Macroeconomics
with Financial Frictions

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Background reading

- “Macroeconomics with Financial Frictions”
  - Brunnermeier, Eisenbach and Sannikov
    - Proceeding of the Econometric Society World Congress in Shanghai, 2010

- “The I Theory of Money”
  - Brunnermeier and Sannikov

- “The Maturity Rat Race”
  - Brunnermeier & Oehmke

- See www.princeton.edu/~markus
Motivation

- Financial crises occur periodically \textsuperscript{1} Kindleberger (1993)
- Financial frictions drive/amplify business cycle
  - Fisher (1933)
  - Keynes (1936)
  - Gurley-Shaw (1955)
  - Minsky (1975)
- Financial sector helps to
  - overcome financing frictions and
  - channels resources
- ... but
  - Credit crunch due to adverse feedback loops & liquidity spirals
    - Non-linear dynamics
Heterogeneous agents

- Lending-borrowing/insuring since agents are different

  - Poor-rich
  - Productive
  - Less patient
  - Less risk averse
  - More optimistic

  - Rich-poor
  - Less productive
  - More patient
  - More risk averse
  - More pessimistic

- Friction \( p_s MRS_s \) different even after transactions

- Wealth distribution matters!

- Financial sector is not a veil
Structuring the Macro-literature on Frictions

1. Persistence, amplification and instability
   a. Persistence: Carlstrom, Fuerst
   b. Amplification: Bernanke, Gertler, Gilchrist
   c. Instability: Brunnermeier, Sannikov

2. Credit quantity constraints through margins
   a. Credit rationing: Stiglitz, Weiss
   b. Margin spirals: Brunnermeier, Pederson
   c. Endogenous constraints: Geanakoplos

3. Demand for liquid assets & Bubbles – “self insurance”
   a. OLG, Aiyagari, Bewley, Krusell-Smith, Holmstroem Tirole,...

4. Financial intermediaries & Theory of Money
Recurring Theme: Liquidity Mismatch

- Instability of financial system arises from the fragility of liquidity

- Asset side
  - Technological liquidity refers to reversibility of investment
  - Market liquidity refers to price impact of capital sale

- Liability side
  - Funding liquidity refers to maturity structure of debt and sensitivity of margins

- The *liquidity mismatch* between assets and liabilities determines the severity of the amplification effects
Amplification & Instability - Overview

  - Perfect (technological) liquidity, but **persistence**
  - Bad shocks erode net worth, cut back on investments, leading to low productivity & low net worth of in the next period

  - Technological/market illiquidity
  - KM: Leverage bounded by margins; BGG: Verification cost (CSV)
  - Stronger **amplification** effects through **prices** (low net worth reduces leveraged institutions’ demand for assets, lowering prices and further depressing net worth)

- Brunnermeier & Sannikov (2010)
  - Instability and volatility dynamics, volatility paradox

- Brunnermeier & Pedersen (2009), Geanakoplos
  - Volatility interaction with margins/haircuts (leverage)
**Amplification & Instability - Overview**

- **Bernanke & Gertler (1989), Carlstrom & Fuerst (1997)**
  - Perfect (technological) liquidity, but persistence
  - Bad shocks erode net worth, cut back on investments, leading to low productivity & low net worth of in the next period

- **Kiyotaki & Moore (1997), BGG (1999)**
  - Technological/market illiquidity
  - KM: Leverage bounded by margins; BGG: Verification cost (CSV)
  - Stronger amplification effects through prices (low net worth reduces leveraged institutions’ demand for assets, lowering prices and further depressing net worth)

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Persistence

- Even in standard real business cycle models, temporary adverse shocks can have long-lasting effects
- Due to feedback effects, persistence is much stronger in models with financial frictions
  - Bernanke & Gertler (1989)
  - Carlstrom & Fuerst (1997)
- Negative shocks to net worth exacerbate frictions and lead to lower capital, investment and net worth in future periods
Costly State Verification

- Key friction in previous models is **costly state verification**, i.e. CSV, a la Townsend (1979)
- Borrowers are subject to an idiosyncratic shock
  - Unobservable to lenders, but can be verified at a cost
- Optimal solution is given by a contract that resembles standard debt
CSV: Contracting

- Competitive market for capital
  - Lender’s expected profit is equal to zero
  - Borrower’s optimization is equivalent to minimizing expected verification cost

- Financial contract specifies:
  - Debt repayment for each reported outcome
  - Reported outcomes that should be verified
CSV: Optimal Contract

- Incentive compatibility implies that
  - Repayment outside of VR is constant
  - Repayment outside of VR is weakly greater than inside
- Maximizing repayment in VR reduces the size and thus the expected verification cost
Output is produced according to $Y_t = A_t f(K_t)$

Fraction $\eta$ of entrepreneurs and $1 - \eta$ of households

- Only entrepreneurs can create new capital from consumption goods

Individual investment yields $\omega_i t$ of capital

- Shock is given by $\omega \sim G$ with $E[\omega] = 1$
- This implies consumption goods are converted to capital one-to-one in the aggregate
- No technological illiquidity!
Households can verify \( \omega \) at cost \( \mu i_t \)
- Optimal contract is debt with audit threshold \( \bar{\omega} \)
- Entrepreneur with net worth \( n_t \) borrows \( i_t - n_t \) and repays \( \min\{\omega_t, \bar{\omega}\} \times i_t \)

Auditing threshold is set by HH breakeven condition
- \[ \left[ \int_0^{\bar{\omega}} (\omega - \mu)dg(\omega) + (1 - G(\bar{\omega}))\bar{\omega} \right] i_t q_t = i_t - n_t \]
- Here, \( q_t \) is the price of capital

No positive interest (within period borrowing) and no risk premium (no aggregate investment risk)
Entrepreneur’s optimization:

\[ \max_{i_t} \int_{\bar{\omega}_t}^{\omega} (\omega - \bar{\omega}_t) dG(\omega) \ i_t \ q_t \]

Subject to HH breakeven constraint

Linear investment rule \( i_t = \psi(q_t) n_t \)

Leverage \( \psi(q_t) \) is increasing in \( q_t \)

Aggregate supply of capital is increasing in

- Price of capital \( q_t \)
- Aggregate net worth \( N_t \)
CF: Demand for Capital

- Return to holding capital:
  \[ R_{t+1}^k = \frac{A_{t+1}f'(K_{t+1})+(1-\delta)q_{t+1}}{q_t} \]

- Risk averse HH have discount factor \( \beta \)
  - Standard utility maximization
  - Budget constraint:
    \[ c_t \leq A_t f'(K_t)k_t + q_t[(1-\delta)k_t - k_{t+1}] \]
  - Euler equation: \( u'(c_t) = \beta E_t[R_{t+1}^k u'(c_{t+1})] \)
CF: Demand for Capital

- Risk-neutral entrepreneurs are less patient, $\beta < \overline{\beta}$
  - Euler equation: $1 = \beta E_t[R_{t+1}^k \rho(q_t)]$
  - Return on internal funds:
    $\rho(q_t) \equiv \int_{\omega_t}^{\infty} (\omega - \overline{\omega}_t) dG(\omega) \psi(q_t) q_t$
- Aggregate demand for capital is decreasing in $q_t$
CF: Persistence & Dampening

- Negative shock in period $t$ decreases $N_t$
  - This increases financial friction and decreases $I_t$
- Decrease in capital supply leads to
  - Lower capital: $K_{t+1}$
  - Lower output: $Y_{t+1}$
  - Lower net worth: $N_{t+1}$
  - Feedback effects in future periods $t + 2, ...$
- Decrease in capital supply also leads to
  - Increased price of capital $q_t$
  - Dampening effect on propagation of net worth shock
Bernanke, Gertler and Gilchrist (1999) introduce *technological illiquidity* in the form of nonlinear adjustment costs to capital.

- Negative shock in period $t$ decreases $N_t$
  - This increases financial friction and decreases $I_t$
- In contrast to the dampening mechanism present in CF, decrease in capital supply leads to
  - Decreased price of capital due to adjustment costs
  - *Amplification* effect on propagation of net worth shock
BGG assume separate investment sector
  - This separates entrepreneurs’ capital decisions from adjustment costs

Φ(·) represents *technological illiquidity*
  - Increasing and concave with Φ(0) = 0
  - \( K_{t+1} = Φ \left( \frac{I_t}{K_t} \right) K_t + (1 - δ)K_t \)

FOC of investment sector
  - \( \max_{I_t} \{ q_t K_{t+1} - I_t \} \Rightarrow q_t = Φ' \left( \frac{I_t}{K_t} \right)^{-1} \)

jump to KM97
Entrepreneurs alone can hold capital used in production

At time $t$, entrepreneurs purchase capital for $t + 1$
- To purchase $k_{t+1}$, an entrepreneur borrows $q_t k_{t+1} - n_t$
- Here, $n_t$ represents entrepreneur net worth

Assume gross return to capital is given by $\omega R^k_{t+1}$
- Here $\omega \sim G$ with $E[\omega] = 1$ and $\omega$ i.i.d.
- $R^k_{t+1}$ is the endogenous aggregate equilibrium return
Entrepreneurs borrow from HH in a CSV framework.

If $R_{t+1}^k$ is deterministic, then threshold satisfies:

$$\left[(1 - \mu) \int_0^{\bar{\omega}} \omega dG(\omega) + (1 - G(\bar{\omega}))\bar{\omega}\right] R_{t+1}^k q_t k_{t+1} = R_{t+1} (q_t k_{t+1} - n_t)$$

Here, $R_{t+1}$ is the risk-free rate.

If there is aggregate risk in $R_{t+1}^k$ then BGG argue that entrepreneurs will insure HH against risk.

- This amounts to setting $\bar{\omega}$ as a function of $R_{t+1}^k$.
- As in CF, HH perfectly diversify against entrepreneur idiosyncratic risk.
Entrepreneurs solve the following problem:

\[
\max_{k_{t+1}} E\left[ \int_0^\infty (\omega - \bar{\omega}) dG(\omega) R_{t+1}^k q_t k_{t+1} \right]
\]

Subject to HH breakeven condition (state-by-state)

Optimal leverage is again given by a linear rule

\[
q_t k_{t+1} = \psi \left( \frac{E[R_{t+1}^k]}{R_{t+1}} \right) n_t
\]

In a log-linearized solution, the remaining moments are insignificant

Aggregate capital supply is increasing in \(E[R_{t+1}^k]\) and aggregate net worth \(N_t\)
Return on capital is determined in a general equilibrium framework

- Gross return to holding a unit of capital

\[ E[R^k_{t+1}] = E \left[ \frac{A_{t+1} f'(K_{t+1}) + q_{t+1}(1-\delta) + q_{t+1} \Phi \left( \frac{I_{t+1}}{K_{t+1}} \right) - \frac{I_{t+1}}{K_{t+1}}}{q_t} \right] \]

- Capital demand is decreasing in expected return \( E[R^k_{t+1}] \)
BGG: Persistence & Amplification

- Shocks to net worth $N_t$ are persistent
  - They affect capital holdings, and thus $N_{t+1}$, ...

- *Technological illiquidity* introduces amplification effect
  1. Decrease in capital leads to reduced price of capital from
     \[ q_t = \Phi' \left( \frac{I_t}{K_t} \right)^{-1} \]
  2. Lower price of capital further decreases net worth
Kiyotaki, Moore (1997) adopt a collateral constraint instead of CSV market illiquidity – second best use of capital

- Durable asset has two roles:
  - Collateral for borrowing
  - Input for production

- Output is produced in two sectors, differ in productivity

- Aggregate capital is fixed, resulting in extreme technological illiquidity
  - Investment is completely irreversible
**KM: Amplification**

- *Static* amplification occurs because fire-sales of capital from productive sector to less productive sector depress asset prices
  - Importance of *market liquidity* of physical capital
- *Dynamic* amplification occurs because a temporary shock translates into a persistent decline in output and asset prices
Two types of infinitely-lived risk neutral agents

- **Mass $\eta$ of productive agents**
  - Constant-returns-to-scale production technology yielding $y_{t+1} = a k_t$
  - Discount factor $\beta < 1$

- **Mass $1 - \eta$ of unproductive agents**
  - Decreasing-returns-to-scale production $y_{t+1} = F(k_t)$
  - Discount factor $\beta \in (\beta, 1)$
KM: Frictions

- Since productive agents are less patient, they will want to borrow $b_t$ from unproductive agents
  - However, friction arises in that each productive agent’s technology requires *his* individual human capital
  - Productive agents cannot pre-commit human capital
- This results in a collateral constraint $Rb_t \leq q_{t+1} k_t$
  - Productive agent will never repay more than the value of his asset holdings, i.e. collateral
Since there is no uncertainty, a *productive agent* will borrow the maximum quantity and will not consume any of the output

- Budget constraint: \( q_t k_t + b_t \leq (a + q_t)k_{t-1} - Rb_{t-1} \)
- Demand for assets: \( k_t = \frac{1}{q_{t-1} + R} [(a + q_t)k_{t-1} - Rb_{t-1}] \)

Unproductive agents are not borrowing constrained

- \( R = \beta^{-1} \) and asset demand is set by equating margins
- Demand for assets: \( R = \frac{F'(k_t) + q_{t+1}}{q_t} \)
With fixed supply of capital, market clearing requires $\eta K_t + (1 - \eta) K_t = \bar{K}$

- This implies $M(K_t) \equiv \frac{1}{R} F' \left( \frac{\bar{K} - \eta K_t}{1 - \eta} \right) = q_t - \frac{1}{R} q_{t+1}$
- Note that $M(\cdot)$ is increasing

Iterating forward, we obtain: $q_t = \sum_{s=0}^{\infty} \frac{1}{R^s} M(K_{t+s})$
In steady state, productive agents use tradable output $a$ to pay interest on borrowing:

This implies that steady state price $q^*$ must satisfy:

$$q^* - \frac{1}{R} q^* = a$$

Further, steady state capital $K^*$ must satisfy:

$$\frac{1}{R} F' \left( \frac{\bar{K} - \eta K^*}{1-\eta} \right) = a$$

This reflects inefficiency since marginal products correspond only to *tradable* output.
Log-linearized deviations around steady state:
- Unexpected one-time shock that reduces production of all agents by factor $1 - \Delta$

Change in assets for given change in asset price:
- $\hat{K}_t = -\frac{\xi}{1+\xi} \left( \Delta + \frac{R}{R-1} \hat{q}_t \right)$, $\hat{K}_{t+s} = \frac{\xi}{1+\xi} \hat{K}_{t+s-1}$
- $\frac{1}{\xi} = \frac{d \log M(K)}{d \log K} \bigg|_{K=K^*}$

Reduction in assets comes from two shocks:
- Lost output $\Delta$
- Capital losses on previous assets $\frac{R}{R-1} \hat{q}_t$
KM: Productivity Shock

- Change in price for given change in assets:
  - Log-linearize the equation $q_t = \sum_{s=0}^{\infty} \frac{1}{R^s} M(K_{t+s})$
  - This provides: $\hat{q}_t = \frac{1}{\xi} \frac{R-1}{R} \sum_{s=0}^{\infty} \frac{1}{R^s} \hat{K}_{t+s}$

- Combining equations:
  - $\hat{K}_t = - \left( 1 + \frac{1}{(\xi+1)(R-1)} \right) \Delta$
  - $\hat{q}_t = - \frac{1}{\xi} \Delta$
We can decompose the previous equations into static and dynamic multiplier effects

- Static effect results from assuming $q_{t+1} = q^*$

- Static multiplier:
  - $\hat{K}_t^S = -\Delta$
  - $\hat{q}_t^S = -\frac{(R-1)}{R} \frac{1}{\xi} \Delta$

- Dynamic multiplier:
  - $\hat{K}_t^D = -\frac{1}{(\xi+1)(R-1)} \Delta$
  - $\hat{q}_t^D = -\frac{1}{R} \frac{1}{\xi} \Delta$
Previous papers only considered log-linearized solutions around steady state

Brunnermeier & Sannikov (2010) build a continuous time model to study full dynamics
- Show that financial system exhibits inherent instability due to highly non-linear effects
- These effects are asymmetric and only arise in the downturn

Agents choose a capital cushion
- Mitigates moderate shocks near steady state
- High volatility away from steady state
**BS: Model overview**

- **Productive**
- **Intermediary**
  - Monitoring
    - Diamond (1984)
    - Holmström-Tirole (1997)
- **Less productive**

- Incentive for entrepreneur to exert effort
- Incentive for intermediaries to monitor
  (have to hold outside equity)
BS: Preview of results

- Full equilibrium dynamics + volatility dynamics
  - Near “steady state”
    - (large) payouts balance profit making
    - intermediaries must be unconstrained and amplification is low
  - Below “steady state”
    - intermediaries constrained, try to preserve capital
      leading to high amplification and volatility

- Crises episodes have significant endogenous risk, correlated asset prices, larger spreads and risk premia

- “Volatility paradox”

- SDF is driven by constraint & $c \geq 0$

- Securitization and hedging of idiosyncratic risks can lead to higher leverage, and greater systemic risk
**BS: ... with volatility dynamics + precaution**

- **Unstable dynamics** away from steady state due to (nonlinear) *liquidity spirals*

- Volatility dynamics leads affects size of “safety cushion”
  - Note: log-linearization with zero probability shocks → no safety cushion

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Shock to capital → Loss of net worth → Precaution + tighter margins
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Fire sales
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volatility price
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**BS: Model details**

- Output: \( y_t = a k_t \) (spend for consumption - investment)

- Capital: \( d k_t = (\Phi(t_t) - \delta) k_t dt + \sigma k_t dZ_t \)

- Agents
  - More productive
    - \( U = E_0 \left[ \int_0^\infty e^{-\rho t} c_t dt \right] \)
    - Production frontier
  - Less productive
    - \( U = E_0 \left[ \int_0^\infty e^{-rt} c_t dt \right] \)
    - Production frontier
      - \( \delta > \delta \)
      - \( t_t = 0 \)

- Endogenous price process for capital
  \[ dq_t = \mu^q_t q_t dt + \sigma^q_t q_t dZ_t \]
  \[ q_t \geq q = \frac{a}{r + \delta} \] if HH limited to buy-hold strategy
BS: Market value of capital/assets $k_t q_t$

- **Capital**
  - $dk_t = g(t)k_t dt + \sigma k_t dZ_t$ “cash flow news” (dividends $a_t$)

- **Price**
  - $dq_t = \mu^q_t q_t dt + \sigma^q_t q_t dZ_t$ “SDF news”

- $k_t q_t$ value dynamics
**BS: Market value of capital/assets $k_t q_t$**

- **Capital**
  
  \[ dk_t = g(i) k_t dt + \sigma k_t dZ_t \]

- **Price**
  
  \[ dq_t = \mu_t q_t dt + \sigma_t q_t dZ_t \]

- **$k_t q_t$ value dynamics**
  
  \[ d(k_t q_t) = (\Phi(i_t) - \delta + \mu_t q_t + \sigma_t q_t)(k_t q_t) dt + (\sigma + \sigma_t q_t)(k_t q_t) dZ_t \]

- **Ito’s Lemma product rule:**
  \[ d(X_t Y_t) = dX_t Y_t + X_t dY_t + \sigma^X \sigma^Y dt \]
Focus on contracts in which agents is required to hold sufficient levered equity stake in projects. The more risk entrepreneur wants to unload, the more they have to be monitored (by someone who takes on exposure).
BS: Microfoundation of contracts (extra)

- Agency problem of entrepreneur
  - Increase capital depreciation rate, private benefit $b$ per $1$ destroyed
  - Incentive constraint: entrepreneur equity stake $\geq b$
- Are these contracts optimal? No
  - Entrepreneur reward depends on $k_t q_t$, but $q_t$ is determined by market – why not hedge $q_t$ to get a better performance?
  - Shocks to $k_t$ are common across entrepreneurs, why not hedge those and get first best?
  - In practice markets aggregate information to determine $k_t q_t$, but hard to distinguish between shocks to $k_t$ (cash flow news) and $q_t$ (SDF news)
- Optimal contracts get first-best, but miss important phenomena
- Same as in Kiyotaki & Moore, BGG, He & Krishnamurthy
Interlinked balance sheets

- **Productive**
  - Debt
  - Equity

- **Intermediary**
  - Monitoring
    - Diamond (1984)
    - Holmström-Tirole (1997)

- **Less productive**
  - Incentive for entrepreneur to exert effort
  - Incentive for intermediaries to monitor
    (have to hold outside equity)

\[
dead \quad debt \quad short-term
\]
\[
equity \quad inside \quad outside
\]
\[
\alpha^I
\]
of total risk

\[
\alpha^E
\]
incentive for entrepreneur to exert effort
Assumption: value of assets $q_t k^i_t$ is contractable, $k^i_t$ not

Agency problem of entrepreneur
  - Can take projects w/ NPV<0, private benefit $b(m)<1$ per $1$ destroyed
  - $m$ is amount of monitoring by intermediary
  - Incentive constraint: $\alpha^E \geq b(m)$, binds in equ. $\Rightarrow \alpha^E (m)$

Agency problem of intermediary
  - Save monitoring cost $c(m)$ per $1$ if shirking
  - Incentive constraint: $\alpha^I \geq c(m)$

Solvency constraint: $n_t \geq 0$ (implied by IC constraints)

Assume $c(m) + b(m)$ is a constant for all $m$
  - Entrepreneurs’ & intermediaries’ net worth are substitutes
  - Special case: if entrepreneurs’ net worth =0, then $m$ s.t. $b(m)=0$
BS: Merging productive HH & Intermediaries

- Productive
  - Intermediary
    - Monitoring
      Diamond (1984)
      Holmström-Tirole (1997)
  - Less productive

\[ \alpha := \alpha^E + \alpha^I \geq b(m) + c(m) \]

“merged experts”

Credit channel
- Lending channel
- Borrowers’ balance sheet channel
Productive entrepreneurs have no capital, $\alpha^E = 0$.

Perfect monitoring required, $b(m) = 0$.

Intermediary can’t issue outside equity, $\alpha^I = 1$ (appropriate choice of $b(m)$, $c(m)$).

BS: Merging productive HH & Intermediaries.
BS: Balance sheet dynamics

- Productive
- Intermediary
- Less productive

\[ \text{assets } k_t q_t \]
\[ \text{debt } d_t \]
\[ \text{equity} = \text{net worth } n_t \]

assume \( \alpha = 1 \) (for today)
**BS: Balance sheet dynamics**

- **Productive**
- **Intermediary**
- **Less productive**

\[ d r_t^k = \left( \frac{a - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma \sigma_t^q \right) dt + (\sigma + \sigma_t^q) dZ_t \]

\[ d n_t = r n_t dt + (d r_t^k - r dt)(k_t q_t) - d c_t = \cdots \]
BS: Intuition – main forces at work

- **Investment**
  - *Scale up*
    - Scalable profitable investment opportunity
    - Higher leverage (borrow at \( r \))
  - *Scale back*
    - **Precaution:** don’t exploit full (GE) debt capacity – “dry powder”
      - Ultimately, stay away from fire-sales prices
      - Debt can’t be rolled over if \( d > k_t q \) (note, price is depressed)
      - Solvency constraint

- **Consumption**
  - Consume *early* and borrow \( r < \rho \)
  - Consume *late* to overcome investment frictions
    
    aggregate leverage!
An equilibrium consists of functions that for each history of macro shocks \( \{Z_s, s \in [0, t]\} \) specify

- \( q_t \) the price of capital
- \( k_t^i, k_t^h \) capital holdings and
- \( dc_t^i, dc_t^h \) consumption of representative expert and households
- \( \iota_t \) rate of internal investment of a representative expert, per unit of capital
- \( r_t \) the risk-free rate

such that

- intermediaries and households maximize their utility, given prices \( q_t \) as given and
- markets for capital and consumption goods clear
1. **Households:** risk free rate of \( r_t = \) households discount rate
   - Makes HH indifferent between consuming and saving, s.t. consumption market clears
   - Required return when their capital >0
     \[
     \frac{a}{q_t} - \delta + \mu_t^q + \sigma \sigma_t^q = r
     \]
     \( r \) is expected return from capital

2. **Experts** choose \( \{k_t, \iota_t, c_t\} \) dynamically to maximize utility
   \[
   \max_{c, \iota, k} E\left[ \int_0^\infty e^{-\rho t} dc_t \right] \quad \text{s.t.}
   \]
   \[
   d n_t = -d c_t + (\Phi(i_t) - \delta + \mu_t^q + \sigma \sigma_t^q)(k_t q_t) dt
   + (\sigma + \sigma_t^q)(k_t q_t) dZ_t + [(a - \iota_t)k_t - rd_t] dt
   \]
   \[
   d n_t \geq 0
   \]

3. Markets clear: total demand for capital is \( K_t \)
BS: Solving for equilibrium

1. Internal investment \textit{(static)}

2. External investment $k_t$
   - Given price dynamics \( dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t \)
   - Solvency constraint \( n_t \geq 0 \)

3. When to consume? \( dc_t \)
   - Bellman equation w/ value function \( \theta_t n_t \) proportional to net worth, atomistic experts have no price impact

payoff experts generate from a dollar of net worth by trading undervalued capital

$$\rho \theta_t n_t dt = \max_{k_t, dc_t} E\left[ dc_t + d(\theta_t n_t) \right]$$
Let value of extra $
\begin{align*}
d\theta_t &= \mu_t \theta_t dt + \sigma_t \theta_t dZ_t \\
\end{align*}
$
recall \, dn_t = ....

Use Ito’s lemma to expand the Bellman equation
\begin{align*}
\rho \theta_t n_t dt &= \max_{k_t, dc_t} E[dc_t + d(\theta_t n_t)] \\
\end{align*}

- Risk free:
  \begin{align*}
  \underbrace{r}_{\text{risk-free}} + \underbrace{\mu_t}_{E[\text{change of investment opportunities}]} &= \underbrace{\rho}_{\text{required return}} \\
  \end{align*}

- Capital:
  \begin{align*}
  \underbrace{\frac{a}{q_t} + g_t + \mu_t^q + \sigma \sigma_t^q - r}_{E[\text{excess return of capital}]} = \underbrace{-\sigma_t^\theta (\sigma + \sigma_t^q)}_{\text{capital risk premium}} \\
  \end{align*}

\[ \theta_t \geq 1, \text{ and } dc_t^i > 0 \text{ only when } \theta_t = 1. \]

\[ e^{-\rho_t \theta_t / \theta_0} \] is the experts’ stochastic discount factor
BS: Scale invariance

- Model is scale invariant
  - $K_t$ total physical capital
  - $N_t$ total net worth of all experts

- Solve $q_t$ and $\theta_t$ as a function of the single state variable $\eta_t$
  - $\eta_t = \frac{N_t}{K_t}$

Mechanic application of Ito’s lemma
Pricing equations get transformed into ordinary differential equations for $q(\eta)$ and $\theta(\eta)$
Start with: \( dK_t = g(q_t)K_t \, dt + \sigma K_t \, dZ_t \)

\( dN_t = r \, N_t \, dt - dC_t + a \, K_t \, dt - \iota(q_t) \, K_t \, dt + K_t \, q_t \left[ (g(q_t) - r + \mu_t^q + \sigma \sigma_t^q) \, dt + (\sigma + \sigma_t^q) \, dZ_t \right] \)

Ito’s lemma \( \Rightarrow d\eta_t = d(N_t/K_t) = (r - g(q_t) + \sigma^2) \left( \eta_t - q_t \right) \, dt + (a - \iota(q_t) + q_t \mu_t^q) \, dt + (q_t(\sigma + \sigma_t^q) - \sigma \eta_t) \, dZ_t \)

\( q_t \sigma_t^q = q'(\eta) \left( q_t(\sigma + \sigma_t^q) - \sigma \eta_t \right) \)

\( \Rightarrow \)

\[ \sigma_t^q = \frac{q'(\eta_t) \sigma(q_t - \eta_t)}{q_t(1 - q'(\eta_t))} \quad \text{and} \quad \sigma_t^\eta = \frac{\sigma(q_t - \eta_t)}{1 - q'(\eta_t)} \]

\( q_t \mu_t^q = q'(\eta) \left( (r - g(q_t) + \sigma^2) \left( \eta_t - q_t \right) + a - \iota(q_t) + p_t \mu_t^q \right) + \frac{1}{2} \left( q_t(\sigma + \sigma_t^q) - \sigma \eta_t \right)^2 q''(\eta) \quad \Rightarrow \)

\[ \mu_t^q = \frac{q'(\eta_t)[(r - g(q_t) + \sigma^2)(\eta_t - q_t) + a - \iota(q_t)] + \frac{1}{2} (\sigma_t^\eta)^2 q''(\eta_t)}{q_t(1 - q'(\eta_t))} \]
(a - \iota(q_t))/q_t + g(q_t) + \mu_t q + \sigma_\sigma q - r = -\theta'(\eta_t)/\theta(\eta_t) \quad \sigma_t^\eta (\sigma + \sigma_t^q) \quad \text{and}

\mu_t^q = \frac{q'(\eta_t)[(r - g(q_t) + \sigma^2)(\eta_t - q_t) + a - \iota(q_t)] + \sigma_t'' q''(\eta_t)}{q_t(1 - \eta'(\eta_t))} \Rightarrow

\frac{a - \iota(q_t)}{q_t} + g(q_t) + \frac{q'(\eta_t)[(r - g(q_t) + \sigma^2)(\eta_t - q_t) + a - \iota(q_t)] + \sigma_t'' q''(\eta_t)}{q_t(1 - \eta'(\eta_t))} + \sigma_\sigma q - r = -\frac{\theta'(\eta_t)}{\theta(\eta_t)} \sigma_t^\eta (\sigma + \sigma_t^q)

(\rho - r) \theta(\eta) = \mu_t^\theta \Rightarrow \quad \text{where} \quad \sigma_t^q = \frac{q'(\eta_t) \sigma(q_t - \eta_t)}{q_t(1 - \eta'(\eta_t))} \quad \text{and} \quad \sigma_t^\eta = \frac{\sigma(q_t - \eta_t)}{1 - \eta'(\eta_t)}

(\rho - r) \theta(\eta) = \theta'(\eta)((r - g(q_t) + \sigma^2)(\eta - q_t) + a - \iota(q_t) + q_t \mu_t^q) + \frac{1}{2} (\sigma_t^\eta)^2 \theta''(\eta_t)

- Boundary conditions: q(0) = a/(r + \delta^*), q'(\eta^*) = 0, \theta(\eta^*) = 1, \theta'(\eta^*) = 0
BS: Equilibrium

- Boundary conditions: $q(0) = q_0$, $\theta(0) = \infty$, $\theta(\eta^*) = 1$, $q'(\eta^*) = \theta'(\eta^*) = 0$
BS: Equilibrium dynamics
BS: Endogenous risk & “Instability”
**BS: Endogenous Risk through Amplification**

- Amplification through prices
  - Adverse shock
  - \( k_t \downarrow \)
  - \( \eta_t \downarrow \) due to leverage
  - \( \eta_t \downarrow \)
  - \( n_t \downarrow \)
  - \( q_t \downarrow \)
  - Capital demand \( \downarrow \)

- Volatility due to endogenous risk

- Key to amplification is \( q'(\eta) \)
  - Depends how constrained experts are

\[
\sigma^q_t = \frac{q'(\eta_t)\sigma(q_t - \eta_t)}{1 - q'(\eta_t)}.
\]
BS: Dynamics near and away from SS

- Intermediaries choose payouts endogenously
  - Exogenous exit rate in BGG/KM
  - Payouts occur when intermediaries are least constrained
    \[ q'(\eta^*) = 0 \]

- Steady state: experts unconstrained
  - Bad shock leads to lower payout rather than lower capital demand
    \[ q'(\eta^*) = 0, \sigma_t^q(\eta^*) = 0 \]

- Below steady state: experts constrained
  - Negative shock leads to lower demand
  - \( q'(\eta^*) \) is high, strong amplification, \( \sigma_t^q(\eta^*) \) is high
  - ... but when \( \eta \) is close to 0,
    \[ q \approx q(\eta_t), q'(\eta) \text{ and } \sigma_t^q(\eta^*) \text{ is low} \]

Note difference to BGG/KM
As $\sigma$ decreases, $\eta^*$ goes down, $q(\eta^*)$ goes up, $\sigma^\eta(\eta^*)$ may go up, max $\sigma^\eta$ goes up
**BS: Ext1: asset pricing (cross section)**

- **Capital**: Correlation increases with $\sigma^q$
  - Extend model to many types $i$ of capital

$$\frac{dk_t^i}{k_t^i} = (\Phi(l_t^i) - \delta)dt + \sigma dZ_t + \sigma' dz_t^i$$

- Experts hold diversified portfolios
  - Equilibrium looks as before, (all types of capital have same price) but
  - Volatility of $q_t k_t$ is $\sigma + \sigma' + \sigma^q$
  - Endogenous risk is perfectly correlated, exogenous risk not
  - For uncorrelated $z^i$ and $z^j$
    - correlation $(q_t^i k_t^i, q_t^j k_t^j)$ is $(\sigma + \sigma^q)/(\sigma + \sigma' + \sigma^q)$
    - which is increasing in $\sigma^q$
Outside equity:
- Negative sknewness
- Excess volatility
- Pricing kernel: $e^{-rt}$
  - Needs risk aversion!

Derivatives:
- Volatility smirk (Bates 2000)
- More pronounced for index options (Driessen et al. 2009)
BS: Ext2: Idiosyncratic jump losses

\[ dk^i_t = gk^i_t dt + \sigma k^i_t dZ_t + k^i_t dJ^i_t \]

- \( J^i_t \) is an idiosyncratic compensated Poisson loss process, recovery distribution \( F \) and intensity \( \lambda(\sigma^q_t) \)
- \( q_t k^i_t \) drops below debt \( d_t \), costly state verification

- Time-varying interest rate spread
- Allows for direct comparison with BGG
\[ dk_t^i = gk_t^i dt + \sigma k_t^i dZ_t + k_t^i dJ_t^i \]

- \( J_t^i \) is an idiosyncratic compensated Poisson loss process, recovery distribution \( F \) and intensity \( \lambda(\sigma_t^q) \)
- \( q_t k_t^i \) drops below debt \( d_t \), costly state verification
- Debt holders’ loss rate
  \[ \lambda(\sigma_p^v) \int_0^{d_v} (\frac{d_v}{v} - x)dF(x) \]
- Verification cost rate
  \[ \lambda(\sigma_p^v) \int_0^{d_v} cxdF(x) \]
- Leverage bounded not only by precautionary motive, but also by the cost of borrowing

\[
\begin{array}{ccc}
\text{Asset} & \text{Liabilities} \\

v_t = k_t q_t & d_t = k_t q_t - n_t \\
& n_t
\end{array}
\]
Experts borrowing rate > $r$
  - Compensates for verification cost
Rate depends on leverage, price volatility
$d\eta_t = \text{diffusion process (without jumps) because losses cancel out in aggregate}$
Experts can contract on shocks $Z_t$ and $dJ_t^i$ directly among each other, zero contracting costs

In principle, good thing (avoid verification costs)

Equilibrium

- experts fully hedge idiosyncratic risks
- experts hold their share (do not hedge) aggregate risk $Z_t$, market price of risk depends on $\sigma_t^\theta (\sigma + \sigma_t^q)$
- with securitization experts lever up more (as a function of $\eta_t$) and bonus payments occur “sooner”
- financial system becomes less stable
- risk taking is endogenous (Arrow 1971, Obstfeld 1994)
Incorporate financial sector in macromodel
  - Higher growth
  - Exhibits instability
    - similar to existing models (BGG, KM) in term of persistence/amplification, but
    - non-linear liquidity spirals (away from steady state) lead to instability

Risk taking is endogenous
  - “Volatility paradox:” Lower exogenous risk leads to greater leverage and may lead to higher endogenous risk
  - Correlation of assets increases in crisis
  - With idiosyncratic jumps: countercyclical credit spreads
  - Securitization helps share idiosyncratic risk, but leads to more endogenous risk taking and amplifies systemic risk

Welfare: (Pecuniary) Externalities
  - excessive exposure to crises events
Overview

- Persistence
- Dynamic Amplification
  - Technological illiquidity  BGG
  - Market illiquidity  KM97
- Instability, Volatility Dynamics, Volatility Paradox
- Volatility and Credit Rationing/Margins/Leverage
- Demand for Liquid Assets
Credit rationing refers to a failure of market clearing in credit

- In particular, an excess demand for credit that fails to increase market interest rate

- Stiglitz, Weiss (1981) show how asymmetric information on risk can lead to credit rationing
Entrepreneurs borrow from competitive lenders at interest rate $r$
- Risky investment projects with $R \sim G(\cdot \mid \sigma_i)$
- Mean preserving spreads, so heterogeneity is only in risk

Assume entrepreneur borrows $B$

Entrepreneur’s payoff is convex in $R$
- $\pi_e(R, r) = \max\{R - (1 + r)B, 0\}$

Lender’s payoff is concave in $R$
- $\pi_l(R, r) = \min\{R, (1 + r)B\}$
Due to convexity, entrepreneur’s expected payoff is increasing in riskiness $\sigma_i$
- Only entrepreneurs with sufficiently risky projects will apply for loans, i.e. $\sigma_i \geq \sigma^*$

Zero-profit condition: $\int \pi_e(R, r) dG(R|\sigma^*) = 0$
- This determines cutoff $\sigma^*$
- Note that $\sigma^*$ is increasing in $r$

Lender’s payoff is not monotonic in $r$
- Ex-post payoff is increasing in $r$
- Higher cutoff $\sigma^*$ leads to riskier selection of borrowers
SW: Credit Rationing

- Lenders will only lend at the profit maximizing interest rate $r$
- Excess demand for funds from borrowers will not increase the market rate
  - There exist entrepreneurs who would like to borrow, willing to pay a rate higher than the prevailing one
- Adverse selection leads to failure of credit markets
For collateralized lending, debt constraints are directly linked to the volatility of collateral
- Constraints are more binding in volatile environments
- Feedback effect between volatility and constraints

These margin spirals force agents to delever in times of crisis
- Collateral runs
- Multiple equilibria
BP: Margins – Value at Risk (VaR)

- Margins give incentive to hold well diversified portfolio
- How are margins set by brokers/exchanges?
  - **Value at Risk**: \( \Pr(-(p_{t+1} - p_t) \geq m) = 1\% \)
Financing a *long position* of \(x_{t}^{j+}>0\) shares at price \(p_{t}^{j}=100\):
- Borrow $90$ dollar per share;
- Margin/haircut: \(m_{t}^{j+}=100-90=10\)
- Capital use: $10 \times x_{t}^{j+}

Financing a *short position* of \(x_{t}^{j-}>0\) shares:
- Borrow securities, and lend collateral of 110 dollar per share
- Short-sell securities at price of 100
- Margin/haircut: \(m_{t}^{j-}=110-100=10\)
- Capital use: $10 \times x_{t}^{j-}

Positions frequently marked to market
- payment of \(x_{t}^{j}(p_{t}^{j}-p_{t-1}^{j})\) plus interest
- margins potentially adjusted – *more later on this*

Margins/haircuts must be financed with capital:

\[
\sum_{j} (x_{t}^{j+} m_{t}^{j+} + x_{t}^{j-} m_{t}^{j-}) \leq W_{t}, \text{ where } x_{t} = x_{t}^{j+} - x_{t}^{j-}
\]

with perfect cross-margining: \(M_{t} (x_{t}^{1}, \ldots, x_{t}^{J}) \leq W_{t}\)
Funding liquidity

- Can’t roll over short term debt
- Margin-funding is recalled
<table>
<thead>
<tr>
<th>A</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Market liquidity</strong></td>
<td><strong>Funding liquidity</strong></td>
</tr>
<tr>
<td>- Can only sell assets at fire-sale prices</td>
<td>- Can’t <strong>roll over</strong> short term debt</td>
</tr>
<tr>
<td></td>
<td>- <strong>Margin</strong>-funding is recalled</td>
</tr>
</tbody>
</table>
**BP: Liquidity Spirals**

- **Borrowers’ balance sheet**
  - **Loss spiral – like in BGG/KM**
    - Net wealth > \( \alpha \times \)
      for asym. info reasons
    - constant or increasing leverage ratio
  - **Margin/haircut spiral**
    - Higher margins/haircuts
    - No rollover
    - redemptions
    - forces to delever

- **Mark-to-market vs. mark-to-model**
  - worsens loss spiral
  - improves margin spiral

- Both spirals reinforce each other

Source: Brunnermeier & Pedersen (2009)
1. Volatility of collateral increases
   - Permanent price shock is accompanied by higher future volatility (e.g. ARCH)
     - Realization how difficult it is to value structured products
   - Value-at-Risk shoots up
   - Margins/haircuts increase = collateral value declines
   - Funding liquidity dries up
   - Note: all “expert buyers” are hit at the same time, SV 92

2. Adverse selection of collateral
   - As margins/ABCP rate increase, selection of collateral worsens
   - SIVs sell-off high quality assets first (empirical evidence)
   - Remaining collateral is of worse quality
BP: Model Setup

- Time: \( t=0,1,2 \)
- One asset with final asset payoff \( v \) (later: assets \( j=1,...,J \))
- Market illiquidity measure: \( \Lambda_t = |E_t(v)-p_t| \)
  (deviation from “fair value” due to selling/buying pressure)
- Agents
  - Initial customers with supply \( S(z,E_t[v]-p_t) \) at \( t=1,2 \)
  - Complementary customers’ demand \( D(z,E_2[v]-p_2) \) at \( t=2 \)
  - Risk-neutral dealers provide *immediacy* and
    - face capital constraint
  - \( \text{cash} \times m(\sigma,\Lambda) \leq W(\Lambda) := \max\{0, B + x_0(E_1[v]-\Lambda)\} \)
    “price” of stock holding
BP: Financiers’ Margin Setting

- Margins are set based on Value-at-Risk
- Financiers do not know whether price move is due to
  - *Likely*, movement in fundamental
  - *Rare*, selling/buying pressure by customers who suffered asynchronous endowment shocks.

\[
m_{1j}^+ = \hat{A}^{-1}(1 \cdot \frac{1}{4})^{3/4} = \frac{3}{4} + \mu_j \Delta p_{1j} = m_{1j}^-
\]
BP: Margin Spiral – Increased Vol.

\[ v_t = v_{t-1} + \Delta v_t = v_{t-1} + \sigma_t \varepsilon_t \]
\[ \sigma_{t+1} = \sigma + \theta |\Delta v_t| \]

Selling pressure
initial customers

complementary
customers
BP: Model Setup in a Figure

\[ v_t = v_{t-1} + \Delta v_t = v_{t-1} + \frac{3}{4} \]

\[ \frac{3}{4} t + 1 = \frac{3}{4} + \mu j \Delta v_t j \]

\[ \gamma = 0.01 \quad \sigma^2 = 16 \quad z_0 = 20 \quad z_1 = 20 \quad v_0 = 140 \quad v_1 = 120 \]
\[ p_0 = 130 \quad k = 10 \quad \alpha = 0.3 \quad \eta_1 = 0 \quad W_0 = 700 \quad x_0 = 0 \]

customers’ supply

\[ x_1 < W_1/m_1 = W_1/(\sigma + \tilde{\theta}|\Delta p_1|) \]

\[ \gamma = 0.025 \quad \sigma^2 = 11 \quad z_0 = 20 \quad z_1 = 20 \quad v_0 = 140 \quad v_1 = 120 \]

\[ p_0 = 130 \quad k = 5 \quad \theta = 0.3 \quad \eta_1 = 0 \quad W_0 = 750 \quad x_0 = 0 \]

\[ x_1 < W_1/m_1 = W_1/(\sigma + \theta|\Delta p_1|) \]

\[
\gamma = 0.025 \quad \sigma^2 = 11 \quad z_0 = 20 \quad z_1 = 20 \quad v_0 = 140 \quad v_1 = 120 \\
p_0 = 130 \quad k = 5 \quad \theta = 0.3 \quad \eta_1 = 0 \quad W_0 = 600 \quad x_0 = 0
\]

customers’ supply

\[x_1 < W_1/m_1 = W_1/(\sigma + \theta|\Delta p_1|)\]
Data Gorton and Metrick (2011)

Haircut Index

“The Run on Repo”
Margins very stable in tri-party repo market

- contrasts with Gorton and Metrick (2011)
- no general run on certain types of collateral

Run (non-renewed financing) only on select counterparties

- Bear Stearns (anecdotally)
- Lehman (in the data)

Like 100% haircut…
(counterparty specific!)
Bilateral and Tri-party Haircuts?

Differences in Median Haircuts

Percent

60

50

40

30

20

10

0

-10

Subprime

Alt-A, Prime MBS

High-Grade Corp Debt

Agency CMO

Treasury

Agency

GSE MBS

Percent

60

50

40

30

20

10

0

-10

Jul-08 Oct-08 Jan-09 Apr-09 Jul-09 Oct-09 Jan-10

Source: FRBNY Calculations
## Tri-party Repo Haircuts April 2011

- This is triparty repo by different asset classes
- Reported by FRBNY

http://www.newyorkfed.org/tripartyrepo/margin_data.html

<table>
<thead>
<tr>
<th>Asset Group</th>
<th>10th Percentile</th>
<th>Median</th>
<th>90th Percentile</th>
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<td>2.0%</td>
<td>5.0%</td>
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<tr>
<td>ABS Non Investment Grade</td>
<td>2.0%</td>
<td>5.5%</td>
<td>8.0%</td>
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<td>Agency CMOs</td>
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<td>3.0%</td>
<td>5.0%</td>
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<tr>
<td>Agency Debentures &amp; Strips</td>
<td>2.0%</td>
<td>2.0%</td>
<td>3.0%</td>
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<tr>
<td>Agency MBS</td>
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<td>2.0%</td>
<td>5.0%</td>
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<td>US Treasuries excluding Strips</td>
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<td>2.0%</td>
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<tr>
<td>US Treasuries Strips</td>
<td>2.0%</td>
<td>2.0%</td>
<td>2.0%</td>
</tr>
</tbody>
</table>
Overview

- Persistence
- Dynamic Amplification
  - Technological illiquidity  BGG
  - Market illiquidity  KM97
- Instability, Volatility Dynamics, Volatility Paradox
- Volatility and Credit Rationing/Margins/Leverage
- Demand for Liquid Assets
- Financial Intermediation
Demand for Liquid Assets

- Technological and market illiquidity create time amplification and instability
  - Fire-sales lead to time varying price of capital
  - Liquidity spirals emerge when price volatility interacts with debt constraints

- Focus on demand for liquid instruments
  - No amplification effects, i.e. reversible investment and constant price of capital $q$
    - Borrowing constraint = collateral constraint
  - Introduce idiosyncratic risk, aggregate risk, and finally amplification
Outline

- Deterministic Fluctuations
  - Overlapping generations
  - Completing markets with liquid asset

- Idiosyncratic Risk
  - Precautionary savings
  - Constrained efficiency

- Aggregate Risk
  - Bounded rationality

- Amplification Revisited
Overlapping Generations

- Samuelson (1958) considers an infinite-horizon economy with two-period lived overlapping agents
  - Population growth rate \( n \)
- Preferences given by \( u(c^t_t, c^t_{t+1}) \)
  - Pareto optimal allocation satisfies \( \frac{u_1}{u_2} = 1 + n \)
- OLG economies have multiple equilibria that can be Pareto ranked
Assume \( u(c_t^t, c_{t+1}^t) = \log c_t^t + \beta \log c_{t+1}^t \)
- Endowment \( y_t^t = e, y_{t+1}^t = 1 - e \)

Assume complete markets and interest rate \( r \)

Agent’s FOC implies that \( \frac{c_{t+1}^t}{\beta c_t^t} = 1 + r \)
- For \( r = n \), this corresponds to the *Pareto solution*
- For \( r = \frac{1-e}{\beta e} - 1 \), agents will consume their endowment

Autarky solution is clearly *Pareto inferior*
OLG: Completion with Durable Asset

- Autarky solution is the **unique** equilibrium implemented in a sequential exchange economy
  - Young agents cannot transfer wealth to next period
- A durable asset provides a **store of value**
  - Effective store of value reflects *market liquidity*
  - Pareto solution can be attained as a competitive equilibrium in which the price level grows at same rate as the population, i.e. \( b_{t+1} = (1 + n)b_t \)
  - Old agents trade durable asset for young agents’ consumption goods
Diamond (1965) introduces a capital good and production
- Constant-returns-to-scale production \( Y_t = F(K_t, L_t) \)
- Optimal level of capital is given by the golden rule, i.e. \( f'(k^*) = n \)
  - Here, lowercase letters signify per capita values
- Individual (and firm) optimization implies that
  - \( \frac{u_1}{u_2} = 1 + r = 1 + f'(k) \)
  - It is possible that \( r < n \Rightarrow k > k^* \Rightarrow \text{Pareto inefficient} \)
Diamond recommends issuing government debt at interest rate $r$

Tirole (1985) introduces a rational bubble asset trading at price $b_t$

- $b_{t+1} = \frac{1+r_{t+1}}{1+n} b_t$

Both solutions *crowd out* investment and increase $r$

- If baseline economy is inefficient, then an appropriately chosen debt issuance or bubble size can achieve Pareto optimum with $r = n$
**OLG: Crowding Out vs. Crowding In**

- Depending on the framework, government debt and presence of bubbles can have two opposite effects
  - **Crowding out** refers to the decreased investment to increase in supply of capital
  - **Crowding in** refers to increased investment due to improved risk transfer
- Woodford (1990) explores both of these effects
Consider a model with two types of agents
- Per capita production $f(k)$
- Alternating endowments $\bar{e} > \underline{e} > 0$
- No borrowing

Stationary solution
- High endowment agents are *unconstrained*, consuming $\bar{c}$ and saving part of endowment
- Low endowment agents are *constrained*, consuming $\underline{c} \leq \bar{c}$ and depleting savings
OLG: Crowding Out

- Euler equations
  - Unconstrained: \( u'(\bar{c}) = \beta (1 + r) u'(\underline{c}) \)
  - Constrained: \( u'(\underline{c}) \geq \beta (1 + r) u'(%c) \)
- Interest rate is lower than discount rate
  - \( f'(k) - 1 = r \leq \beta^{-1} - 1 \equiv \rho \Rightarrow \) Pareto inefficient
- Increasing debt provides market liquidity
  - This increases interest rate and reduces capital stock
  - With \( r = \rho \Rightarrow \underline{c} = \bar{c} \) (full insurance)
Assume agents now have alternating *opportunities* (instead of endowments)

- Unproductive agents can only hold government debt
- Productive agents can hold debt *and* capital

**Stationary solution**

- Unproductive agents are *unconstrained*, consuming $\bar{c}$ and saving part of endowment (as debt)
- Productive agents are *constrained*, consuming $c \leq \bar{c}$ and investing savings and part of endowment in capital
OLG: Crowding In

- Euler equations
  - Unconstrained: \( u'(\bar{c}) = \beta(1 + r)u'(c) \)
  - Constrained: \( u'(c) = \beta f'(k)u'(\bar{c}) \)
  - Interest rate satisfies \( 1 + r \leq f'(k) \)

- Increasing debt provides *market liquidity*
  - This increases \( r \) and \( k \) since \( \beta(1 + r) = \frac{1}{\beta f'(k)} \)
  - Transfer from unproductive periods to productive periods
  - Increase debt until both agents are unconstrained
Precautionary Savings

- Consumption smoothing implies that agents will save in high income states and borrow in low income states.
  - If markets are incomplete, agents may not be able to efficiently transfer consumption between these outcomes.

- Additional precautionary savings motive arises when agents cannot insure against uncertainty.
  - Shape of utility function: $u'''$
  - Borrowing constraint: $a_t \geq -b$
Utility maximization $E_0[\sum_{t=0}^{\infty} \beta^t u(c_t)]$

- Budget constraint: $c_t + a_{t+1} = e_t + (1 + r)a_t$
- Standard Euler equation: $u'(c_t) = \beta(1 + r)E_t[u'(c_{t+1})]$

If $u''' > 0$, then Jensen’s inequality implies:

- $\frac{1}{\beta(1+r)} = \frac{E_t[u'(c_{t+1})]}{u'(c_t)} > \frac{u'(E_t[c_{t+1}])}{u'(c_t)}$
- Marginal value is greater due to uncertainty in $c_{t+1}$
- Difference is attributed to precautionary savings

**Prudence** refers to curvature of $u'$, i.e. $P = -\frac{u'''}{u''}$
Idiosyncratic Risk

- With *incomplete markets* and *borrowing constraints*, agents engage in precautionary savings in the presence of *idiosyncratic income shocks*

- Following Bewley (1977), mean asset holdings $E[a]$ result from individual optimization

\[
\rho \quad r \quad \text{mean}[a] \quad -b
\]
In an exchange economy, aggregate supply of assets must be zero

- Huggett (1993)

Equilibrium interest rate always satisfies \( r < \rho \)
Aiyagari (1994) combines the previous setup with standard production function $F(K, L)$

- Constant aggregate labor $L$

Demand for capital is given by $f'(k) - \delta = r$

- Efficient level of capital $f'(k^*) - \delta = \rho \Rightarrow k^* < k$
Aiyagari (1995) shows that a tax on capital earnings can address this efficiency problem

- This decreases the net interest rate received by agents

**Government debt does not work “perfectly”**

- No finite amount of government debt will achieve \( r = \rho \)
Bewley-Aiyagari economies result in competitive allocations that are not only Pareto inefficient, but are also constrained Pareto inefficient.

- Social planner can achieve a Pareto superior outcome even facing same market incompleteness.

This result can be attributed to pecuniary externalities.

- In competitive equilibrium, agents take prices as given whereas a social planner can induce wealth transfers by affecting relative prices.
- Stiglitz (1982), Geanakoplos-Polemarcharkis (1986)
Davila, Hong, Krusell, Rios-Rull (2005) consider welfare increasing changes in Aiyagari setting

- Higher level capital leads to higher wages and lower interest rates
  - Higher wage amplifies contemporaneous effect of labor endowment shock
  - Lower interest rate dampens impact of endowment shock in following periods
CI: Amplification

- Two period setting with \( t \in \{0, 1\} \)
  - Initial wealth \( y \)
  - Labor endowment \( e \in \{e_1, e_2\} \) (i.i.d)
  - Aggregate labor: \( L = \pi e_1 + (1 - \pi)e_2 \)
  - Production function \( f(K, L) \)

- Agent consumption plan given by \( \{c_0, c_1, c_2\} \)
  - \( c_i \leq e_i w + K(1 + r) \)
  - \( \frac{dU}{dK} = \{-u'(c_0) + \beta(1 + r)[\pi u'(c_1) + (1 - \pi)u'(c_2)]\} + \)
  - \( \beta[\pi u'(c_1)K + (1 - \pi)u'(c_2)K] \frac{dr}{dK} + \)
  - \( \beta[\pi u'(c_1)e_1 + (1 - \pi)u'(c_2)e_2] \frac{dw}{dK} \)
The first expression is zero from agent’s FOC

- Agents take prices as given, i.e. assume \( \frac{dw}{dK} = \frac{dr}{dK} = 0 \)

In a competitive equilibrium \( \frac{dr}{dK} = f_{KK} \) and \( \frac{dw}{dK} = f_{KL} \)

- \( f \) linearly homogeneous implies \( Kf_{KK} + Lf_{KL} = 0 \)

This provides:

- \( \frac{dU}{dK} = \beta \pi (1 - \pi) \frac{Kf_{KK}}{L} (u'(c_1) - u'(c_2))(e_2 - e_1) < 0 \)

- Reducing level of capital improves ex-ante utility
Cl: Dampening

- Consider addition of third period \( t = 2 \)
  - Same labor endowment \( e \in \{e_1, e_2\} \)
- Effect of change in capital level at \( t = 1 \) depends on realization of labor endowment
  - \( \Delta = \beta \pi (1 - \pi) \frac{K f_{KK}}{L} (u'(c_1) - u'(c_2))(e_2 - e_1) < 0 \)
  - \( \frac{dU_i}{dK} = \beta \left[ \Delta + \beta (\pi u'(c_{i1})) + (1 - \pi)u'(c_{i2}) \right] (K_i - K) f_{KK} \)
- Second term is positive if and only if \( K_i < K \)
  - Increasing capital more desirable for low endowment agents and less desirable for high endowment agents
Krusell, Smith (1998) introduce aggregate risk into the Aiyagari framework

- Aggregate productivity shock that follows a Markov process $z_t$ and $Y_t = z_tF(K_t, L_t)$

- Aggregate capital stock determines equilibrium prices $r_t, w_t$
  - However, the evolution of aggregate stock is affected by the distribution of wealth since poor agents may have a much higher propensity to save
  - Tracking whole distribution is practically impossible
AR: Bounded Rationality

- Krusell, Smith assume agents are boundedly rational and approximate the distribution of capital by a finite set of moments $M$
  - Regression $R^2$ is relatively high even if $\#M = 1$
- This result is strongly dependent on low risk aversion and low persistence of labor shocks
  - Weak precautionary savings motive except for poorest agents
  - Since wealth-weighted averages are relevant, this has a negligible effect on aggregate quantities
Amplification Revisited

- **Investment possibility shocks**
  - Production possibilities: Scheinkman & Weiss (1986)
  - Investment possibilities: Kiyotaki & Moore (2008)

- **Interim liquidity shocks**
  - Endogenous shock: Shleifer & Vishny (1997)

- **Preference shocks**
  - Aggregate risk: Allen & Gale (1994)
Scheinkman & Weiss

- Two types of agents with perfectly negatively correlated idiosyncratic shocks
  - No aggregate risk, but key difference is that labor supply is now elastic

- Productivity reflects *technological liquidity*
  - Productivity switches according to a Poisson process
  - Productive agents can produce consumption goods

- No capital in the economy
  - Can only save by holding money (fixed supply)
  - Productive agents exchange consumption goods for money from unproductive agents
SW: Aggregate Dynamics

- Aggregate fluctuations due to elastic labor supply
- Price level is determined in equilibrium
  - As productive agents accumulate money, wealth effect induces lower labor supply
  - Aggregate output declines and price level increases
- Effect of changes in money supply depends on distribution of money between agent types
  - Increase in money supply will reduce (increase) aggregate output when productive agents hold less (more) than half the money supply, i.e. when output is high (low)
Two types of agents, entrepreneurs and households

- Entrepreneurs can invest, but only when they have an investment opportunity
- Opportunities correspond to technological liquidity

Investment opportunities arrive i.i.d. and cannot be insured against

- Entrepreneur can invest with probability $\pi$

Agents can hold equity or fiat money
KM: Financing

- Entrepreneurs have access to 3 sources of capital
  - New equity claims, but a fraction $1 - \theta$ must be held by entrepreneur for at least one period
  - Existing equity claims, but only a fraction $\phi_t$ of these can be sold right away
  - Money holdings, with no frictions
- Capital frictions represent *illiquidity*
KM: Entrepreneurs

- Budget constraint:
  - \[ c_t + i_t + q_t(n_{t+1} - i_t) + p_t(m_{t+1} - m_t) = r_t n_t + q_t(1 - \delta)n_t \]
  - Equity holdings net of investment \( n_{t+1} - i_t \)
  - Price of equity/capital \( q_t \) can be greater than 1 due to limited investment opportunities

- Liquidity constraint:
  - \[ n_{t+1} \geq (1 - \theta)i_t + (1 - \phi_t)(1 - \delta)n_t \]
  - Limits on selling new and existing equity place lower bound on future equity holdings
For low $\theta$, $\phi_t$, liquidity constraints are binding and money has value

An entrepreneur with an investment opportunity will spend all of his money holding

- Budget constraint can be rewritten as $c_t^i + q_t^R n_{t+1}^{i} = r_t n_t + (\phi_t q_t + (1 - \phi_t)q^R_t)(1 - \delta)n_t + p_t m_t$
- Replacement cost of capital: $q_t^R \equiv \frac{1-\theta_q t}{1-\theta}$
- Can create new equity holdings at cost $q_t^R < q_t$, but this reduces value of remaining unsold holdings
Entrepreneur without investment opportunity decides on allocation between equity (depends on opportunity at \( t + 1 \)) and money

- Return to money: \( R_{t+1}^m \equiv \frac{p_{t+1}}{p_t} \)

- No opportunity: \( R_{t+1}^s \equiv \frac{r_{t+1} + q_{t+1}(1-\delta)}{q_t} \)

- Opportunity: \( R_{t+1}^i \equiv \frac{r_{t+1} + (\phi_{t+1}q_{t+1} + (1-\phi_{t+1})q_{t+1}^R(1-\delta))}{q_t} \)
KM: Logarithmic Utility

- Under logarithmic utility, entrepreneurs will consume $1 - \beta$ fraction of wealth.

- Around steady-state, aggregate level of capital is smaller than in first-best economy, i.e. $K_{t+1} < K^*$
  - Expected return on capital $E_t[f'(K_{t+1}) - \delta] > \rho$

- Conditional liquidity premium arises since $E_t[R_{t+1}^m] < E_t[R_{t+1}^s] < 1 + \rho$
  - Unconditional liquidity premium may also arise (but is smaller) since $E_t[R_{t+1}^i] < E_t[R_{t+1}^m]$
KM: Real Effects

- Negative shocks to market liquidity $\phi_t$ of equity have aggregate effects
  - Shift away from equity into money
  - Drop in price $q_t$ and increase in $p_t$
  - Decrease in investment and capital

- Shock to financing conditions feeds back to real economy as a reduction in output
  - KM find that government can counteract effects by buying equity and issuing new money (upward pressure on $q_t$ and downward pressure on $p_t$)
Three period model with \( t \in \{0,1,2\} \)

Entrepreneurs with initial wealth \( A \)
- Investment of \( I \) returns \( RI \) in \( t = 2 \) with probability \( p \)
- Interim funding requirement \( \rho I \) at \( t = 1 \) with \( \rho \sim G \)
- Extreme technological illiquidity, as investment is worthless if interim funding is not provided

Moral hazard problem
- Efficiency requires \( \rho \leq \rho_1 \equiv pR \Rightarrow \) continuation
- Only \( \rho \leq \rho_0 < \rho_1 \) of funding can be raised due to manager’s private benefit, i.e. principal-agent conflict
HT: Optimal Contracting

- Optimal contract specifies:
  - Investment size $I$
  - Continuation cutoff $\hat{\rho}$
  - Division of returns contingent on realized $\rho$

- Entrepreneurs maximize expected surplus, i.e.
  $$\max_{I,\hat{\rho}} \left\{ I \int_{0}^{\hat{\rho}} (\rho_1 - \rho) dG(\rho) - I \right\}$$

- Households can only be promised $\rho_0$ at $t = 1$
  - Breakeven condition: $I \int_{0}^{\hat{\rho}} (\rho_0 - \rho) dG(\rho) = I - A$

- Solution provides cutoff $\hat{\rho} \in [\rho_0, \rho_1]$
Without a storage technology, liquidity must come from financial claims on real assets

- *Market liquidity* of claims becomes crucial

If there is no aggregate uncertainty, the optimal contract can be implemented:

- Sell equity
- Hold part of market portfolio
- Any surplus is paid to shareholders as dividends
HT: Aggregate Risk

- With aggregate risk, optimal contract may not be implementable
  - Market liquidity of equity is affected by aggregate state
- Consider perfectly correlated projects
  - Liquidity is low when it is needed (bad aggregate state)
  - Liquidity is high when it is not needed (good state)
- This introduces a role for government to provide a store of wealth
Fund managers choose how aggressively to exploit an arbitrage opportunity

Mispricing can widen in interim period
- Investors question investment and withdraw funds
- Managers must unwind position when mispricing is largest, i.e. most profitable
- Low market liquidity due to limited knowledge of opportunity

Fund managers predict this effect, and thus limit arbitrage activity
- Keep buffer of liquid assets to fund withdrawals
Three period model with $t \in \{0,1,2\}$

- Continuum of ex-ante identical agents
  - Endowment of 1 in $t = 0$
  - Idiosyncratic preference shock, i.e. probability $\lambda$ that agent consumes in $t = 1$ and probability $1 - \lambda$ that agent consumes in $t = 2$

- Preference shock is not observable to outsiders
  - Not insurable, i.e. incomplete markets
DD: Investment

- Good can be stored without cost
  - Payoff of 1 in any period
- Long term investment project
  - Payoff of $R > 1$ in $t = 2$
  - Salvage value of $r \leq 1$ if liquidated early in $t = 1$
  - Market for claims to long-term project at price $p$
- Trade-off between return and liquidity
  - Investment is subject to *technological illiquidity*, i.e. $r \leq 1$
  - Market liquidity is represented by interim price $p$
Investing $x$ induces contingent consumption plan:

- $c_1 = px + (1 - x)$
- $c_2 = Rx + \frac{R(1-x)}{p}$

In equilibrium, we require $p = 1$

- If $p < 1$, then agents would store the asset and purchase project at $t = 1$
- If $p > 1$, then agents would invest and sell project at $t = 1$
With interim markets, any investment plan leads to $c_1 = 1, c_2 = R$

- If $r < 1$, fraction $1 - \lambda$ of aggregate wealth must be invested in project (market clearing)
- Since $p > r$, then asset’s *market liquidity* is greater than its *technological liquidity*

This outcome is clearly superior to autarky, with $c_1' = r, c_2' = R$ or $c_1'' = c_2'' = 1$
AG extend DD framework by adding aggregate risk
   - Here, $\lambda = \lambda_H$ with probability $\pi$ and $\lambda = \lambda_L < \lambda_H$ with probability $1 - \pi$

Agents observe realization of aggregate state and idiosyncratic preference shock at $t = 1$
   - After resolution of uncertainty, agents can trade claims to long-term project at $p_s \in \{p_H, p_L\}$
   - Asset's market liquidity will vary across states

For simplicity, assume $r = 0$
AG: Prices

- Market clearing requires $p_s \leq R$
  - Late consumers stored goods: $\lambda_s(1-x)$
  - Early consumers invested goods: $\lambda_s x$

- Cash-in-the-market pricing
  - $p_s = \min\left\{ R, \frac{(1-\lambda_s)(1-x)}{\lambda_s x} \right\}$
  - This implies that $p_H \leq p_L$, i.e. market liquidity is weaker when there are a large proportion of early consumers

- Despite deterministic project payoffs, there is volatility in prices
Overview

- Persistence
- Dynamic Amplification
  - Technological illiquidity  BGG
  - Market illiquidity  KM97
- Instability, Volatility Dynamics, Volatility Paradox
- Volatility and Credit Rationing/Margins/Leverage
- Demand for Liquid Assets
- Financial Intermediation
Gross Shadow Banking and Commercial Banking Liabilities

$ Trillion

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