}^2)$ is a function that we have assumed (without loss of generality) to be smooth. The goal is to find an energy estimate of the form

$$\int_U E^\epsilon(u)\, dx \geq \pi\ \ln(1/\epsilon)\ \deg(u; \partial U) - C,$$

for some constant C, independent of ϵ and u. We want this to hold for all $u \in H^1(U; \mathbb{R}^2)$ and $\epsilon \in (0, 1]$. However, there are problems with this estimate as it is stated. The function u may have a zero on the boundary of U. Then, the degree of u around ∂U is not even be defined. Also, when u has a zero very close to the boundary, most of the energy could be outside the domain U. These possibilities indicate that we have make either an assumption about the boundary behavior of u, or to modify the statement of the lower bound. The latter is better suited for the later use of these estimate as it makes the lower bound a "local" one. However the statement of this local lower bound is rather technical. To explain the main idea we first outline the proof under assumptions on the boundary behavior. Remarkably, the proof technique of this "easier" lower bound carried over the more technical one with very little change.

We start with a brief discussion of the degree.

2.1 Degree and Jacobian

In our arguments, we will use the degree and the Jacobian repeatedly. For that reason we recall their definition.

Let $V \subset U \subset \mathbb{R}^2$ and $|u| \neq 0$ on ∂V. Then, u admits a local representation $u = |u|e^{i\phi}$ on ∂V, and the degree of u around ∂V is given by

$$\deg(u; \partial V) = \frac{1}{2\pi} \int_{\partial V} \nabla\phi \cdot \vec{t}\, d\mathcal{H}^1(x),$$

where as in the Introduction, $\vec{t}$ is the unit tangent and $\int_{\partial V} \cdots d\mathcal{H}^1$ is the line integral around ∂V. For future reference, we note that

$$\deg(u; \partial V) = \deg(u/|u|; \partial V).$$

The Jacobian Ju of u satisfies

$$Ju = \frac{1}{2} \ \nabla \times j(u),$$

where

$$j(u) = u \times Du = det(u; Du).$$

Hence by the Stokes' theorem

$$\int_V Ju \ dx = \int_{\partial V} j(u) \cdot \vec{t} \ d\mathcal{H}^1. \tag{2.1}$$

Generalizations to the case when $U \subset I\!R^n$ will be discussed later.

For $u = |u|e^{i\varphi}$, we directly calculate that $j(u) = |u|^2 \nabla\varphi$. Hence

$$\nabla\varphi = j(v) = j(u)/|u|^2, \qquad v = u/|u|,$$

and by the Stokes' theorem

$$\deg(u; \partial V) = \frac{1}{2\pi} \int_{\partial V} \frac{j(u) \cdot \vec{t}}{|u|^2} \ d\mathcal{H}^1 = \frac{1}{2\pi} \int_{\partial V} j(v) \cdot \vec{t} \ d\mathcal{H}^1, \tag{2.2}$$

for all u which do not vanish on ∂V.

2.2 Covering argument

In this subsection we outline a covering technique developed by Jerrard to prove energy lower bounds [12]. Similar techniques were also used by Sandier [25]. To simplify the presentation, we assume that

$$|u(x)| > \frac{1}{2}, \text{ whenever } x \in U_{r_0}, \quad U_{r_0} := \{ \ x \in U \ | \ \ dist(x, \partial U) \geq r_0 \} \tag{2.3}$$

for some constant $r_0 > 0$. We also assume that

$$\deg(u; \partial U) \neq 0.$$

Theorem 2.1 (Jerrard [12]) *There exists a constant C, such that for all $\epsilon \in (0,1]$, and for all $u \in H^1$ satisfying above assumptions,*

$$\int_U E^\epsilon(u) \ dx \geq \pi \ \ln(r_0/\epsilon) - C.$$

A more general result which do not assume (2.3) will be proved later in this section.

We introduce some notation and definitions, taken from [12].

We let S denote the set on which $|u|$ is small, that is,

$$\{x \in U \ : \ |u(x)| \leq 1/2\}. \tag{2.4}$$

We define the *essential* part S_E of S to be

$$S_E := \cup\{\text{components } S_i \text{ of } S \ : \ \deg(u; \partial S_i) \neq 0.\}. \tag{2.5}$$

For any subset $V \subset U$ such that $\partial V \cap S_E \neq \emptyset$, we define the generalized degree

$$\mathrm{dg}(u; \partial V) := \sum \{\deg(u; \partial S_i) \mid \text{components } S_i \text{ of } S_E \text{ such that } S_i \subset V\}. \tag{2.6}$$

Notice that if u is nonzero on the boundary of V, the generalized degree agrees with the degree of u around ∂V. Hence the generalization of the degree is only relevant when u has zeroes on ∂V. But in this case, we could remove these zeroes by slightly modifying u and with small change in the energy. Hence, in view of the Ginzburg-Landau energy these zeroes are removable and this justifies the definition of S_E and the generalized degree.

In view of 2.3,

$$U_{r_0} \cap S_E \neq \emptyset,$$

and by the definition of the generalized degree and the assumption that the degree of u around ∂U is nonzero,

$$\mathrm{dg}(u; \partial U_r) = \deg(u; \partial U_r) = \deg(u; \partial U) \neq 0, \quad \forall\ r \in (0, r_0].$$

Our strategy for proving Theorem 2.1 will be to find a collection of balls with a good lower bound for the Ginzburg-Landau energy on each ball. We then show that the sum of the radii of the balls is bounded below by $r_0/2$, hence obtaining a lower bound for the total Ginzburg-Landau energy in terms of this quantity. This will be done in several steps.

1. First cover of S_E.

We find the collection of balls by starting from an initial collection of small balls that cover S_E^ϵ, then letting these balls grow by expanding them and combining them. The first step is thus to establish the existence of the initial collection of small balls. This is the content of

Proposition 2.2 *There is a collection of closed, pairwise disjoint balls $\{B_i^*\}_{i=1}^k$ with radii r_i^* such that*

$$S_E \subset \cup_{i=1}^k B_i^*, \tag{2.7}$$

$$r_i^* \geq \epsilon \qquad \forall i. \tag{2.8}$$

$$\int_{B_i^* \cap U} E^\epsilon(u) dx \ \geq \ \frac{c_0}{\epsilon} r_i^*. \tag{2.9}$$

Proof. This is proved in [12]. Let $\{S_i\}_{\{i=1,\dots,N\}}$ be the disjoint components of S_E. Choose $x_i \in S_i$ for each i, and set

$$\rho_i := \inf\{r > 0 \ : \ \partial B_r(x_i) \cap S_i \neq \emptyset\}, \quad r_i := \max\{\epsilon \ ; \ \rho_i\}.$$

Set $B_i := B_{r_i}(x_i)$ so that in view of the definition of r_i

$$S_i \subset B_i \cap U.$$

Note that $|u| = 1/2$ on ∂S_i, and

$$E^\epsilon(u) \geq \frac{1}{2}|Du|^2 \geq |Ju|.$$

By (2.1) and (2.2),

$$\begin{aligned}
\int_{B_i \cap U} E^\epsilon(u) \ dx \ &\geq \ \left| \int_{S_i} Ju \ dx \right| \\
&\geq \ \left| \int_{\partial S_i} j(u) \cdot \vec{t} d\mathcal{H}^1 \right| \\
&\geq \ \frac{1}{2\pi} \left| \deg(u; \partial S_i) \right| \\
&\geq \ \frac{1}{2\pi}.
\end{aligned}$$

Moreover,

$$\partial B_r(x_i) \cap S_i \neq \emptyset, \quad \forall \ r \in (0, \rho_i].$$

This means that for every $r \in (0, \rho_i]$, there is a point $x^* \in \partial B_r(x_i)$ such that $|u(x^*)| < 1/2$. Due to the potential term $(1 - |u|^2)^2/4\epsilon^2$ term in the energy E^ϵ, near x^*, is large and in Lemma 2.7 below we will show that

$$\int_{\partial B_r(x_i) \cap U} E^\epsilon(u) \ d\mathcal{H}^1 \geq C \ \frac{1}{\epsilon}, \qquad \forall \ r \in [\epsilon, \rho_i],$$

for some constant C. Assume that $r_i \geq \epsilon$, and integrate this estimate over $[\epsilon, r_i]$. The result is

$$\int_{B_i \cap U} E^\epsilon(u)\ dx \geq C \frac{(r_i - \epsilon)^+}{\epsilon}.$$

Combining the two estimates,

$$\int_{B_i \cap U} E^\epsilon(u)\ dx \geq C\ \max\{\ \frac{(r_i - \epsilon)^+}{\epsilon}\ ;\ 1\ \} \geq \frac{C}{2}\ \frac{r_i}{\epsilon}.$$

The balls constructed above may not be disjoint. If two or more of these balls intersect, they can be combined into larger balls, relabeling as necessary. One can use the Besicovitch Covering Theorem to control the overlap and show that the larger balls still satisfy (2.9). The details of this argument appear in [12]. $\square$

2. Annulus estimate.

In the previous step, we did not attempt to make the covering as large as possible. In particular, they could be off the size ϵ and when we add them we will not get the desired energy estimate.

In this step, we obtain an estimate which will be used when we extend the balls in our covering.

Suppose that $x^* \in U$, and $\epsilon \leq r_0 < r_1$ satisfy

$$[B_{r_1}(x^*) \setminus B_{r_0}(x^*)] \cap S_E = \emptyset,$$

and

$$\mathrm{dg}(u; \partial B_{r_1}(x^*)) \neq 0.$$

Then, for all $r \in [r_0, r_1]$,

$$\mathrm{dg}(u; \partial B_r(x^*)) = \mathrm{dg}(u; \partial B_{r_1}(x^*)) \neq 0.$$

Set

$$\lambda^\epsilon(r) = \min_{m \in [0,1]} \left[\frac{m^2 \pi}{r} + \frac{(1-m)^2}{c_0 \epsilon} \right], \qquad \Lambda^\epsilon(r) := \int_0^r \lambda^\epsilon(s) \wedge \frac{c_1}{\epsilon} ds \tag{2.10}$$

for certain constants c_0, c_1 whose choice is discussed below.

Lemma 2.3 *There are constants c_0 and c_1 so that*

$$(2.11) \qquad \int_{B_{r_1}(x_0)\setminus B_{r_0}(x_0)} E^\epsilon(u)dx \geq [\Lambda^\epsilon(r_1) - \Lambda^\epsilon(r_0)].$$

Proof. This is Proposition 3.2, [12]. The key estimate is

$$(2.12) \qquad \int_{\partial B_r(x^*)\cap U} E^\epsilon(u)\ d\mathcal{H}^1 \geq \lambda^\epsilon(r), \quad \forall r \in [r_0, r_1].$$

We then obtain (2.11) by integrating (2.12) over $r \in [r_0, r_1]$.

Set

$$m := \min_{\partial B_r(x^*)} \{ \ |u(x)|\ \}.$$

Then, as in Lemma 2.7 below, we can prove that

$$\int_{\partial B_r(x^*)\cap U} E^\epsilon(u)\ d\mathcal{H}^1 \geq \frac{(1-m)^2}{c_0 \epsilon}.$$

For $u = |u|e^i\varphi$,

$$E^\epsilon(u) = \frac{1}{2}\ |u|^2|\nabla\varphi|^2 + \frac{1}{2}\ |\nabla|u||^2\,.$$

Since

$$|j(u)| = |u|^2|\nabla\varphi|,$$

$$E^\epsilon(u) = \frac{1}{2}\ \frac{|j(u)|^2}{|u|^2} + \frac{1}{2}\ |\nabla|u||^2 \geq \frac{1}{2}\ |u|^2\ |j(u/|u|)|^2 \geq \frac{m^2}{2}\ |j(u/|u|)|^2.$$

Suppose that $m > 0$. Then, for $r \in [r_0, r_1]$, $\deg(u; \partial[B_r^* \cap U]) = \mathrm{dg}(u; \partial[B_r^* \cap U]) \neq 0$, and

$$\begin{aligned}
\int_{\partial[B_r(x^*)\cap U]} E^\epsilon(u)\ d\mathcal{H}^1 \quad &\geq \quad \frac{m^2}{2} \int_{\partial[B_r(x^*)\cap U]} |j(u/|u|)|^2\ d\mathcal{H}^1 \\
&\geq \quad \frac{m^2}{4\pi r} \left| \int_{\partial[B_r(x^*)\cap U]} j(u/|u|)\ d\mathcal{H}^1 \right|^2 \\
&= \quad \frac{\pi\ m^2}{r}\ |\deg(u; \partial[B_r(x^*)\cap U]|^2 \\
&\geq \quad \frac{\pi\ m^2}{r}.
\end{aligned}$$

Combining the two preceeding estimates we obtain (2.12). $\square$

3. Properties of λ^ϵ.

The following elementary estimates are proved in Propositions 3.1 and 3.2 in [12]:

$$\Lambda^\epsilon(r_1 + r_2) \le \Lambda^\epsilon(r_1) + \Lambda^\epsilon(r_2) \ , \tag{2.13}$$

$$s \mapsto \frac{1}{s}\ \Lambda^\epsilon(s) \quad \text{is non increasing} \ , \qquad \frac{1}{s}\Lambda^\epsilon(s) \le \frac{c_1}{\epsilon} \quad \forall s \tag{2.14}$$

and $\Lambda^\epsilon(r) \ge \pi \ln(r/\epsilon) - c_2$ for some constant c_2. Also, clearly, $\lambda^\epsilon(r) \le \pi/r$, and therefore, by redefining c_2 if necessary,

$$|\Lambda^\epsilon(r) - \pi \ln(r/\epsilon)\ | \le c_2 \qquad \forall\ r \ge \epsilon \ . \tag{2.15}$$

4. Amalgamation.

Our next result is Lemma 3.1 in [12]. It is used below when we allow the small balls to grow and merge, to form large balls. For the sake of completeness, here we state it and give its short proof.

Lemma 2.4 *Given any finite collection of closed balls in* $I\!R^k$*, say* $\{C_i\}_{i=1}^N$*, we can find a collection* $\{\tilde{C}_i\}_{i=1}^{\tilde{N}}$ *of pairwise disjoint balls such that*

$$\bigcup_{i=1}^N C_i \subset \bigcup_{i=1}^{\tilde{N}} \tilde{C}_i,$$

$$\sum_{C_j \subset \tilde{C}_i} diam C_j = diam \tilde{C}_i,$$

$$\tilde{N} \le N, \qquad \textit{with strict inequality unless } \{C_i\}_{i=1}^N \textit{ is pairwise disjoint.}$$

Proof. Replace pairs of intersecting balls C_i, C_j by larger single balls $\tilde{C}$ such that $C_i \cup C_j \subset \tilde{C}$ and diam $\tilde{C}$ = diam C_i+diam C_j, continuing until a pairwise disjoint collection is reached. This collection has the stated properties. □

5. Cover of S_E.

We next show that, starting from the initial collection of balls, we can let them grow in such a way that each ball continues to satisfy a good lower bound.

As above let $\{B_i^*\}$ denote the balls found in Proposition 2.2, with radii r_i^* and generalized degree $d_i^* := \text{dg}(u; \partial[B_i^* \cap U])$. Define

$$\sigma^* := \min\{ \, r_i^* \mid d_i^* \neq 0 \, \} \; .$$

The idea is to extend the balls with the smallest radius with nonzero degree until they hit each other or the boundary of U. When they hit each other we use the amalgamation lemma and continue the process until one of the balls with nonzero degree hits the boundary of U. However, here we follow the presentation of Sandier and Serfaty [27]. Although this is slightly more technical than the outline procedure, it extends very easily to more general situations.

Proposition 2.5 *For every $\sigma \geq \sigma^*$, there exists a collection of disjoint, closed balls $\mathcal{B}(\sigma) = \{B_k^\sigma\}_{k=1}^{k(\sigma)}$ satisfying $r_k^\sigma \geq \epsilon$,*

$$S_E \subset \cup_k \; B_k^\sigma \; , \tag{2.16}$$

$$\int_{U\cap B_k^\sigma} E^\epsilon(u) \; dx \geq \frac{r_k^\sigma}{\sigma} \; \Lambda^\epsilon(\sigma) \; , \tag{2.17}$$

$$r_k^\sigma \geq \sigma \qquad \textit{whenever } B_k^\sigma \cap \partial U = \emptyset \; \textit{, and } d_k^\sigma \neq 0, \tag{2.18}$$

where r_k^σ is the radius and $d_k^\sigma = dg(u; \partial[B_k^\sigma \cap U])$ is the generalized degree.

Proof.

Let C be the set of all $\sigma \geq \sigma^*$ for which such a collection exists.

1. We first claim that $\sigma^* \in C$. Indeed $\{B_k^*\}$ be the collection of balls constructed in Proposition 2.2. Set $\mathcal{B}(\sigma^*) := \{B_k^*\}$. The definition 2.10 of Λ^ϵ easily implies that $\Lambda^\epsilon(\sigma)/\sigma \leq c_1/\epsilon$ forall σ, so Proposition 2.2 implies that this collection satisfies (2.16) and (2.17). Also, (2.18) is satisfied due to the definition of σ^*.

2. In step we will show that C is closed. Let $\{\sigma^n\}_n$ be a sequence in C and suppose that σ^n converges to σ_0 as n tends to infinity. Since the balls are disjoint, and their radii are at least ϵ, the total number of balls $k(\sigma_n)$ is uniformly bounded in n. Therefore by passing to a subsequence we may assume that $k(\sigma_n)$ is equal to a constant k_0 independent of n. By passing to a further subsequence, we may assume that the radii $r_k^{\sigma_n}$ and the centers $a_k^{\sigma_n}$ converge to r_k^0 and a_k^0, respectively, for each $k \leq k_0$. Let $B_{k,0}$ be the closed ball centered at a_k^0 with radius r_k^0. It is clear that this collection of balls satisfies (2.16), (2.17), and (2.18). If the balls are disjoint , we set $\mathcal{B}(\sigma_0) := \{B_{k,0}\}_{k=1}^{k_0}$. If they are not disjoint, we apply the amalgamation process outlined in Lemma 2.4. Let $\{B_j^{\sigma_0}\}$ be the resulting balls and $r_j^{\sigma_0}$ be their radius. Then, by Lemma 2.4

$$r_j^{\sigma_0} \; = \sum_{B_{k,0} \subset B_j^{\sigma_0}} r_{k,0}. \tag{2.19}$$

Since $\{B_{k,0}\}$ satisfies (2.17), this implies that

$$
\begin{aligned}
\int_{U\cap B_j^{\sigma_0}} E^\epsilon(u)\,dx &\geq \sum_{B_{k,0}\subset B_j^{\sigma_0}} \int_{U\cap B_{k,0}} E^\epsilon(u)\,dx \\
&\geq \sum_{B_{k,0}\subset B_j^{\sigma_0}} \frac{r_{k,0}}{\sigma_0}\,\Lambda^\epsilon(\sigma_0) \\
&= \frac{r_j^{\sigma_0}}{\sigma_0}\,\Lambda^\epsilon(\sigma_0)\;.
\end{aligned}
$$

Hence, $\mathcal{B}(\sigma_0) := \{B_j^{\sigma_0}\}$ satisfies (2.17). Moreover,

$$
\left|d_j^{\sigma_0}\right| = \left|\sum_{B_{k,0}\subset B_j^{\sigma_0}} d_{k,0}\right| \leq \sum_{B_{k,0}\subset B_j^{\sigma_0}} |d_{k,0}|.
$$

Hence if $B_j^{\sigma_0}\cap\partial U=\emptyset$ and $d_j^{\sigma_0}\neq 0$, then $B_{k,0}\cap\partial U=\emptyset$ for all $B_{k,0}\subset B_j^{\sigma_0}$ and at least one $d_{k^*,0}\neq 0$. Since $B_{k^*,0}$ satisfies (2.18),

$$
r_j^{\sigma_0}\geq r_{k^*,0}\geq\sigma_0.
$$

This implies that the balls in the collection $\mathcal{B}(\sigma_0)$ satisfy (2.18).

3. Suppose that $\sigma_1\in C$. We will show that there is $\delta>0$ such that $[\sigma_1,\sigma_1+\delta]\subset C$. Indeed, let

$$
K_1 := \{\ k \mid B_k^{\sigma_1}\cap\partial U=\emptyset \ \text{ and } \ d_k^{\sigma_1}\neq 0\ \},
$$

and set

$$
s_1 := \min_{k\in K_1}\ \{\ r_k^{\sigma_1}\ \}.
$$

By (2.18), $\sigma_1\leq s_1$. If this inequality is strict, we set $\mathcal{B}(\sigma)=\mathcal{B}(\sigma_1)$ for all $\sigma\in[\sigma_1,s_1]$. It is clear that this collection of balls satisfies (2.16), and (2.18). Also (2.17) follows from (2.14). So let us assume that $s_1=\sigma_1$, and let

$$
K_2 := \{\ k\in K_1 \mid s_1=r_k^{\sigma_1}\ \}.
$$

For $\sigma\geq\sigma_1$, set

$$
r_k^\sigma := \begin{cases} r_k^{\sigma_1}\ , & \text{if } k\notin K_2\ , \\[2ex] \frac{\sigma}{\sigma_1}\ r_k^{\sigma_1}\ , & \text{if } k\in K_2\ . \end{cases}
$$

Let B_k^σ be the closed ball with radius r_k^σ with the same center as $B_k^{\sigma_1}$ and let $\mathcal{B}(\sigma)$ be the collection of these balls. Since $\{B_k^{\sigma_1}\}_k$ are disjoint closed sets, there is $\delta_1 > 0$ such that for all $\sigma \in [\sigma_1, \sigma_1 + \delta_1]$ B_k^σ's are disjoint and

$$K_1(\sigma) := \{ \ k \mid B_k^\sigma \cap \partial U = \emptyset \text{ and } d_k^\sigma \neq 0 \ \} = K_1 \ .$$

Then, for $k \in K_2$,

$$\frac{r_k^\sigma}{\sigma} = \frac{r_k^{\sigma_1}}{\sigma_1} = 1.$$

and for $k \notin K_2$,

$$\frac{r_k^\sigma}{\sigma} = \frac{\sigma_1}{\sigma} \ \frac{r_k^{\sigma_1}}{\sigma_1} \ .$$

Since for $k \notin K_2$, $r_k^{\sigma_1} > \sigma_1$, there is $0 < \delta \leq \delta_1$ such that (2.18) is satisfied by the collection $\mathcal{B}(\sigma)$. Since $r_k^\sigma \geq r_k^{\sigma_1}$, (2.16) is also satisfied.

To verify (2.17), we observe that for $k \notin K_2$, $B_k^\sigma = B_k^{\sigma_1}$ and (2.17) is satisfied in light of (2.14). If, however, $k \in K_2$, then

$$d_k^\sigma = d_k^{\sigma_1} \ , \qquad\qquad r_k^\sigma = \sigma \ , \tag{2.20}$$

and

$$[B_k^\sigma \setminus B_k^{\sigma_1}] \cap S^E = \emptyset \ .$$

Then by (2.11),

$$\begin{aligned}
\int_{B_k^\sigma} E^\epsilon(u) \ dx \quad &= \quad \int_{B_k^{\sigma_1}} E^\epsilon(u) \ dx + \int_{B_k^\sigma \setminus B_k^{\sigma_1}} E^\epsilon(u) \ dx \\
&\geq \quad \frac{r_k^{\sigma_1}}{\sigma_1} \ \Lambda^\epsilon(\sigma_1) + [\Lambda^\epsilon \left(r_k^\sigma\right) - \Lambda^\epsilon \left(r_k^{\sigma_1}\right)] \\
&= \quad \Lambda^\epsilon \left(r_k^{\sigma_1}\right) + [\Lambda^\epsilon \left(r_k^\sigma\right) - \Lambda^\epsilon \left(r_k^{\sigma_1}\right)] \\
&= \quad \Lambda^\epsilon \left(r_k^\sigma\right) \\
&= \quad \frac{r_k^\sigma}{\sigma} \ \Lambda^\epsilon \left(\sigma\right) \ .
\end{aligned}$$

Here we repeatedly used the identities (2.20) and the fact that $B_k^{\sigma_1}$ satisfies (2.17). Hence $\mathcal{B}(\sigma)$ also satisfies (2.17) for all $\sigma \in [\sigma_1, \sigma_1 + \delta]$.

4. We have shown that C is closed set including σ^* and for every $\sigma \in C$, there exists $\delta > 0$ such that $[\sigma, \sigma + \delta] \subset C$. Hence, $C = [\sigma^*, \infty)$. $\square$

6. Proof of Theorem 2.1

Let σ^* be as in the previous Lemma and let r_0 be as in (2.3).

1. First suppose that $r_0 < 2\sigma^*$. The opposite inequality will be treated later in the proof.

Consider the balls $\{B_k^*\}$ constructed in Proposition 2.2. Set

$$K^* := \{ \ k \mid \text{and } d_k^* \neq 0 \ \}.$$

Recall that $d_k^* = \mathrm{dg}(u; \partial[B_k^* \cap U])$. Since

$$0 \neq \deg(u; \partial U) = \sum_k \ \mathrm{dg}(u; B_k^*)$$

K^* is nonempty. Then, by (2.9) and the definition of σ^*,

$$\begin{aligned}
\int_U E^\epsilon(u)\ dx \quad &\geq \quad \sum_k \int_{U \cap B_k^*} E^\epsilon(u)\ dx \geq \sum_k \frac{c_1}{\epsilon}\ r_k^* \\
&\geq \quad \frac{c_1 \sigma^*}{\epsilon} \sum_{k \in K^*} 1 \ \geq \ c_1 \frac{r_0}{2\epsilon}\ |K^*| \ \geq \ c_1 \frac{r_0}{2\epsilon} \\
&\geq \quad \Lambda^\epsilon \left(\frac{r_0}{2} \right).
\end{aligned}$$

In view of (2.15), this gives the desired lower bound.

2. We now assume that $r_0 \geq 2\sigma^*$. Set $\bar{\sigma} := r_0/2$ and consider the collection of balls $\mathcal{B}(\bar{\sigma})$ provided by Proposition 2.5. Let

$$\bar{K} := \{ \ k \mid d_k^{\bar{\sigma}} \neq 0 \ \}.$$

Since $\deg(u; \partial U) \neq 0$, $\bar{K}$ is nonempty. We claim that

$$r_k^{\bar{\sigma}} \geq \bar{\sigma} = \frac{r_0}{2}, \qquad \forall\ k \in \bar{K}.$$

Indeed if $B_k^{\bar{\sigma}} \cap \partial U = \emptyset$ for some $k \in \bar{K}$, then this follows from (2.18). Suppose that $B_k^{\bar{\sigma}} \cap \partial U \neq \emptyset$ for some $k \in \bar{K}$. Since $d_k^{\bar{\sigma}} \neq 0$, $B_k^{\bar{\sigma}}$ contains a zero of u, and by (2.3),

$$B_k^{\bar{\sigma}} \cap [U \setminus U_{r_0}] \neq \emptyset.$$

Since by assumption $B_k^{\bar{\sigma}} \cap \partial U \neq \emptyset$, this implies that the diameter of $B_k^{\bar{\sigma}}$ is at least r_0, proving the claim.

3. By the previous step

$$\sum_{k \in \bar{K}} r_k^{\bar{\sigma}} \geq \bar{\sigma} = \frac{r_0}{2}.$$

Hence by (2.17),

$$\begin{aligned}\int_U E^\epsilon(u)\,dx &\geq \sum_{k\in\bar K}\int_{U\cap B_k^{\bar\sigma}} E^\epsilon(u)\,dx\\ &\geq \sum_{k\in\bar K} r_k^{\bar\sigma}\,\frac{\Lambda^\epsilon(\bar\sigma)}{\bar\sigma}\\ &\geq \Lambda^\epsilon(\bar\sigma)\ .\end{aligned}$$

□

2.3 Main lower bound

In this section, we prove a generalization of the lower bound. This sharper lower bound is taken from [13] and its proof is very similar to that of Theorem 2.1. Here we repeat the arguments of the previous section with minor modifications for the sake of completeness.

Let U be a bounded open subset of $I\!R^2$, and $u \in H^1(U; I\!R^2)$ be a function that we have assumed (without loss of generality) to be smooth. In addition, ϕ is a nonnegative Lipschitz test function that vanishes on ∂U.

Throughout this section we will use the notation

$$T \;=\; \|\phi\|_\infty \;=\; \max_U \phi(x). \tag{2.21}$$

Given $\phi \in C_c^{0,1}(U)$ we use the notation

$$\mathrm{Reg}(\phi) \;:=\; \left\{t\in[0,T] : \partial\Omega(t) = \phi^{-1}(t), \partial\Omega(t) \text{ is rectifiable}, \mathcal{H}^1(\partial\Omega(t)) \;<\; \infty\right\}. \tag{2.22}$$

The coarea formula implies that $\mathrm{Reg}(\phi)$ is a set of full measure. For every $t \in \mathrm{Reg}(\phi)$, $\partial\Omega(t)$ is a union of finite Jordan curves $\Gamma_i(t)$, i.e.,

$$\partial\Omega(t) = \cup_i\ \Gamma_i(t)\ , \qquad \forall\ t\in \mathrm{Reg}(\phi).$$

In particular this holds for almost every t. For $t \in \mathrm{Reg}(\phi)$ we define

$$\Gamma(t) = \cup\left\{\text{ components } \Gamma_i(t) \text{ of } \partial\Omega(t) \mid \min_{x\in\Gamma_i(t)} |u(x)| > 1/2\right\}. \tag{2.23}$$

We also define $\gamma(t) = \partial\Omega(t)\setminus\Gamma(t)$,

$$\gamma(t) = \cup\left\{\text{ components } \Gamma_i(t) \text{ of } \partial\Omega(t) \mid \min_{x\in\Gamma_i(t)} |u(x)| \leq 1/2\right\}. \tag{2.24}$$

When we want to indicate explicitly the dependence of $\Gamma(t)$ on ϕ and u, we will write $\Gamma_{\phi,u}(t)$.

We will also use the notation

$$t_\epsilon := \epsilon\|\nabla\phi\|_\infty \tag{2.25}$$

For any positive integer d, let

$$D_d^\epsilon \ := \ \{t \in \mathrm{Reg}(\phi) \ : \ t \geq t_\epsilon, \Gamma(t) \text{ is nonempty, and } \ |\mathrm{deg}(u;\Gamma(t))| \geq d\}\,. \tag{2.26}$$

The main result of this section is the following theorem. We follow very closely arguments introduced in [12].

Theorem 2.6 (Jerrard & Soner [13]) *If $u : U \to I\!R^2$ is a smooth function and ϕ is a nonnegative Lipschitz function such that $\phi = 0$ on ∂U, then for any positive integer d,*

$$\int_{spt(\phi)} E^\epsilon(u) \ \geq \ d\Lambda^\epsilon\left(\frac{|D_d^\epsilon|}{2d\|\nabla\phi\|_\infty}\right).$$

For any $t_2 > t_1$ the ratio $(t_2 - t_1)/\|\nabla\phi\|_\infty$ is a lower bound for the distance between $\partial\Omega(t_2)$ and $\partial\Omega(t_1)$. This explains the role of $\|\nabla\phi\|_\infty$ in the estimate.

Similar results were proven in [12] under more or less the assumption that D_d^ϵ is an interval; and in [7] in the case $d = 1$. Related results have also appeared in Sandier [26].

Note that the case covered in the statement of the theorem, $\{x : \phi(x) > 0\} \subset U$, can be reduced to the case $\{x : \phi(x) > 0\} = U$, if we replace U by $\tilde{U} := \{x \in U \ : \ \phi(x) > 0\}$. So we will henceforth assume for notational simplicity that this holds, so that $\mathrm{spt}(\phi) = \bar{U}$.

For the proof of Theorem 2.6 we define

$$S_E^\epsilon := \cup\{\text{components } S_i \text{ of } S_E \ : \ S_i \subset \Omega(t_\epsilon)\}. \tag{2.27}$$

If $x \in \Omega(t_\epsilon)$ and $y \in \partial U$, then

$$|x - y| \ \|\nabla\phi\|_\infty \ \geq \ |\phi(x) - \phi(y)| = |\phi(x)| \ \geq \ t_\epsilon \ = \epsilon\|\nabla\phi\|_\infty.$$

In particular,

$$\mathrm{dist}(x, \partial U) \geq \epsilon \qquad \text{for all } x \in S_E^\epsilon. \tag{2.28}$$

Note also that if $V \subset \Omega(t_\epsilon)$ and $\partial V \cap S_E = \emptyset$, then

$$\mathrm{dg}(u;\partial V) := \sum\{\mathrm{deg}(u;\partial S_i) \mid \text{components } S_i \text{ of } S_E^\epsilon \text{ such that } S_i \subset\subset V\}\,.$$

In other words, for such sets V we can ignore $S_E \setminus S_E^\epsilon$ when computing $\text{dg}(u; \partial V)$. In the proof of Theorem 1 below we will always be concerned with subsets $V \subset \Omega(t_\epsilon)$, so this will always be the case.

Our strategy for proving Theorem 2.6 is very similar to the method we used to prove Theorem 2.1. We will first find a collection of balls such we have a good lower bound for the Ginzburg-Landau energy on each ball. We then show that the sum of the radii of the balls is bounded below by $|D_d^\epsilon|/(2\|\nabla\phi\|_\infty)$, hence obtaining a lower bound for the total Ginzburg-Landau energy in terms of this quantity. We start with a technical step.

1. If $\gamma(t) \neq \emptyset$.

In this step, we prove an estimate in the case when one of the level sets intersects with the zero set. In this case, $|u|$ falls below $1/2$ on $\gamma(t)$ and we expect the Ginzburg-Landau energy to be large on $\gamma(t)$. The following technical lemma proves this under the assumption that $\gamma(t)$ is not too small.

Lemma 2.7 *Suppose that*

$$\mathcal{H}^1(\gamma(t)) \geq \epsilon \ .$$

Then

$$\int_{\partial\Omega(t)} E^\epsilon(u)\ d\mathcal{H}^1 \geq \frac{1}{25\epsilon} \ . \tag{2.29}$$

Proof.

This is very similar to Lemma 2.3 in [12]. Fix a connected component $\Gamma_i(t)$ of $\gamma(t)$ and set $\rho := |u|$ and

$$\gamma_i := \int_{\Gamma_i(t)} \frac{1}{2}|\nabla\rho|^2\ d\mathcal{H}^1 \ .$$

By the definition (2.24) of $\gamma(t)$ there is a point $x_{min} \in \Gamma_i(t)$ such that $\rho(x_{\min}) \leq 1/2$. Parametrize $\Gamma_i(t)$ by arclength so that

$$\Gamma_i(t) = \{\ x(s) \mid s \in [0, G_i]\}\ , \qquad G_i := \mathcal{H}^1(\Gamma_i(t))$$

with $x_{\min} = x(0) = x(G_i)$. Then since $|\dot{x}(s)| = 1$,

$$\begin{aligned} \rho(x(s)) &= \rho(x(0)) + \int_0^s \nabla\rho(x(r)) \cdot \dot{x}(r)\ dr \\ &\leq \frac{1}{2} + s^{1/2}\left(\int_0^s |\nabla\rho(x(r))|^2 dr\right)^{1/2} \\ &\leq \frac{1}{2} + \sqrt{\gamma_i\ s} \leq \frac{3}{4}\ , \end{aligned}$$

provided that $s \le \sigma_i := [G_i \wedge 1/(16\gamma_i)]$. Then, for $s \in [0, \sigma_i]$, $(1-\rho^2(x(s)))^2/4 \ge 1/25$. Therefore,

$$
\begin{aligned}
\int_{\Gamma_i(t)} E^\epsilon(u)\ d\mathcal{H}^1 &\ge \gamma_i + \int_{\Gamma_i(t)} \frac{1}{4\epsilon^2}(1-\rho^2)^2\ d\mathcal{H}^1 \\
&\ge \gamma_i + \frac{\sigma_i}{25\ \epsilon^2}\ .
\end{aligned}
$$

By calculus,

$$
\gamma_i + \frac{\sigma_i}{25\ \epsilon^2} = \gamma_i + \frac{G_i \wedge (1/4\gamma_i)}{25\ \epsilon^2} \ge \frac{1}{25\epsilon}\ \left[\frac{G_i}{\epsilon} \wedge 5\right]\ .
$$

Thus

$$
\int_{\Gamma_i(t)} E^\epsilon(u)\ d\mathcal{H}^1 \ge \frac{1}{25\epsilon}\ \left[\frac{G_i}{\epsilon} \wedge 5\right]\ .
$$

Since

$$
\mathcal{H}^1(\gamma(t)) = \sum_{\{i|\Gamma_i(t) \text{ is a component of } \gamma(t)\}} \mathcal{H}^1(\Gamma_i(t)) = \sum_i\ G_i \ge \epsilon\ ,
$$

we can sum over components $\Gamma_i(t)$ of $\gamma(t)$ to conclude that

$$
\int_{\partial\Omega(t)} E^\epsilon(u)\ d\mathcal{H}^1 \ge \frac{1}{25\epsilon}\ .
$$

$\square$

2. First Cover.

This is very similar to Step 1 of the previous subsection.

We find the collection of balls by starting from an initial collection of small balls that cover S_E^ϵ, then letting these balls grow by expanding them and combining them. The first step is thus to establish the existence of the initial collection of small balls. This is the content of

Proposition 2.8 *There is a collection of closed, pairwise disjoint balls $\{B_i^*\}_{i=1}^k$ with radii r_i^* such that*

$$
S_E^\epsilon \subset \cup_{i=1}^k B_i^*, \tag{2.30}
$$

$$
r_i^* \ge \epsilon \qquad \forall i. \tag{2.31}
$$

$$
\int_{B_i^* \cap U} E^\epsilon(u)dx\ \ge\ \frac{c_0}{\epsilon} r_i^*\ \ge\ \Lambda^\epsilon(r_i^*) \tag{2.32}
$$

This is essentially proved in the previous subsection; Proposition 2.2 and in Proposition 3.3 in [12].

Proposition 2.8 differs from Proposition 2.2 in that in the latter, S_E appears in place of S_E^ϵ in the counterpart of (2.30).

3. Cover of S_E^ϵ.

This step is similar to Step 5 of the previous section.

Starting from the initial collection of balls, we can let them grow in such a way that each ball continues to satisfy a good lower bound.

As above let $\{B_i^*\}$ denote the balls found in Proposition 2.8, with radii r_i^* and generalized degree $d_i^* := \mathrm{dg}(u; \partial[B_i^* \cap U])$. Define

$$\sigma^* := \min\{ \ \frac{r_i^*}{|d_i^*|} \ | \ d_i^* \neq 0 \ \} \ .$$

Proposition 2.9 *For every $\sigma \geq \sigma^*$, there exists a collection of disjoint, closed balls $\mathcal{B}(\sigma) = \{B_k^\sigma\}_{k=1}^{k(\sigma)}$ satisfying $r_k^\sigma \geq \epsilon$,*

$$S_E^\epsilon \subset \cup_k \ B_k^\sigma \ , \tag{2.33}$$

$$\int_{U \cap B_k^\sigma} E^\epsilon(u) \ dx \geq \frac{r_k^\sigma}{\sigma} \ \Lambda^\epsilon(\sigma) \ , \tag{2.34}$$

$$r_k^\sigma \geq \sigma |d_k^\sigma| \qquad \textit{whenever } B_k^\sigma \cap \partial U = \emptyset \ , \tag{2.35}$$

where r_k^σ is the radius and d_k^σ is the generalized degree.

The proof of this proposition is very is very similar to that of Proposition 2.5. The only difference is we consider the ratio $r_k^\sigma/|d_k^\sigma|$ to decide which balls to expand.

Proof.

Let C be the set of all $\sigma \geq \sigma^*$ for which such a collection exists.

1. We first claim that $\sigma^* \in C$. Indeed $\{B_k^*\}$ be the collection of balls constructed in Proposition 2.8. Set $\mathcal{B}(\sigma^*) := \{B_k^*\}$. The definition 2.10 of Λ^ϵ easily implies that $\Lambda^\epsilon(\sigma)/\sigma \leq c_1/\epsilon$ forall σ, so Proposition 2.8 implies that this collection satisfies (2.33) and (2.34). Also, (2.35) is satisfied due to the definition of σ^*.

2. In step we will show that C is closed. Let $\{\sigma^n\}_n$ be a sequence in C and suppose that σ^n converges to σ_0 as n tends to infinity. Since the balls are disjoint, and their radii are at least ϵ, the total number of balls $k(\sigma_n)$ is uniformly bounded in n. Therefore by passing to a subsequence we may assume that $k(\sigma_n)$ is equal to a constant k_0 independent of n. By passing to a further subsequence, we may assume that the radii $r_k^{\sigma_n}$ and the centers $a_k^{\sigma_n}$ converge to r_k^0 and a_k^0, respectively, for each $k \le k_0$. Let $B_{k,0}$ be the closed ball centered at a_k^0 with radius r_k^0. It is clear that this collection of balls satisfies (2.33), (2.34), and (2.35). If the balls are disjoint , we set $\mathcal{B}(\sigma_0) := \{B_{k,0}\}_{k=1}^{k_0}$. If they are not disjoint, we apply the amalgamation process outlined in Lemma 2.4. Let $\{B_j^{\sigma_0}\}$ be the resulting balls and $r_j^{\sigma_0}$ be their radius. Then, by Lemma 2.4

$$r_j^{\sigma_0} = \sum_{B_{k,0} \subset B_j} r_{k,0} \ge \sum_{B_{k,0} \subset B_j} \sigma_0 |d_{k,0}|. \tag{2.36}$$

Since $\{B_{k,0}\}$ satisfies (2.34), this implies that

$$\begin{aligned} \int_{U \cap B_j} E^\epsilon(u)\, dx &\ge \sum_{B_{k,0} \subset B_j} \int_{U \cap B_{k,0}} E^\epsilon(u)\, dx \\ &\ge \sum_{B_{k,0} \subset B_j} \frac{r_{k,0}}{\sigma_0} \Lambda^\epsilon(\sigma_0) \\ &\ge \frac{r_j^{\sigma_0}}{\sigma_0} \Lambda^\epsilon(\sigma_0)\ . \end{aligned}$$

Hence, $\mathcal{B}(\sigma_0) := \{B_j^{\sigma_0}\}$ satisfies (2.34). Moreover,

$$|d_j^{\sigma_0}| = \left| \sum_{B_{k,0} \subset B_j} d_{k,0} \right| \le \sum_{B_{k,0} \subset B_j} |d_{k,0}|,$$

and this together with (2.36) implies that the balls in the collection $\mathcal{B}(\sigma_0)$ satisfy (2.35).

3. Suppose that $\sigma_1 \in C$. We will show that there is $\delta > 0$ such that $[\sigma_1, \sigma_1 + \delta] \subset C$. Indeed, let K_1 be the set of indices k such that $B_k^{\sigma_1} \cap \partial U = \emptyset$ and set

$$s_1 := \min_{k \in K_1} \frac{r_k^{\sigma_1}}{|d_k^{\sigma_1}|}\ .$$

By (2.35), $\sigma_1 \le s_1$. If this inequality is strict, we set $\mathcal{B}(\sigma) = \mathcal{B}(\sigma_1)$ for all $\sigma \in [\sigma_1, s_1]$. It is clear that this collection of balls satisfies (2.33), and (2.35). Also (2.34) follows from (2.14). So let us assume that $s_1 = \sigma_1$, and let $K_2 \subset K_1$ be the indices k which minimize the ratio $r_k^{\sigma_1}/d_k^{\sigma_1}$. For $\sigma \ge \sigma_1$, set

$$r_k^\sigma := \begin{cases} r_k^{\sigma_1}\ , & \text{if } k \notin K_2\ , \\ \frac{\sigma}{\sigma_1}\, r_k^{\sigma_1}\ , & \text{if } k \in K_2\ . \end{cases}$$

Let B_k^σ be the closed ball with radius r_k^σ with the same center as $B_k^{\sigma_1}$ and let $\mathcal{B}(\sigma)$ be the collection of these balls. Since $\{B_k^{\sigma_1}\}_k$ are disjoint closed sets, there is $\delta_1 > 0$ such that for all $\sigma \in [\sigma_1, \sigma_1 + \delta_1]$ B_k^σ's are disjoint and

$$K_1(\sigma) := \{\ k \mid B_k^\sigma \cap \partial U = \emptyset\ \} = K_1\ .$$

Then, for $k \in K_2$,

$$\frac{r_k^\sigma}{\sigma} = \frac{r_k^{\sigma_1}}{\sigma_1} = |d_k^{\sigma_1}| = |d_k^\sigma|\ ,$$

and for $k \notin K_2$,

$$\frac{r_k^\sigma}{\sigma} = \frac{\sigma_1}{\sigma}\ \frac{r_k^{\sigma_1}}{\sigma_1}\ .$$

Since for $k \notin K_2$, $r_k^{\sigma_1}/\sigma_1 > |d_k^{\sigma_1}|$, there is $0 < \delta \leq \delta_1$ such that (2.35) is satisfied by the collection $\mathcal{B}(\sigma)$. Since $r_k^\sigma \geq r_k^{\sigma_1}$, (2.33) is also satisfied.

To verify (2.34), we observe that for $k \notin K_2$, $B_k^\sigma = B_k^{\sigma_1}$ and (2.34) is satisfied in light of (2.14). If, however, $k \in K_2$, then

$$d_k^\sigma = d_k^{\sigma_1}\ , \qquad\qquad r_k^\sigma = \sigma|d_k^\sigma|\ , \tag{2.37}$$

and

$$[B_k^\sigma \setminus B_k^{\sigma_1}] \cap S^E = \emptyset\ .$$

Then by Lemma 2.3

$$\begin{aligned}
\int_{B_k^\sigma} E^\epsilon(u)\ dx &= \int_{B_k^{\sigma_1}} E^\epsilon(u)\ dx + \int_{B_k^\sigma \setminus B_k^{\sigma_1}} E^\epsilon(u)\ dx \\
&\geq \frac{r_k^{\sigma_1}}{\sigma_1}\ \Lambda^\epsilon(\sigma_1) + |d_k^{\sigma_1}| \left[\Lambda^\epsilon\left(\frac{r_k^\sigma}{|d_k^\sigma|}\right) - \Lambda^\epsilon\left(\frac{r_k^{\sigma_1}}{|d_k^{\sigma_1}|}\right)\right] \\
&= |d_k^{\sigma_1}|\ \Lambda^\epsilon\left(\frac{r_k^{\sigma_1}}{|d_k^{\sigma_1}|}\right) + |d_k^{\sigma_1}| \left[\Lambda^\epsilon\left(\frac{r_k^\sigma}{|d_k^\sigma|}\right) - \Lambda^\epsilon\left(\frac{r_k^{\sigma_1}}{|d_k^{\sigma_1}|}\right)\right] \\
&= |d_k^{\sigma_1}|\ \Lambda^\epsilon\left(\frac{r_k^\sigma}{|d_k^\sigma|}\right) \\
&= \frac{r_k^\sigma}{\sigma}\ \Lambda^\epsilon\left(\sigma\right)\ .
\end{aligned}$$

Here we repeatedly used the identities (2.37) and the fact that $B_k^{\sigma_1}$ satisfies (2.34). Hence $\mathcal{B}(\sigma)$ also satisfies (2.34) for all $\sigma \in [\sigma_1, \sigma_1 + \delta]$.

4. We have shown that C is closed set including σ^* and for every $\sigma \in C$, there exists $\delta > 0$ such that $[\sigma, \sigma + \delta] \subset C$. Hence, $C = [\sigma^*, \infty)$. □

We are now ready for the

Proof of Theorem 2.6

Set $R := |D^\epsilon_d|/(2\|\nabla\phi\|_\infty)$ and $\bar\sigma := R/d$. Let σ^* be as in the previous Lemma. We suppose that D^ϵ_d is nonempty as there is nothing to prove otherwise .

1. First suppose that $\bar\sigma < \sigma^*$. The opposite inequality will be treated later in the proof.

Consider the balls $\{B^*_k\}$ constructed in Proposition 2.8. By (2.32) and the definition of σ^*,

$$
\begin{aligned}
\int_U E^\epsilon(u)\,dx &\geq \sum_k \int_{U\cap B^*_k} E^\epsilon(u)\,dx \\
&\geq \sum_k \frac{c_1}{\epsilon}\,r^*_k \;\geq\; \frac{c_1\sigma^*}{\epsilon}\sum_k |d^*_k| \;\geq\; c_1\frac{R}{d\epsilon}\sum_k |d^*_k|\;.
\end{aligned}
$$

Let $t_0 \in D^\epsilon_d$. Then the definition (2.26) of D^ϵ_d implies that $d \leq |\deg(u;\Gamma(t_0))|$ and by definition, (2.23), $|u| > 1/2$ on $\Gamma(t_0)$. Hence $d \leq |\mathrm{dg}(u;\Gamma(t_0))|$. Moreover, by (2.6) and (2.30),

$$
d \leq |\mathrm{dg}(u;\Gamma(t_0))| \;\leq \sum_{\{k\;:\;B^*_k\cap\Omega(t_0)\neq\emptyset\}} |d^*_k| \;\leq\; \sum_k |d^*_k|\;.
$$

Hence by (2.14),

$$
\int_U E^\epsilon(u)\,dx \geq c_1\frac{R}{d\epsilon}\,d \geq d\Lambda^\epsilon\left(\frac{R}{d}\right)
$$

which is what we needed to prove.

2. We now assume that $\bar\sigma \geq \sigma^*$. Consider the collection of balls $\mathcal{B}(\bar\sigma)$ provided by Proposition 2.9. Assume towards a contradiction that

$$
\sum_k r^{\bar\sigma}_k < R\;. \tag{2.38}
$$

Set

$$
C := \{\; t\in(0,\|\phi\|_\infty) \mid \Gamma(t)\cap[\cup_k B^{\bar\sigma}_k] \neq \emptyset \;\}\;.
$$

The definitions imply that $C \subset \cup_k\, \phi(B^{\bar\sigma}_k)$, and as a consequence

$$
|C| \leq 2\|\nabla\phi\|_\infty \sum_k r^{\bar\sigma}_k < 2\|\nabla\phi\|_\infty R = |D^\epsilon_d|\;.
$$

Hence $D_d^\epsilon \setminus C \neq \emptyset$.

3. Let $t_0 \in D_d^\epsilon \setminus C$. The definition of D_d^ϵ implies that $|\mathrm{dg}(u;\Gamma(t_0))| = |\mathrm{deg}(u;\Gamma(t_0)| \geq d$. On the other hand, the definition of C implies that $\Gamma(t_0) \cap (\cup_k B_k^{\bar\sigma}) = \emptyset$, so (2.33) and the additivity of the degree yield

$$\begin{aligned} d \;\leq\; |dg(u;\Gamma(t_0))| \;&\leq \sum_{\{k\ :\ B_k^{\bar\sigma} \subset \Omega(t_0)\ \}} |d_k^{\bar\sigma}| \\ &\leq \sum_{\{k\ :\ B_k^{\bar\sigma} \cap \partial U = \emptyset\ \}} |d_k^{\bar\sigma}| \\ &\leq \sum_{\{k\ :\ B_k^{\bar\sigma} \cap \partial U = \emptyset\ \}} \frac{r_k^{\bar\sigma}}{\bar\sigma} \ , \end{aligned}$$

by (2.35). On the other hand by (2.38),

$$d = \frac{R}{\bar\sigma} > \sum_k \frac{r_k^{\bar\sigma}}{\bar\sigma} \ .$$

Therefore we conclude that (2.38) is false.

4. By the previous step $\sum_k r_k^{\bar\sigma} \geq R = d\bar\sigma$. Hence by (2.34),

$$\begin{aligned} \int_U E^\epsilon(u)\ dx \;&\geq\; \sum_k \int_{U \cap B_k^{\bar\sigma}} E^\epsilon(u)\ dx \\ &\geq\; \sum_k r_k^{\bar\sigma} \frac{\Lambda^\epsilon(\bar\sigma)}{\bar\sigma} \\ &\geq\; d\ \Lambda^\epsilon(\bar\sigma)\ . \end{aligned}$$

□

3 Jacobian and the GL Energy

In this section, we show that the Jacobian is bounded by the GL energy. This estimate is the crucial step in a Gamma convergence result.

Results of this section are taken from [13].

3.1 Jacobian estimate

The chief result of this section is the following estimate of the Jacobian in terms of the Ginzburg Landau energy. This estimate will be the main ingredient in the compactness result. We give a more precise version of the estimate at the end of the section.

We use the notation introduced in the previous section and set

$$\mu^\epsilon(V) \ : \ = \ \mu_u^\epsilon(V) \ = \ \frac{1}{\ln(1/\epsilon)} \int_V E^\epsilon(u) \ dx.$$

Theorem 3.1 (Jerrard & Soner [13]) *Suppose $\phi \in C_c^{0,1}(U)$ and $u \in H^1(U; I\!R^2)$. For any $\lambda \in (1,2]$, and $\epsilon \in (0,1]$,*

$$\left| \int_U \phi \ Ju \ dx \right| \leq \pi d_\lambda \|\phi\|_\infty + \|\phi\|_{C^{0,1}} h^\epsilon(\phi, u, \lambda) \tag{3.1}$$

where

$$d_\lambda = \left\lfloor \frac{\lambda}{\pi} \mu_u^\epsilon(spt(\phi)) \right\rfloor , \tag{3.2}$$

$\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and

$$h^\epsilon(\phi, u, \lambda) \leq C\epsilon^{\alpha(\lambda)}(1 + \mu_u^\epsilon(spt(\phi)))(1 + Leb^2(spt(\phi))) \tag{3.3}$$

where $\alpha(\lambda) = \frac{\lambda-1}{12\lambda}$ and C is a constant independent of u, ϕ, ϵ, λ and U.

Note that h^ϵ depends on ϕ only through the support of ϕ, and on u only through its (linear) dependence on $\mu_u^\epsilon(\mathrm{spt}(\phi))$.

It suffices to consider nonnegative test functions, since we can decompose an arbitrary function ϕ into its positive and negative parts. So we will assume that $\phi \geq 0$.

By an approximation argument, we may also assume that u is smooth.

The main idea behind the above estimate is the following identity, which relies on the co-area formula, integration by parts, and the identity $Ju = \nabla \times j(u)/2$:

$$\int_U \phi \ Ju \ dx = \frac{1}{2} \int_0^T \int_{\partial\Omega(t)} j(u) \cdot \vec{t} \ d\mathcal{H}^1 \ dt \ , \tag{3.4}$$

where as before

$$\Omega(t) = \{ \ x \in U \mid \phi(x) > t \ \} \ ,$$

$$\vec{t} = \text{ unit tangent to } \partial\Omega(t) \ = \ \frac{\nabla \times \phi}{|\nabla \times \phi|}.$$

The proof shows that

$$\int_{\partial\Omega(t)} j(u)\cdot\vec{t}\,d\mathcal{H}^1 \;\approx\; 2\pi\,\deg(u;\partial\Omega(t))$$

for most values of t. The other main point is then to prove that the set of t such that $\deg(u;\partial\Omega(t)) > d_\lambda$ has Leb^1 measure that can be controlled by $\mu^\epsilon(\text{spt}(\phi))$. This last point is similar in spirit to results established in [7, 12, 25] for example. We use the lower bounds obtained in the previous section to achieve this.

We start the proof of Theorem 3.1 with two simple estimates.

Lemma 3.2 *For any set A,*

$$\int_A \left| \int_{\partial\Omega(t)} j(u)\cdot\vec{t}\,d\mathcal{H}^1 \right| dt \le \frac{|A|}{2}\int_{spt(\phi)} E^\epsilon(u)\;dx\;. \tag{3.5}$$

For any nonnegative function f,

$$\int_0^T \int_{\partial\Omega(t)} f(x)\;d\mathcal{H}^1\;dt \le \|\nabla\phi\|_\infty \int_{spt(\phi)} f(x)\;dx\;. \tag{3.6}$$

Proof.

For any $t \in \text{Reg}(\phi)$, Stokes' Theorem yields

$$\int_{\partial\Omega(t)} j(u)\cdot\vec{t}\,d\mathcal{H}^1 = \frac{1}{2}\int_{\Omega(t)} Ju\;dx\;.$$

Since $|Ju| \le \frac{1}{2}|\nabla u|^2 \le E^\epsilon(u)$, (3.5) follows from the above identity.

For (3.6), we calculate by using the coarea formula,

$$\begin{aligned}\int_0^T \int_{\partial\Omega(t)} f\;d\mathcal{H}^1\;dt &= \int_{spt(\phi)} f\;|\nabla\phi|\;dx\\ &\le \|\nabla\phi\|_\infty \int_{spt(\phi)} f\;dx.\end{aligned}$$

□

We are now in a position to prove Theorem 3.1. In the proof we repeatedly absorb logarithmic factors by using the fact that if $\beta < \alpha$ then

$$\epsilon^\alpha \ln(1/\epsilon) \le C\epsilon^\beta$$

for some $C = C(\alpha,\beta)$ independent of $\epsilon \in (0,1]$.

Proof of Theorem 3.1.

1. Recall that we are writing $T = \|\phi\|_\infty$. Fix $\lambda \in (1,2]$ and define $d_\lambda := \lfloor \frac{\lambda}{\pi}\mu^\epsilon(\mathrm{spt}(\phi))\rfloor$. We define sets $A, B \subset [0,T]$ by

$$B := \{t \in \mathrm{Reg}(\phi) \ : \ |\deg(u;\Gamma(t))| \geq d_\lambda + 1 \text{ or } \mathcal{H}^1(\gamma(t)) \geq \epsilon\}, \tag{3.7}$$

$$A = \mathrm{Reg}(\phi) \setminus B. \tag{3.8}$$

Because almost every t belongs to $A \cup B = \mathrm{Reg}(\phi)$, (3.4) implies that

$$\begin{aligned}
\int_U \phi \ Ju \ dx &= \frac{1}{2}\int_A \int_{\Gamma(t)} j(u)\cdot \vec{t} \, d\mathcal{H}^1 \ dt \qquad (3.9)\\
&\quad + \frac{1}{2}\int_A \int_{\gamma(t)} j(u)\cdot \vec{t} \, d\mathcal{H}^1 \ dt + \frac{1}{2}\int_B \int_{\partial\Omega(t)} j(u)\cdot \vec{t} \, d\mathcal{H}^1 \ dt\\
&= I_{A,\Gamma} + I_{A,\gamma} + I_B. \qquad (3.10)
\end{aligned}$$

2. Estimate of $I_{A,\Gamma}$

Suppose $t \in A$. On $\Gamma(t)$, $|u| \geq 1/2$ by the definition (2.23), and we set $v := u/|u|$, so that $j(v) = j(u)/|u|^2$, and

$$\int_{\Gamma(t)} j(v)\cdot \vec{t} \, d\mathcal{H}^1 = 2\pi \ \deg(u;\Gamma(t)).$$

Then

$$\int_{\Gamma(t)} j(u)\cdot \vec{t} \, d\mathcal{H}^1 = 2\pi \ \deg(u;\Gamma(t)) + \int_{\Gamma(t)} j(u) \ \frac{|u|^2-1}{|u|^2}\cdot \vec{t} \, d\mathcal{H}^1 \ .$$

Since $|j(u) \leq |u| \ |\nabla u|$, and since $|u| \geq 1/2$ on $\Gamma(t)$, Cauchy's inequality and (3.6) imply that

$$\begin{aligned}
\int_A \left| \int_{\Gamma(t)} j(u)\cdot \vec{t} \, d\mathcal{H}^1 \ - 2\pi \deg(u;\Gamma(t)) \right| dt &\leq \int_A \int_{\Gamma(t)} |\nabla u| \ \left| \frac{|u|^2-1}{|u|} \right| \, d\mathcal{H}^1\\
&\leq 4\epsilon \int_A \int_{\Gamma(t)} E^\epsilon(u) \ d\mathcal{H}^1\\
&\leq 4\epsilon \ln(1/\epsilon) \ \|\nabla\phi\|_\infty \ \mu^\epsilon(spt(\phi)) \ . \qquad (3.11)
\end{aligned}$$

Clearly $A \subset [0,T]$ has measure less than $T = \|\phi\|_\infty$. Also, by definition of A, if $t \in A$ and $\Gamma(t)$ is nonempty, then $|\deg(u;\Gamma(t))| \leq d_\lambda$. It follows that

$$|I_{A,\Gamma}| \leq \ \pi\|\phi\|_\infty d_\lambda + C\epsilon^{1/2}\|\nabla\phi\|_\infty \ \mu^\epsilon(spt(\phi)). \tag{3.12}$$

3. Estimate of $I_{A,\gamma}$

Using Cauchy's inequality and the elementary fact that $x \le \frac{1}{b}(1-x)^2 + (1+\frac{b}{4})$ for all $x \in I\!R$ and $b > 0$, we have

$$|j(u)| \;\le\; |u||\nabla u| \;\le\; \frac{a}{2}\left(|\nabla u|^2 + \frac{1}{a^2}|u|^2\right) \le \frac{a}{2}\left(|\nabla u|^2 + \frac{(1-|u|^2)^2}{a^2 b}\right) + \frac{1}{2a}(1+\frac{b}{4})$$

for every $a, b > 0$. We select $a = \epsilon^\alpha$ for $\alpha \in (0,1)$ and $b = \epsilon^{2-2\alpha}$ to find

$$|j(u)| \le C\epsilon^\alpha E^\epsilon(u) + C\epsilon^{-\alpha} \tag{3.13}$$

The definition (3.8) of A implies that $|A| \le T = \|\phi\|_\infty$ and that $\mathcal{H}^1(\gamma(t)) < \epsilon$ for every $t \in A$, so we can take $\alpha = 1/2$ and use (3.6) to find

$$\begin{aligned} |I_{A,\gamma}| \;&\le\; C\int_A\int_{\gamma(t)} \sqrt{\epsilon} E^\epsilon(u) d\mathcal{H}^1(x)dt \;\; + C\int_A\int_{\gamma(t)} \frac{C}{\sqrt{\epsilon}} d\mathcal{H}^1(dx)dt \\ &\le\; C\ln(1/\epsilon)\,\sqrt{\epsilon}\,\mu^\epsilon(\mathrm{spt}(\phi))\|\nabla\phi\|_\infty \;\; + \;\; C\sqrt{\epsilon}\|\phi\|_\infty. \end{aligned}$$

4. Estimate of I_B

To estimate I_B we prove that B has small measure. Toward this end we define

$$B_1 := \{t \in \mathrm{Reg}(\phi) \;\; : \mathcal{H}^1(\gamma(t)) \ge \epsilon\},$$

$$B_2 := \{t \in \mathrm{Reg}(\phi) \;:\; \Gamma(t) \text{ is nonempty, and } |\deg(u;\Gamma(t))| \ge d_\lambda + 1\}. \tag{3.14}$$

The estimate of B_2 is deferred to the end of this subsection, where we prove

Proposition 3.3 *For every $\lambda \in (1,2]$, $\epsilon \in (0,1]$, smooth $u : U \to I\!R^2$, and nonnegative test function $\phi \in C_c^{0,1}(U)$,*

$$|B_2| \;\le\; C\epsilon^{1-\frac{1}{\lambda}}\|\nabla\phi\|_\infty(d_\lambda + 1) \;\;\le\;\; C\epsilon^{1-\frac{1}{\lambda}}\|\nabla\phi\|_\infty(1 + \mu^\epsilon(spt(\phi)). \tag{3.15}$$

For the time being we assume this fact and use it to complete the proof of the theorem.

The measure of B_1 is easily estimated: using (3.6) and Lemma 2.7,

$$\begin{aligned} \frac{1}{25\epsilon}|B_1| \;&\le\; \int_{t\in B_1}\int_{\partial\Omega(t)} E^\epsilon(u) d\mathcal{H}^1 dt \\ &\le\; \|\nabla\phi\|_\infty \ln(\frac{1}{\epsilon})\mu^\epsilon(\mathrm{spt}(\phi)). \end{aligned} \tag{3.16}$$

Clearly $|B| \le |B_1| + |B_2|$, so by combining (3.16) and (3.15) we obtain

$$|B| \le C\epsilon^{\frac{\lambda-1}{2\lambda}} \|\nabla\phi\|_\infty (1 + \mu^\epsilon(\mathrm{spt}(\phi))). \tag{3.17}$$

Finally, we use (3.5) to estimate

$$|I_B| \ \le \ C\epsilon^{\frac{\lambda-1}{3\lambda}} \|\nabla\phi\|_\infty (1 + \mu^\epsilon(\mathrm{spt}(\phi)))\mu^\epsilon(\mathrm{spt}(\phi)). \tag{3.18}$$

5. The previous three steps imply that

$$\left|\int \phi Ju \ dx\right| \le d_\lambda \|\phi\|_\infty + \|\phi\|_{C^1} h_0^\epsilon(\phi, u, \lambda)$$

for

$$h_0^\epsilon(\phi, u, \lambda) \le C\epsilon^{4\alpha(\lambda)} \left(1 + \mu^\epsilon(\mathrm{spt}(\phi)) + (\mu^\epsilon(\mathrm{spt}(\phi)))^2\right), \qquad \alpha(\lambda) = \frac{\lambda - 1}{12\lambda}.$$

To complete the proof of the Theorem, note that by (3.6) and (3.13) (with $\alpha = 2\alpha(\lambda)$)

$$\begin{aligned}
\left|\int \phi Ju \ dx\right| \ &\le \ \int_0^T \int_{\partial\Omega(t)} |j(u)| d\mathcal{H}^1 dt \\
&\le \ C\|\nabla\phi\|_\infty \int_{\mathrm{spt}(\phi)} \epsilon^{2\alpha(\lambda)} E^\epsilon(u) + \epsilon^{-2\alpha(\lambda)} \ dx \\
&\le \ C\|\phi\|_{C^1} h_1^\epsilon(\phi, u, \lambda),
\end{aligned}$$

for $h_1^\epsilon = \epsilon^{\alpha(\lambda)}\mu^\epsilon(\mathrm{spt}(\phi)) + \epsilon^{-2\alpha(\lambda)}\mathrm{Leb}^2(\mathrm{spt}(\phi))$. We define $h^\epsilon(\phi, u, \lambda) := \min\{h_0^\epsilon, h_1^\epsilon\}$, so that (3.1) clearly holds. It thus suffices to verify that (3.3) holds, that is,

$$h^\epsilon(\phi, u, \lambda) = \min\{h_0^\epsilon, h_1^\epsilon\} \le C\epsilon^{\alpha(\lambda)}(1 + \mu^\epsilon(\mathrm{spt}(\phi))(1 + \mathrm{Leb}^2(\mathrm{spt}(\phi)))$$

for some appropriately large constant C. This follows immediately from the definition of h_0^ϵ if $\mu^\epsilon(\mathrm{spt}(\phi)) \le \epsilon^{-3\alpha(\lambda)}$, and if not, it follows directly from the definition of h_1^ϵ. □

Note that the result we have proved is in fact somewhat sharper than Theorem 3.1 as stated, in that it not only provides an upper bound for $\int \phi Ju$, but in fact gives an approximate value for the integral. The following corollary states a small technical modification of this sharper estimate.

Corollary 3.4 *Let U be a bounded, open subset of $\mathbb{R}^2$, and suppose that $\phi \in C_c^{0,1}(U)$ and $u \in H^1(U; \mathbb{R}^2)$. Define $Reg(\phi)$, $\Gamma(t)$ and $\gamma(t)$ as in (2.22), (2.23) and (2.24) respectively.*

Then for any $\lambda \in (1, 2]$ and $\epsilon \in (0, 1]$, there exists a set $A = A(\phi, u, \lambda, \epsilon) \subset (0, \|\phi\|_\infty)$ such that

$$|A| \;\geq\; \|\phi\|_\infty - C\epsilon^{\alpha(\lambda)}\|\nabla\phi\|_\infty(1+\mu^\epsilon(spt(\phi))); \tag{3.19}$$

$$\Gamma(t) \textit{ is nonempty, and } |deg(u;\Gamma(t))| \leq d_\lambda \qquad \forall t \in A; \;\; \textit{and} \tag{3.20}$$

$$\left| \int \phi Ju \;\; - \;\; \pi \int_{t\in A} deg(u;\Gamma(t))\; dt \right| \;\;\leq\;\; \|\phi\|_{C^1} h^\epsilon(\phi, u, \lambda), \tag{3.21}$$

where h^ϵ is defined in (3.3) and d_λ is defined in (3.2).

Proof. We cannot take A to be the set defined in (3.8), as we have now imposed the additional condition that $\Gamma(t) \neq \emptyset$ for $t \in A$. So we let $\tilde{A}$ be the set formerly known as A, defined in (3.8), and we define

$$A = \{t \in \tilde{A} \;:\; \Gamma(t) \text{ is nonempty.}\}$$

Then (3.20) follows from the definition of $\tilde{A}$, and (3.21) follows from (3.11). We claim moreover that $\tilde{A} \setminus A$ has measure at most $\epsilon\|\nabla\phi\|_\infty$. In view of (3.17) and (3.8), this will suffice to establish (3.19), and thus to complete the proof of the Corollary.

To prove our claim, note first that for every $t \in \tilde{A}$, $\mathcal{H}^1(\gamma(t)) < \epsilon$. If $t \in \tilde{A} \setminus A$, then $\Gamma(t)$ is empty, and so $\mathcal{H}^1(\phi^{-1}(t)) = \mathcal{H}^1(\gamma(t)) < \epsilon$ for all $t \in \tilde{A} \setminus A$. On the other hand, let $x_0 \in U$ be a point such that $\phi(x_0) = \|\phi\|_\infty$. If $|y - x_0| \leq \epsilon$ then $\phi(y) \;\geq\; \|\phi\|_\infty - \epsilon\; \|\nabla\phi\|_\infty$. It follows that $B_\epsilon(x_0) \subset \Omega(t)$ for all $t < \|\phi\|_\infty - \epsilon\; \|\nabla\phi\|_\infty$. Thus the isoperimetric inequality implies that $\mathcal{H}^1(\phi^{-1})(t) \geq 2\pi\epsilon$.

We conclude that if $t \in \tilde{A} \setminus A$, then $t \geq \|\phi\|_\infty - \epsilon\|\nabla\phi\|_\infty$, which proves the claim. □

We now use Theorem 2.6 and the facts about Λ^ϵ to give the proof of Proposition 3.3.

Recall that for Proposition 3.3 we want to estimate the measure of a set $B_2 \subset \mathrm{Reg}(\phi)$, and from the definition (3.14) of B_2 we see that

$$D^\epsilon_{d^*_\lambda} = B_2 \;\cap\; \{t \;:\; t \geq t_\epsilon\}, \qquad \text{for } d^*_\lambda \;:=\; \lfloor \frac{\lambda}{\pi}\mu^\epsilon(\mathrm{spt}(\phi)) \rfloor + 1 \;\geq\; \frac{\lambda}{\pi}\mu^\epsilon(\mathrm{spt}(\phi)). \tag{3.22}$$

Proof of Proposition 3.3

We need to show that

$$|B_2| \;\leq\; C\epsilon^{1-\frac{1}{\lambda}}\|\nabla\phi\|_\infty d^*_\lambda, \qquad\qquad d^*_\lambda := d_\lambda + 1.$$

Let $R := \frac{|D^\epsilon_{d^*_\lambda}|}{2\|\nabla\phi\|_\infty}$. From (3.22) and the definition (2.25) of t_ϵ it suffices to show that

$$\frac{R}{d^*_\lambda} \leq C\epsilon^{1-\frac{1}{\lambda}}.$$

We may assume that $\frac{R}{d_\lambda^*} \geq \epsilon$, as otherwise the conclusion is obvious. Then (2.15), Theorem 2.6, and the choice (3.22) of d_λ^* imply that

$$\begin{aligned} \ln\left(\frac{R}{d_\lambda^*}\right) &= \frac{1}{\pi}\left[\pi \ln\left(\frac{R}{\epsilon d_\lambda^*}\right) - \pi \ln\left(\frac{1}{\epsilon}\right)\right] \\ &\leq \frac{1}{\pi}\Lambda^\epsilon\left(\frac{R}{d_\lambda^*}\right) + C - \ln\left(\frac{1}{\epsilon}\right) \\ &\leq \frac{1}{\pi d_\lambda^*}\ln\left(\frac{1}{\epsilon}\right)\mu^\epsilon(\mathrm{spt}(\phi)) + C - \ln\left(\frac{1}{\epsilon}\right) \\ &\leq \left(\frac{1}{\lambda} - 1\right)\ln\left(\frac{1}{\epsilon}\right) + C. \end{aligned}$$

. □

3.2 Compactness in two dimensions

In this section we consider a sequence of functions $u^\epsilon \in H^1(U; I\!R^2)$, where U is a bounded open subset of $I\!R^2$ and the renormalized Ginzburg-Landau energy is uniformly bounded:

$$K_U := \sup_{\epsilon \in (0,1]} \mu^\epsilon(U) < \infty \;, \qquad \mu^\epsilon := \mu^\epsilon_{u^\epsilon}. \tag{3.23}$$

We will show that under this assumption, the Jacobian is compact in the dual norm $(C^{0,\beta})^*$ for every $\beta \in (0, 1]$. We refer to §5 and to [13] for a compactness in higher dimensions.

We introduce the Jacobian (signed) measure

$$Ju^\epsilon(E) := \int_E \det(\nabla u^\epsilon)\ dx \;, \qquad E \subset U.$$

Since $\det(\nabla u^\epsilon) = \frac{1}{2}\nabla \times j(u^\epsilon)$ for $j(u^\epsilon) := u^\epsilon \times \nabla u^\epsilon$,

$$\int_{I\!R^2} \phi\ dJu^\epsilon = \frac{1}{2}\int_{I\!R^2} \nabla \times \phi(x) \cdot j(u^\epsilon)(x)\ dx \;, \qquad \forall \phi \in C^1_c(U),$$

where for a scalar function ϕ, we write $\nabla \times \phi := (\phi_{x_2}, -\phi_{x_1})$.

Theorem 3.5 *Let $\{u^\epsilon\} \subset H^1(U; I\!R^2)$ satisfy (3.23). Then there exists a subsequence ϵ_n converging to zero and a signed Radon measure J such that Ju^{ϵ_n} converges to J in the dual norm $\left(C_c^{0,\beta}\right)^*$ for every $\beta \in (0, 1]$. Moreover, there are $\{a_i\}_{i=1}^N \subset U$ and integers k_i such that*

$$J = \pi \sum_{i=1}^N k_i\ \delta_{a_i} \;, \qquad \textit{and} \qquad |J|(U) = \pi \sum_i |k_i| \leq K_U \;.$$

Finally, if μ^ϵ converges weakly to a limit μ, then $J \ll \mu$, and $\frac{dJ}{d\mu}(x) \le 1$ for μ almost every x.

We will first prove

Proposition 3.6 *Assume (3.23). Then, Ju^ϵ can be written in the form*

$$Ju^\epsilon = J_0^\epsilon + J_1^\epsilon$$

where J_0^ϵ and J_1^ϵ are signed measures such that

$$\|J_0^\epsilon\|_{(C^0)^*} \le C, \text{ and } \qquad \|J_1^\epsilon\|_{(C_c^{0,1})^*} \le C\epsilon^\alpha \tag{3.24}$$

for some $\alpha > 0$ and a constant C depending only on the constant K_U in (3.23).

Proof.

1. In light of the assumption $\mu^\epsilon(U) \le K$, Theorem 3.1 (with $\lambda = 2$ and $\alpha = 1/24$, for example) implies that

$$\int \phi Ju^\epsilon \le C\|\phi\|_\infty + C\epsilon^\alpha\|\nabla\phi\|_\infty \qquad \text{for all } \phi \in C_c^{0,1}(U). \tag{3.25}$$

We write $\delta = \epsilon^\alpha$, and we define $U_\delta = \{x \in U : \mathrm{dist}(x, \partial U) > \delta\}$. Let

$$\chi_\delta = \begin{cases} 1 & \text{if } x \in U_{2\delta} \\ 0 & \text{if not.} \end{cases}$$

We define $J_0^\epsilon := \chi_\delta(\eta^\delta * Ju^\epsilon)$, where η^δ is a standard mollifier with support in $B_\delta(0)$. We then define $J_1^\epsilon := Ju^\epsilon - J_0^\epsilon$.

Suppose that ϕ is a C^1 test function vanishing on ∂U, and note that

$$\int \phi \; J_0^\epsilon dx = \int \eta^\delta * (\chi_\delta\phi) Ju^\epsilon dx.$$

We write $\phi^\delta := \eta^\delta * (\chi_\delta\phi)$. It is clear that ϕ^δ is compactly supported in U, and one easily checks that

$$\|\phi^\delta\|_\infty \le \|\chi_\delta\phi\|_\infty \le \|\phi\|_\infty, \qquad \|\nabla\phi^\delta\|_\infty \le \frac{C}{\delta}\|\chi_\delta\phi\|_\infty \le \frac{C}{\delta}\|\phi\|_\infty.$$

Then (3.25) implies that

$$\int \phi \ J_0^\epsilon dx \leq C\|\phi\|_\infty.$$

2. We now estimate J_1^ϵ. Given $\phi \in C_0^1(U)$, write

$$\phi_1 := \min\{\phi, 2\delta\|\nabla\phi\|_\infty\}, \qquad \phi_2 := \phi - \phi_1.$$

It is clear that $\phi \leq 2\delta\|\nabla\phi\|_\infty$ in $U \setminus U_{2\delta}$, so ϕ_2 is supported in $U_{2\delta}$.

From the definitions,

$$\int \phi_1 J_1^\epsilon \, dx = \int (\phi_1 - \eta^\delta * (\chi_\delta \phi_1)) J u^\epsilon \, dx.$$

It is clear that

$$\|\phi_1\|_\infty \leq 2\delta\|\nabla\phi\|_\infty, \qquad \|\nabla\phi_1\|_\infty \leq \|\nabla\phi\|_\infty.$$

Similarly, $\eta^\delta * (\chi_\delta\phi_1)$ satisfies

$$\|\eta^\delta * (\chi_\delta\phi_1)\|_\infty \leq 2\delta\|\nabla\phi\|_\infty, \qquad \|\nabla\eta^\delta * (\chi_\delta\phi_1)\|_\infty \leq \frac{C}{\delta}\|\phi_1\|_\infty \leq C\|\nabla\phi\|_\infty.$$

So (3.25) implies that

$$\int \phi_1 J_1^\epsilon \, dx \leq C\delta\|\nabla\phi\|_\infty = C\epsilon^\alpha\|\nabla\phi\|_\infty.$$

Finally, since ϕ_2 is supported in $U_{2\delta}$,

$$\int \phi_2 J_1^\epsilon \, dx = \int (\phi_2 - \eta^\delta * (\chi_\delta\phi_2)) J u^\epsilon \, dx = \int (\phi_2 - \eta^\delta * \phi_2) J u^\epsilon \, dx.$$

It is easy to check that

$$\|\phi_2 - \eta^\delta * \phi_2\|_\infty \leq C\delta\|\nabla\phi\|_\infty, \qquad \|\nabla(\phi_2 - \eta^\delta * \phi_2)\|_\infty \leq C\|\nabla\phi\|_\infty.$$

So we again use (3.25) to conclude

$$\int \phi_2 J_1^\epsilon \, dx \leq C\delta\|\nabla\phi\|_\infty = C\epsilon^\alpha\|\nabla\phi\|_\infty.$$

□

Once we have the above decomposition, the compactness of the sequence Ju^ϵ follows from soft arguments.

Lemma 3.7 *If ν is a Radon measure on U, then*

$$\|\nu\|_{(C_c^{0,\alpha})^*} \le C\|\nu\|^{\alpha}_{(C_c^{0,1})^*}\|\nu\|^{1-\alpha}_{(C_c^0)^*} \ . \tag{3.26}$$

Proof. Since U is bounded and we are considering compactly supported functions, the Hölder seminorm is in fact a norm and is topologically equivalent to the usual $C^{0,\alpha}$ norm. So for this lemma we set

$$\|\phi\|_{C_c^{0,\alpha}(U)} \ := \ [u]_{C^{0,\alpha}} \ = \ \sup_{x\neq y}\frac{|\phi(x)-\phi(y)|}{|x-y|^{\alpha}}, \qquad \alpha\in(0,1].$$

Fix $\phi\in C_c^{0,\alpha}$, and let $\tilde{\phi}^{\epsilon}=\eta^{\epsilon}*\phi$, where η^{ϵ} is a smoothing kernel and ϵ will be chosen later. Then one easily checks that

$$\|\tilde{\phi}^{\epsilon}\|_{C^{0,1}} \le C\epsilon^{\alpha-1}\|\phi\|_{C^{0,\alpha}} := M_{\epsilon}, \qquad \|\phi-\tilde{\phi}^{\epsilon}\|_{C^0}\le C\epsilon^{\alpha}\|\phi\|_{C^{0,\alpha}} \tag{3.27}$$

In particular, $|\tilde{\phi}^{\epsilon}|\le C\epsilon^{\alpha}\|\phi\|_{C^{0,\alpha}}$ on ∂U.

We next modify $\tilde{\phi}^{\epsilon}$ so that it vanishes on ∂U while continuing to satisfy the above estimates. Let

$$u(x)=\sup_{y\in\partial U}\Big(\tilde{\phi}^{\epsilon}(y)-M_{\epsilon}|x-y|\Big)^{+}, \qquad v(x)=\sup_{y\in\partial U}\Big(\tilde{\phi}^{\epsilon}(y)+M_{\epsilon}|x-y|\Big)^{-}.$$

Then one easily checks that $\tilde{\phi}^{\epsilon}=u-v$ on ∂U. Moreover, if we define $\phi^{\epsilon}:=\tilde{\phi}^{\epsilon}-u+v$, then ϕ^{ϵ} satisfies the estimates in (3.27) and also vanishes on ∂U.

So

$$\begin{aligned}
\int\phi d\nu \ &= \ \int\phi^{\epsilon}\,d\mu+\int(\phi-\phi^{\epsilon})\,d\nu \\
&\le \ \|\phi^{\epsilon}\|_{C^{0,1}}\|\nu\|_{(C_c^{0,1})^*}+\|\phi-\phi^{\epsilon}\|_{C^0}\|\nu\|_{(C^0)^*} \\
&\le \ C\|\phi\|_{C^{0,\alpha}}\Big(\epsilon^{\alpha-1}\|\nu\|_{(C_c^{0,1})^*}+\epsilon^{\alpha}\|\nu\|_{(C^0)^*}\Big).
\end{aligned}$$

Taking $\epsilon=\|\nu\|_{(C_c^{0,1})^*}/\|\nu\|_{(C^0)^*}$ gives the conclusion of the lemma. □

Lemma 3.8 *If $\alpha>0$, then $(C^0)^*\subset\subset(C^{0,\alpha})^*$.*

Proof. The Arzela-Ascoli Theorem implies that any sequence that is bounded on $C^{0,\alpha}$ is precompact in C^0. The lemma follows by duality.

More concretely: given a sequence of measures bounded in $(C^0)^*$, we can extract a subsequence, say μ_n that converges to a limit μ in the weak-* topology. We must show that this sequence converges in norm in $(C^{0,\alpha})^*$. If not, then we can find a sequence of functions ψ_n with $\|\psi_n\|_{C^{0,\alpha}} \leq 1$ such that

$$\int \psi_n d(\mu_n - \mu) \geq c_0 > 0 \tag{3.28}$$

for all n. However, the Arzela-Ascoli theorem implies that, upon passing to a subsequence, ψ_n converges to some limit ψ uniformly, whence (3.28) is impossible. $\square$

We now prove

Theorem 3.9 *Assume (3.23). Then Ju^ϵ is strongly precompact in $(C^{0,\beta})^*$ for all $\beta > 0$.*

Proof. By Proposition 3.6 we can write $Ju^\epsilon = J_0^\epsilon + J_1^\epsilon$, where the two measures on the right-hand side satisfy (3.24).

Fix any $\beta \in (0,1]$. Lemma 3.8 implies that $\{J_0^\epsilon\}$ is precompact in $(C^{0,\beta})^* \subset (C_c^{0,\beta})^*$.

Also, it is clear from the definitions that

$$\|J_1^\epsilon\|_{(C^0)^*} \leq \|Ju^\epsilon\|_{L^1} + \|J_0^\epsilon\|_{(C^0)^*} \leq C\|\nabla u^\epsilon\|_{L^2}^2 + C \leq K \ln(\frac{1}{\epsilon}).$$

So together with (3.24) and the interpolation inequality (3.26) this implies that $\|J_1^\epsilon\|_{(C_c^{0,\beta})^*} \to 0$ as $\epsilon \to 0$. $\square$

Remark 3.10 The above result is sharp in the sense that Ju^ϵ need not be precompact, or even weakly precompact, in $(C^0)^*$. To see this, consider the sequence of functions

$$u^\epsilon(x,y) = (1,0) + \epsilon^2(\ln(\frac{1}{\epsilon}))^{1/2}(\cos(\frac{x}{\epsilon^2}), \sin(\frac{y}{\epsilon^2}))$$

on the open unit disk D in the plane. One easily verifies that $\mu^\epsilon(D) \leq C$, and that $\|Ju^\epsilon\|_{(C^0)^*} = \|Ju^\epsilon\|_{L^1} \geq c^{-1}\ln(\frac{1}{\epsilon})$. In particular, since $\|Ju^\epsilon\|_{(C^0)^*}$ is unbounded, the Uniform Boundedness Principle implies that the sequence cannot converge weakly in $(C^0)^*$.

Remark 3.11 Suppose ν^ϵ is any sequence of measures on a bounded open set $U \subset I\!R^m$, and that

$$|\nu^\epsilon|(U) \leq K \ln(\frac{1}{\epsilon}), \qquad \int \phi d\nu^\epsilon \leq C\|\phi\|_\infty + C\epsilon^\alpha\|\nabla\phi\|_\infty$$

for some $\alpha > 0$. The arguments given above then show, with essentially no change, that $\{\nu^\epsilon\}$ is precompact in $(C^{0,\beta})^*$ for all $\beta \in (0,1]$.

We are now in a position to give the

Proof of Theorem 3.5. Suppose $\{u^\epsilon\}_{\epsilon\in(0,1]} \subset H^1(U;\mathbb{R}^2)$ is a sequence satisfying (3.23). By an approximation argument, we may assume that in fact each u^ϵ is smooth. In view of Theorem 3.9, we can find a measure J and a subsequence ϵ_n such that $Ju^{\epsilon_n} \to J$ in $(C_c^{0,\beta})^*$ for every $\beta \in (0,1]$.

1. Since μ^{ϵ_n} is a sequence of uniformly bounded, nonnegative Radon measures, we may assume upon passing to a further subsequence (still labeled ϵ_n) that there is a Radon measure μ such that

$$\mu_n := \mu^{\epsilon_n} \stackrel{*}{\rightharpoonup} \mu \ ,$$

in the weak* topology of Radon measures in U. For $x \in U$, set

$$\Theta(x) := \lim_{r\downarrow 0} \ \mu\left(B_r(x)\cap U\right) \ .$$

We first claim that J is supported only on the points with $\Theta(x) \geq \pi$.

Indeed, suppose that $\Theta(x_0) < \pi$ at some $x_0 \in U$. Then there exists some $r_0 > 0$ and a number $\alpha < \pi$ such that

$$\mu_n(B_{r_0}(x_0)) \ \leq \ \alpha < \pi$$

for all sufficiently large n. Then Theorem 3.1 with $\lambda = \frac{a+\pi}{2a} > 1$ immediately implies that

$$\int \phi \ dJ(x) = \lim_{n\to\infty} \int \phi \ Ju^{\epsilon_n} dx \quad = 0$$

for all smooth ϕ with support in $B_{r_0}(x_0)$, since $d_\lambda = 0$ for such ϕ. Thus $x_0 \notin \mathrm{spt}(J)$.

Since μ is bounded on U, there are finitely many points $\{a_i\}_i \subset U$ such that

$$\Theta(a_i) \geq \pi \ .$$

Therefore there are constants c_i such that the limit measure J satisfies

$$J = \pi \ \sum_i \ c_i \ \delta_{a_i} \ .$$

We need to prove that c_i's are integers and that $\pi|c_i| \leq \Theta(a_i)$ for all i; this will immediately imply all the remaining conclusions of Theorem 3.5.

2. Choose $r_1 \leq 1$ so that $B_{r_1}(a_1)$ does not intersect $\{a_i\}_{i>1} \cup \partial U$. We may also assume, taking r_1 smaller if necessary, that there exists some $\lambda > 1$ and an integer N_0 such that

$$
(3.29) \qquad d_\lambda := \lfloor \frac{\lambda}{\pi}\mu_n(B_{r_1}(a_1))\rfloor \le \frac{1}{\pi}\Theta(a_1) \qquad \forall n \ge N_0.
$$

We first apply Theorem 3.4 to the function $\phi(x) := (r_1 - |x-a_1|)^+$, which is supported in $B_{r_1}(a_1)$. Let $A^n = A(\phi, u^{\epsilon_n}, \lambda, \epsilon_n)$ be the set whose existence is asserted in Theorem 3.4. Note that if $t \in A^n$, then $\Gamma_{\phi,u^{\epsilon_n}}(t)$ is nonempty, which is to say that there is a component of $\phi^{-1}(t)$ on which $\min|u| \ge 1/2$. However, $\phi^{-1}(t) = \partial B_{r_1-t}(a_1)$ is connected, so in fact $\Gamma_{\phi,u^{\epsilon_n}}(t) = \partial B_{r_1-t}(a_1)$ for all $t \in A^n$. So for every $t \in A^n$ and $n \ge N_0$, Theorem 3.4 and the choice of λ imply that

$$
\min_{x\in\partial B_{r_1-t}(a_1)} |u^{\epsilon_n}| \ge \frac{1}{2}, \qquad |\deg(u^{\epsilon_n}; \partial B_{r_1-t}(a_1))| \le d_\lambda \le \frac{1}{\pi}\Theta(a_1).
$$

It follows that for all such n there is an integer $d(n)$ such that the set

$$
S_n^{d(n)} := \{r \in [0, r_1] \ : \ \min_{\partial B_r(a_i)} |u^{\epsilon_n}| > \frac{1}{2}, \ \deg(u^{\epsilon_n}; \partial B_r) = d(n)\}
$$

has measure at least $k_0 := \frac{r_1}{3d_\lambda}$. Note also that $S_n^{d(n)}$ is open, since u^{ϵ_n} is by assumption continuous (indeed, smooth). We can therefore find an open set $\Sigma_n \subset S_n^{d(n)}$ such that $|\Sigma_n| = k_0$.

3. We now define new test functions ψ^n as follows. First let

$$
f^n(r) = |[r, r_1] \cap \Sigma_n|
$$

We then define $\psi^n(x) = f^n(|x-a_1|)$. One can then check that t is a regular value of ψ^n if and only if

$$
(\psi^n)^{-1}(t) = \partial B_r(a_1) \quad \text{ for some } r \in \Sigma_n.
$$

In particular, $\deg(u; (\psi^n)^{-1}(t)) = d(n)$ for a.e. $0 < t < \|\psi^n\|_\infty = k_0$.

One can then easily check, using Theorem 3.4, that

$$
\int \psi^n J u^{\epsilon_n} dx = \pi d(n) k_0 + O(\epsilon^\alpha).
$$

On the other hand, since the functions ψ^n are uniformly bounded in $C_c^{0,1}$ and since $Ju^{\epsilon_n} \to J = \pi \sum c_i \delta_{a_i}$ in $C_c^{0,1}(U)^*$

$$
0 = \lim_n \left| \int \psi^n J u^{\epsilon_n} dx - \pi c_1 \psi^n(a_1) \right| = \lim_n \left| \int \psi^n J u^{\epsilon_n} dx - \pi c_1 k_0 \right| \ .
$$

Comparing the last two equations, we find that $d(n) = c_1$ for all sufficiently large n. In particular, c_1 is an integer and $|c_1| \le d_\lambda \le \frac{1}{\pi}\Theta(a_1)$, which is what we needed to show. □

4 Gamma Limit

Suppose that a sequence of functionals J_n on a Banach space X is given. We assume that they are bounded from below and by adding a constant, if necessary, we may assume that these functionals are nonnegative, i.e.,

$$J_n : X \mapsto [0, \infty].$$

The Gamma limit, J, of these functional in the topology of X is roughly given by

$$J(x) := \liminf\{ \ J_n(x_n) \ |(n, x_n) \to (\infty, x) \ \}.$$

More precisely, the Gamma limit is a nonnegative functional

$$J : X \mapsto [0, \infty],$$

satisfying the following two conditions:

- Let $\{x_n\} \subset X$ be a convergent sequence with limit $x \in X$, i.e.,

  $$\lim_{n\to\infty} \ \|x_n - x\|_X = 0.$$

  Then,

  $$\liminf_{n\to\infty} \ J_n(x_n) \ \geq \ J(x).$$

- For any $x \in X$, there exists a sequence $\{x_n^*\}$ satisfying

  $$\lim_{n\to\infty} \ \|x_n^* - x\|_X = 0,$$

  and

  $$\lim_{n\to\infty} \ J_n(x_n^*) \ = \ J(x).$$

Generally an accompanying compactness result is useful. Such a compactness result states that if for a sequence $\{x_n\} \subset X$

$$\sup_n \ J_n(x_n) < \infty, \tag{4.1}$$

then the set $\{x_n\}$ is compact in X.

An immediate corollary to a Gamma limit and to the compactness result is this: Let y_n^* be a minimizer of J_n, and they satisfy (4.1). Then, by compactness on a subsequence, denoted by n again, y_n converges to a point $y^* \in X$. By the first condition on the Gamma limit we know that

$$J(y^*) \leq \liminf_n \ J_n(y_n).$$

Let x be any point in X and let $\{x_n^*\}$ be the sequence in the second condition on the Gamma limit. Since y_n's minimize J_n,

$$\begin{aligned} J(x) &= \lim_n \ J_n(x_n) \\ &\geq \limsup_n \ J_n(y_n) \\ &\geq J(y^*). \end{aligned}$$

Hence any limit y^* of the minimizing sequence y_n is a minimizer of the Gamma limit J. In practice, this method is used to a construct minimizer of a given functional J. Given J, we construct a "regular" functionals J_n such that the Gamma limit of this sequence is J. Then, we obtain minimizers of J_n by standard methods and then apply the above method to construct a minimizer of J. The sequence J_n is often called a *relaxation* of J.

We refer to the book of Dal Maso [8] for more information.

In this section we calculate the Gamma limit of

$$I^\epsilon(u) \quad := \quad \frac{1}{\ln(1/\epsilon)} \int E^\epsilon(u) \ dx.$$

First we introduce a function space which is needed in order to state the Gamma limit.

4.1 Functions of BnV.

Motivated by the analysis of I_ϵ and the central role of BV in the scalar case, Jerrard and the author introduced and studied a class of functions called BnV in [14]; a short summary is provided in [15]. It turns out that the Gamma limit of the functional I^ϵ is finite only for functions $u \in B2V$. For that reason, we give a brief discussion of BnV. We refer to [14, 15] for more information.

Briefly, function $u \in W^{1,n-1}(U; I\!R^n)$, for $U \subset I\!R^m, m \geq n$, is said to belong to BnV if the weak determinants of all n by n submatrices of the gradient matrix ∇u are signed Radon measures. Here we only we give the precise definition of $B2V$ and refer to [14] for higher dimensional case.

For $U \subset I\!R^m \to I\!R^2$ with $m \geq 2$ we view the Jacobian as a measure taking values in the exterior algebra $\Lambda^2 I\!R^m$. For every n (and in particular for $n = 2$) we endow $\Lambda^n I\!R^m$ with the natural inner product structure, which we denote $(\cdot, \cdot)$, and for a multivector $v \in \Lambda^n I\!R^m$ we write $|v| = (v, v)^{1/2}$. If $u \in W^{1,1}(U; I\!R^2)$ we define

(4.2) $$j(u) = \sum_{i=1}^{m} \ u \times u_{x_i} \ dx^i \ ,$$

and if $j(u) \in L^1_{\text{loc}}$, we define

(4.3) $$Ju = \frac{1}{2} \ d \ j(u) \qquad \text{in the sense of distributions}$$

where d is the exterior derivative. Thus if $u \in H^1_{\text{loc}}$, then

$$Ju \ = \ \sum_{i<j} J^{ij} u \ dx^i \wedge dx^j \ = \ \frac{1}{2} \sum_{i,j} J^{ij} u \ dx^i \wedge dx^j,$$

where

$$J^{ij} u \ = \ -J^{ji} u \ = \ u_{x_i} \times u_{x_j} \ = \ \det(u_{x_i}, u_{x_j}).$$

For sufficiently differentiable $u : I\!R^m \to I\!R^n$ one can define in a similar way Ju as a measure taking values in $\Lambda^n I\!R^m$. We omit the most general definition as we will not need it here.

Here we give the precise definition of $B2V$. Let $U \subset I\!R^m$ with $m \geq 2$.

Definition. *We say that u belongs to $B2V(U; I\!R^2)$ if both of the conditions are satisfied*

- $j(u) \in L^1_{loc}(U; I\!R^2)$,
- *Ju is a Radon measure with values in $\Lambda^2 I\!R^m$.*

A priori Ju is only a distribution; we say that $u \in B2V$ if it happens to be a measure. Also there are several conditions that ensure that $j(u) \in L^1$. For instance if $u \in W^{1,1} \cap L^\infty$ $j(u) \in L^1$.

The class BnV is very closely related to the *Cartesian Currents* of Giaquinta, Modica and Soucek [10, 11]. This connection is discussed in detail in [14].

In [14] it is shown that if $u \in B2V(I\!R^m; S^1)$, then the Jacobian measures Ju is supported on an $m-2$ dimensional rectifiable set. In particular, if $u \in B2V(U; S^1)$ and $U \subset I\!R^2$, then there are $\{a_i\} \subset U$ and integers k_i such that

$$Ju = \pi \ \sum_{i} \ k_i \ \delta_{a_i} \ .$$

This is interpreted as encoding the location and degree of the topological singularities of u.

Here we outline only the proof in the case of $u \in B2V(U; S^1)$ with $U \subset I\!R^2$, and refer to [14] for the higher dimensional result.

Lemma 4.1 *Let $U \subset I\!R^2$ and $u \in B2V(U; I\!R^2)$. Then,*

$$Ju = \pi \sum_j k_j \; \delta_{a_j} \; , \tag{4.4}$$

for finite collections of points $\{a_j\} \subset U$ and integers k_j.

Outline of Proof:

In this case, Ju is a scalar valued, signed Radon measure. Hence for every $x \in U$, the following limit exists:

$$d(x) := \lim_{r \downarrow 0} \; d_r(x), \qquad d_r(x) := Ju(B_r(x)).$$

We claim that for every $x \in U$, and for almost every $r > 0$, $d_r(x)/\pi$ is an integer. Indeed, since $Ju = \nabla \times j(u)$, for any smooth function $\phi \in C_c^\infty(U; I\!R^1)$, by integration by parts

$$\int_U \phi Ju(dx) = \int_U \nabla\phi \cdot j(u) \; dx.$$

Formally if we take ϕ to be the characteristic function of the set $B_r(x)$, we obtain

$$\begin{aligned} Ju(B_r(x)) &= \int_{B_r(x))} Ju(dx) \\ &= \frac{1}{2} \int_{\partial B_r(x)} j(u) \; \cdot \vec{t} \; d\mathcal{H}^1. \end{aligned}$$

By approximation, we may show that the above identity holds if u is sufficiently smooth. For $u \in B2V$, in [14], the above identity is proved for almost every $r > 0$.

Moreover, since $u \in W^{1,1}(U; S^1)$, for almost every $r > 0$, $u \in W^{1,1}(B_r(x); S^1)$. Hence, for these values of r, u is absolutely continuous with values in S^1. Then, by the degree formulae discussed earlier,

$$\frac{1}{2} \int_{\partial B_r(x)} j(u) \; \cdot \vec{t} \; d\mathcal{H}^1 = \pi \; \deg(u; \partial B_r(x)).$$

Therefore $d_r(x)/\pi$ is an integer for almost every r. Since the limit of $d_r(x)$ exists, we conclude that for every $x \in$ there exists $r(x) > 0$ such that

$$d_r(x) = d(x), \qquad \forall \; r \in (0, r(x)],$$

and

$$d(x)/\pi \in Z, \qquad \forall\, x \in U.$$

Since $|Ju|(U)$ is finite, by a simple covering argument these yield the desired result. $\square$

4.2 Gamma limit of I^ϵ

Let U be an open bounded subset of $I\!R^2$ with a smooth boundary.

In this section we study the Γ limit of the functionals

$$I_\epsilon(u) := \frac{1}{\ln(1/\epsilon)} \int_U \frac{1}{2}|\nabla u|^2 + \frac{1}{4\epsilon^2}(1-|u|^2)^2 \; dx \ ,$$

as ϵ tends to zero and show that the limiting functional is

$$I(u) := \begin{cases} |Ju|(U) = \pi \ \sum_i |k_i| \ , & \text{if } u \in B2V(U; S^1) \ , \\ +\infty \ , & \text{if } u \notin B2V(U; S^1) \ . \end{cases}$$

Theorem 4.2 *The Γ limit of I_ϵ in the topology of $W^{1,1}(U; I\!R^2)$ is equal to I, i.e., for every sequence u^ϵ converging to u in $W^{1,1}(U; I\!R^2)$,*

$$\liminf_{\epsilon \to 0} I_\epsilon(u^\epsilon) \geq I(u) \ , \tag{4.5}$$

and for every $u \in B2V(U; S^1)$, there exist functions u^ϵ converging to u in $W^{1,1}(U; I\!R^2)$ satisfying

$$\liminf_{\epsilon \to 0} I_{\epsilon_n}(u^\epsilon) = I(u) \ . \tag{4.6}$$

A similar result also holds in higher dimensions [13].

The above result is proved independently in [1], and in [13]

Remark 4.3 In view of our compactness result and our introductory discussion of Gamma Limit, the Banach space $W^{1,1}$ is not appropriate. Since we do not have a compactness result in $W^{1,1}$, and the only compactness result is for the Jacobian, it is more natural to consider the Gamma Limit in the space of equivalent classes of functions with a topology equivalent to the convergence of the Jacobian. However, here and in [13], we chose to work with $W^{1,1}$ as it is a standard Banach space.

Proof. We start with the proof of (4.5). Suppose that u^ϵ converges to u in $W^{1,1}(U; I\!R^2)$. We assume that

$$\liminf_{\epsilon} I_\epsilon(u^\epsilon) < \infty \ ,$$

as there would be nothing to prove otherwise.

1. By the Compactness Theorem 3.5, there exists a subsequence ϵ_n converging to zero such that the Jacobian measure Ju^{ϵ_n} converges to a Radon measure J in $(C_c^{0,\beta})^*$ for all $\beta > 0$. We claim that $J = Ju$. In particular this will show that $u \in B2V(U; S^1)$.

To simplify the notation, set $u_n := u^{\epsilon_n}$.

2. We directly estimate that

$$\begin{aligned}
\left| j(u_n) - \frac{j(u_n)}{|u_n|^2 \wedge 1} \right| &\leq |u_n||\nabla u_n| \left| \frac{|u_n|^2 \wedge 1 - 1}{|u_n|^2 \wedge 1} \right| \\
&= |\nabla u_n| \ \frac{|1 - |u_n|^2|}{|u_n|} \ \chi_{|u_n| \geq 1} \\
&\leq \epsilon_n \left[\frac{1}{2} |\nabla u_n|^2 + \frac{1}{2\epsilon_n^2} (1 - |u_n|^2)^2 \right] \ .
\end{aligned}$$

Hence,

$$\lim_{n \to \infty} \int_U \left| j(u_n) - \frac{j(u_n)}{|u_n|^2 \wedge 1} \right| \ dx = 0 \ .$$

3. Set $v_n := u_n / (|u_n|^2 \wedge 1)$ so that

$$\begin{aligned}
\frac{1}{|u_n|^2 \wedge 1} \ j(u_n) - j(u) &= v_n \times \nabla u_n - u \times \nabla u \\
&= v_n \times (\nabla u_n - \nabla u) + (v_n - u) \times \nabla u \ .
\end{aligned}$$

Hence

$$\begin{aligned}
\left| \frac{1}{|u_n|^2 \wedge 1} \ j(u_n) - j(u) \right| &\leq |v_n||\nabla u_n - \nabla u| + |v_n - u||\nabla u| \\
&\leq |\nabla u_n - \nabla u| + |v_n - u||\nabla u| \ .
\end{aligned}$$

Since u_n converges to u in $W^{1,1}(U; I\!R^2)$, there exists a subsequence, denoted by n again, so that u_n converges to u almost everywhere. Hence $|v_n - u||\nabla u|$ converges to zero almost everywhere and also it is less than $2|\nabla u|$. So we may use the dominated convergence theorem to conclude that

$$\lim_{n \to \infty} \int_U \left| \frac{1}{|u_n|^2 \wedge 1} \ j(u_n) - j(u) \right| \ dx = 0 \ .$$

4. Steps 2 and 3 imply that on a subsequence $j(u_n)$ converges to $j(u)$ in L^1. Hence, Ju^{ϵ_n} converges to Ju in the sense of distributions. This implies that $J = Ju$. Since by Theorem 3.5, J is a Radon measure, so is Ju and therefore $u \in B2V(U; I\!R^2)$. It is also clear that $|u| = 1$ almost everywhere. Hence, $u \in B2V(U; S^1)$.

5. The Jacobian estimate (3.1) implies that

$$\begin{aligned} \left| \int_U \phi \ Ju(dx) \right| &= \lim_{n\to\infty} \left| \int_U \phi \ Ju_n(dx) \right| \\ &\leq \lambda \|\phi\|_\infty \liminf_{n\to\infty} I^{\epsilon_n}(u_n) \ , \end{aligned}$$

for every $\lambda > 1$. Hence,

$$\begin{aligned} \liminf_{n\to\infty} I^{\epsilon_n}(u_n) &\geq \sup \left\{ \left| \int_U \phi \ Ju(dx) \right| \ : \ \|\phi\|_\infty \leq 1 \right\} \\ &= |Ju|(U) \\ &= I(u) \ . \end{aligned}$$

This proves (4.5).

6. We continue by proving the Γ-limit upper bound (4.6). Fix $u \in B2V(U; S^1)$. As remarked above, it is shown in [14] that Ju must have the form

$$Ju = \pi \sum_j k_j \ \delta_{a_j} \ ,$$

It suffices to show that, given any sufficiently small $\delta > 0$, there exists a sequence of functions $\{v^\epsilon\} \subset H^1(U; I\!R^2)$ such that

$$I_\epsilon(v^\epsilon) \to \pi \sum |k_j|, \qquad \limsup_\epsilon \|v^\epsilon - u\|_{W^{1,1}(U)} \leq C\delta.$$

To do this, fix some small $\delta > 0$. Let $r_0 > 0$ be a number such that the balls $\{B_{2r}(a_j)\}$ are pairwise disjoint and do not intersect ∂U, whenever $r \leq r_0$, and select some $r > 0$ such that

$$\sum_j \int_{B_{2r}(a_j)} |\nabla u| \ dx \ \leq \ \delta, \qquad r \leq \min\{r_0, \delta\}. \tag{4.7}$$

For any $s > 0$, let U_s denote $U \setminus \cup_j B_s(a_j)$. Demengel [9] proves that if V is an open subset of $I\!R^2$, then smooth functions taking values in S^1 are dense in the subspace $\{w \in W^{1,1}(V; S^1) \ : \ Jw = 0\}$. Since $Ju = 0$ on U_r, this implies that there exists a function $v \in C^\infty(U_r, S^1)$ such that

$$\|u - v\|_{W^{1,1}(U_r)} \leq \delta. \tag{4.8}$$

Demengel's proof in fact shows that we may also assume that

$$\|j(u) - j(v)\|_{L^1(U_r)} \leq \delta. \tag{4.9}$$

7. Clearly (4.7) and (4.8) imply that

$$\sum_j \int_r^{2r} \int_{\partial B_s(a_j)} |\nabla v(x)| \; d\mathcal{H}^1(x) ds \;=\; \sum_j \int_{B_{2r} \setminus B_r(a_j)} |\nabla v| \; dx \;\leq\; 2\delta.$$

So for each j we can find some number $r_j \in [r, 2r]$ such that

$$\int_{\partial B_{r_j}(a_j)} |\nabla v(x)| \; d\mathcal{H}^1(x) \leq \frac{2\delta}{r} \tag{4.10}$$

We also claim that

$$\deg(v; \partial B_{r_j}(a_j)) = k_j \tag{4.11}$$

if δ is sufficiently small. Indeed, since v is smooth and S^1-valued it is clear that $s \mapsto \deg(v; \partial B_s(a_j))$ is constant for $s \in [r, 2r_0]$, so we only need to verify that this constant must equal k_j. To do this, note that if ϕ is any function of the form $\phi(x) = \bar{\phi}(|x - a_j|)$ that is constant on $B_r(a_j)$ and has its support in $B_{2r_0}(a_j)$, then

$$\frac{1}{2} \int \nabla \times \phi \cdot j(v) \; dx \;=\; \frac{1}{2} \int_0^\infty \deg(v; \phi^{-1}(s)) \; ds \;=\; \pi \phi(a_j) \deg(v; \partial B_{r_j}(a_j))$$

and

$$\frac{1}{2} \int \nabla \times \phi \cdot j(u) \; dx \;=\; \int \phi \; dJu \;=\; \pi \phi(a_j) k_j.$$

If δ is small enough, (4.11) follows from these two identities and (4.9), since $\nabla \times \phi$ is supported in U_r.

8. We claim that for each j there exists smooth functions v_j^ϵ, defined in $B_{r_j}(a_j)$ such that $v_j^\epsilon(x) = v(x)$ for $x \in \partial B_{r_j}(a_j)$,

$$\int_{B_{r_j}(a_j)} |\nabla v_j^\epsilon| dx \;\leq\; C\delta, \qquad \text{and} \qquad \lim_{\epsilon \to 0} \frac{1}{|\ln \epsilon|} \int_{B_{r_j}(a_j)} E^\epsilon(v_j^\epsilon) dx = \pi |k_j|. \tag{4.12}$$

To see this, fix some j. We may assume without loss of generality that $a_j = 0$, and due to (4.11) we can write

$$v(x) = \exp[i(k_j \theta + \alpha_j + \psi(x))] \qquad \text{for } x \in \partial B_{r_j}$$

where α_j is a constant, ψ is a smooth, single-valued function on ∂B_{r_j}, and θ as usual satisfies $\frac{x}{|x|} = (\cos\theta, \sin\theta)$. We are identifying $\mathbb{R}^2 \cong \mathbb{C}$ in the usual way. We extend ψ to be homogeneous of degree zero on $\mathbb{R}^2 \setminus \{0\}$, and we define

$$v_j^\epsilon(x) \;=\; \exp[i(k_j\theta + \alpha_j + \frac{2|x| - r_j}{r_j}\psi(x))] \quad \text{if} \quad \frac{1}{2}r_j \le |x| \le r_j.$$

For $|x| \le \frac{1}{2}r_j$ we define $v_j^\epsilon(x)$ to be a minimizer of

$$\int_{B_{r_j/2}} E^\epsilon(w)\ dx$$

subject to the boundary conditions $w = \exp[i(k_j\theta + \alpha_j)]$ on $\partial B_{r_j/2}$.

Since v_j^ϵ restricted to the annulus $B_{r_j} \setminus B_{r_j/2}$ is just a fixed smooth function of unit modulus, independent of ϵ, it is clear that

$$\lim_\epsilon \frac{1}{|\ln\epsilon|}\int_{B_{r_j}\setminus B_{r_j/2}} E^\epsilon(v_j^\epsilon)\ dx \;=\; \lim_\epsilon \frac{1}{|\ln\epsilon|}\int_{B_{r_j}\setminus B_{r_j/2}} \frac{1}{2}|\nabla v_j^\epsilon|^2\ dx \;=\; 0.$$

Also, using (4.10) one can check that

$$\int_{B_{r_j}\setminus B_{r_j/2}(a_j)} |\nabla v_j^\epsilon| dx \;\le\; C\delta.$$

Finally, the book of Bethuel, Brezis, and Hélein gives a detailed description of the asymptotics of Ginzburg-Landau energy-minimizers, and their results imply that

$$\lim_\epsilon \frac{1}{|\ln\epsilon|}\int_{B_{r_j/2}} E^\epsilon(v_j^\epsilon)\ dx \;=\; \pi|k_j|, \qquad \limsup_\epsilon \int_{B_{r_j/2}} |\nabla v_j^\epsilon|\ dx \;\le\; Cr_j \le C\delta.$$

Putting these facts together we find that the sequence $\{v_j^\epsilon\}$ has the properties specified in (4.12).

9. Finally we define

$$v^\epsilon(x) = \begin{cases} v(x) & \text{if } x \in U \setminus \left(\cup_j B_{r_j}(a_j)\right) \\ v_j^\epsilon(x) & \text{if } x \in B_{r_j}(a_j) \end{cases}$$

Since v is a fixed smooth function and $|v| \equiv 1$, $\frac{1}{|\ln\epsilon|}E^\epsilon(v) = \frac{1}{|\ln\epsilon|}|\nabla v|^2$ tends to zero uniformly as $\epsilon \to 0$. Thus it is clear from (4.12) that

$$\lim_{\epsilon\to 0} \frac{1}{|\ln\epsilon|}\int_U E^\epsilon(v^\epsilon) dx \;=\; \sum_j \lim_{\epsilon\to 0}\frac{1}{|\ln\epsilon|}\int_{B_{r_j}(a_j)} E^\epsilon(v_j^\epsilon) dx = \pi\sum_j |k_j|.$$

Also,

$$\|u - v^\epsilon\|_{W^{1,1}(U)} \;\leq\; \|u - v\|_{W^{1,1}(U_r)} + \sum_j \left(\|u\|_{W^{1,1}(B_{r_j}(a_j))} + \|v_j^\epsilon\|_{W^{1,1}(B_{r_j}(a_j))} \right) \leq C\delta$$

by (4.7), (4.8), and (4.12). So the sequence $\{v^\epsilon\}$ has all the required properties. □

5 Compactness in Higher Dimensions

Now suppose U is a bounded, open subset of $I\!R^m$ with $m \geq 3$.

In this section we will show that if $\{u^\epsilon\}_{\epsilon \in (0,1]} \subset H^1(U; I\!R^2)$ is a sequence of functions such that the normalized Ginzburg-Landau energy measure $\mu^\epsilon(U)$ is uniformly bounded, then the Jacobians Ju^ϵ are precompact in $(C^{0,\beta})^*$ for all $\beta > 0$, and any limit is rectifiable. In addition, we prove that

$$|\bar{J}|(U) \leq \liminf \mu^\epsilon(U).$$

This is not a full Γ-convergence result, but it shows that the mass of the Jacobian is a reasonable candidate for the Γ-limit. We also believe that the compactness result and the upper bound for the Jacobian (ie, lower bound for the energy) are interesting and will be useful in other contexts.

We start by defining some of the terms used above. We remark that good general references for this material include Giaquinta *et. al* [10] and Simon [28].

Let $\Lambda^2 I\!R^m$, $j(u)$ and Ju be as in subsection 4.1.

A set $M \subset I\!R^m$ is said to be a k-dimensional rectifiable set if there are Lipschitz functions $f_j : I\!R^k \to I\!R^m$ and measurable subsets A_j of $I\!R^k$ such that

$$M = M_0 \cup \left(\cup_{j=1}^\infty f_j(A_j) \right), \qquad\qquad \mathcal{H}^k(M_0) = 0.$$

Thus, in a precise measure theoretic sense, a k-dimensional rectifiable set is not much worse than a k-dimensional Lipschitz submanifold. Rectifiable sets can also be characterized by the fact that they have k-dimensional approximate tangent spaces $\mathcal{H}^k$ almost everywhere; see [28] or [10].

Suppose that M is an oriented, rectifiable $(m-n)$-dimensional subset of $I\!R^m$, and for $\mathcal{H}^{m-n}$ almost every $x \in M$, let $\nu(x) \in \Lambda^n I\!R^m$ be the unit n-vector representing the appropriately oriented normal space to M. (It is more convenient for our purposes to work with normal spaces rather than tangent spaces.) Suppose also that $\theta : M \to I\!N$ is a $\mathcal{H}^{m-n}$-integrable function. One can define a measure J taking values in $\Lambda^n I\!R^m$ by

$$(5.1) \qquad \int \phi(x) J(dx) \;=\; \int_M \phi(x) \cdot \nu(x) \theta(x) \mathcal{H}^{m-n}(dx) \qquad \forall \phi \in C^0(I\!R^m; \Lambda^n I\!R^m).$$

We say that a measure J taking values in $\Lambda^n I\!R^m$ is $(m-n)$-dimensional integer multiplicity rectifiable (or more briefly, integer multiplicity rectifiable) if it has the form (5.1) for some rectifiable set M and an integer-valued function θ as above.

The class of functions for which Ju is a measure is denoted $BnV(U, I\!R^n)$ and was defined and studied in [14]. In particular we prove there that if $u \in BnV(I\!R^m, S^{n-1})$ then Ju/ω_n is integer multiplicity rectifiable, where ω_n is the volume of the unit ball in $I\!R^n$. This is deduced as a consequence of a more general rectifiability criterion which we recall here.

Let J be a measure on a subset $U \subset I\!R^m$ taking values in $\Lambda^n I\!R^m$, where $n \leq m$. We can write J in the form $J = \nu|J|$, where $|J|$ is a nonnegative Radon measure, and ν is a $|J|$-measurable function taking values in $\Lambda^n I\!R^m$ such that $|\nu(x)| = 1$ at $|J|$-a.e. $x \in U$.

Suppose that $e_1, ..., e_m$ is an orthonormal basis for $I\!R^m$. Given any point $x \in I\!R^m$, we write $y_i = x \cdot e_i$ if $i = 1, ..., m-n$; and $z_i = x \cdot e_{m-n+i}$ if $i \in 1, \ldots, n$. We write $I\!R^{m-n}_y$ to denote the span of $\{e_i\}_{i=1}^{m-n}$. Similarly, $I\!R^n_z = \mathrm{span}\{e_i\}_{i=m-n+1}^m$. Thus we identify points $x \in I\!R^m$ with corresponding $(y,z) \in I\!R^{m-n}_y \times I\!R^n_z$. Let $dz := dz^1 \wedge \ldots \wedge dz^n$, and let J^z denote the scalar signed measure defined by $J^z := (dz, \nu)|J|$.

We say that J^z is *locally represented by slices* $J_y(dz)$ if, given any open set $O \subset U$ of the form $O = O_y \times O_z$, with $O_y \subset I\!R^{m-n}_y$ and $O_z \subset I\!R^n_z$, there exist signed Radon measures $J_y(dz)$ on O_z for a.e. $y \in O_y$, such that

$$\int \phi J^z = \int_{O_y} \int_{O_z} \phi(y,z) J_y(dz)\ dy \tag{5.2}$$

for all continuous ϕ with compact support in O.

We say that a statement holds for *a.e.* $J_y(dz)$ if, for every open set O as above, it is valid for a.e. $y \in O_y$.

In [14] we prove the following

Theorem 5.1 *Suppose that J is a Radon measure on $U \subset I\!R^m$ taking values in $\Lambda^n I\!R^m$, and also that $dJ = 0$ in the sense of distributions. Suppose also that for every choice on an orthonormal basis $\{e_i\}_{i=1}^m$ (determining a decomposition of $I\!R^m$ into $I\!R^{m-n}_y \times I\!R^n_z$) J^z is represented locally by slices, and that for a.e. $y \in O_y$ these slices have the form*

$$J_y(dz) = \sum_{i=1}^{K} d_i \delta_{a_i}(dz)$$

for an integers K and d_i, and points $a_i \in O_z$.

Then J is rectifiable.

A much more general version of this result was later established by Ambrosio and Kirchheim [2]. A similar theorem in somewhat different and very general setting was proved independently (and slightly earlier) by White [34].

We will need Theorem 5.1 to prove

Theorem 5.2 (Jerrard & Soner [13]) *Let $U \subset I\!R^m$, and suppose that $\{u^\epsilon\}_{\epsilon\in(0,1]}$ is a collection of functions in $W^{1,2}(U; I\!R^2)$ such that $\mu^\epsilon(U) \leq K_U < \infty$ for all ϵ. Then there exists a subsequence $\epsilon_n \to 0$ such that*

(i): Ju^{ϵ_n} converges to a limit $\bar{J}$ in the $(C^{0,\alpha})^*$ norm for every $\alpha > 0$;

(ii): For any choice of basis $\{e_i\}$ for $I\!R^m$ (determining a decomposition of $I\!R^m$ into $I\!R^{m-2}_y \times I\!R^2_z$), $\bar{J}^z$ is represented locally by slices $\bar{J}_y(dz)$, and for a.e. y these slices have the form $\bar{J}_y(dz) = \pi \sum_{i=1}^K d_i \delta_{a_i}$, with $d_i \in Z\!\!\!Z$ for all i.

(iii): $d\bar{J} = 0$ in the sense of distributions, and $\bar{J}/\pi$ is integer multiplicity rectifiable;

(iv): Finally, if $\bar{\mu}$ is any weak limit of a subsequence of μ^{ϵ_n}, then $|\bar{J}| \ll \bar{\mu}$, and $\frac{d|\bar{J}|}{d\bar{\mu}} \leq 1$. In particular, $|\bar{J}|(U) \leq K_U$.

Remark 5.3 For any $\bar{J}$ as above, (iii) and the definition of rectifiability imply that a lower density bound:

$$\liminf_{r\to 0} \frac{|\bar{J}|(B_r(x))}{\mathcal{H}^{m-2}(B_r(x))} \geq \pi$$

for $|\bar{J}|$ almost every x. Also, if $\bar{\mu}$ is as in (iv), then clearly the $m-2$-dimensional density of μ is greater than $m-2$-dimensional density of $\bar{J}$. In particular,

$$\liminf_{r\to 0} \frac{\bar{\mu}(B_r(x))}{\mathcal{H}^{m-2}(B_r(x))} \geq \pi$$

for $|\bar{J}|$ almost every x.

The basic idea of the proof is to decompose a component of Ju^ϵ, for example $J^{m-1,m}u^\epsilon$, into two-dimensional slices, say $J^\epsilon_y(dz)$, and to use the two-dimensional estimates on each slice. Arguing in this fashion, it is quite easy to obtain uniform estimates for $J^{m,m-1}u^\epsilon$ in certain weak spaces, and these imply (i) by results of the previous section.

To prove (ii), it is convenient to view the sliced measures $J^\epsilon_y(dz)$ as constituting a function mapping $I\!R^{m-2}_y$ into $C^1_c(I\!R^2_z)^*$; the latter is a space that contains measures on $I\!R^2_z$ and is endowed with a rather weak topology. Claim (i) can be seen as assertion that the function $y \mapsto J^\epsilon_y(dz)$ is precompact in some weak sense. What one would like to do is to show that in fact $y \mapsto J^\epsilon_y(dz)$ is precompact in some stronger sense, for example in $L^1(I\!R^{m-2}_y; (C^1_c(I\!R^2_z)^*)$, so that one can extract a subsequence that converges to some limiting function $y \mapsto \bar{J}_y(dz)$ in L^1. In particular, after passing to a further subsequence we could then assume that $J^{\epsilon_n}_y(dz) \to \bar{J}_y(dz)$ for almost every y. In addition, by our two-dimensional results, for almost every y, one can find a subsequence $\epsilon_{n_m} \to 0$ (in general depending on y) such that $J^{\epsilon_{n_m}}_y(dz)$ converges to some limit that has the form sought in (ii). By combining these results one can hope to show that in fact $\frac{1}{\pi}\bar{J}_y(dz)$ is a sum of point masses with integer multiplicities.

The key point is then to establish some sort of strong compactness of the sequence of functions $y \mapsto J^\epsilon_y(dz)$ as $\epsilon \to 0$. We do this using the observation from [14, 15] that the total variation of $y \mapsto J^\epsilon_y(dz)$ in the $(C^{0,1}_c)^*$ norm can be estimated by controlling "orthogonal" components of Ju^ϵ, which is already done in the proof of (i). Using this one can argue that the functions $y \mapsto J^\epsilon_y(dz)$ have uniformly bounded variation in $(C^{0,1}_c)^*$, modulo terms that vanish in still weaker norms, and this gives the necessary strong convergence. (The terms involving weaker norms force us to work with test functions that are C^2 instead of C^1 in much of the proof.)

The remaining points follow quite directly from (ii) and the rectifiability criterion of Theorem 5.1, and from the two-dimensional results.

6 Dynamic Problems: Evolution of Vortex Filaments

The stationary results, in particular the energy lower bounds and the compactness of the Jacobian are very useful tools in the asymptotic study of the evolution problems related to I^ϵ. In these note we only consider the parabolic equation which is the gradient flow of I^ϵ:

(6.1) $$u^\epsilon_t - \Delta u^\epsilon = \frac{u^\epsilon}{\epsilon^2}\,(1 - |u^\epsilon|^2), \quad t > 0,\ x \in I\!R^n,$$

where the unknown function

$$u^\epsilon : I\!R^n \mapsto I\!R^2,$$

satisfies an initial condition

(6.2) $$u^\epsilon(0,x) = u^\epsilon_0(x), \qquad x \in I\!R^n,$$

where u^ϵ_0 is a given function.

Here we do not consider the corresponding nonlinear Schrodinger and the nonlinear heat equations. Instead we refer to the notes of Rubinstein [23] for an introduction and the description of formal results and to the paper of Colliander and Jerrard [7] for a rigorous study of the nonlinear, planar Schrodinger equation.

Also here we only consider the case $n \geq 3$. Results for $n = 2$ are outlined in the Introduction.

The asymptotic analysis of these equations initiated by the seminal paper of Rubinstein, Sternberg and Keller [24]. Much has been done for the scalar equation since then. We refer to the survey article of the author [29] and the references therein for the scalar equation, which is also called as the Cahn-Allen equation in the literature.

The vector valued Ginzburg-Landau equation with $u^\epsilon : I\!R^n \mapsto I\!R^m$ is studied first by Struwe [33]. He considered this equation as a relaxation of the heat flow for harmonic maps. Under the assumption that original (not rescaled) energy is uniformly bounded in ϵ, he obtained deep partial regularity results. A monotonicity result is one of the interesting results of [33]. Our results differ from [33] in that we only consider the case $u^\epsilon \in I\!R^2$ but we allow for singularities to form and study the time evolution of the singular structures.

Here we outline the results of [17] and [4]. This technique is also outlined in the lecture notes of the author [31] for the scalar equations. As discussed in the Introduction, in the following proof, we will use the compactness of the Jacobian to provide a shorter proof.

6.1 Energy identities

Set

$$e_\epsilon := e_\epsilon(t,x) = E^\epsilon(u^\epsilon(t,x)).$$

Recall that the energy density E^ϵ is given as

$$E^\epsilon(u) := \frac{1}{2}|\nabla u|^2 + \frac{(1-|u|^2)^2}{4\epsilon^2}.$$

By calculus and (6.1),

$$(e_\epsilon)_t = -|u_t^\epsilon|^2 + \mathrm{div} p^\epsilon, \tag{6.3}$$

$$\nabla e_\epsilon = -p^\epsilon + \mathrm{div}(\nabla u^\epsilon \otimes \nabla u^\epsilon), \tag{6.4}$$

$$p^\epsilon := \nabla u^\epsilon \ u_t^\epsilon$$

To localize the energy estimates, let $\eta \geq 0$ be a smooth compactly supported test function. Multiply (6.3) by η and (6.4) by $\nabla\eta$ and subtract the two identities. Then use the resulting identity to compute the time derivative of the integral of ηe_ϵ. The result is:

$$\frac{d}{dt}\int \eta\ e_\epsilon = \int (\eta_t - \Delta\eta)\ e_\epsilon + D^2\eta\nabla u^\epsilon\cdot\nabla u^\epsilon - \int \eta\ |u_t^\epsilon|^2. \tag{6.5}$$

Although we will not use it in these notes, let us mention that if we add the two identities instead of subtracting them, we obtain the following identity:

$$\begin{aligned} \frac{d}{dt}\int \eta\ e_\epsilon \ &= \ \int (\eta_t + \Delta\eta)\ e_\epsilon - D^2\eta\nabla u^\epsilon\cdot\nabla u^\epsilon \\ &\quad + \frac{|\nabla\eta\cdot\nabla u^\epsilon|^2}{\eta} - \int \eta\left|u_t^\epsilon - \frac{\nabla\eta\cdot\nabla u^\epsilon}{\eta}\right|^2. \end{aligned} \tag{6.6}$$

In [33], Struwe used the above identity with the special choice for η:

$$\eta(t,x) = \frac{1}{(4(t_0-t))^{(n-m)/2}}\ \exp\left(-\frac{|x-x_0|^2}{4(t_0-t)}\right), \qquad t<t_0,\ x\in I\!R^n,$$

where $t_0 > 0$ and $x_0 \in I\!R^n$ are arbitrary, and m is the dimension of the range of u.

The special case of (6.5) with $\eta \equiv 1$ yields the classical energy estimate,

$$\int e^\epsilon(t,x)\ dx + \int_0^t\ |u_t^\epsilon|^2\ dxdt = \int e^\epsilon(0,x)\ dx. \tag{6.7}$$

6.2 Mean curvature flow and the distance function

The distance and the square distance functions to a smooth manifold can be used to describe all the relevant geometric quantities. These functions were first used in [17] in the convergence proofs.

Let $\{\Gamma_t\}_{t\in[0,T]}$ be a smooth solution of co-dimension two mean curvature flow. Then the square distance function $\eta(t,x)$ satisfies

$$\eta_t - \Delta\eta = -2, \qquad \text{on } \Gamma_t, \tag{6.8}$$

$$\nabla\left[\eta_t - \Delta\eta\right] = 0, \qquad \text{on } \Gamma_t, \tag{6.9}$$

$$D^2\eta \le I_n, \qquad \text{on whenever it is smooth.} \tag{6.10}$$

The equation (6.9) is an observation of DeGiorgi. It can be viewed as the definition of the codimension two mean curvature flow. We refer to [3] for information on the mean curvature flow in any codimension. However, (6.8), and (6.10) are the properties of any square distance function to a smooth codimension two manifold.

Since Γ_t is smooth, η is smooth in a tubular neighborhood of Γ_t. (6.8) and (6.9) imply that in this neighborhood,

$$|\eta_t - \Delta\eta + 2| \le C\ \eta, \tag{6.11}$$

for some constant C. We extend η smoothly to all of $[0,T]\times I\!R^n$ so that the extension satisfies (6.10), (6.11), and

$$\eta \ge 0, \quad \text{and } \eta(t,x) = 0, \quad \text{if and only if} \quad x\in\Gamma_t. \tag{6.12}$$

In (6.5) we choose η to be the square distance function, modified as above. Set

$$\alpha^\epsilon(t) := \frac{1}{\ln(1/\epsilon)}\int\ \eta\ e_\epsilon(t,x)\ dx,$$

so that by (6.5), (6.10) and (6.11),

$$
\begin{aligned}
\frac{d}{dt}\alpha^\epsilon(t) &\leq \frac{1}{\ln(1/\epsilon)} \int \left[C\ \eta_t\ e_\epsilon - 2e_\epsilon + |\nabla u^\epsilon|^2\right] - \frac{1}{\ln(1/\epsilon)} \int |u_t^\epsilon|^2\ dx \\
&\leq C\ \alpha^\epsilon(t) - \frac{1}{\ln(1/\epsilon)} \int |u_t^\epsilon|^2\ dx.
\end{aligned} \tag{6.13}
$$

6.3 Convergence.

In this subsection, we prove the convergence of the solutions of 6.1 to smooth solutions of the mean curvature flow. This is first proved in [17], and here we follow a new approach using the compactness result on the Jacobians. The convergence to weak solutions of the mean curvature flow is still not known. A first step in this direction is proved in [4].

Here we assume that the initial data concentrates on a smooth co-dimension two manifold Γ_0 and that there exists a smooth solution $\{\Gamma_t\}_{t\in[0,T]}$ of the codimension two mean curvature flow. then, we will prove that the energy measure concentrates on the smooth solution.

Set

$$
\mu_t^\epsilon(V) := \frac{1}{\ln(1/\epsilon)} \int_V E^\epsilon(u^\epsilon(t,x))\ dx,
$$

be the rescaled Ginzburg-Landau energy. Precisely, we assume that u_0^ϵ is such that

$$
\mu_0^\epsilon \stackrel{\star}{\rightharpoonup} \pi\ \mathcal{H}^{n-2} \lfloor \Gamma_0,
$$

in the weak* topology of Radon measure, where $\mathcal{H}^{n-2}\lfloor \Gamma_0$ is the $n-2$ dimensional Hausdorff measure restricted to the codimension two manifold Γ_0, i.e, it is the surface area measure of Γ_0.

A simple, consequence of this assumption is that

$$
C^* := \sup_\epsilon\ \mu_0^\epsilon(I\!R^n)\ <\ \infty.
$$

This together with the energy estimate (6.7) imply

$$
\sup\{\ \mu_t^\epsilon(I\!R^n)\mid t\in[0,T],\ \epsilon>0\ \}\ \leq\ C^*.
$$

Using an argument due to Brakke (see [17] for details), this implies that there exists a subsequence, denoted by ϵ again, such that

$$
\mu_t^\epsilon \stackrel{\star}{\rightharpoonup} \mu_t, \qquad t\in[0,T],
$$

to some nonnegative Radon measures $\{\mu_t\}_{t\in[0,T]}$.

Theorem 6.1 *Suppose that* $\{\Gamma(t)\}_{t\in[t_0,t_1]}$ *is a collection of compact sets which is a classical solution of the codimension two mean curvature flow. Let* μ_t *be a weak* limit of the rescaled energy measure* μ_t^ϵ*. Then,*

$$\mu_t \geq \pi \mathcal{H}^1 \lfloor \Gamma_t, \qquad t \in [0,T],$$

and

$$\textit{support } \mu_t = \Gamma_t, \qquad t \in [0,T].$$

Proof.

Since each connected component of the solution can be studied separately, without loss of generality, we may assume that Γ_t is conneceted with no boundary.

1. Let $\alpha^\epsilon(t)$ be as in the previous subsection. Then by the convergence of μ_t^ϵ,

$$\lim_{\epsilon \to 0} \alpha^\epsilon(t) = \alpha(t) := \int \eta \; \mu_t(dx).$$

In view of (6.12) and our assumption on μ_0,

$$\alpha(0) = 0.$$

Then by (6.13) and the Gronwald's inequality

$$\alpha(t) \leq \alpha(0) \; e^{Ct} = 0.$$

Since η is nonnegative, this implies that the support of μ_t is included in the zero set of η. Hence by (6.12),

$$\text{support } \mu_t \subset \Gamma_t, \qquad t \in [0,T].$$

2. Fix $t \in [0,T]$ and let $J_t^\epsilon := Ju^\epsilon(t,\cdot)$. Since the rescaled Ginzburg-Landau energy $\mu_t^\epsilon(I\!R^n)$ is uniformly bounded in ϵ, we may apply the compactness result proved earlier. Then, on a subsequence denoted by ϵ again, J_t^ϵ converges to a signed Radon measure J_t in the topology of $(C^{0,\alpha})^*$ for every $\alpha > 0$. Moreover,

$$\frac{d|J_t|}{d\mu_t} \leq 1, \qquad \mu_t \text{ a.e.},$$

where $|J_t|$ is the total variation of the vector valued measure J_t and μ_t is the weak* limit of μ_t^ϵ.

Hence, by step 1,

$$\text{support } J_t \subset \text{support } \mu_t \subset \Gamma_t.$$

3. Since

$$J_t^\epsilon = dj(u^\epsilon(t,\cdot)),$$

$$dJ_t^\epsilon = 0, \quad \Rightarrow \quad dJ_t = 0.$$

If we see J_t as a current, the above states that it has no boundary. By the previous step J_t is included in a smooth manifold Γ_t which has no boundary. In Lemma 6.2 below, we will show that these fact imply that the density of $|J_t|$ on Γ_t

$$\Theta_t := \frac{d|J_t|}{d\mathcal{H}^{n-2} \llcorner \Gamma_t}$$

is constant.

We postpone the proof of this result to Lemma 6.2 and complete the proof of this theorem.

In view of our compactness result, this constant has to be an integer multiple of π, i.e.,

$$\theta_t \equiv n_t\ \pi, \quad \mathcal{H}^{n-2} \llcorner \Gamma_t \text{ a.e.},$$

for some integer n_t.

4. It suffices to show that $n_t \equiv 1$ for $t \in [0,T]$. Indeed this implies that

$$|J_t| = \pi\ \mathcal{H}^{n-2} \llcorner \Gamma_t.$$

Since $|J_t| \leq \mu_t$,

$$\mu_t \geq \pi\ \mathcal{H}^{n-2} \llcorner \Gamma_t.$$

and

$$\Gamma_t \subset \text{support } \mu_t.$$

The opposite inclusion is proved in Step 1.

5. To prove that $n_t \equiv 1$, consider the space-time Jacobian J^ϵ of u^ϵ on $[0,T] \times I\!R^n$. The space-time Ginzburg-Landau energy is

$$\mathcal{E}^\epsilon := \frac{1}{2}\left[|\nabla u^\epsilon|^2 + |u_t^\epsilon|^2\right] + \frac{(1-|u^\epsilon|^2)^2}{4\epsilon^2},$$

and in view of (6.7),

$$\sup_{\epsilon} \{ \frac{1}{\ln(1/\epsilon)} \int_0^T \int_{I\!R^n} \mathcal{E}^\epsilon \ dx \ dt \ \} < \infty.$$

Hence we may apply our compactness result to J^ϵ, concluding that on a subsequence, denoted by ϵ again, it converges to a Radon measure J^*. Moreover

$$|J^*|([0,T] \times I\!R^n) \leq \mu([0,T] \times I\!R^n),$$

where μ is the weak* limit of the space-time energy. Then,

$$\mu([0,T] \times I\!R^n) = \int_0^T \ \mu_t(I\!R^n) \ dt + \nu([0,T] \times I\!R^n),$$

where ν is the weak* limit of

$$\nu^\epsilon(dt \times dx) := \frac{1}{\ln(1/\epsilon)} \ |u_t^\epsilon|^2 \ dt \times dx.$$

Let η and α be as in Step 1. By (6.5), and the properties of η,

$$\frac{d}{dt}\alpha^\epsilon(t) = C \ \alpha^\epsilon(t) - \frac{1}{2\ln(1/\epsilon)} \int_{I\!R^n} \ \eta(t,x) \ |u_t^\epsilon|^2 \ dx.$$

By the Gronwald's inequality

$$\begin{aligned} \alpha^\epsilon(T) \ &\leq \ \alpha^\epsilon(0) \ e^{CT} - \frac{1}{2\ln(1/\epsilon)} \int_0^T \int_{I\!R^n} \ e^{C(T-t)}\eta(t,x) \ \ |u_t^\epsilon|^2 \ dx \ dt \\ &\leq \ \alpha^\epsilon(0) \ e^{CT} - \frac{1}{2\ln(1/\epsilon)} \int_0^T \int_{I\!R^n} \ \eta(t,x) \ \nu^\epsilon(dt \times dx). \end{aligned}$$

Since α^ϵ tends to zero as ϵ approaches to zero. In the limit we obtain

$$\int_0^T \int_{I\!R^n} \ \eta \ \nu = 0.$$

Hence, the support of ν is included in the graph Γ of $\{\Gamma_t\}_{t\in[0,T]}$:

$$\Gamma := \{ \ (t,x) \in [0,T] \times I\!R^n \ | \ x \in \Gamma_t \ \}.$$

Therefore,

$$\text{support } J^* \subset \Gamma.$$

Using Lemma 6.2, we conclude that

$$|J^*| = \pi\ \mathcal{H}^{n-1} \lfloor \Gamma,$$

and consequently

$$|J_t| = \pi\ \mathcal{H}^{n-2} \lfloor \Gamma_t, \qquad t \in [0,T].$$

□

In the following lemma we assume J is a Radon measure of the form in Theorem 5.2. In particular it satisfies (5.1):

$$\int \phi(x) J(dx) \;=\; \int_M \phi(x) \cdot \nu(x)\theta(x)\mathcal{H}^{m-2}(dx) \qquad \forall \phi \in C^0(I\!R^m; \Lambda^2 I\!R^m).$$

Lemma 6.2 *Let J be as above. Further assume that $M \subset I\!R^n$ be a codimension two, smooth manifold with no boundary. Then, θ is constant on M.*

Proof.

Let $\tilde{J}$ be the same as J but with $\theta \equiv 1$, i.e.,

$$\int \phi(x) \tilde{J}(dx) \;=\; \int_M \phi(x) \cdot \nu(x)\mathcal{H}^{m-2}(dx) \qquad \forall \phi \in C^0(I\!R^m; \Lambda^2 I\!R^m).$$

Since M has no boundary and since M is smooth, we directly calculate that $d\tilde{J} = 0$. We also know that $dJ = 0$. Using these two facts we calculate that

$$\begin{aligned} 0 = d\ J &= d\ [\theta\ \tilde{J}] = \theta\ d\ \tilde{J} + d\ \theta \wedge \tilde{J} \\ &= d\ \theta \wedge \tilde{J}. \end{aligned}$$

Hence

$$\nabla_{tan} \theta = 0, \quad \text{on } M,$$

where ∇_{tan} is the tangential derivative on M. Since M is connected, this implies that θ is constant on M.

□

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