Persuasion with Unknown Beliefs

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First draft: Summer 2017
This draft: January 2020

Abstract

A sender designs a signal structure to persuade a receiver to take an action. The sender is ignorant about the receiver’s prior, and evaluates each signal structure using the receiver’s prior that is the worst for the sender. I characterize the optimal signal structures in this environment. I show that there exists an optimal signal with two realizations, characterize the support of the signal and provide a formula that the signal must satisfy on the support, showing that the optimal signal is a hyperbola. The lack of knowledge of the receiver’s prior causes the sender to hedge her bets: the optimal signal induces the high action in more states than in the standard model, albeit with a lower probability. Increasing the sender’s ignorance can hurt both the sender and the receiver.

1 Introduction

When trying to persuade someone, one finds it useful to know the beliefs the target of persuasion holds. Yet often such beliefs are unknown to the persuader. How should persuasion be designed when knowledge of prior beliefs is limited?

The following example illustrates an application of the model. A pharmaceutical company commissions an experiment on the safety of a drug that it would like to persuade

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*I am indebted to my advisor, Wolfgang Pesendorfer, for his encouragement and support in developing this project. I am also grateful to Faruk Gul and Germán Gieczewski for insightful conversations. For helpful comments I would like to thank David Dillenberger, Pietro Ortoleva, Kris Ramsay, Evgenii Safonov, Juan Pablo Xandri, Leeat Yariv, the audiences at the 2019 meetings of the American Economic Association (Atlanta, Georgia), the 29th International Conference on Game Theory at Stony Brook University (2018), the 2019 Midwest Economic Theory Conference (Bloomington, Indiana), Young Economists Symposium 2019 (Columbia University) and seminar audiences at Princeton.

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the Food and Drug Administration (the FDA) to approve. The company does not know the exact prior belief of the regulator and instead only knows that the FDA believes that the drug is sufficiently safe with a high enough probability.

The company is designing the experiment in an environment where the receiver can take one of two actions. For instance, the FDA can approve a drug or not. The sender wishes to convince the receiver to take the high action in all states. Thus a pharmaceutical company aims to convince the FDA to approve the drug regardless of its quality. The receiver takes the action desired by the sender only if the receiver’s expectation of the state given his information is above a threshold and takes the other action otherwise. We call this threshold a *threshold of doubt*. In line with this reasoning, the FDA only approves the drugs that it believes to be sufficiently safe.

In the standard Bayesian persuasion model, the sender and the receiver have a common prior belief about the state. An optimal signal in that model recommends the high action with probability one in all states above a threshold and recommends the low action with probability one in all states below the threshold. We call this threshold a *threshold of action*. The threshold of action is below the threshold of doubt, so that the receiver takes the high action on a greater range of states than he would under complete information. If the sender and the receiver have commonly known heterogeneous priors, the optimal signal is not necessarily threshold but is still partitional: the high action is recommended either with probability one or with probability zero given a state. As the results in this paper will establish, when the receiver’s beliefs are unknown, the optimal signal is very different.

Towards establishing the results, let us now describe the setting this paper focuses on. I model the sender’s ignorance by assuming that the sender believes that Nature chooses the receiver’s prior from a set of priors to minimize the sender’s payoff. The sender has a known prior over the states and designs an experiment to maximize her payoff in the worst case scenario.

I focus on the case where the sender knows that the receiver’s prior assigns no less than a certain probability to each state but is otherwise ignorant about the receiver’s prior. Formally, the set of the receiver’s priors is all priors that put a mass of at least \((1 - a)g(w)\) on each state \(w\). This set of priors has the advantage of allowing me to flexibly model the lack of the sender’s knowledge about the receiver’s prior. Because the higher \(a\) is, the smaller the mass the receiver’s prior has to put on each state, \(a\) measures the sender’s ignorance or, equivalently, the flexibility that Nature has in choosing the receiver’s prior. If \(g\) is higher on one subset of states than on the other, then the sender has more information about the
receiver’s prior on the first subset.

The main contribution of this paper is a characterization of the optimal signal structure in a model of persuasion when the receiver’s beliefs are unknown. The contribution has several parts. First, I show that there exists a sender-optimal signal structure with only two signal realizations. Second, I establish that the distribution of the receiver’s actions induced by any sender-optimal signal structure is essentially unique. Third, I characterize the support of the signal realization recommending the high action, showing that it consists of all states such that an *importance* index exceeds a threshold. Fourth, I provide a formula for the probability that the high action is recommended on the support of the signal, showing that the optimal signal is a *hyperbola*. Fifth, I analyze comparative statics on the optimal signal structure with respect to the sender’s knowledge of the receiver’s prior.

I show that a state $w$ is in the support of the signal realization recommending the high action if the importance index $\frac{f_s(w)}{g(w)\sqrt{w^* - w}}$ exceeds a threshold, where $f_s$ is the density of the sender’s prior. Thus the optimal signal is more likely to induce the high action with a strictly positive probability in a state if the sender’s prior assigns a high probability to the state, Nature has more freedom in choosing the probability that the receiver assigns to this state, and the state is close to the threshold of doubt. I also establish that if the sender is sufficiently ignorant, then the optimal signal recommends the high action with a strictly positive probability in every state, which shows that full support is a robust property of the signal chosen by an ignorant sender.

Because the optimal signal is a hyperbola on the support, it is inversely proportional to the distance between the state and the threshold. In our example of the FDA, for instance, the probability that a drug is approved is then inversely proportional to the drug’s morbidity. Moreover, it is convex on any interval where the high action is recommended with an interior probability. This has consequences for the sender’s welfare, implying that the sender benefits whenever his prior $F_s$ becomes more dispersed on such an interval.

I also consider a special case of the above general setup where the sender’s knowledge is detail-free, and yet the sender is still able to design a persuasion mechanism inducing the receiver to act where he would not have acted in absence of persuasion. In particular, I consider the case where the sender only knows that the receiver believes that the probability the state is greater than $\alpha$ is at least $\beta$. In this environment, the persuasion mechanism that the sender designs is continuous in the state and recommends the high action with a strictly positive probability in every state.
I provide comparative statics on the optimal signal structure with respect to the
sender’s knowledge of the receiver’s prior. I show that the more ignorant the sender is,
the more she hedges her bets and spreads out on the states the probability with which
the high action is recommended. Formally, if we increase $a$, thereby decreasing the weight
$(1 - a)g(w)$ that the receiver’s prior has to put on each state $w$, then the support of the
optimal signal expands, so that the high action is recommended in more states, but the
probability with which the high action is recommended decreases.

The results thus change the way we think about Bayesian persuasion: unlike the
intuition in the standard model, it is not optimal to pool all sufficiently high states together
and give up on persuading the receiver in the lower states. Instead, the sender must allow
persuasion to fail with some probability on some of the high states and is able to persuade the
receiver with a positive probability on the low states. The model thus makes clear the impact
of the sender’s lack of knowledge about the receiver’s prior on the sender-optimal signal: the
lack of knowledge causes her to hedge her bets and spread out the probability with which
the high action is recommended. This reveals the nature of persuasion to be fundamentally
local: oftentimes, for small increases in the state the sender is able to increase the probability
with which the receiver is persuaded only by a small amount.

I next consider the welfare implications of the sender’s ignorance. I show that the
impact of increasing the sender’s ignorance on the receiver’s welfare is ambiguous: it can
either benefit or hurt the receiver. Because greater ignorance always hurts the sender, this
implies that the sender’s ignorance about the receiver’s prior can hurt both the sender and
the receiver. I also show that the receiver strictly prefers to face an ignorant sender rather
than a sender that is perfectly informed about the receiver’s prior. Finally, I show that if
the sender-optimal signal recommends the high action with a strictly positive probability
in every state, then greater sender’s ignorance benefits the receiver. These results have
important implications for the optimal transparency requirements for the FDA, showing
that full transparency is never optimal and that if there are bounds on the transparency
requirements that can be enforced, the optimal transparency level may be interior.

We say that the receiver makes a mistake in state $\omega$ if, conditional on the state being
$\omega$, the receiver takes an action different from the one he would take if he knew the state. The
pattern of mistakes in the model with unknown beliefs differs from the one in the standard
model. Because the optimal signal recommends the high action with a strictly positive probability
in more states in the robust model, mistakes happen with a strictly positive probability in more states. This happens because in the model with unknown beliefs the
sender’s ability to gain credibility by pooling sufficiently favorable states together is limited: if a signal realization recommends the high action in low states with a probability that is too high, then Nature is able to choose the receiver’s prior that causes the receiver to take the low action after this signal realization. On the other hand, because in the standard model the sender pools sufficiently favorable states together, when the receiver’s beliefs are known, mistakes only happen on the interval of intermediate states between the threshold of action and the threshold of doubt.

We see that when the receiver’s beliefs are unknown especially pernicious outcomes are possible. For instance, there are parameters under which the FDA approves even the most unsafe drugs with a strictly positive probability, whereas if the receiver’s prior is known, the probability that the most unsafe drugs are approved is zero. Thus a model of persuasion with unknown beliefs can rationalize the occurrence of adverse outcomes that cannot be explained by the standard model.

The final contribution of the paper lies in solving a mechanism design problem to which the revelation principle does not apply. Solving such problems tends to be challenging. I show that the model in the present paper can be solved by the means of using a fixed-point argument to define the receiver’s prior chosen by Nature in response to the sender choosing a signal structure. Importantly, the set of priors that Nature can choose from has to be sufficiently rich to enable Nature to match the signal structure chosen by the sender, and my analysis highlights the correct notion of richness for this problem.

The rest of the paper proceeds as follows. Section 2 introduces the model. Section 3 presents the characterization of the optimal signal structure, discusses in greater detail the important special case of the model where the sender’s knowledge is detail-free and explains the intuition for the main result. Section 3 also provides examples of optimal signal structures in other special cases and compares the robust signal to the signal optimal in the standard common prior model. Section 4 contains results on the comparative statics of the optimal signal structure with respect to the sender’s ignorance, conducts welfare analysis and considers the convergence of the optimal signal structure to the case of a known prior belief of the receiver. Section 5 provides counterexamples, showing that if the assumptions on the parameters made in the present paper are violated, then there may not exist an optimal signal structure with two realizations. Section 6 provides a sketch of the proof. Section 7 reviews the related literature. Section 8 concludes.
2 Model

2.1 Payoffs

The state space is an interval $\Omega = [l, h]$ such that $l > 0$. The sender’s preferences are state-independent. The sender gets a utility of $u(E)$ if the receiver’s expected value of the state given the receiver’s prior and the signal realization is $E$. The sender’s utility function $u$ is

$$u(E) = \begin{cases} 
0 & \text{if } E \in [l, w^*) \\
1 & \text{if } E \in [w^*, h]
\end{cases}$$

The model can be interpreted as one where the receiver can take one of the two actions, 0 or 1, and he takes action 1 if and only if his expectation of the state is weakly greater than $w^*$.

2.2 Priors

The sender has a known prior over the states, while the receiver has a set of priors. This is in contrast to the standard model of Bayesian persuasion, where the sender and the receiver have a known common prior. Let $\varphi$ denote the set of all CDFs of probability measures on $\Omega$. The sender’s prior is a probability measure on $\Omega$ with a CDF $F_s$. I assume that $F_s$ admits a density $f_s$ that is $C^1$. For the sake of convenience, I also assume that $f_s$ has full support.\(^2\)

Let $\mu_G$ be a probability measure on $[l, h]$ with a CDF $G$ that admits a $C^1$ density $g$. For convenience, in most of the paper I also assume that $g$ has full support. Given $a \in (0, 1)$, the receiver’s set of priors is\(^4\)

$$C_{a,g} = \{ F \in \varphi : \mu_F(A) \geq (1 - a)\mu_G(A) \text{ for all } A \in \mathcal{B}([l, h]) \}$$

\(^1\)The assumption that $l > 0$ is without loss of generality and is made for convenience.

\(^2\)Some of the results in this paper do not require this assumption. In particular, the result that there exists an optimal signal with only two realizations does not depend on it, nor does the characterization in Corollary 2.1 of the optimal signal structure in the case when the sender only knows that the receiver’s prior puts a mass of at least $\beta$ on states above $\alpha$.

\(^3\)All results generalize to the case when the sender’s prior does not have full support.

\(^4\)Here $\mathcal{B}([l, h])$ denotes the Borel sigma-algebra on $[l, h]$.
That is, the receiver’s set of priors is the set of all priors that put on each Borel set \(A\) a mass that is at least as large as the mass that the measure \((1 - a)\mu_G\) puts on \(A\).

To understand the assumption on the set of priors, consider a version of the model in which the state space is discrete. Then the receiver’s set of priors consists of all priors that assign a probability of at least \((1 - a)g(w)\) to each state \(w\). Thus the sender knows that the receiver believes that the probability of each state \(w\) is at least \((1-a)g(w)\), but does not know what exactly this probability is. The fact that \(a \in (0, 1)\) implies that \(\int_{[l,h]} (1-a)g(w)dw < 1\), which ensures that the receiver’s prior is not completely pinned down by this requirement.

Observe that \(a = 1 - (1-a)\int_{[l,h]} g(w)dw\) is the difference between 1 and the mass put on the state space \([l,h]\) by the measure \((1-a)\mu_G\). Thus \(a\) measures the flexibility that Nature has in choosing the receiver’s prior: the larger \(a\) is, the more freedom Nature has. Consequently, \(a\) can also be interpreted as a measure of the sender’s ignorance. The larger \(a\) is, the more ignorant the sender is.

I assume that \(\int_{[w^*,h]} g(w)dw > 0\) and \((1-a)\int_{[l,h]} wg(w)dw + al < w^*\). These assumptions ensure that the sender’s problem is non-trivial. In particular, the assumption that \(\int_{[w^*,h]} g(w)dw > 0\) ensures that there exists a feasible signal structure that induces the receiver to take action 1 with a strictly positive probability. The assumption that \((1-a)\int_{[l,h]} g(w)wdw + al < w^*\) ensures that if no information is provided, then the receiver will take action 0.

I next introduce special classes of sets of the receiver’s priors. Definition 1 says that the sender’s knowledge is detail-free if the measure \(\mu_G\) puts zero mass on states below the receiver’s threshold – where the sender and the receiver disagree about the optimal action.

**Definition 1.** The sender’s knowledge is said to be detail-free if \(\mu_G([l,w^*)) = 0\).

The reason that a set of priors satisfying the condition in definition 1 represents detail-free knowledge is that \(\mu_G\) quantifies the limited knowledge that the sender has about the receiver’s prior. If \(g(w)\) varies on the states where the sender and the receiver disagree, then the sender knows more about the receiver’s prior on some such states than on others: in particular, the sender knows more about the prior on the states where \(g(w)\) is high, since Nature has less freedom in choosing the prior there. If, on the other hand, \(g(w)\) is 0 on all states below the threshold, then Nature can move a given probability mass freely below the threshold, and the sender is equally ignorant about the receiver’s prior on all states in which there is a conflict of interest. Thus the sender only has a vague idea of how pessimistic the receiver can be – the receiver’s prior can put no more than a certain mass on states below
the threshold, but the sender lacks knowledge of the details of how the receiver’s prior may vary on the states where there is disagreement.

I also consider the following important class of the sets of the receiver’s priors. Given \( \alpha \in \Omega \) and \( \beta \in (0, 1) \), define a set of CDFs \( C(\alpha, \beta) \) as follows:

\[
C(\alpha, \beta) = \left\{ F \in \varphi : 1 - \lim_{w \uparrow \alpha} F(w) \geq \beta \right\}
\]

If the set of the receiver’s priors is \( C(\alpha, \beta) \), then the sender knows that there exists a state \( \alpha \in \Omega \) such that the mass that the receiver’s prior puts on states above \( \alpha \) is at least \( \beta \) for some \( \beta \in (0, 1) \). Figure 1 illustrates the set of the receiver’s priors: the set consists of all CDFs below the CDF drawn in bright blue. This class of sets of priors is a special case of a setting where the sender’s knowledge is detail-free. This is because under this set of priors Nature can move the mass of \( 1 - \beta \) freely below the receiver’s threshold, and, in addition, the sender’s knowledge about the prior above the threshold takes a particularly simple form.

### 2.3 Information Structures and Evaluation of Payoffs

The order of moves is as follows. First, the sender commits to a signal structure \( \pi \). Next, Nature chooses the receiver’s prior \( F \in C_{a,g} \) to minimize the sender’s payoff. Then the state is realized (from the sender’s perspective, the state is drawn from the distribution \( F_s \)). After this, the signal is realized according to the signal structure \( \pi \). Then, having seen a signal realization \( \sigma \), the receiver forms an expectation of the state given that the receiver’s prior is \( \mu_F = (1 - \beta)\delta_l + \beta \delta_\alpha \), where \( \alpha > w^* \).

5Technically, this is not a special case of the general model introduced above because here the conditions are imposed on the CDFs and not on the densities. The set of priors that is a special case of the general model and induces the same optimal signal with two realizations as in the case with \( C(\alpha, \beta) \) described in Corollary 2.1 is \( C_{a,g} \) with \( \mu_G = \delta_\alpha \) and \( 1 - a = \beta \).

6The assumptions on the set of priors that ensure that the sender’s problem is non-trivial translate to this special case as follows. The first assumption translates to \( \alpha > w^* \), so that the receiver’s prior puts a mass of at least \( \beta \) on states strictly above \( w^* \). This assumption ensures that the sender can induce the receiver to take action 1 with a strictly positive probability. Because the worst prior the receiver can have if the sender provides no information is \( \mu_F = (1 - \beta)\delta_l + \beta \delta_\alpha \), the second assumption translates to \( (1 - \beta)l + \beta \alpha < w^* \). This assumption ensures that the receiver with the worst prior takes action zero if he does not receive any information.

7A signal structure is a Markov kernel \( \pi \). Informally, we can interpret \( \pi(\sigma | \omega) \) as the probability of signal realization \( \sigma \) given that the state is \( \omega \). Formally, letting \( \mathcal{B}(M) \) and \( \mathcal{B}(\Omega) \) denote the Borel sigma-algebras on the message space \( M \) and the state space \( \Omega \) respectively, a Markov kernel \( \pi \) is defined as a mapping \( \pi : \Omega \times \mathcal{B}(M) \to [0, 1] \) such that for every \( \omega \in \Omega \), \( B \mapsto \pi(B | \omega) \) is a probability measure on \( M \) and for every \( B \in \mathcal{B}(M) \), \( \omega \mapsto \pi(B | \omega) \) is \( \mathcal{B}(\Omega) \)-measurable. See Pollard (2002) for more details.
prior is $F$ and that the signal structure is $\pi$. 

I restrict my attention to signal structures that have a finite number of signal realizations given each state. I conjecture, but do not have a proof, that my results hold for all signal structures.

If the sender chooses a signal structure $\pi$ and a receiver with a prior $F$ sees signal realization $\sigma$, then the receiver’s expectation of the state is $E_{F,\pi} [\omega | \sigma]$. Then, if the sender chooses a signal structure $\pi$ and Nature chooses $F \in C_{a,g}$, the sender’s payoff is

$$
\int_{\Omega} \sum_{\sigma} \mathbb{1}_{E_{F,\pi}[\omega | \sigma] \geq w^*} \pi(\sigma | w) dF_s(w)
$$

Thus the sender’s payoff from choosing a signal structure $\pi$ is

$$
U(\pi) = \min_{F \in C_{a,g}} \int_{\Omega} \sum_{\sigma} \mathbb{1}_{E_{F,\pi}[\omega | \sigma] \geq w^*} \pi(\sigma | w) dF_s(w)
$$

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8If $g$ does not have full support, then we need to specify how the receiver updates his beliefs after observing signal realizations that have zero probability under the receiver’s prior. In this case, reasonable updating rules such as the receiver not changing his prior belief or putting mass one on the lowest state in the support of the signal realization ensure that the results in the present paper hold.
and the sender’s equilibrium payoff is

$$\max_\pi \min_{F \in \mathcal{C}_{a,g}} \int_\Omega \sum_{\sigma} 1_{E_{F,\pi}[\omega|\sigma] \geq w^*} \pi(\sigma|w) dF_s(w)$$

(1)

3 The Optimal Signal

This section characterizes the optimal signal. I show that there is an optimal signal with two realizations. Under this signal, the probability of the signal realization recommending the high action is 1 above the threshold $w^*$ and is a hyperbola on the support below $w^*$. The support of this signal realization below $w^*$ is the set of all states such that an importance index exceeds a threshold $t$. There is a tradeoff between adding more states to the support (by increasing the threshold $t$) and recommending the high action with a greater probability (by increasing the constant $c$ scaling the hyperbola), and the optimal signal balances these considerations. I start by defining a class of distributions over the receiver’s actions that have the above form.

Letting $t = \min_{w \in [l, w^*)} \frac{f_s(w)}{g(w)(w^*-w)}$, I define a class of distributions $S^{t,c,z}$ over the receiver’s actions as follows: given constants $t \geq t$, $c$, $z \geq 0$, the probability of action 1 in state $\omega$ is given by

$$S^{t,c,z}(\omega) = \begin{cases} 
1 & \text{for } \omega \in [w^*, h] \\
\min \left\{ \frac{c}{w^*-\omega}, 1 \right\} & \text{for } \omega \in \Omega(t, z) \\
0 & \text{for } \omega \in [l, w^*) \setminus \Omega(t, z) 
\end{cases}$$

where

$$\Omega(t, z) = \left\{ w \in [l, w^*) : \frac{f_s(w)}{g(w)(w^*-w)} > t \right\} \cup \left\{ w \in [l, w^*) : \frac{f_s(w)}{g(w)(w^*-w)} = t \text{ and } w \geq z \right\}$$

A signal is said to be sender-optimal if it solves the sender’s problem [1]. Theorem 1 describes a sender-optimal signal and the distribution over the receiver’s actions induced by it.

**Theorem 1.** There exist $t \geq t$, $c$, $z \geq 0$ and a sender-optimal signal structure that induces a distribution $s$ over the receiver’s actions satisfying $s = S^{t,c,z}$.

A sender-optimal signal structure inducing the distribution $s$ is given by $\pi(\sigma|\omega) = s(\omega)$,
\[ \pi(\sigma_0|\omega) = 1 - s(\omega) \text{ for all } \omega \in [l, h]. \]

Theorem 1 says that there exists a sender-optimal signal structure with two realizations, \( \sigma \) and \( \sigma_0 \). The receiver takes action 1 after seeing signal \( \sigma \) and action zero after seeing signal \( \sigma_0 \). If the state is in \([w^*, h]\), the receiver takes action 1 with probability one, if the state is in a set \( \Omega(t, z) \), the receiver takes action 1 with probability \( \min \left\{ \frac{c}{w^* - \omega}, 1 \right\} \) for some constant \( c \), and if the state is in a set \([l, w^*) \setminus \Omega(t, z)\), the receiver takes action 1 with probability zero.

### 3.1 Uniqueness of the Optimal Signal

To ensure the essential uniqueness of the distribution of the high action induced by a sender-optimal signal structure, I make use of the following assumption.

**Assumption 1.** \( \mu \left( \left\{ w \in [l, w^*) : \frac{f_s(w)}{g(w)(w^* - w)} = c_0 \right\} \right) = 0 \) for all \( c_0 > 0 \).

Assumption 1 says that the set of states for which the importance index \( \frac{f_s(w)}{g(w)(w^* - w)} \) is equal to a constant \( c_0 \) is of measure zero for all constants \( c_0 > 0 \). Lemma 1 in the Appendix shows that a sufficient condition for Assumption 1 to be satisfied is that \( f_s \) and \( g \) are real-analytic functions on \([l, h]\), and the function \( w \mapsto \frac{f_s(w)}{g(w)(w^* - w)} \) is not a constant function on \([l, w^*)\).

**Theorem 2.** If Assumption 1 is satisfied, then there exist unique \( t \geq t_0 \), \( c \geq 0 \) such that any sender-optimal signal structure induces a distribution \( s \) over the receiver’s actions satisfying \( s = S_{t, c, l} \mu_{F_s} \)-a.e.

Theorem 2 implies that, provided that Assumption 1 is satisfied, any sender-optimal signal structure induces a distribution over the actions of the receiver that is unique up to sets that have measure zero under the sender’s prior.

### 3.2 The Support of the Optimal Signal

Theorem 2 shows that, given a sender-optimal signal structure with two realizations, provided that Assumption 1 is satisfied, the support \( \Omega(t, z) \) below the threshold \( w^* \) of the...
signal realization recommending the high action is

\[ \Omega_1(t) = \left\{ w \in [l, w^*) : \frac{f_s(w)}{g(w)(w^*-w)} \geq t \right\} \]

\( \Omega_1(t) \) consists of all states \( w \) below the threshold \( w^* \) such that the importance index \( \frac{f_s(w)}{g(w)(w^*-w)} \) is greater than some threshold \( t \). This is the importance index because it puts higher weight on the states that are more important to the sender. Thus the sender is more likely to induce the receiver to take the high action in state \( w \) if state \( w \) is more important to the sender because the sender believes that this state is very likely, if the conflict of interest between the sender and the receiver at state \( w \) is not too large because the distance \( w^*-w \) between the state and the threshold of doubt \( w^* \) is small, and if Nature has greater freedom in choosing the probability that the receiver assigns to state \( w \) — because \( g(w) \) is small.

It is immediate that if \( g(w) = 0 \) for some state \( w \), then this state is in the support of the signal realization recommending the high action. That is, the sender never gives up on the states that the receiver may consider impossible. This is true regardless of what the sender’s prior belief is.

Recall that the sender’s knowledge is detail-free if the measure \( \mu_G \) puts mass zero on the states below the threshold \( w^* \). Proposition 1 shows that, more generally, the distribution over the receiver’s actions induced by an optimal signal structure is independent of the sender’s prior if and only if the sender’s knowledge is detail-free. That is, the distribution is independent of the sender’s prior if and only if the receiver may consider all states on which he disagrees with the sender with regard to the optimal action impossible.

**Proposition 1.** A sender-optimal signal structure induces a distribution over the receiver’s actions that is independent of the sender’s prior \( F_s \) if and only if the sender’s knowledge is detail-free.

A natural question to ask is, when is the support of the optimal signal connected? The support of the optimal signal is connected if the function \( w \mapsto \frac{f_s(w)}{g(w)(w^*-w)} \) is increasing on \([l, w^*)\). Thus, for example, the sender’s prior \( f_s \) and the density \( g \) being equal is sufficient to ensure that the support of the optimal signal is connected. The sender’s prior \( F_s \) dominating the prior \( G \) in the monotone likelihood ratio property sense is also sufficient for the support to be connected. Figure 2 shows an example of the optimal information policy when \( w \mapsto \frac{f_s(w)}{g(w)(w^*-w)} \) is increasing.

Interestingly, the support of the optimal signal consists of exactly two intervals of the
Figure 2: Optimal Information Policy: Connected Support of the Signal, 
\( w \mapsto \frac{f_s(w)}{g(w)(w^*-w)} \) is Increasing

\[ \pi(\sigma|\omega) \]

\[ \omega \]

\[ w^* \]

form \([l, w'] \cup [w^*, h] \) if the function \( w \mapsto \frac{f_s(w)}{g(w)(w^*-w)} \) is decreasing on \([l, w^*]\). That is, the optimal signal cuts out an interval in the middle if the sender perceives the lower states to be sufficiently more likely and the receiver perceives the lower states to be sufficiently less likely. Observe that the function \( w \mapsto \frac{f_s(w)}{g(w)(w^*-w)} \) being decreasing on \([l, w^*]\) requires that \( f_s \) converges to zero as \( w \) converges to \( w^* \), so that the sender’s prior puts very little mass on the states right below the threshold \( w^* \). Figure 3 shows an example of the optimal information policy when \( w \mapsto \frac{f_s(w)}{g(w)(w^*-w)} \) is decreasing.

As a final example of the support of the optimal signal, observe that the support consists of two intervals of the form \([l, w'] \cup [w'', h] \) for \( w'' \leq w^* \) if the function \( w \mapsto \frac{f_s(w)}{g(w)(w^*-w)} \) is single-dipped on \([l, w^*]\). Figure 4 shows an example of the optimal information policy when \( w \mapsto \frac{f_s(w)}{g(w)(w^*-w)} \) is single-dipped.

Let \( t^* \) denote the fixed point of the following fixed point equation:

\[ t = \frac{\int_{\Omega_1(t) \cap [l, w^*-c(t)]} f_s(w) \frac{w^*-w}{g(w)(w^*-w)} dw}{\frac{a}{1-a} + \int_{\Omega_1(t) \cap [l, w^*-c(t)]} g(w) dw} \]

where \( t \mapsto c(t) \) is a strictly increasing function such that \( s = S^{t, c(t)} \).
Figure 3: Optimal Information Policy: A Gap in the Middle, 
\[ w \mapsto \frac{f_s(w)}{g(w)(w^*-w)} \] is Decreasing

Figure 4: Optimal Information Policy: A Gap in the Middle, 
\[ w \mapsto \frac{f_s(w)}{g(w)(w^*-w)} \] is Single-Dipped
Lemma 13 in the Appendix shows that the threshold $t \geq \underline{t}$ is unique and is given by $	ext{max}\{t^*,\underline{t}\}$.\footnote{This is true provided that Assumption 1 is satisfied.}

Let $\overline{A} = \Omega_1(t) \cap [w^* - c(t), h]$ denote the part of the support of the signal on which the high action is recommended with probability one and let $A = \Omega_1(t) \cap [l, w^* - c(t)]$ denote the part of the support of the signal on which the high action is recommended with probability strictly between zero and one.

Then we can equivalently write the fixed point equation as

$$t = \frac{E \left[ \frac{f_s(w)}{w^* - w} | w \in A \right] \mu(A)}{\frac{a}{1-a} + \mu_G(A)}$$

where $E$ is the expectation with respect to the uniform distribution and $\mu$ is the measure corresponding to the uniform distribution.

Observe that if the sender’s prior $F_s$ puts higher weight on the states below the threshold, this increases the right hand side of the fixed point equation (2) and thus increases the threshold $t$, making the support of the signal smaller.

The final result I provide in this section describes how the support of the signal behaves as the sender becomes very ignorant.

**Proposition 2.** $\lim_{a \to 1} \Omega_1(t) = [l, w^*)$. Moreover, if there exists $m > 0$ such that $f_s(w) \geq m$ for all $w \in [l, w^*)$, then there exists $\overline{a} \in (0,1)$ such that for all $a \in (\overline{a}, 1)$ we have $\Omega_1(t) = [l, w^*)$.

Proposition 2 shows that, as the ignorance of the sender approaches its maximal value of 1, in the limit the optimal signal recommends the high action with a strictly positive probability in all states. Moreover, if the density of the sender’s prior is bounded away from zero, then the signal has full support not just in the limit but for all sufficiently high levels of the sender’s ignorance. This clarifies the sense in which recommending the high action with a strictly positive probability in all states is a robust property of the signal chosen by an ignorant sender.
3.3 The Formula for the Optimal Signal

Theorem 1 shows that on the support \( \Omega(t, z) \) the probability that the high action is recommended is a hyperbola (provided that the constraint that \( \pi(\sigma|w) \leq 1 \) does not bind). In particular, this probability is proportional to \( \frac{1}{w^* - w} \) with a constant of proportionality \( c \) that is independent of the state \( w \). In other words, the probability is inversely proportional to the distance between the state and the threshold. Going back to our example of the FDA, if we interpret the state as the number of deaths per one thousand people who took the drug, the probability that a drug is approved should be inversely proportional to its morbidity.

The fact that the probability with which the high action is recommended is a hyperbola has consequences for the sender’s welfare: because this probability is convex on any interval in the support of the signal, if the sender’s prior is more dispersed in the sense of a mean-preserving spread on such an interval, then the sender’s equilibrium payoff is higher.\(^{11}\)

We can show that if Assumption 1 is satisfied, then, given any conjectured threshold \( t \), there exists a unique the constant of proportionality \( c \). We now make the dependence of the constant of proportionality \( c \) on the threshold \( t \) explicit by writing \( c(t) \).

The proof of Theorems 1 and 2 shows that \(^{12}\) \( c(t) \) is given by the fixed point equation

\[
c(t) = \frac{\int_{[w^*, a]} g(w)(w - w^*)dw - \int_{\Omega(t) \cap [w^* - c(t), w^*]} g(w)(w^* - w)dw}{\frac{a}{1-a} + \int_{\Omega(t) \cap [l, w^* - c(t)]} g(w)dw}
\]

We can equivalently write this as

\[
c(t) = \frac{E_G[w - w^*|w \in A] \mu_G(A)}{\frac{a}{1-a} + \mu_G(A)} \quad (3)
\]

Because the states on which the constraint that \( \pi(\sigma|w) \leq 1 \) binds depend on \( c(t) \), in general, we are only able to characterize \( c(t) \) implicitly. However, in some cases \( c(t) \) can be computed explicitly. I next consider one such special case – the case in which the optimal signal structure is such that the constraint that the probability with which the high action is

\(^{11}\)This is immediate if the parameters are such that the optimal signal structure is independent of the sender’s prior. In general, consider two sender’s priors, \( F^1_s \) and \( F^2_s \), such that \( F^2_s \) is more dispersed than \( F^1_s \) in the sense of a mean-preserving spread on an interval in the support of the signal \( \pi_1 \) that is optimal under \( F^1_s \). Then if the sender with prior \( F^2_s \) chooses \( \pi_1 \), her payoff is strictly higher. This implies that the sender’s payoff under the signal optimal given the prior \( F^2_s \) must also be strictly higher.

\(^{12}\)This is true provided that Assumption 1 is satisfied.
recommended in a state is weakly less than 1 does not bind at any state below the threshold \( w^* \).

### 3.4 Intuition

I next provide the intuition for the hyperbolic functional form of the optimal signal. In the standard Bayesian persuasion model where the sender knows the receiver’s prior the sender-optimal signal has a threshold structure, recommending action 1 with probability one if the state is above a threshold and recommending action 0 otherwise. When a prior common to the sender and the receiver is fixed, recommending action 1 in higher states below the threshold of doubt \( w^* \) yields a strictly greater benefit to the sender than recommending action 1 in lower states, so the sender recommends action 1 in all sufficiently high states such that the receiver’s expectation upon seeing the signal realization \( \sigma \) is exactly \( w^* \).

The threshold signal structure is not optimal when the receiver’s beliefs are unknown. The case in which the sender’s knowledge is detail-free and the set of priors is \( C(\alpha, \beta) \) is convenient for providing the intuition for the non-optimality of a threshold signal. Towards this end, consider a signal structure with two realizations, \( \sigma \) and \( \sigma_0 \), satisfying \( \pi(\sigma|\alpha) = 1 \) and the receiver’s prior that puts a mass of \( \beta \) on \( \alpha \) and a mass of \( 1 - \beta \) on some state \( \omega \) below the threshold \( w^* \). Then the receiver’s expectation conditional on seeing the signal realization \( \sigma \) is \( E[w|\sigma] = \frac{\omega(1-\beta)\pi(\sigma|\omega) + \alpha\beta}{(1-\beta)\pi(\sigma|\omega) + \beta} \). In order for the receiver to take action 1 after seeing \( \sigma \), we need \( E[w|\sigma] \geq w^* \), which turns out to be equivalent to \( \pi(\sigma|\omega) \leq \frac{\beta(\alpha-w^*)}{(1-\beta)(w^*-\omega)} \).

Note then that Nature moves after the sender chose the signal structure and can put a mass of \( 1 - \beta \) on any state \( \omega \) below the threshold \( w^* \). Thus if the probability of signal realization \( \sigma \) exceeds the bound \( \frac{\beta(\alpha-w^*)}{(1-\beta)(w^*-\omega)} \) at any state below the threshold, then by putting a mass of \( 1 - \beta \) on this state Nature can ensure that the receiver never takes action 1. Therefore, \( \pi(\sigma|\omega) \) must be below the bound in all states. On the other hand, the sender’s payoff is increasing in the probability that action 1 is taken, implying that it is best for the sender to maximize \( \pi(\sigma|\omega) \) subject to the constraint that it be below the bound. Thus setting \( \pi(\sigma|\omega) \) equal to the bound in all states yields the signal that is optimal (in the class of all signals with two realizations) when the receiver’s beliefs are unknown.

If the state is below \( w^* \), then \( \pi(\sigma|\omega) \), the probability with which the optimal signal recommends action 1, is increasing in \( \alpha \) and \( \beta \). Thus if the sender knows that the receiver’s prior puts a positive mass on states above a higher threshold or that the mass that the receiver’s prior puts on states above a certain threshold is higher, the sender is able to
induce the receiver to take action 1 with a higher probability.

3.5 Examples

3.5.1 The Optimal Signal with Non-Binding Constraints

Consider the parameters under which an optimal signal structure induces a distribution over the receiver’s actions such that \( s(\omega) \), the probability of action 1 in state \( \omega \), satisfies \( s(\omega) < 1 \) for all \( \omega \in [l, w^*] \). A sufficient condition for this to obtain is that the sender’s prior puts zero weight on states below \( w^* \) that are sufficiently close to \( w^* \) and that the mass that \( \mu_G \) puts on the states above \( w^* \) is sufficiently low. Then it follows from Theorems 1 and 2 that there exists a unique threshold \( t \geq 0 \) such that

\[
\begin{align*}
\frac{c(t)}{t} &= \int_{[w^*, h]} (w - w^*) g(w) \, dw \\
\int_{\Omega_1(t)} g(w) \, dw &\geq t \\
\end{align*}
\]

where \( \Omega_1(t) = \left\{ w \in [l, w^*] : \frac{f_s(w)}{g(w)(w^*-w)} \geq t \right\} \mu_F \text{-a.e.} \)

Here \( c(t) = \int_{[w^*, h]} (w - w^*) g(w) \, dw \). The reason that \( c(t) \) can be computed explicitly in this case is that the constraint that \( \pi(\sigma|w) \leq 1 \) never binds, and the only way that \( c(t) \) impacts the right hand side of the fixed point equation (3) is through determining the states on which the constraint binds.

We can see that \( c(t) \), and thus the probability that the high action is recommended on a state below the threshold, is likely to be higher if \( \mu_G \) puts more weight on the states above \( w^* \) and less weight on the states below \( w^* \), if the states above the threshold that the receiver considers possible are higher, and if the mass \( a \) that Nature can move freely is lower.

3.5.2 A Uniform Example

To illustrate the solution of the model, here I solve for the sender-optimal signal given simple parametric assumptions on the sender’s prior and the receiver’s set of priors. I suppose that both \( f_s \) and \( g \) are uniform. Proposition 3 provides the formulas characterizing

\[\text{Here we suppose that Assumption 1 is satisfied.}\]
Figure 5: Numerical Example: Dependence of the Optimal Signal on the Sender’s Ignorance

(a) Constant of proportionality and the threshold

(b) The value of the receiver

The parameters in the numerical example are $f_s(w) = \frac{1}{h-l}$ for all $w \in [l, h]$, $g(w) = \frac{1}{h-l}$ for all $w \in [l, h]$, $l = 0$, $h = 1000$, $w^* = 800$. The prior of the receiver is $\mu_F = 0.9\mu_G + 0.1\delta_h$.

Proposition 3. Suppose that $f_s$ and $g$ are uniform. Then there exists a threshold $w' \in [l, w^*)$ and a constant $c$ such that there is a sender-optimal signal structure $\pi$ with two realizations satisfying $\{w \in [l, w^*): \pi(\sigma|w) > 0\} = [w', w^*)$ and $\pi(\sigma|w) = \min \left\{ \frac{c}{w^*-w}, 1 \right\}$ for all $w \in [w', w^*)$ with

$$-c^2 + 2c \left( \frac{a}{1-a} (h-l) + (w^*-w') \right) - (h-w^*)^2 = 0$$

and either

$$\ln(w^*-w') - \ln(c) = 1 + \frac{a(h-l) - c}{w^*-w'}$$

or $w' = l$ and $\ln(w^*-w') - \ln(c) < 1 + \frac{a(h-l)-c}{w^*-w'}$ for all $w \in [l, w^*)$.

If $f_s$ and $g$ are uniform, then $w \mapsto \frac{f_s(w)}{g(w)(w^*-w)}$ is increasing, which implies that the support of the optimal signal is connected and is an interval from some state $w' \geq l$ to the highest state $h$. The threshold $w'$ and the constant $c$ are determined by the two equations in Proposition 3.

Figure 5 provides a numerical example for the case in which both $f_s$ and $g$ are uniform, showing how the optimal signal depends on the sender’s ignorance $a$. The graph on the left plots the constant of proportionality $c$ and the threshold $w'$. We see that the signal realization recommending the high action has full support for $a$ greater than 0.75. The graph on the right plots the payoff of the receiver with prior $\mu_F = 0.9\mu_G + 0.1\delta_h$ as a function of the sender’s ignorance. The graph shows that in this example the payoff of the receiver is strictly increasing in the sender’s ignorance.
3.5.3 The Optimal Signal when the Sender’s Knowledge is Detail-Free

I next consider a special case of detail-free knowledge: I suppose that the receiver’s set of priors is \( C(\alpha, \beta) \). That is, I suppose that the sender only knows that there exists a state \( \alpha \) such that the mass that the receiver’s prior puts on states above \( \alpha \) is at least \( \beta \).

**Corollary 2.1.** Suppose that the set of the receiver’s priors is \( C(\alpha, \beta) \). Then there exists a sender-optimal signal structure that induces a distribution over the receiver’s actions under which the probability of action 1 in state \( \omega \) is given by

\[
s(\omega) = \begin{cases} 
1 & \text{if } \omega \in [w^*, h] \\
\min \left\{ \frac{\beta \omega}{(1-\beta)(w^*-\omega)}, 1 \right\} & \text{if } \omega \in [l, w^*)
\end{cases}
\]

The optimal signal described in Corollary 2.1 recommends action 1 with probability one if the state is above a certain threshold. If the state is below this threshold, the sender recommends action 1 with a strictly positive probability that follows a hyperbola. Figure 6 illustrates this information policy.
3.6 Comparison with the Standard Model

In this section I introduce the standard model of Bayesian persuasion and compare my results in the case where the sender’s knowledge is detail-free to the optimal signal in the standard model. The standard model has the same payoff structure as the model with unknown beliefs but the sender and the receiver have a common prior $F_s \in C(\alpha, \beta)$ over the states. I assume that the receiver takes action 0 if no information is provided, that is, that $E_{F_s}[\omega] < w^*$.

It is known that in this case the solution of the standard persuasion model has a threshold structure: there is an optimal signal $p$ with two signal realizations, $\sigma$ and $\sigma_0$, such that $p(\sigma|w) = 1$ for $w \in [w', h]$, $p(\sigma|w) = 0$ for $w \in [l, w')$ for some action threshold $w' \in [l, w^*)$ and the receiver takes action 1 if and only if the realized signal is $\sigma$.

Recall that the equilibrium probability of action 1 in the model with unknown beliefs when the receiver’s set of priors is $C(\alpha, \beta)$ is given by $s(w) = \min \left\{ \frac{\beta(\alpha-w^*)}{(1-\beta)(w^*-w)}, 1 \right\}$ on states $w$ such that $w < w^*$ and is given by $s(w) = 1$ on states $w$ such that $w \geq w^*$. Thus the function $w \mapsto s(w)$ is strictly increasing on the states $w$ such that $\frac{\beta(\alpha-w^*)}{(1-\beta)(w^*-w)} \leq 1$ and is constant on the states $w$ such that $\frac{\beta(\alpha-w^*)}{(1-\beta)(w^*-w)} > 1$.

Let $\alpha$ denote the state such that $w \mapsto s(w)$ is strictly increasing on the states below $\alpha$ and is constant on states above $\alpha$. That is, define

$$\alpha = w^* - \frac{\beta}{1-\beta}(\alpha - w^*)$$

The optimal signal in the model with unknown beliefs when the sender’s knowledge is detail-free recommends action 0 with a strictly positive probability on states below $\alpha$ and recommends action 0 with probability zero on states above $\alpha$.

**Proposition 4.** $\alpha > w'$.

Proposition 3 shows that we must have $\alpha > w'$, that is, that the state at which the robust signal starts recommending action 1 with probability one is higher than the state at which the signal in the standard persuasion model starts recommending action 1 with probability one. Figure 7 illustrates Proposition 4.

This comparison highlights why the signal that is optimal in the standard model is not optimal in the model with unknown beliefs. Because $\alpha > w'$, the interval $[w', \alpha]$ is non-empty. The signal optimal in the standard model recommends action 1 with probability
one on the interval \([w', \alpha]\). On the other hand, the signal optimal in the model with unknown beliefs when the receiver’s set of priors is \(C(\alpha, \beta)\) recommends action 1 with probability that is strictly less than one on this interval. If instead action 1 was recommended with a greater probability on the interval, then Nature would be able to choose the receiver’s prior ensuring that the receiver’s expectation given the signal realization \(\sigma\) is strictly less than \(w^*\), causing the receiver to never take action 1.

### 3.7 A Simpler Model: Convincing Groups with Diverse Beliefs

The model in the present paper has Nature choose the receiver’s prior after the sender commits to a signal structure but before the signal is realized. We could imagine a simpler model, where Nature chooses the receiver’s prior after the signal is realized. This model is simpler because if there is a feasible prior such that the receiver with this prior takes action 0 after observing a signal, Nature can choose this prior after this signal. Thus if action 1 is taken after a signal realization in equilibrium, it must be that a receiver with any feasible prior takes action 1 after seeing the signal realization. This implies that there is no loss in pooling the signals after which action 1 gets taken into one signal. Therefore, there is an optimal signal with only two signal realizations in the simpler model.

An application of this simpler model is persuasion of a large group with diverse beliefs. The members of the group hold diverse prior beliefs. It is known that all members of the group have beliefs that put a mass of at least \((1 - a)g(w)\) on each state \(w\). Every such belief is represented in the group, so that for every prior that puts a mass of at least \((1 - a)g(w)\)
on each state $w$, we can find a group member which holds this belief.\footnote{The following is an example of a situation to which the above logic may apply. In 1995, the Clinton administration introduced the Project Excellence in Leadership (XL) that provided opportunities for firms to engage in experimentation with the goal of developing technological innovations leading to greater environmental benefits. The firms were to reach an agreement on their experimental projects with the Environmental Protection Agency (EPA) and stakeholders. The firms’ concern about litigation by environmental protection groups was a salient feature of the implementation of Project XL: the firms were conscious of the need to design their experiments in such a way as to prevent costly litigation (Fiorino 2006: 141). This strategic situation can be described as follows. A company wishes to convince environmental NGOs that the technology it plans to use is sufficiently environmentally friendly. The company can design an experiment that will reveal the characteristics of the technology with some probability. The NGOs hold diverse prior beliefs about environmental friendliness of the technology. If an NGO believes that the technology is sufficiently harmful, the NGO will challenge the company’s use of the technology in court. In order to prevent litigation, the company needs to convince all the NGOs that the technology is sufficiently safe.}

I observe that the optimal signal in the model in the present paper is the same as in the simpler model described above. This follows from the fact that a signal that is optimal in the model of persuasion with unknown beliefs has two realizations, $\sigma$ and $\sigma_0$, and a receiver with any feasible prior takes action 1 if and only if he sees the signal realization $\sigma$. Thus Nature cannot do any better when choosing the receiver’s prior after, as compared to before, the signal is realized – because no matter which prior Nature chooses, the receiver takes the same action after the signal realization, making Nature indifferent among all feasible priors.

As the counterexamples in section 5 show, the equivalence need not hold in general. In particular, the equivalence may fail for different sets of priors and different utility functions.

4 Comparative Statics, Welfare and Convergence to Known Prior Beliefs

Given a signal structure $\pi$ with two realizations, I let $\Omega_1^\pi = \{w \in [l, w^*) : \pi(\sigma|w) > 0\}$ denote the set of states below the threshold $w^*$ such that the signal realization $\sigma$ recommending the high action has a strictly positive probability given those states.

Proposition 5 shows that if we take two sets of the receiver’s priors, one putting a density $(1 - a)g$ on the states and one putting a density $(1 - a')g$ for some $0 < a' < a < 1$ on the states, then, provided that the sender-optimal signal for the second set of priors does not have full support, the support of the sender-optimal signal for the second set contains the support of the sender-optimal signal for the first set and the probability with which the high action is induced on the support of the signal is higher for the first set. Figure 8 illustrates
the comparative statics described in Proposition 5.

**Proposition 5.** Suppose that Assumption 1 is satisfied. Let $\pi_a$ and $\pi_{a'}$ denote the sender-optimal signal structures with two realizations given that the receiver’s sets of priors are $C_{a,g}$ and $C_{a',g}$ respectively. Let $t_a \geq t$ denote the threshold such that $\Omega_{1a} = \Omega_1(t_a) \mu_{F_s}$-a.e. If $0 < a' < a < 1$ and $\mu(\Omega_{1a}) < \mu([l, w^*])$, then

$$\Omega_{1a'} \subset \Omega_{1a} \mu_{F_s}$-a.e.$ and $c(t_{a'}) > c(t_a)$

Observe that increasing $a$ reduces the sender’s ignorance about the receiver’s prior. We can, therefore, interpret Proposition 5 as saying that increasing the sender’s ignorance has two effects: the support effect and the probability effect. The support effect is that the range of states on which the sender’s signal induces the receiver to take action 1 with a positive probability expands, and the probability effect is that the probability with which the receiver takes action 1 on these states decreases. Thus greater ignorance about the receiver’s prior makes the sender “hedge her bets” and spread out the probability with which the high action is recommended on the states below the threshold.
Next, consider two sets of priors, $C_{a,g}$ and $C_{a',g}$, for $0 < a' < a < 1$, so that the sender is more uncertain under $C_{a,g}$ than under $C_{a',g}$. Proposition 6 shows how the receiver’s equilibrium payoff changes as the sender becomes more ignorant.

**Proposition 6.** Suppose that the receiver’s prior is $F \in C_{a,g}, C_{a',g}$ for $0 < a' < a < 1$. Let $V(a)$ denote the equilibrium payoff of the receiver when the set of priors is $C_{a,g}$. Let $\pi_a$ and $\pi_{a'}$ denote the sender-optimal signal structures with two realizations given that the receiver’s sets of priors are $C_{a,g}$ and $C_{a',g}$ respectively. Then

1. If $V(a') = 0$, then $V(a) > 0$.

2. Suppose that $\Omega_1^{\pi_a} = \Omega_1^{\pi_{a'}} = [l, w^*)$. Then $V(a) \geq V(a')$. Moreover, if $\mu_F([l, w^*)) = 0$, then $V(a) = V(a')$, and if $g$ has full support, then $V(a) > V(a')$.

3. There exist parameters such that $0 < V(a) < V(a')$.

Proposition 6 shows that if we take the true receiver’s prior that is contained in both sets of priors, $C_{a,g}$ and $C_{a',g}$, and if the receiver’s value of information is zero when the sender is less uncertain, the receiver’s value of information must become strictly positive as the sender becomes more uncertain, which implies that in this case greater sender’s ignorance benefits the receiver. Because the receiver’s value of information is zero in the standard Bayesian persuasion model where the receiver’s prior is common knowledge, this implies that, in particular, the receiver strictly prefers to face an ignorant sender rather than the sender that knows the receiver’s prior.

Proposition 6 next considers the case in which the sender-optimal signal recommends the high action with a strictly positive probability in every state. The Proposition shows that in this case an increase in the sender’s ignorance weakly increases the receiver’s payoff. Moreover, if the receiver considers the states below the threshold $w^*$ impossible, then the receiver’s payoff is constant, while if the receiver considers all states below the threshold possible, then the increase in the receiver’s payoff is strict.

The intuition for these results is as follows. An increase in the sender’s ignorance has two effects: it expands the support of the signal and lowers the probability with which the high action is recommended on the support. If the signal already has full support, then only the second effect is present: senders that are more ignorant recommend the high action with a lower probability below the threshold. Because the receiver does not want to take the action anywhere below the threshold, this is unambiguously good for the receiver. In general, as the next part of Proposition 6 shows, either effect can dominate and the impact of greater sender’s ignorance on the receiver’s payoff is ambiguous.
The last part of Proposition 6 shows that the receiver’s equilibrium payoff can be strictly lower when the sender is more uncertain. Because ignorance always hurts the sender, an implication of this is that an increase in sender’s ignorance can hurt both the sender and the receiver.

The reason that the sender’s ignorance can hurt the receiver is that the sender designs the signal structure to make the worst types in the set of priors just indifferent between taking the action and not. If a receiver with some prior strictly prefers to act upon seeing the signal recommending the action, so that the receiver’s expectation given the signal is strictly above the threshold of action, the sender’s payoff is not affected by the distance between the expectation and the threshold. Because of this, the impact of the change in the signal structure due to the greater ignorance of the sender on the agents with priors that are not the worst in the set of priors that the sender considers possible can be ambiguous.

To better understand the intuition for the possible non-monotonicity of the receiver’s payoff, recall the two comparative statics effects that increasing the sender’s ignorance has, the support effect and the probability effect. The support effect is bad for the receiver because expanding support means that more bad drugs are approved. The probability effect is good for the receiver because a decreasing probability of approval on the support means that bad drugs are approved with a lower probability. My analysis shows that the support effect can dominate, causing greater ignorance of the sender to reduce the receiver’s payoff.

One reason why the support effect can dominate is that the priors of the sender and the receiver may differ, and the sender makes the decision as to which states to add to the support based in part on her own prior $F_s$. Thus if, for example, the sender’s prior puts a sufficiently high weight on state $w$, the sender will add this state to the support no matter how much she has to reduce the probability with which approval is recommended in other states. If adding $w$ to the support reduces the probability of approval in other states a lot, then the probability effect dominates and the receiver’s payoff increases, while if it reduces the probability of approval only a little bit, then the support effect dominates and the receiver’s payoff decreases.

The final observation is that complete ignorance of the sender is the best possible circumstance for the receiver. This is because as the sender’s ignorance $a$ converges to 1, the sender becomes extremely cautious and recommends that the receiver takes the high actions only on the states where the sender and the receiver agree. Thus as a pharmaceutical company becomes very ignorant about the FDA’s prior, it recommends that the FDA approves a drug if and only if the FDA would approve this drug under complete information,
which is the first-best outcome for the FDA.

The result has important implications for the design of the optimal transparency requirements for the FDA. The FDA can change the level of knowledge that pharmaceutical companies have about its prior belief by changing the rules it uses to assign to the review teams for new drug applications. If the companies know the likely composition of the review teams for their drugs, they would have a reasonably good idea of the prior beliefs of the agents making the decision, while if the composition of the review teams is hard to predict, there is significant uncertainty about the priors. Interpreting greater transparency requirements as a reduction in the uncertainty of the pharmaceutical companies about the FDA’s beliefs in this manner, the result implies that full transparency is never optimal. The result also implies that in a setting where the FDA can enforce transparency requirements within some bounds, the optimal level of transparency may be interior.

Figure 9 provides a numerical example for the case in which $f_s$ is uniform, and $g$ is uniform on the intervals $[l, w_0]$ and $[w_0, h]$, with $g$ having a higher density $H$ on the interval $[l, w_0]$. The graph on the left plots the constant of proportionality $c$ and the threshold $w'$. We see that the threshold $w'$ is strictly decreasing between 0.1 and 0.12, constant and equal to $w_0$ for $a$ from 0.12 to 0.7, strictly decreasing from 0.7 to $a = 0.9$, and is constant and equal to $l$ for $a$ greater than 0.9. The reason there is an interval on which the threshold is constant and strictly greater than $l$ is that $g$ in this example is discontinuous. The graph on the right plots the payoff of the receiver with prior $\mu_F = 0.9\mu_G + 0.1\delta_h$ as a function of the sender’s ignorance. The graph shows that the payoff of the receiver is non-monotone in the sender’s ignorance: the payoff is increasing until $a = 0.7$, then decreasing until $a = 0.9$, then increasing again. The reason the payoff starts decreasing at $a = 0.75$ is that the sender starts adding to the support states with the high density $g(w) = H$ which are relatively cheap for the sender because Nature has less freedom in choosing the prior on them. The payoff starts increasing again at $a = 0.9$ because this is the point at which the signal attains full support, and further increases in the sender’s ignorance only cause the sender to recommend the high action with a lower probability.

Proposition 7 characterizes the sender-optimal signal structure when the receiver has a commonly known prior with a density $g$ (and the sender has a commonly known prior with a density $f_s$).

**Proposition 7.** Suppose that $a = 0$. Then the optimal signal is given by $\pi(\sigma|w) = 1$ for $w \in \Omega_1^f \cup [w^*, h]$, $\pi(\sigma|w) = 0$ for $w \in [l, w^*) \setminus \Omega_1^f$, where $\Omega_1^f = \Omega_1(t)$ for some $t \geq 0$ such that $E_{G,\pi}[w|\sigma] = w^*$.
Suppose that Assumption 7 is satisfied. Then, as $a \to 0$, the signal that is optimal under an unknown prior converges to $\pi^{15}$.

Proposition 7 shows that when the receiver’s prior is known the sender-optimal signal structure is partitional but not necessarily monotone. The sender-optimal signal structure is monotone (and so threshold) in the special case where the sender’s prior $F_s$ and the receiver’s prior $G$ coincide.

Proposition 7 also shows that, as we decrease to zero the mass $a$ that Nature can move freely on the state space, the sender-optimal signal in the robust model converges to the partitional (but not necessarily monotone) signal in the model with known heterogeneous priors.

5 Counterexamples

In this section I provide counterexamples, showing that there may not exist an optimal signal structure with only two realizations if the assumptions on the sender’s utility function and on the set of the receiver’s priors made in the present paper fail.

I first show that there exist utility functions of the sender such that any sender-optimal signal structure must have more than two realizations. Suppose that the state space $\Omega = $
{0, 1} is binary, so that the receiver’s prior \( f \) is the probability that the state is 1. Suppose also that the sender’s utility \( u(E) \) as a function of the receiver’s posterior belief \( E \) is \( u(E) = 1 \) for \( E = w^* \) for some \( w^* \in (0, 1) \) and \( u(E) = 0 \) for \( E \in [0, 1] \setminus w^* \). The set of the receiver’s priors is a finite set \( \{f_1, \ldots, f_k\} \) satisfying \( 0 < f_1 < \ldots < f_k < 1 \).

Observe that the set of the signal realizations of a sender-optimal signal structure in this setting must have the cardinality no smaller than the cardinality of the set of priors. In particular, letting \( \sigma_i \) denote a signal realization such that, upon observing this signal realization, a receiver with prior \( f_i \) has the posterior belief \( w^* \), we find that the set of signal realizations must contain the set \( \{\sigma_i\}_{i \in \{1, \ldots, k\}} \). To see why, consider a signal structure that does not contain \( \sigma_i \) for some \( i \in \{1, \ldots, k\} \). Then if Nature chooses the receiver’s prior \( f_i \), the receiver’s posterior is not equal to \( w^* \) after any signal realization, implying that the sender obtains a payoff of zero. However, the sender can do strictly better by choosing a signal structure that sends the signal realization \( \sigma_i \) with a strictly positive probability.

Next, I provide an example showing that in a setting with multiple states where the utility functions are the same as in the model in the present paper, if the receiver’s set of priors does not satisfy the assumptions in the present paper, then there may not exist an optimal signal with only two realizations.

Suppose that there are five states, 0, 1, 2, 3 and 4. The receiver takes the high action if and only if the receiver’s expectation given the signal realization is weakly greater than \( w^* = 2.5 \). Two priors of the receiver are possible: \( f_1 = (\frac{1}{4}, \frac{1}{24}, \frac{1}{2}, \frac{5}{24}, 0) \) and \( f_2 = (\frac{1}{4}, \frac{1}{2}, \frac{3}{16}, 0, \frac{1}{16}) \). The sender’s prior \( f_s = f_2 \) coincides with the second possible prior of the receiver.

Consider a signal structure \( \pi \) with three realizations, \( \sigma_1, \sigma_2 \) and \( \sigma_0 \), such that \( \pi(\sigma_2|1) = 1 \), \( \pi(\sigma_2|3) = 1 \) and \( \pi(\sigma_1|2) = 1 \), \( \pi(\sigma_1|4) = 1 \). Note that \( E_{F_1, \pi}[\omega|\sigma_2] \geq w^* \), which implies that the sender’s payoff given \( \pi \) and the receiver’s prior \( f_1 \) is greater than \( \pi(\sigma_2|1)f_s(1) + \pi(\sigma_2|3)f_s(3) = f_s(1) + f_s(3) = f_2(1) + f_2(3) = \frac{1}{2} \). On the other hand, \( E_{F_2, \pi}[\omega|\sigma_2] < w^* \) and \( E_{F_2, \pi}[\omega|\sigma_1] \geq w^* \), which implies that the sender’s payoff given \( \pi \) and \( f_s \) is \( \pi(\sigma_1|2)f_s(2) + \pi(\sigma_1|4)f_s(4) = f_s(2) + f_s(4) = f_2(2) + f_2(4) = \frac{1}{4} \).

Then, faced with the signal structure \( \pi \), Nature strictly prefers to choose the second prior of the receiver \( f_2 \), which yields a payoff of \( \frac{1}{4} \) to the sender. Lemma 15 in the Appendix proves that there do not exist signal structures with two realizations that guarantee a payoff of \( \frac{1}{4} \) to the sender. This shows that in this example there does not exist a sender-optimal signal structure with two realizations. Importantly, the set of the receiver’s priors in the example is non-convex, whereas the conditions imposed in the present paper ensure the
6 Sketch of the Proof

In this section I sketch the proof of Theorems 1 and 2. The proof has two parts. The first part establishes that there exists a sender-optimal signal structure with only two realizations. The second part characterizes this signal structure. I will provide a sketch of the proof of the first part.

I show that for all signal structures there exists a feasible receiver’s prior $F$ such that the sum of the probabilities of signal realizations recommending the high action (provided that the receiver’s prior is $F$) given each state $\omega$ below the threshold $w^*$ is bounded above by an expression depending on $\omega$. The proof of the lemma uses a fixed-point argument outlined in greater detail below. I then use the above result to show that there exists a signal structure with two realizations that gives to the sender a payoff equal to the payoff from any sender-optimal signal structure with more than two realizations.

6.1 Example: Closed-Form Solution for Receiver’s Prior

I first present a simple example which explains how Nature can choose the receiver’s prior to keep the sender’s payoff below the bound state-by-state.

Suppose that the set of the receiver’s priors is $C(\alpha, \beta)$ and that the sender chose a signal structure $\pi$ with three realizations $\sigma_1$, $\sigma_2$ and $\sigma_0$. Suppose further that the signal realizations $\sigma_1$ and $\sigma_2$ have support on exactly one state below the threshold. That is, suppose that $\pi(\sigma_1|w_1) > 0$, $\pi(\sigma_1|w) = \gamma > 0$ for all $w \in [\alpha, h]$, $\pi(\sigma_1|w) = 0$ for all $w \in [l, \alpha) \setminus w_1$ and $\pi(\sigma_2|w_2) > 0$ for all $w \in [\alpha, h]$, $\pi(\sigma_2|w) = 1 - \gamma$, $\pi(\sigma_2|w) = 0$ for all $w \in [l, \alpha) \setminus w_2$.

We would like to find a feasible prior of the receiver that Nature can choose to keep the probability of the high action in each state $w$ lower than $w^*$ below the bound $\min \left\{ \frac{\beta}{1-\beta}, \frac{\alpha - w^*}{w^*}, 1 \right\}$.

Consider a prior given by $f(w_1) = \gamma(1-\beta)$, $f(w_2) = (1-\gamma)(1-\beta)$, $f(\alpha) = \beta$. We will show that this prior meets the above requirement. This prior is feasible because it puts a mass of exactly $\beta$ on $\alpha$ and a mass of $1 - \beta$ on states below $w^*$.

\[ \text{Convexity of the set of priors.}^{16} \]

---

\[ ^{16} \text{I conjecture, but have not been able to prove, that if the set of the receiver’s prior beliefs is convex, then there exists an optimal signal structure with only two realizations.} \]
Note that the probability of the high action in state $w_i$ is $\pi(\sigma_i|w_i)$ if $E_{F,\pi}[\omega|\sigma_i] \geq w^*$ and is zero otherwise. Thus we want to show that the prior $F$ is such that whenever $E_{F,\pi}[\omega|\sigma_i] \geq w^*$, we have $\pi(\sigma_i|w_i) \leq \min \left\{ \frac{\beta}{1-\beta} \frac{\alpha-w^*}{w^*-w_i}, 1 \right\}$ for $i = 1, 2$.

Observe next that $\pi(\sigma|w) \leq \min \left\{ \frac{\beta}{1-\beta} \frac{\alpha-w^*}{w^*-w}, 1 \right\}$ is equivalent to $E_{F,\pi}[\omega|\sigma] \geq w^*$ if $\pi(\sigma|\alpha) = 1$ and $f(w) = 1 - \beta$, that is, if the signal realization $\sigma$ is sent with probability one above the threshold and Nature puts the entire mass $1 - \beta$ that it can move freely below the threshold on the state $w$. Then it is sufficient to show that the prior $F$ keeps the conditional expectations given $\sigma_1$ and $\sigma_2$ equal to the conditional expectation under the signal structure with only one signal realization $\sigma$ recommending the high action provided that Nature chooses the receiver’s prior with two-point support on states $w$ and $\alpha$. It can be easily verified that this holds because the prior $f$ satisfies

$$E_{F,\pi}[\omega|\sigma_1] = \frac{\pi(\sigma_1|w_1)w_1f(w_1) + \gamma \alpha \beta}{\pi(\sigma_1|w_1)f(w_1) + \gamma \beta} = \frac{\pi(\sigma_1|w_1)w_1(1 - \beta) + \alpha \beta}{\pi(\sigma_1|w_1) (1 - \beta) + \beta}$$

$$E_{F,\pi}[\omega|\sigma_2] = \frac{\pi(\sigma_2|w_2)w_2f(w_2) + (1 - \gamma) \alpha \beta}{\pi(\sigma_2|w_2)f(w_2) + (1 - \gamma) \beta} = \frac{\pi(\sigma_2|w_2)w_2(1 - \beta) + \alpha \beta}{\pi(\sigma_2|w_2) (1 - \beta) + \beta}$$

Note that Nature faces a tradeoff between decreasing $E_{F,\pi}[\omega|\sigma_1]$ and $E_{F,\pi}[\omega|\sigma_2]$ in choosing the receiver’s prior: decreasing one conditional expectation must necessarily come at the expense of increasing the other one. The prior $f$ resolves this tradeoff and keeps the conditional expectations equal to each other and to the abovementioned conditional expectation under the signal structure with two realizations.

Thus under the simple set of priors $C(\alpha, \beta)$, if the sender chooses a signal structure such that the signal realizations recommending the high action have disjoint support on exactly one state below the threshold, then there is a closed-form solution for the feasible prior of the receiver that Nature can choose to keep the sender’s payoffs below the bound: this prior is given by $f(w_i) = \pi(\sigma_i|\alpha)(1 - \beta)$. That is, on each state $w_i$ below the threshold the prior puts a mass proportional to the probability with which the signal realization supported on $w_i$ is sent in state $\alpha$. In general, however, if the sender chooses more complicated signal structures, there may not be a closed-form solution for the feasible prior that Nature can choose.

Observe that this example uses the assumption that the set of priors is $C(\alpha, \beta)$: in particular, it is important that the set of priors contains convex combinations of the priors putting mass $1 - \beta$ on exactly one state below $w^*$. In general, it is important that the set of
priors Nature can choose from is rich enough so that Nature can tailor the prior it chooses to the features of the possibly complicated signal structure designed by the sender.

6.2 Receiver’s Prior as a Fixed Point

I next show how to construct the receiver’s prior in the general case. For each signal realization $\sigma_i$ recommending the high action, I define the influence of the signal $\sigma_i$ as

$$
\epsilon_i = \frac{\int_{[w^*, h]} (w - w^*) \pi(\sigma_i | w) g(w) dw - \int_{[l, w^*]} (w^* - w) \pi(\sigma_i | w) g(w) dw}{\int_{[w^*, h]} (w - w^*) g(w) dw - \int_{[l, w^*]} (w^* - w) \left( \sum_j \pi(\sigma_j | w) \right) g(w) dw}
$$

where in the denominator we sum over the signal realizations $\sigma_j$ recommending the high action.

Note that $\epsilon_i$ is a ratio of the expected payoff of the receiver from taking the high action given the signal $\sigma_i$ to the expected payoff from action 1 given that a signal recommends action 1, where the expectation is taken with respect to the measure $\mu_G$. That is, $\epsilon_i = \frac{E_{G}[|w - w^*| | \sigma_i]}{E_{G}[|w - w^*| | \sigma \neq \sigma_0]}$. The higher this ratio is, the larger the expected payoff of the receiver from action 1 after signal $\sigma_i$, as compared to other signals recommending action 1. We can thus view $\epsilon_i$ as the influence of the signal $\sigma_i$. Observe that the influence of $\sigma_i$ is defined independently of the prior that Nature chooses.

I suppose that Nature chooses the receiver’s prior $F$ that places a mass of $(1 - a) \mu_G$ on states $[l, h]$ and, in addition, for all signal realizations $\sigma_i$ recommending the high action, places the mass of

$$
\nu_\omega^i \epsilon_i a
$$

on each state $\omega$ in a certain finite collection of states below the threshold.\(^{18}\)

The weights $\nu$ satisfy $\nu_\omega^i \in [0, 1]$, $\sum_\omega \nu_\omega^i = 1$ and are defined as the fixed point of the mapping $H$:

$$
H_{iw}(\nu) = \frac{\pi(\sigma_i | w)(w^* - w) \left[ \sum_k \nu_\omega^k \epsilon_k \right]}{\sum_\omega \pi(\sigma_i | \omega)(w^* - \omega) \left[ \sum_k \nu_\omega^k \epsilon_k \right]}
$$

\(^{17}\)A signal realization recommends the high action if it has a strictly positive probability under the signal structure $\pi$ given some state weakly above the threshold $w^*$.

\(^{18}\)It is enough to consider a finite collection of states because I restrict my attention to signal structures with a finite number of realizations conditional on each state.
Note that the expected payoff of the receiver from action 1 can be written as a sum of terms depending on the states. The right hand side of the above equation is a ratio of the term associated with state $w$ to the sum of the terms over all states below $w^*$. That is, $\nu^i_w = \frac{|w^*-w|\pi(\sigma_i|w)P_{w^*|w}}{E_{F}[|w^*-w|\pi, w<w^*]}$. The higher this ratio is, the more the term associated with state $w$ contributes to the expectation given the signal $\sigma_i$. We can thus view the weight $\nu^i_w$ as the influence of the state $w$ for the signal $\sigma_i$. Observe that the influence of $w$ given $\sigma_i$ depends on the prior $F$ that Nature chooses.

Recall that Nature chooses a prior $F$ that places a mass of $\nu^i_\omega \epsilon_i a$ on state $\omega$. Because $\nu^i_\omega$ is the influence of $w$ given $\sigma_i$ and $\epsilon_i$ is the influence of $\sigma_i$, if we sum $\nu^i_\omega \epsilon_i$ over all signals, we would obtain the influence of the state $w$. Thus the greater the influence of $w$ is, the larger the mass that Nature has to put on $w$ to counteract its influence.

Figure 10 illustrates the construction of the prior. The black intervals denote the supports of different signal realizations. $\omega_1$, $\omega_2$ and $\omega_3$ are the states below the threshold such that there is a signal realization recommending the high action that has a strictly positive probability given at least one of these states. The red selection in figure 10 shows the set of all signal indices $i$ such that $\pi(\sigma_i|\omega_1) > 0$. The blue selection in figure 10 shows the set of all state indices $j$ such that $\pi(\sigma_3|\omega_j) > 0$. $\nu^i_j$ is the weight corresponding to the state $\omega_j$ and signal $\sigma_i$. Figure 10 shows the weights $\nu^3_1$, $\nu^3_2$ and $\nu^3_3$ corresponding to the signal $\sigma_3$ and states $\omega_1$, $\omega_2$ and $\omega_3$ respectively.
7 Related Literature

The present paper is related to two strands of literature: the literature on Bayesian persuasion and the literature on robust mechanism design. Early papers on Bayesian persuasion include Brocas and Carillo (2007), Ostrovsky and Schwarz (2010) and Rayo and Segal (2010). Kamenica and Gentzkow (2011) introduce a general model of Bayesian persuasion and provide a characterization of the sender’s value in this model. Alonso and Camara (2016) consider a model of Bayesian persuasion where the sender and the receiver have heterogeneous priors. Alonso and Camara (2016b) and Chan et al. (2019) study the persuasion of voters.

Kolotilin (2017) and Kolotilin et al. (2017), among others, consider models of Bayesian persuasion with an infinite state space. Kolotilin et al. (2017) prove an equivalence result concerning public and private persuasion of a privately informed receiver. Guo and Shmaya (2019) also consider persuasion of a privately informed receiver and show that the optimal mechanism takes the form of nested intervals. Perez-Richet (2014) and Hedlund (2017) consider a Bayesian persuasion model with a privately informed sender. Ely (2017), among others, studies a dynamic Bayesian persuasion model.

One strand of the Bayesian persuasion literature studies models where the distribution of the state is endogenous: after observing the signal structure chosen by the sender, the agent can take an action affecting the distribution of the state. Thus Rosar (2017) studies the design of tests when privately informed agents can opt out of taking a test. Boleslavsky and Kim (2018) and Rodina (2017) consider models where the prior distribution of the state depends on the agent’s effort. Perez-Richet and Skreta (2018) characterize information structures optimal for the receiver when the persuader can manipulate the state. These papers are similar to the present paper in that there is an agent that can affect the distribution of the state, which changes the prior belief of the receiver (in my model, this agent is Nature). The papers differ in the objective that the agent has, the way in which the agent can change the prior, as well as the characterization of the optimal signal.

Several papers consider models related to ambiguity in Bayesian persuasion. Laclau and Renou (2017) consider a model of publicly persuading receivers with heterogeneous priors under the unanimity rule. Their model is equivalent to a model of persuading a single receiver who has multiple priors where, after the sender commits to a signal structure and after a signal realization from this signal structure is realized, Nature chooses the receiver’s prior to minimize the sender’s payoff. In contrast, in the present paper Nature chooses the
receiver’s prior after the sender commits to a signal structure but before a signal realization from this signal structure is realized. The difference has several important implications. First, the model of Laclau and Renou (2017) has a concave closure characterization, which means that standard methods can be applied to solve it. In contrast, the model in the present paper, in general, does not have a concave closure characterization. In order to solve my model, I develop a novel fixed-point argument. Second, the model in the present paper has the interpretation of the sender not knowing the receiver’s beliefs and designing a signal structure that is robust to this lack of knowledge, while the model of Laclau and Renou (2017) does not have this interpretation.

A paper by Hu and Weng (2018) features a sender persuading a receiver who will receive private information unknown by the sender. The sender is ambiguity averse with regard to this private information and believes that Nature chooses the distribution of the private information to minimize the sender’s payoff. The differences from the present paper are as follows. First, in the paper by Hu and Weng (2018) Nature chooses a distribution of the receiver’s private information (that is, a distribution over the receiver’s posteriors subject to the constraint that the expectation is equal to the common prior), whereas in the present paper Nature chooses the prior of the receiver. This leads to significant differences in the results: because in Hu and Weng (2018) Nature engages in information design to maximize the negative of the sender’s payoff, unlike in the present paper, the solution to the Nature’s problem is a concave closure of the negative of the sender’s induced utility function (which is equal to the convex closure of the sender’s induced utility function). Second, Hu and Weng (2018) solve for the optimal signal structure in the example with two states (and two actions), whereas I am able to characterize the optimal signal structure in a setting with a continuum of states. Moreover, the nature of the optimal signal structure in the present paper differs markedly from that in the example in Hu and Weng (2018). Finally, the proof techniques differ: the novel fixed-point technique is unique to the present paper.

Beauchêne et al. (2019) consider a model in which the sender and the receiver share a common prior, but the sender can commit to ambiguous information structures, and both the sender and the receiver are ambiguity averse. They show that the sender can benefit from committing to ambiguous signals. The model of Beauchêne et al. (2019) cannot be interpreted as one where the sender does not know the receiver’s beliefs.

The literature on robust mechanism design studies the design of optimal mechanisms in the environments where the designer does not know the distribution of agents’ types and designs a mechanism to maximize his utility in the worst case scenario. Carroll (2015)
shows that linear contracts have attractive robustness properties. Carroll (2017) shows that the robust multidimensional screening mechanism sells objects separately. Carrasco et al. (2017) study robust selling mechanisms under moment conditions on the distribution of the buyer’s types. Chassang (2013) considers robust dynamic contracts. To the best of the author’s knowledge, the present paper is the first one to consider a model of robust Bayesian persuasion in which the prior belief of the receiver is unknown to the sender.

8 Conclusion

This paper analyzes a model of Bayesian persuasion with unknown beliefs and characterizes the optimal signal. The paper finds that there is an optimal signal with two realizations. The support below $w^*$ of the signal realization recommending the high action is given by all states below $w^*$ with an importance index exceeding a threshold. On the support, the probability that the signal recommends the high action is a hyperbola. More ignorant senders induce the high action in more states but with a lower probability. An increase in the sender’s ignorance can hurt both the sender and the receiver. If the sender’s knowledge is detail-free, the optimal signal recommends the high action with a strictly positive probability in every state, with the probability that the high action is recommended changing gradually as the state changes.

These distinct features of the robust signal can potentially be useful for empirically distinguishing between senders who know the prior of the receiver and those who do not have this knowledge. When the sender is ignorant, especially pernicious outcomes are possible: even the least safe drugs are approved with a positive probability. Moreover, the results have important implications for the optimal transparency requirements for the FDA. While full transparency is never optimal, maximal obfuscation may not be a good choice either: an intermediate level of transparency may be the best.
Appendix

B Notation and Definitions

B.1 Signal Structures and Priors

Define
\[ \mathcal{D} = \{ \pi : |\text{supp } \pi(\cdot|\omega)| < \infty \text{ for all } \omega \in [l, h] \} \]

\( \mathcal{D} \) is the set of signal structures that have a finite number of signal realizations conditional on each state.

Define
\[ \mathcal{F}_{a,g} = \{ F \in \varphi : \mu_F = \mu_{\tilde{F}} + (1-a)\mu_G, \text{supp } \mu_{\tilde{F}} \subseteq [l, w^*) \} \]

\( \mathcal{F}_{a,g} \) is the set of all CDFs \( F \) associated with probability measures that are a sum of some measure \( \mu_{\tilde{F}} \) on \([l, w^*)\) and the measure \((1-a)\mu_G\).

Given \( \omega \in [l, h] \), define
\[ \mu_{F_{\omega}} = (1-a)\mu_G + a\delta_{\omega} \]

\( F_{\omega} \) is the feasible prior of the receiver that puts a mass of \((1-a)\mu_G\) on states \([l, h]\) and puts a mass of \(a\) on state \(\omega\).

B.2 Sets of Signal Indices

Given a signal structure \( \pi \in \mathcal{D} \) and a state \( \omega \), define
\[ S_{\omega} = \{ k : \pi(\sigma_k|\omega) > 0 \} \]

\( S_{\omega} \) is the set of indices of signal realizations that have a strictly positive probability in state \(\omega\) under the signal structure \(\pi\).

Given a signal structure \( \pi \in \mathcal{D} \), define
\[ R(\pi) = \left\{ i : \int_{w^*}^{h} \pi(\sigma_i|w)g(w)dw > 0 \right\} \]

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$R(\pi)$ is the set of indices of signal realizations such that the support of $\pi(\sigma_i|w)$ above the threshold $w^*$ has a strictly positive measure under $\mu_G$.

Given a signal structure $\pi \in \mathcal{D}$ and a prior $F$, define

$$R(F, \pi) = \{ i \in R(\pi) : E_{F,\pi}[\omega|\sigma_i] \geq w^* \}$$

$R(F, \pi)$ is the set of signal indices in $R(\pi)$ such that the expectation given the prior $F$ and the signal realization $\sigma_i$ is greater than $w^*$.

### B.3 Parts of the Support

Given $\pi \in \mathcal{D}$, define

$$\Omega_1^\pi = \{ w \in [l, w^*) : \pi(\sigma_i|w) > 0 \text{ for some } i \in R(\pi) \}$$

$\Omega_1^\pi$ is the set of all states below $w^*$ such that there exists a signal realization with an index in $R(\pi)$ that has a strictly positive probability given the state.

Given $\pi \in \mathcal{D}$, define

$$\Omega_2^\pi = \left\{ w \in \Omega_1^\pi : \sum_{i \in R(\pi)} \pi(\sigma_i|w) = 1 \right\}$$

$\Omega_2^\pi$ is the set of all states in $\Omega_1^\pi$ such that the sum of the probabilities of signal realizations with indices in $R(\pi)$ is exactly one given the state.

Define a function $t \mapsto \Omega_1(t)$ as

$$\Omega_1(t) = \left\{ w \in [l, w^*) : \frac{f_s(w)}{g(w)(w^*-w)} \geq t \right\}$$

$\Omega_1(t)$ is the set of all states in $[l, w^*)$ such that the importance index $\frac{f_s(w)}{g(w)(w^*-w)}$ weakly exceeds the threshold $t$. 

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Define a function \((t, z) \mapsto \Omega_1(t, z)\) as
\[
\Omega(t, z) = \left\{ w \in [l, w^*] : \frac{f_s(w)}{g(w)(w^* - w)} > t \right\} \cup \left\{ w \in [l, w^*] : \frac{f_s(w)}{g(w)(w^* - w)} = t \text{ and } w \geq z \right\}
\]

Define a correspondence \(t \mapsto \overline{\Omega}_1(t)\) as
\[
\overline{\Omega}_1(t) = \left\{ w \in [l, w^*] : \frac{f_s(w)}{g(w)(w^* - w)} > t \right\} \cup \Omega_0 \in \mathcal{B} \left( \left\{ w \in [l, w^*] : \frac{f_s(w)}{g(w)(w^* - w)} = t \right\} \right)
\]

\(\overline{\Omega}_1(t)\) is the collection of all Borel subsets of \(\Omega_1(t)\) containing the set of states on which the importance index \(\frac{f_s(w)}{g(w)(w^* - w)}\) strictly exceeds the threshold \(t\).

### B.4 The Bound and Other Notation

Given a signal structure \(\pi\) with two realizations, \(\sigma\) and \(\sigma_0\), such that \(\pi(\sigma_0|w) = 0\) \(\mu_{F_s}\)-a.e. \(w \in [w^*, h]\) and a state \(w' \in [l, w^*]\), define
\[
Z_\pi(w') = \min \left\{\int_{[w^*, h]} (w - w^*) g(w) dw - \int_{\Omega_1^c} g(w) \pi(\sigma|w)(w^* - w) dw, 1\right\}
\]

Define a function \(\Omega_1 \mapsto \hat{c}(\Omega_1)\) as
\[
\hat{c}(\Omega_1) = \frac{\int_{[w^*, h]} g(w)(w - w^*) dw - \int_{\Omega_1 \cap [w^* - \hat{c}(\Omega_1), w^*]} g(w)(w^* - w) dw}{\frac{a}{1-a} + \int_{\Omega_1 \cap [l, w^* - \hat{c}(\Omega_1)]} g(w) dw}
\]

Lemma 11 shows that \(\Omega_1 \mapsto \hat{c}(\Omega_1)\) is a well-defined function.

Define a function \(t \mapsto c(t)\) as
\[
c(t) = \frac{\int_{[w^*, h]} g(w)(w - w^*) dw - \int_{\Omega_1(t) \cap [w^* - c(t), w^*]} g(w)(w^* - w) dw}{\frac{a}{1-a} + \int_{\Omega_1(t) \cap [l, w^* - c(t)]} g(w) dw}
\]

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Define a function \((c, t) \mapsto x(c, t)\) as

\[
x(c, t) = \int_{[w^*, h]} (w - w^*) g(w) dw - \int_{\Omega_1(t) \cap [w^* - c, w^*]} g(w)(w^* - w) dw - c \left( \frac{a}{1 - a} + \int_{\Omega_1(t) \cap [l, w^* - c]} g(w) dw \right)
\]

Note that, because \(c(t) = \int_{[w^*, h]} g(w)(w - w^*) dw - \int_{\Omega_1(t) \cap [w^* - c(t), w^*]} g(w)(w^* - w) dw\), \(t \mapsto c(t)\) is the mapping satisfying \(x(c, t) = 0\). Lemma 11 shows that \(t \mapsto c(t)\) is a continuous and strictly increasing function.

Define functions \(t \mapsto J(t)\) and \(t \mapsto y(t)\) as

\[
J(t) = \frac{\int_{\Omega_1(t) \cap [l, w^* - c(t)]} \frac{f_s(w)}{w^* - w} dw}{\frac{a}{1 - a} + \int_{\Omega_1(t) \cap [l, w^* - c(t)]} g(w) dw}
\]

\[
y(t) = t \left( \frac{a}{1 - a} + \int_{\Omega_1(t) \cap [l, w^* - c(t)]} g(w) dw \right) - \int_{\Omega_1(t) \cap [l, w^* - c(t)]} \frac{f_s(w)}{w^* - w} dw
\]

Note that \(y(t) = 0\) is equivalent to \(J(t) = t\).

Define a correspondence \(t \mapsto \mathcal{J}(t)\) as

\[
\mathcal{J}(t) = \frac{\int_{\Omega_1 \cap [l, w^* - \hat{c}(\Omega_1)]} \frac{f_s(w)}{w^* - w} dw}{\frac{a}{1 - a} + \int_{\Omega_1 \cap [l, w^* - \hat{c}(\Omega_1)]} g(w) dw}, \quad \Omega_1 \in \overline{\Omega_1(t)}
\]

Define a correspondence \(t \mapsto \mathcal{y}(t)\) as

\[
\mathcal{y}(t) = t \left( \frac{a}{1 - a} + \int_{\Omega_1 \cap [l, w^* - \hat{c}(\Omega_1)]} g(w) dw \right) - \int_{\Omega_1 \cap [l, w^* - \hat{c}(\Omega_1)]} \frac{f_s(w)}{w^* - w} dw, \quad \Omega_1 \in \overline{\Omega_1(t)}
\]

Note that \(t = j\) for some \(j \in \mathcal{J}(t)\) is equivalent to \(0 \in \mathcal{y}(t)\).

Define \(\hat{t} = \min_{w \in [l, w^*]} \frac{f_s(w)}{g(w)(w^* - w)}\).
B.5 Expectations

$E_{F,n}[\omega|\sigma]$ denotes the receiver’s expectation of the random variable $\omega$ with a prior distribution $F$ after the receiver observes signal realization $\sigma$ given that the signal structure is $\pi$. When no confusion can result, I suppress the dependence on $\pi$ and denote this expectation by $E_{F}[\omega|\sigma]$.

C Proofs

**Lemma 1.** Suppose that $f_s$ and $g$ are real-analytic functions on $[l, h]$, and the function $w \mapsto \frac{f_s(w)}{g(w)(w^*-w)}$ is not a constant function on $[l, w^*)$. Then Assumption 1 is satisfied.

**Proof of lemma 1.**

Because the ratio of analytic functions is analytic whenever the denominator is not zero, $w \mapsto \frac{f_s(w)}{g(w)(w^*-w)}$ is an analytic function on $[l, w^*) \setminus \{w \in [l, h] : g(w) = 0\}$.

Because $g$ is a continuous function, $\{w \in [l, h] : g(w) = 0\}$ is a closed set. Let $A = [l, w^*) \setminus \{w \in [l, h] : g(w) = 0\}$. Consider a countable collection $\{A_n\}_{n=1}^{\infty}$ of compact sets such that $A_n \subseteq A$ for all $n$ and $\bigcup_{n=1}^{\infty} A_n = A$. Observe that, for all $n$, $w \mapsto \frac{f_s(w)}{g(w)(w^*-w)}$ is a non-constant real-analytic function on $A_n$.

Recall that a non-zero real-analytic function on a compact set has only a finite number of zeros. Therefore, because $A_n$ is a compact set for all $n$ and the function $w \mapsto \frac{f_s(w)}{g(w)(w^*-w)}$ is a non-constant real-analytic function on $A_n$, $\{w \in A_n : \frac{f_s(w)}{g(w)(w^*-w)} - c_0 = 0\}$ is a finite set for all $c_0 > 0$ and for all $n$. Because $\{A_n\}_{n=1}^{\infty}$ is countable, the set $\{w \in [l, w^*) : \frac{f_s(w)}{g(w)(w^*-w)} - c_0 = 0\} = \bigcup_{n=1}^{\infty} \{w \in A_n : \frac{f_s(w)}{g(w)(w^*-w)} - c_0 = 0\}$ is countable for all $c_0 > 0$. Thus $\mu\left(\left\{w \in [l, w^*) : \frac{f_s(w)}{g(w)(w^*-w)} = c_0\right\}\right) = 0$ for all $c_0 > 0$, as required.

**Lemma 2.** $\sum_{i \in R(\pi)} \int_{[w^*, h]} (w - w^*) g(w) \pi(\sigma_i|w) dw \leq \int_{[w^*, h]} (w - w^*) g(w) dw$. Moreover, if $\sum_{i \in R(\pi)} \pi(\sigma_i|w) = 1$ for $\mu_{F_s}$-a.e. $w \in [w^*, h]$ and $f_s$ has full support, then $\sum_{i \in R(\pi)} \int_{[w^*, h]} (w - w^*) g(w) \pi(\sigma_i|w) dw = \int_{[w^*, h]} (w - w^*) g(w) dw$. 

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Proof of lemma 2. We have

\[
\sum_{i \in R(\pi)} \int_{[w^*,h]} (w - w^*) g(w) \pi(\sigma_i|w) dw = \int_{[w^*,h]} g(w) (w - w^*) \sum_{i \in R(\pi)} \pi(\sigma_i|w) dw \\
\leq \int_{[w^*,h]} g(w) (w - w^*) dw
\]

where the inequality follows from the fact that \(\sum_{i \in R(\pi)} \pi(\sigma_i|w) \leq 1\) for all \(w \in [w^*, h]\).

The claim that if \(\sum_{i \in R(\pi)} \pi(\sigma_i|w) = 1\) for \(\mu_{F_s}-a.e.\) \(w \in [w^*, h]\) and \(f_s\) has full support, then \(\sum_{i \in R(\pi)} \int_{[w^*,h]} (w - w^*) g(w) \pi(\sigma_i|w) dw = \int_{[w^*,h]} (w - w^*) g(w) dw\) is then immediate. ■

Lemma 3. Suppose that there exists a signal structure \(\pi \in \mathcal{D}\) such that

1. \(|R(\pi)| > 1\)
2. \(\sum_{i \in R(\pi)} \pi(\sigma_i|w) = 1\) \(\mu_{F_s}-a.e.\) \(w \in [w^*, h]\)
3. for any \(i \in R(\pi)\) there exists \(F \in \mathcal{C}_{a,g}\) such that \(E_{F,\pi}[w|\sigma_i] \geq w^*\) and Nature finds it optimal to choose \(F\).

Then there exists \(F \in \mathcal{F}_{a,g}\) such that for all \(w' \in [l, w^*)\) we have

\[
\sum_{i \in R(F,\pi)} \pi(\sigma_i|w') \leq \frac{\int_{[w^*,h]} (w - w^*) g(w) dw - \int_{\Omega_i^*} g(w) \left(\sum_{i \in R(F,\pi)} \pi(\sigma_i|w)\right) (w^* - w) dw}{(w^* - w') \frac{a}{1-a}}
\]

Proof of lemma 3.

Without loss of generality, suppose that for all \(i \in R(\pi)\) we have \(\pi(\sigma_i|\omega) > 0\) for some \(\omega \in [l, w^*)\).

Step 1: Partitioning the set \(\Omega_i^*\).

Given \(\pi \in \mathcal{D}, A \subseteq R(\pi)\) and \(\omega \in [l, w^*)\), define

\[
Z^{A}_\pi (\omega) = \frac{\int_{[w^*,h]} (w - w^*) g(w) dw - \int_{\Omega_i^*} g(w) \left(\sum_{i \in A} \pi(\sigma_i|w)\right) (w^* - w) dw}{(w^* - \omega) \frac{a}{1-a}}
\]
For each $\omega \in [l, w^*)$, define

$$C(\omega) = \left\{ A \subseteq R(\pi) : \sum_{i \in A} \pi(\sigma_i|\omega) > Z^A_{\pi}(\omega) \right\}$$

$C(\omega)$ is the set of all sets $A$ of indices in $R(\pi)$ such that the sum of the probabilities of signals with indices in $A$ given the state $\omega$ is strictly greater than $Z^A_{\pi}(\omega)$.

Define a finite number of sets $Y_1, \ldots, Y_J$ such that $Y_i \subseteq \Omega^\pi_1$ for all $i \in \{1, \ldots, J\}$ as follows:

1. $\omega, \omega' \in Y_i$ implies that, for all $j \in R(\pi)$, if $\pi(\sigma_j|\omega) > 0$, then $\pi(\sigma_j|\omega') > 0$;
2. $\omega, \omega' \in Y_i$ implies that $C(\omega) = C(\omega')$;
3. $\omega_i \in Y_i, \omega_j \in Y_j$ for $i \neq j$ implies that
   
   (a) either there exists a signal realization $\sigma_l$ such that $l \in R(\pi)$ and either $\pi(\sigma_l|\omega_i) > 0$, $\pi(\sigma_l|\omega_j) = 0$ or $\pi(\sigma_l|\omega_j) > 0$ and $\pi(\sigma_l|\omega_i) = 0$
   
   (b) or $C(\omega_i) \neq C(\omega_j)$

Note that $\Omega^\pi_1$ is the subset of $[l, w^*)$ such that for every state $\omega$ in $\Omega^\pi_1$ there exists a signal realization $\sigma_i$ with an index $i \in R(\pi)$ that has a strictly positive probability in state $\omega$ given the signal structure $\pi$. The sets $Y_1, \ldots, Y_J$ form the coarsest partition of $\Omega^\pi_1$ such that for each set $Y_i$ in the partition the signal realizations in $R(\pi)$ that have a strictly positive probability under $\pi$ given $\omega \in Y_i$ are the same and the sets $C(\omega)$ are the same for each $\omega \in Y_i$.

Next, we choose a finite set of states as follows. From each $Y_i, i \in \{1, \ldots, J\}$, choose exactly one $\omega_i$. Let $W$ denote the set of states $\{\omega_i\}_{i=1}^J$ chosen in this manner.

**Step 2: Some notation and reference states.**

Define

$$O_i = \{ \omega \in W : \pi(\sigma_i|\omega) > 0 \}$$

$O_i$ is the set of states in $W$ such that the probability of signal realization $\sigma_i$ given each such state is strictly positive.

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For all \( w \in W \), define
\[
b_w = w^* - w
\]

\( b_w \) is the difference between the state \( w^* \) and the state \( w \in W \). Note that, because \( w \in [l, w^*) \) for all \( w \in W \), we have \( b_w > 0 \) for all \( w \in W \).

For each \( i \in R(\pi) \), choose a reference state \( \omega_{j(i)} \in W \) such that \( \pi(\sigma_i | \omega_{j(i)}) > 0 \) (note that, because for all \( i \in R(\pi) \), we have \( \pi(\sigma_i | \omega) > 0 \) for some \( \omega \in [l, w^*) \), such a state exists). Observe that we have \( \omega_{j(i)} \in O_i \) by construction. Given \( i \in R(\pi) \) with a reference state \( \omega_{j(i)} \) and \( w \in W \), define
\[
\lambda_{iw} = \frac{\pi(\sigma_i | w)}{\pi(\sigma_i | \omega_{j(i)})}
\]

\( \lambda_{iw} \) is the ratio of the probability of the signal realization \( \sigma_i \) given state \( w \) to the probability of the signal realization \( \sigma_i \) given the reference state \( \omega_{j(i)} \). Observe that \( \lambda_{iw} \) is well-defined because \( \pi(\sigma_i | \omega_{j(i)}) > 0 \). Observe also that, for all \( i \in R(\pi) \) and \( w \in W \), we have \( \pi(\sigma_i | w) = \lambda_{iw} \pi(\sigma_i | \omega_{j(i)}) \).

Finally, for each \( i \in R(\pi) \), we define
\[
Q_i = \int_{\Omega_i} b_w \pi(\sigma_i | w) g(w) dw \quad Q = \sum_{i \in R(\pi)} Q_i \quad \eta_i = \frac{Q_i}{Q}
\]

**Step 3:** \( \int_{[w^*, l]} (w - w^*) g(w) \pi(\sigma_i | w) dw - \eta_i Q \geq 0 \) for all \( i \in R(\pi) \).

Suppose for the sake of contradiction that there exists \( i \in R(\pi) \) such that \( \int_{[w^*, l]} (w - w^*) g(w) \pi(\sigma_i | w) dw - \eta_i Q < 0 \). This is equivalent to
\[
\frac{\int_{\Omega_i} w \pi(\sigma_i | w) g(w) dw + \int_{[w^*, l]} g(w) w \pi(\sigma_i | w) dw}{\int_{\Omega_i} \pi(\sigma_i | w) g(w) dw + \int_{[w^*, l]} g(w) \pi(\sigma_i | w) dw} < w^*
\]

Consider \( F \in C_{a,g} \) such that \( \mu_F = (1 - a) \mu_G + a \mu_{\bar{F}} \) for some probability measure \( \mu_{\bar{F}} \) satisfying \( \text{supp} \mu_{\bar{F}} \subseteq [l, w^*) \). Note that we may, without loss of generality, assume that all priors that Nature finds it optimal to choose have this form.

Observe that \( E_{F,w}[w | \sigma_i] = \frac{a \int_{\Omega_i} w \pi(\sigma_i | w) dF(w) + \int_{[w^*, l]} \pi(\sigma_i | w) (1-a) g(w) dw + \int_{[w^*, l]} (1-a) g(w) \pi(\sigma_i | w) dw}{a \int_{\Omega_i} \pi(\sigma_i | w) dF(w) + \int_{[w^*, l]} \pi(\sigma_i | w) (1-a) g(w) dw + \int_{[w^*, l]} (1-a) g(w) \pi(\sigma_i | w) dw} = \)
\[ \frac{\alpha}{\pi} \int_{\Omega_1} w \pi(w) dF(w) + \int_{\Omega_1} w \pi(w) \pi_i(w) g(w) dw + \int_{[w^*, l]} g(w) \pi(w) d\tilde{F}(w) - \frac{\alpha}{\pi} \int_{\Omega_1} \pi_i(w) dF(w) + \int_{\Omega_1} \pi_i(w) g(w) dw + \int_{[w^*, l]} g(w) \pi_i(w) dw \]. Then, because \( \Omega_1 \subseteq [l, w^*) \), (4) implies that \( E_{\pi,F}[w \mid \sigma_i] < w^* \) for any prior \( F \) that Nature finds it optimal to choose.

Recall that, by the hypothesis, \( \pi \) is such that for any \( i \in R(\pi) \) there exists \( F \in C_{a,g} \) such that \( E_{F,\pi}[w \mid \sigma_i] \geq w^* \) and Nature finds it optimal to choose \( F \). Then the fact that \( E_{F,\pi}[w \mid \sigma_i] < w^* \) for any prior \( F \) that Nature finds it optimal to choose contradicts the hypothesis of the lemma.

**Step 4: Defining weights for signal realizations.**

In this step I define a collection of weights \( \{\epsilon_i\}_{i \in R(\pi)} \) for signal realizations with indices in \( R(\pi) \) satisfying \( \epsilon_i \in [0, 1] \) for all \( i \in R(\pi) \) and \( \sum_{i \in R(\pi)} \epsilon_i = 1 \).

For each \( i \in R(\pi) \), define a weight \( \epsilon_i \) as

\[ \epsilon_i = \frac{\int_{[w^*, h]} g(w)(w - w^*) \pi(w) \pi_i(w) dw - \eta_i Q}{\int_{[w^*, h]} g(w)(w - w^*) dw - Q} \]

Note that Step 3 implies that \( \epsilon_i \geq 0 \).

Recall that, by the hypothesis, \( \pi \) satisfies \( \mu_{F_s} \)-a.e. \( w \in [w^*, h] \).

\[ \sum_{i \in R(\pi)} \pi(w) = 1 \] (5)

Note that lemma 2 implies that if \( \sum_{i \in R(\pi)} \pi(w) = 1 \) for \( \mu_{F_s} \)-a.e. \( w \in [w^*, h] \) and \( f_s \) has full support, then \( \sum_{i \in R(\pi)} \int_{[w^*, h]} (w - w^*) \pi_i(w) g(w) \pi(w) dw = \int_{[w^*, h]} (w - w^*) g(w) dw \). Then, because \( \sum_{i \in R(\pi)} \pi(w) = 1 \) for \( \mu_{F_s} \)-a.e. \( w \in [w^*, h] \) holds by (5) and, by assumption, \( f_s \) has full support, we have

\[ \sum_{i \in R(\pi)} \int_{[w^*, h]} (w - w^*) g(w) \pi_i(w) dw = \int_{[w^*, h]} (w - w^*) g(w) dw \] (6)
Observe that

$$\sum_{i \in R(\pi)} \epsilon_i = \sum_{i \in R(\pi)} \frac{\int_{[w^*, h]} (w - w^*) g(w) \pi(\sigma_i | w) dw - Q \eta_i}{\int_{[w^*, h]} (w - w^*) g(w) dw - Q} = \frac{\sum_{i \in R(\pi)} \int_{[w^*, h]} (w - w^*) g(w) \pi(\sigma_i | w) dw - Q \sum_{i \in R(\pi)} \eta_i}{\int_{[w^*, h]} (w - w^*) g(w) dw - Q} = \frac{\int_{[w^*, h]} (w - w^*) g(w) dw - Q}{\int_{[w^*, h]} (w - w^*) g(w) dw} = 1$$

(7)

where the penultimate equality follows from the fact that $\sum_{i \in R(\pi)} \int_{[w^*, h]} (w - w^*) g(w) \pi(\sigma_i | w) dw = \int_{[w^*, h]} (w - w^*) g(w) dw$ by (6) and the fact that $\sum_{i \in R(\pi)} \eta_i = \sum_{i \in R(\pi)} \frac{Q_i}{Q} = \frac{\sum_{i \in R(\pi)} Q_i}{Q} = 1$.

**Step 5: Nature’s strategy.**

Suppose that Nature chooses the receiver’s prior $\mu_F$ that places $(1 - a) \mu_G$ on states $[l, h]$ and, in addition, for all $i \in R(\pi)$, places the mass of $\nu^i_j$ on $a$ on each $\omega \in O_i$ such that $\nu^i_j \in [0, 1]$ for all $\omega \in W$ satisfying $\omega \in O_i$ and $\sum_{\omega \in O_i} \nu^i_j = 1$.

Note that, by construction, $O_i \neq \emptyset$ for all $i \in R(\pi)$. Then we have

$$\sum_{i \in R(\pi)} \sum_{\omega \in O_i} \nu^i_j \epsilon_i a = \sum_{i \in R(\pi)} \epsilon_i a = a$$

where the last equality follows from the fact that $\sum_{i \in R(\pi)} \epsilon_i = 1$ by (7).

Thus Nature places the mass of $a + (1 - a) \int_{[l, w^*]} g(w) dw$ on states in $[l, w^*]$. Observe that $\mu_F \in F_{a, g}$.

**Step 6:** $E_F[\omega | \sigma_i] \geq w^*$ is equivalent to

$$\frac{\int_{[w^*, h]} g(w) w \pi(\sigma_i | w) dw + \frac{a}{1 - a} \sum_{\omega \in O_i} \pi(\sigma_i | \omega) \sum_{k \in S_{\omega \cap R(\pi)}} \nu^i_k \epsilon_i k + \int_{[l, w^*]} w \pi(\sigma_i | w) g(w) dw}{\int_{[w^*, h]} g(w) \pi(\sigma_i | w) dw + \frac{a}{1 - a} \sum_{\omega \in O_i} \pi(\sigma_i | \omega) \sum_{k \in S_{\omega \cap R(\pi)}} \nu^i_k \epsilon_i k + \int_{[l, w^*]} \pi(\sigma_i | w) g(w) dw} \geq w^*.$$
Fix $i \in R(\pi)$. We have

$$\int_{\Omega} w\pi(\sigma_i|w)dF(w) = \sum_{\omega \in O_i} \omega\pi(\sigma_i|\omega) \sum_{k \in S_{\omega \cap R(\pi)}} \nu_{\omega}^{k} \epsilon_{k} + \int_{\Omega}^{\pi} w\pi(\sigma_i|w)(1-a)g(w)dw + \int_{[\omega^*,h]} (1-a)g(w)w\pi(\sigma_i|w)dw$$

Observe that

$$\int_{[I,w^*)} \pi(\sigma_i|w)dF(w) = \sum_{\omega \in O_i} \pi(\sigma_i|\omega) \sum_{k \in S_{\omega \cap R(\pi)}} \nu_{\omega}^{k} \epsilon_{k} + \int_{\Omega}^{\pi} \pi(\sigma_i|w)(1-a)g(w)dw$$

Then

$$\int_{\Omega} \pi(\sigma|w)dF(w) = \int_{[I,w^*)} \pi(\sigma_i|w)dF(w) + \int_{[\omega^*,h]} (1-a)g(w)\pi(\sigma_i|w)dw = \sum_{\omega \in O_i} \pi(\sigma_i|\omega) \sum_{k \in S_{\omega \cap R(\pi)}} \nu_{\omega}^{k} \epsilon_{k} + \int_{\Omega}^{\pi} \pi(\sigma_i|w)(1-a)g(w)dw + \int_{[\omega^*,h]} (1-a)g(w)\pi(\sigma_i|w)dw$$

$E_F[\omega|\sigma_i] \geq w^*$ is equivalent to

$$\frac{\int_{\Omega} w\pi(\sigma_i|w)dF(w)}{\int_{\Omega} \pi(\sigma_i|w)dF(w)} \geq w^*$$

(8)

The above observations imply that [8] is equivalent to

$$\frac{\int_{[\omega^*,h]} g(w)\pi(\sigma_i|w)dw + \frac{\alpha}{1-a} \sum_{\omega \in O_i} \pi(\sigma_i|\omega) \sum_{k \in S_{\omega \cap R(\pi)}} \nu_{\omega}^{k} \epsilon_{k} + f_{\omega^*}^{\omega^*} \pi(\sigma_i|w)d\omega(w))}{\int_{[\omega^*,h]} g(w)\pi(\sigma_i|w)dw + \frac{\alpha}{1-a} \sum_{\omega \in O_i} \pi(\sigma_i|\omega) \sum_{k \in S_{\omega \cap R(\pi)}} \nu_{\omega}^{k} \epsilon_{k} + f_{\omega^*}^{\omega^*} \pi(\sigma_i|w)d\omega(w)} \geq w^*.$$

Step 7: $E_F[\omega|\sigma_i] \geq w^*$ given Nature’s strategy.
Simplifying the formula above, we obtain

\[
\int_{[w^*, h]} g(w) w \pi(\sigma_i | w) dw + \frac{a}{1 - a} \sum_{\omega \in O_i} \omega \pi(\sigma_i | \omega) \sum_{k \in S_{\omega} \cap R(\pi)} \left[ \nu_k^v \epsilon_k \right] + \int_{\Omega_i}^{*} w \pi(\sigma_i | w) g(w) dw \geq w^* \int_{[w^*, h]} g(w) \pi(\sigma_i | w) dw + \frac{a}{1 - a} w^* \sum_{\omega \in O_i} \pi(\sigma_i | \omega) \sum_{k \in S_{\omega} \cap R(\pi)} \left[ \nu_k^v \epsilon_k \right] + w^* \int_{\Omega_i}^{*} \pi(\sigma_i | w) g(w) dw
\]

The above is equivalent to

\[
\int_{[w^*, h]} (w - w^*) g(w) \pi(\sigma_i | w) dw - \int_{\Omega_i}^{*} (w - w^*) \pi(\sigma_i | w) g(w) dw \geq \frac{a}{1 - a} \sum_{\omega \in O_i} \pi(\sigma_i | \omega) \sum_{k \in S_{\omega} \cap R(\pi)} \left[ \nu_k^v \epsilon_k \right]
\]

Because \(w^* - \omega = b_\omega\), we can write this as

\[
\int_{[w^*, h]} (w - w^*) g(w) \pi(\sigma_i | w) dw - \int_{\Omega_i}^{*} b_\omega \pi(\sigma_i | w) g(w) dw \geq \frac{a}{1 - a} \sum_{\omega \in O_i} \pi(\sigma_i | \omega) b_\omega \sum_{k \in S_{\omega} \cap R(\pi)} \left[ \nu_k^v \epsilon_k \right]
\] (9)

Because \(\pi(\sigma_i | \omega) = \lambda_i \omega \pi(\sigma_i | \omega_{j(i)})\), we can write (9) as

\[
\int_{[w^*, h]} (w - w^*) g(w) \pi(\sigma_i | w) dw - \int_{\Omega_i}^{*} b_\omega \pi(\sigma_i | w) g(w) dw \geq \frac{a}{1 - a} \sum_{\omega \in O_i} \lambda_i \omega b_\omega \sum_{k \in S_{\omega} \cap R(\pi)} \left[ \nu_k^v \epsilon_k \right]
\]

Dividing both sides by \(\sum_{\omega \in O_i} \lambda_i \omega b_\omega \frac{a}{1 - a} \sum_{k \in S_{\omega} \cap R(\pi)} \left[ \nu_k^v \epsilon_k \right]\), we find that this is equivalent to

\[
\pi(\sigma_i | \omega_{j(i)}) \leq \frac{\int_{[w^*, h]} (w - w^*) g(w) \pi(\sigma_i | w) dw - \int_{\Omega_i}^{*} b_\omega \pi(\sigma_i | w) g(w) dw}{\sum_{\omega \in O_i} \lambda_i \omega b_\omega \sum_{k \in S_{\omega} \cap R(\pi)} \left[ \nu_k^v \epsilon_k \right]}
\]

Because \(\pi(\sigma_i | v) = \lambda_i v \pi(\sigma_i | \omega_{j(i)})\), for all \(v \in O_i\) we have \(\pi(\sigma_i | \omega_{j(i)}) = \frac{\pi(\sigma_i | v)}{\lambda_i v}\). Then

\[
\pi(\sigma_i | v) \leq \frac{\int_{[w^*, h]} (w - w^*) g(w) \pi(\sigma_i | w) dw \lambda_i v - \left( \int_{\Omega_i}^{*} b_\omega \pi(\sigma_i | w) g(w) dw \right) \lambda_i v}{\sum_{\omega \in O_i} \lambda_i \omega b_\omega \sum_{k \in S_{\omega} \cap R(\pi)} \left[ \nu_k^v \epsilon_k \right]}
\] (10)

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Because $Q_i = \eta_i Q$ for all $i \in R(\pi)$, (10) is equivalent to
\[
\pi(\sigma_i|v) \leq \frac{\int_{[w^*,h_i]} (w - w^*) g(w) \pi(\sigma_i|w) dw \lambda_{iv} - \eta_i Q \lambda_{iv}}{\sum_{\omega \in O_i} \lambda_{iw} b_{\omega} \sum_{k \in S_w \cap R(\pi)} \nu^k_{\omega} \epsilon_k}
\] (11)

**Step 8: Defining the receiver’s prior.**

Given $w \in W$, let
\[
z_w = \sum_{k \in S_w \cap R(\pi)} \nu^k_w \epsilon_k
\]

For each $i \in R(\pi)$, choose $\nu^i_w$ such that
\[
\frac{\epsilon_i \lambda_{iw} b_{w}}{\sum_{\omega \in O_i} \lambda_{iw} b_{\omega} g_{\omega}} = \frac{\nu^i_w \epsilon_i}{z_w}
\] (12)

This implies that we set
\[
\nu^i_w = \frac{\lambda_{iw} b_{w} z_w}{\sum_{\omega \in O_i} \lambda_{iw} b_{\omega} z_w}
\] (13)

**Step 9: The receiver’s prior is well-defined.**

Next, we will show that the collection of weights $\{\nu^i_w\}_{i \in R(\pi), w \in O_i}$ is well-defined. In particular, we will show that there exists a collection of weights satisfying (13) such that $\nu^i_w \in [0, 1]$ for all $i \in R(\pi)$ and for all $w \in W$ satisfying $w \in O_i$, and $\sum_{w \in O_i} \nu^i_w = 1$.

Observe that
\[
\sum_{w \in O_i} \nu^i_w = \frac{\sum_{w \in O_i} \sum_{\omega \in O_i} \lambda_{iw} b_{\omega} z_{w}}{\sum_{w \in O_i} \lambda_{iw} b_{\omega} z_{w}} = \frac{\sum_{w \in O_i} \lambda_{iw} b_{w} z_{w}}{\sum_{w \in O_i} \lambda_{iw} b_{\omega} z_{w}} = 1
\]

This shows that we have $\sum_{w \in O_i} \nu^i_w = 1$.

Fix $w \in W$ and $i \in S_w \cap R(\pi)$. Then (13) is equivalent to
\[
\nu^i_w = \frac{\lambda_{iw} b_{w} \left[ \sum_{k \in S_w \cap R(\pi)} \nu^k_{\omega} \epsilon_k \right]}{\sum_{\omega \in O_i} \lambda_{iw} b_{\omega} \left[ \sum_{k \in S_w \cap R(\pi)} \nu^k_{\omega} \epsilon_k \right]}
\] (14)
Let 

\[ T = \times_{i \in R(\pi)} \Delta(O_i) \]

where \( \Delta(O_i) \) denotes the simplex over \( O_i \) and \( \times \) denotes the Cartesian product.

Let \( \nu = \{\nu^i_w\}_{i \in R(\pi), w \in O_i} \). Define a mapping \( H : T \to T \) by

\[
H_{iw}(\nu) = \frac{\lambda_{iw} b_w \left[ \sum_{k \in S_w \cap R(\pi)} \nu^k_w \epsilon_k \right]}{\sum_{\omega \in O_i} \lambda_{i\omega} b_{\omega} \left[ \sum_{k \in S_\omega \cap R(\pi)} \nu^k_\omega \epsilon_k \right]} \]

Then we can write (14) as

\[
\nu^i_w = H_{iw}(\nu)
\]

Thus the weights \( \{\nu^i_w\}_{i \in R(\pi), w \in O_i} \) are defined by the equation

\[
\nu = H(\nu) \quad (15)
\]

Observe that \( \Delta(O_i) \) is a compact and convex set. Because the Cartesian product of convex sets is convex and the Cartesian product of compact sets is compact, this implies that \( T = \times_{i \in R(\pi)} \Delta(O_i) \) is a compact and convex set. Thus \( H(\cdot) \) is a continuous self-map on a compact and convex set \( T \). Then the Brouwer fixed point theorem implies that \( H \) has a fixed point.

**Step 10: Bound on** \( \sum_{i \in S_w \cap R(\pi)} \left( \frac{\epsilon_i \lambda_{iw}}{\sum_{\omega \in O_i} \lambda_{i\omega} b_{\omega} z_{\omega}} \right) \).

Define

\[
\tau_{iw} = \frac{\epsilon_i \lambda_{iw}}{\sum_{\omega \in O_i} \lambda_{i\omega} b_{\omega} z_{\omega}}
\]

Then (12) implies that \( \tau_{iw} b_w = \frac{\nu^i_w \epsilon_i}{z_w} \), which is equivalent to

\[
\tau_{iw} = \frac{\nu^i_w \epsilon_i}{z_w b_w}
\]

Thus

\[
\sum_{i \in S_w \cap R(\pi)} \tau_{iw} = \sum_{i \in S_w \cap R(\pi)} \frac{\nu^i_w \epsilon_i}{z_w b_w} = \frac{1}{b_w} \sum_{i \in S_w \cap R(\pi)} \frac{\nu^i_w \epsilon_i}{z_w} = \frac{1}{b_w} \quad (16)
\]
because \( \sum_{i \in S_w \cap R(\pi)} \frac{\nu_i w_i}{z_w} = 1 \) by definition of \( z_w \).

Then (16) implies that

\[
\sum_{i \in S_w \cap R(\pi)} \tau_{iw} = \frac{1}{b_w}
\]

(17)

Step 11: Bound on the sum of signal probabilities \( \sum_{i \in S_w \cap R(\pi)} \pi(\sigma_i|\omega) \).

Suppose that Nature chooses the weights defined by the fixed-point equation (15). By construction, since \( E_{F,\pi}[w|\sigma_i] \geq w^* \) for all \( i \in R(F, \pi) \), we have that (11) holds for all \( w \in O_i \), for all \( i \in R(F, \pi) \). (11) is equivalent to

\[
\pi(\sigma_i|w) \leq \tau_{iw} \frac{1}{1-a} \left( \frac{\int_{[w^*, h]} (w - w^*) g(w) \pi(\sigma_i|w) dw - \eta_i Q}{\int_{[w^*, h]} (w - w^*) g(w) dw - Q} \right)
\]

\[
= \tau_{iw} \frac{1}{1-a} \left( \int_{[w^*, h]} (w - w^*) g(w) \pi(\sigma_i|w) dw - \eta_i Q \right)
\]

\[
= \tau_{iw} \frac{1}{1-a} \left( \int_{[w^*, h]} (w - w^*) g(w) dw - Q \right)
\]

(18)

where the first equality follows from substituting in the formula for \( \epsilon_i \).

Observe that the fact that \( \int_{[w^*, h]} (w - w^*) g(w) \pi(\sigma_i|w) dw - \eta_i Q \geq 0 \) for all \( i \in R(\pi) \) by Step 3 and the fact that \( \int_{[w^*, h]} (w - w^*) g(w) dw - Q = \sum_{i \in R(\pi)} \left( \int_{[w^*, h]} (w - w^*) g(w) \pi(\sigma_i|w) dw - \eta_i Q \right) \) imply that

\[
\int_{[w^*, h]} (w - w^*) g(w) dw - Q \geq 0
\]

(19)
Then we have
\[
\sum_{i \in S \cap R(F, \pi)} \pi(\sigma_i | w) \leq \sum_{i \in S \cap R(F, \pi)} \tau_{iw} \frac{1}{1-a} \left( \int_{[w^*, h]} (w - w^*)g(w)dw - Q \right)
\]
\[
\leq \left( \sum_{i \in S \cap R(F, \pi)} \tau_{iw} \right) \frac{1}{1-a} \left( \int_{[w^*, h]} (w - w^*)g(w)dw - Q \right)
\]
\[
= \frac{1}{b_w} \frac{1}{1-a} \left( \int_{[w^*, h]} (w - w^*)g(w)dw - Q \right)
\]
where the first inequality follows from summing \[18\] over \(R(F, \pi)\), the second inequality follows from the fact that \(R(F, \pi) \subseteq R(\pi)\), \(\tau_{iw} \geq 0\) for all \(i \in R(\pi)\) and \(w \in O_i\) and \(\int_{[w^*, h]} (w - w^*)g(w)dw - Q \geq 0\) by \[19\], and the equality follows from the fact that \(\sum_{i \in S \cap R(\pi)} \tau_{iw} = \frac{1}{b_w}\) by \[17\].

Thus \[20\] implies that
\[
\sum_{i \in R(F, \pi)} \pi(\sigma_i | w) \leq \frac{\int_{[w^*, h]} (w - w^*)g(w)dw - Q}{\frac{a}{1-a} b_w}
\]
for all \(w \in W\) such that \(w \in O_i\) for some \(i \in R(F, \pi)\).

Observe also that \[21\] holds for all \(w \in W\) such that \(w \notin O_i\) for any \(i \in R(F, \pi)\), because in this case \(\pi(\sigma_i | w) = 0\) for all \(i \in R(F, \pi)\). Therefore, \[21\] holds for all \(w \in W\).

Let \(\overline{Q} = \sum_{i \in R(F, \pi)} Q_i\). Then, because \(R(F, \pi) \subseteq R(\pi)\) and \(Q_i \geq 0\) for all \(i \in R(\pi)\), we have \(\overline{Q} = \sum_{i \in R(F, \pi)} Q_i \leq \sum_{i \in R(\pi)} Q_i = Q\). Therefore, \[21\] implies that for all \(w \in W\) we have
\[
\sum_{i \in R(F, \pi)} \pi(\sigma_i | w) \leq \frac{\int_{[w^*, h]} (w - w^*)g(w)dw - \overline{Q}}{\frac{a}{1-a} b_w}
\]
\[
(22)
\]

**Step 12: Conclusion of the proof.**

Suppose for the sake of contradiction that for some \(w \in [l, w^*]\) we had \(\sum_{i \in R(F, \pi)} \pi(\sigma_i | w) > \frac{\int_{[w^*, h]} (w - w^*)g(w)dw - \overline{Q}}{\frac{a}{1-a} b_w}\). By definition of \(C(w)\), this is equivalent to saying that \(R(F, \pi) \in C(w)\).

Let \(Y(w)\) denote the element \(Y_i\) of the partition \(Y_1, \ldots, Y_J\) such that \(w \in Y_i\). Observe that, because \(\sum_{i \in R(F, \pi)} \pi(\sigma_i | w) > \frac{\int_{[w^*, h]} (w - w^*)g(w)dw - \overline{Q}}{\frac{a}{1-a} b_w}\), \[22\] implies that \(w \notin W\). Let \(w^*\)
denote the element of \( W \) such that \( w^* \in Y(w) \). Then, because \( C(\omega) = C(\omega') \) for all \( \omega, \omega' \in Y(w) \), it must be the case that \( R(F, \pi) \in C(w^*) \). By definition of \( C(w^*) \), this is equivalent to \( \sum_{i \in R(F, \pi)} \pi(\sigma_i | w^*) > \frac{f_{w^*, \lambda}(w-w^*)g(w)dw}{\frac{a}{1-a}b_w} \). However, this contradicts (22).

Therefore, we must have \( \sum_{i \in R(F, \pi)} \pi(\sigma_i | w^*) \leq \frac{f_{w^*, \lambda}(w-w^*)g(w)dw}{\frac{a}{1-a}b_w} \) for all \( w \in [l, w^*) \).

Equivalently,

\[
\sum_{i \in R(F, \pi)} \pi(\sigma_i | w') \leq \frac{f_{w^*, \lambda}(w-w^*)g(w)dw - \int_{\Omega} \left( \sum_{i \in R(F, \pi)} \pi(\sigma_i | w) \right) g(w)(w^* - w)dw}{\frac{a}{1-a}(w^* - w')}
\]

for all \( w' \in [l, w^*) \), as required.

**Lemma 4.** Let \( \mathcal{F} \) denote the set of the extreme points of \( \mathcal{F}_{a,g} \). Suppose that \( \pi \) is a signal structure with two realizations, \( \sigma \) and \( \sigma_0 \). If \( E_{F,\pi}[\omega|\sigma] \geq w^* \) for all \( F \in \mathcal{F} \), then \( E_{F,\pi}[\omega|\sigma] \geq w^* \) for all \( F \in \mathcal{F}_{a,g} \).

**Proof of lemma 4.**

Fix \( F \in \mathcal{F}_{a,g} \). Because \( \mathcal{F} \) is the set of the extreme points of \( \mathcal{F}_{a,g} \), we have \( \mathcal{F} = \{F_w : w \in [l, w^*)\} \). Then there exists a collection \( \{\alpha_w\}_{w \in [l, w^*)} \) such that \( \int_{w \in [l, w^*)} \alpha_w dw = 1 \), \( \mu_F = \int_{w \in [l, w^*)} \alpha_w \mu_{F_w} dw \) and \( \alpha_w \in [0, 1], \mu_{F_w} = (1 - a)\mu_G + a\delta_w \) for all \( w \in [l, w^*) \).

Because \( E_{F,\pi}[\omega|\sigma] \geq w^* \) for all \( F_w \in \mathcal{F} \), we have that for all \( \omega \in [l, w^*) \),

\[
\frac{\int_{\Omega} w \pi(\sigma | w)d\mu_{F_w}(w)}{\int_{\Omega} \pi(\sigma | w)d\mu_{F_w}(w)} \geq w^*	ag{23}
\]

Then we have

\[
\frac{\int_{\Omega} w \pi(\sigma | w)d\mu_{F_w}(w)}{\int_{\Omega} \pi(\sigma | w)d\mu_{F_w}(w)} = \frac{\int_{\Omega} w \pi(\sigma | w)d \left( \int_{w \in [l, w^*)} \alpha_w \mu_{F_w}(w)dw \right)}{\int_{\Omega} \pi(\sigma | w)d \left( \int_{w \in [l, w^*)} \alpha_w \mu_{F_w}(w)dw \right)}
\]

\[
= \frac{\int_{\Omega} \pi(\sigma | w)d \left( \int_{w \in [l, w^*)} \alpha_w \mu_{F_w}(w)dw \right)}{\int_{w \in [l, w^*)} \alpha_w \left( \int_{\Omega} \pi(\sigma | w)d\mu_{F_w}(w) \right)dw}
\]

\[
\geq \min_{\omega \in [l, w^*)} \frac{\alpha_w \int_{\Omega} \pi(\sigma | w)d\mu_{F_w}(w)}{\int_{\Omega} \pi(\sigma | w)d\mu_{F_w}(w)} = \min_{\omega \in [l, w^*)} \frac{\int_{\Omega} w \pi(\sigma | w)d\mu_{F_w}(w)}{\int_{\Omega} \pi(\sigma | w)d\mu_{F_w}(w)} \geq w^*	ag{24}
\]

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Lemma 5. If \( \pi \) is a signal structure with two realizations satisfying \( \pi(\sigma|w) = 1 \) \( \mu_{F_s} \)-a.e., \( w \in [w^*, h] \), then, given \( w' \in [l, w^*] \), \( E_{F_{w',\pi}}[\omega|\sigma] \geq w^* \) is equivalent to \( \pi(\sigma|w') \leq \min \left\{ \frac{\int_{[w^*,h]}(w-w^*)g(w)dw - \int_{\Omega_1^s,\Omega_2^s} g(w)\pi(\sigma|w)(w^*-w)dw - \int_{\Omega_2^s} g(w)(w^*-w)dw}{(w^*-w')^{1-a}}, 1 \right\} \).

If \( \pi \) sender-optimal in the class of all signal structures with two realizations, then it satisfies \( \pi(\sigma|w') \leq \min \left\{ \frac{\int_{[w^*,h]}(w-w^*)g(w)dw - \int_{\Omega_1^s,\Omega_2^s} g(w)\pi(\sigma|w)(w^*-w)dw - \int_{\Omega_2^s} g(w)(w^*-w)dw}{(w^*-w')^{1-a}}, 1 \right\} \) for all \( w' \in \Omega_1^s \) and \( \pi(\sigma|w) = 1 \) \( \mu_{F_s} \)-a.e. \( w \in [w^*, h] \).

Proof of lemma 5.

Fix \( w' \in [l, w^*] \). Because \( \pi(\sigma|w) = 1 \) \( \mu_{F_s} \)-a.e., \( w \in [w^*, h] \) and \( f_s \) has full support, \( E_{F_{w',\pi}}[\omega|\sigma] \geq w^* \) is equivalent to

\[
\int_{\Omega_1^s \setminus \Omega_2^s} (1 - a)g(w)\pi(\sigma|w)wdw + \pi(\sigma|w')w'\sigma + \int_{[w^*, h]}(1 - a)g(w)wdw \geq w^* \]

Equivalently, because \( \pi(\sigma|w) = 1 \) for all \( w \in \Omega_2^s \) by definition of \( \Omega_2^s \), we have

\[
\int_{\Omega_1^s \setminus \Omega_2^s} (1 - a)g(w)\pi(\sigma|w)wdw + \int_{\Omega_2^s}(1 - a)g(w)wdw + \pi(\sigma|w')w'\sigma + \int_{[w^*, h]}(1 - a)g(w)wdw \geq w^* \]

Rearranging the above inequality, we obtain \( \pi(\sigma|w') \leq \min \left\{ \frac{\int_{[w^*,h]}(w-w^*)g(w)dw - \int_{\Omega_1^s,\Omega_2^s} g(w)\pi(\sigma|w)(w^*-w)dw - \int_{\Omega_2^s} g(w)(w^*-w)dw}{(w^*-w')^{1-a}}, 1 \right\} \).

For the proof of the second part of the lemma observe that, if \( \pi \) is sender-optimal in the class of all signal structures with two realizations, we have \( \pi(\sigma|w) = 1 \) \( \mu_{F_s} \)-a.e. \( w \in [w^*, h] \) because the sender’s payoff \( \int_{[l,h]} \pi(\sigma|w)f_s(w)dw \) is increasing in \( \pi(\sigma|w) \). Moreover, because \( \pi \) is sender-optimal in the class of all signal structures with two realizations, we must have \( E_{F_{w',\pi}}[\omega|\sigma] \geq w^* \) for all \( w' \in \Omega_1^s \).

Lemma 6. Suppose that a signal structure \( \pi \) with two realizations satisfies \( \pi(\sigma|w') \leq \pi(\sigma|w) \leq Z_{\pi}(w') \) for all \( w' \in [l, w^*], \pi(\sigma|w') = 1 \) \( \mu_{F_s} \)-a.e. \( w' \in [w^*, h], \) and \( \pi(\sigma_0|w) = 1 - \pi(\sigma|w) \). Then \( U(\pi) = \int_{[l,h]} \pi(\sigma|w)f_s(w)dw \).
Proof of lemma 6.

By lemma 5, a signal structure \( \pi \) satisfying \( \pi(\sigma|w) = 1 \) \( \mu_{F^*} \)-a.e. \( w \in [w^*, h] \) satisfies \( \pi(\sigma|w') \leq Z_\pi(w') \) for all \( w' \in [l, w^*] \) if and only if \( \pi \) satisfies \( E_{F^*, \pi}[\omega|\sigma] \geq w^* \) for all priors \( F_{w'} \) of the form \( \mu_{F_{w'}} = (1 - a)\mu_G + a\delta_{w^*} \) for some \( w' \in [l, w^*] \).

Observe that priors of the form \( \mu_{F_{w'}} = (1 - a)\mu_G + a\delta_{w^*} \) for some \( w' \in [l, w^*] \) are the extreme points of the set of priors \( \mathcal{F}_{a,g} \). Because \( E_{F_{w'}, \pi}[\omega|\sigma] \geq w^* \) for all priors \( F \) in the set of the extreme points of \( \mathcal{F}_{a,g} \), lemma 4 implies that \( E_{F, \pi}[\omega|\sigma] \geq w^* \) for all priors \( F \) in \( \mathcal{F}_{a,g} \).

Observe next that if \( E_{F, \pi}[\omega|\sigma] \geq w^* \) for all \( F \in \mathcal{F}_{a,g} \), then \( E_{F, \pi}[\omega|\sigma] \geq w^* \) for all \( F \in \mathcal{C}_{a,g} \).

Then the receiver takes action 1 after seeing the signal realization \( \sigma \) for all priors in the feasible set of priors \( \mathcal{C}_{a,g} \). Therefore, \( U(\pi) = \int_{[l,h]} \pi(\sigma|w) f_s(w) dw \), as required. \( \blacksquare \)

Lemma 7. For all \( \pi_0 \in \mathcal{D} \) there exists a signal structure \( \pi \) with two realizations satisfying \( U(\pi) \geq U(\pi_0) \).

Moreover, if \( \pi_0 \) is sender-optimal, then \( \pi \) can be chosen such that

\[
U(\pi) = \int_{[l,h]} \left( \sum_{i : i \in R(\bar{F}, \pi_0) \text{ for all } \bar{F} \in \mathcal{C}_{a,g}} \pi_0(\sigma_i|w) \right) f_s(w) dw
\]

and \( \pi(\sigma|w) = \sum_{i \in R(\bar{F}, \pi_0) \text{ for all } \bar{F} \in \mathcal{C}_{a,g}} \pi_0(\sigma_i|w) \) for all \( w \in [l, h] \).

Proof of lemma 7.

Observe that if \( \pi_0 \in \mathcal{D} \) does not satisfy \( \sum_{i \in R(\pi_0)} \pi_0(\sigma_i|w) = 1 \) \( \mu_{F^*} \)-a.e. \( w \in [w^*, h] \) or the requirement that for any \( i \in R(\pi_0) \) there exists \( \bar{F} \in \mathcal{C}_{a,g} \) such that \( E_{\bar{F}, \pi_0}[w|\sigma_i] \geq w^* \) and Nature finds it optimal to choose \( \bar{F} \), then there exists a \( \pi_1 \in \mathcal{D} \) that satisfies these requirements and gives a payoff to the sender that is at least as high as the payoff from \( \pi_0 \). Then it is enough to prove the lemma for the signal structures satisfying the above requirements.

Fix \( \pi_0 \in \mathcal{D} \) satisfying \( |R(\pi_0)| > 1 \), \( \sum_{i \in R(\pi_0)} \pi_0(\sigma_i|w) = 1 \) \( \mu_{F^*} \)-a.e. \( w \in [w^*, h] \) and the requirement that for any \( i \in R(\pi_0) \) there exists \( \bar{F} \in \mathcal{C}_{a,g} \) such that \( E_{\bar{F}, \pi_0}[w|\sigma_i] \geq w^* \) and Nature finds it optimal to choose \( \bar{F} \). Then \( \pi_0 \) satisfies the conditions in the hypothesis of
lemma 3. By lemma 3, there exists $F \in C_{a,g}$ such that for all $w' \in [l, w^*), \pi_0$ satisfies

$$
\sum_{i \in R(F, \pi_0)} \pi_0(\sigma_i | w') \leq \frac{\int_{[w^*, h]} (w - w^*) g(w) dw - \int_{\Omega_{\pi_0}} \left( \sum_{i \in R(F, \pi_0)} \pi_0(\sigma_i | w) \right) g(w)(w^* - w) dw}{\frac{a}{1-a} (w^* - w')} 
$$

(25)

We define a signal structure $\pi$ with two realizations, $\sigma$ and $\sigma_0$, as follows. We set $\pi(\sigma | w) = \sum_{i \in R(F, \pi_0)} \pi_0(\sigma_i | w)$ for all $w \in [l, w^*), \pi(\sigma | w) = 1$ for all $w \in [w^*, h]$, and set $\pi(\sigma_0 | w) = 1 - \pi(\sigma | w)$ for all $w \in [l, h]$.

Observe that (25) implies that $\pi$ satisfies

$$
\pi(\sigma | w') \leq \frac{\int_{[w^*, h]} (w - w^*) g(w) dw - \int_{\Omega_{\pi_0}} \sum_{i \in R(F, \pi_0)} \pi_0(\sigma_i | w) g(w)(w^* - w) dw}{(w^* - w') \frac{a}{1-a}} 
$$

for all $w' \in [l, w^*)$.

Then

$$
U(\pi) = \int_{[l, h]} \pi(\sigma | w) f_s(w) dw \geq \int_{[l, h]} \left( \sum_{i \in R(F, \pi_0)} \pi_0(\sigma_i | w) \right) f_s(w) dw
$$

(26)

where the equality follows from lemma 6 because because $\pi(\sigma | w') \leq Z_\pi(w')$ for all $w' \in [l, w^*)$ and $\pi(\sigma | w') = 1$ for all $w' \in [w^*, h]$, and the inequality follows from the way in which we defined $\pi$.

Observe that $U(\pi_0) \leq \min_{F \in C_{a,g}} \int_{[l, h]} \left( \sum_{i \in R(F, \pi_0)} \pi_0(\sigma_i | w) \right) f_s(w) dw$. In particular, $U(\pi_0) \leq \int_{[l, h]} \left( \sum_{i \in R(F, \pi_0)} \pi_0(\sigma_i | w) \right) f_s(w) dw$. Then (26) implies that $U(\pi_0) \leq U(\pi)$, as required.

Claim 7.1. If $\pi_0$ is sender-optimal (in $D$), then $R(F, \pi_0) = R(\pi_0)$.

Proof of claim 7.1

Suppose for the sake of contradiction that $R(F, \pi_0) \subset R(\pi_0)$. Let $S_0 = R(\pi_0) \setminus R(F, \pi_0)$.

Define a signal structure $\pi_2$ with three signal realizations, $\sigma$, $\sigma_k$ and $\sigma_0$, as follows. Set $\pi_2(\sigma | w) = \pi(\sigma | w)$ for all $w \in [l, h]$. Set $\pi_2(\sigma_k | w) = \sum_{i \in S_0} \pi_0(\sigma_i | w)$ for $w \in [w^*, h]$ and $\pi_2(\sigma_k | w) = 0$ for $w \in [l, w^*)$.

Observe that, by construction, $E_{\tilde{F}, \pi_2}[\omega | \sigma_k] \geq w^*$ for all $\tilde{F} \in C_{a,g}$. Moreover, the fact that $E_{\tilde{F}, \pi}[\omega | \sigma] \geq w^*$ for all $\tilde{F} \in C_{a,g}$ and $\pi_2(\sigma | w) = \pi(\sigma | w)$ for all $w \in [l, h]$ implies that
Claim 7.2. If \( \pi \) is sender-optimal, then for all \( \tilde{F} \in C_{a,g} \),

\[
E_{\tilde{F},\pi}[\omega|\sigma] \geq w^* \text{ for all } \tilde{F} \in C_{a,g}.
\]

Thus we have

\[
U(\pi_2) = \int_{[l,h]} (\pi_2(\sigma|w) + \pi_2(\sigma_k|w))f_s(w)dw > \int_{[l,h]} \pi_2(\sigma|w)f_s(w)dw = \int_{[l,h]} \pi(\sigma|w)f_s(w)dw = U(\pi).
\]

Because \( \pi_0 \) is sender-optimal and, by the first part of the lemma, \( U(\pi) \geq U(\pi_0) \), \( \pi \) is also sender-optimal. Thus \( U(\pi_2) > U(\pi) \) contradicts the sender-optimality of \( \pi \).

\[\blacksquare\]

Claim 7.3. If \( \pi_0 \) is sender-optimal, \( \pi \) can be chosen such that \( \pi(\sigma|w) = \sum_{i \in R(\pi_0)} \pi_0(\sigma_i|w) \) for all \( w \in [l, h] \) for all \( w \in [l, h] \).

Proof of claim 7.2.

Note that, because \( \pi(\sigma|w) = 1 \) for all \( w \in [w^*, h] \) and \( \sum_{i \in R(\pi_0)} \pi_0(\sigma_i|w) = 1 \) \( \mu_F \)-a.e. \( w \in [w^*, h] \), we have \( \pi(\sigma|w) = \sum_{i \in R(\pi_0)} \pi_0(\sigma_i|w) \) \( \mu_{F_0} \)-a.e. \( w \in [w^*, h] \). Observe if \( \tilde{\pi} \) satisfies \( \tilde{\pi}(\sigma|w) = \sum_{i \in R(\pi_0)} \pi_0(\sigma_i|w) \) for all \( w \in [w^*, h] \), then \( U(\tilde{\pi}) = U(\pi) \). Then we may assume that \( \pi(\sigma|w) = \sum_{i \in R(\pi_0)} \pi_0(\sigma_i|w) \) for all \( w \in [w^*, h] \). Because \( R(\pi_0) = R(F, \pi_0) \) by claim 7.1, we then have \( \pi(\sigma|w) = \sum_{i \in R(F,\pi_0)} \pi_0(\sigma_i|w) \) for all \( w \in [w^*, h] \).

Thus, because \( \pi(\sigma|w) = \sum_{i \in R(F,\pi_0)} \pi_0(\sigma_i|w) \) for all \( w \in [l, w^*] \), we have \( \pi(\sigma|w) = \sum_{i \in R(F,\pi_0)} \pi_0(\sigma_i|w) \) for all \( w \in [l, h] \). Then the fact that \( R(F, \pi_0) = R(\pi_0) \) by claim 7.1 implies that \( \pi(\sigma|w) = \sum_{i \in R(\pi_0)} \pi_0(\sigma_i|w) \) for all \( w \in [l, h] \).

\[\blacksquare\]

Claim 7.3. If \( \pi_0 \) is sender-optimal, then \( R(\tilde{F}, \pi_0) = R(\pi_0) \) for all \( \tilde{F} \in C_{a,g} \).

Proof of claim 7.3.

By claim 7.2 we have

\[
U(\pi) = \int_{[l,h]} \pi(\sigma|w)f_s(w)dw = \int_{[l,h]} \sum_{i \in R(\pi_0)} \pi_0(\sigma_i|w)f_s(w)dw.
\]

The fact that \( \pi_0 \) is sender-optimal and \( U(\pi) \geq U(\pi_0) \) imply that \( U(\pi_0) = U(\pi) \), so that

\[
U(\pi_0) = \int_{[l,h]} \sum_{i \in R(\pi_0)} \pi_0(\sigma_i|w)f_s(w)dw \tag{27}
\]

Suppose for the sake of contradiction that \( R(\tilde{F}, \pi_0) \subset R(\pi_0) \) for some \( \tilde{F} \in C_{a,g} \). Then if Nature chooses \( \tilde{F} \), the sender’s payoff from \( \pi_0 \) satisfies

\[
U(\pi_0) \leq \int_{[l,h]} \left( \sum_{i \in R(F,\pi_0)} \pi_0(\sigma_i|w) \right) f_s(w)dw < \int_{[l,h]} \left( \sum_{i \in R(\pi_0)} \pi_0(\sigma_i|w) \right) f_s(w)dw = U(\pi_0)
\]

where the inequality follows from the fact that \( R(\tilde{F}, \pi_0) \subset R(\pi_0) \) and the last equality follows from (27).
Then $U(\pi_0) < U(\pi_0)$, a contradiction. ■

Claim 7.4. If $\pi_0$ is sender-optimal, $\pi$ can be chosen such that $\pi(\sigma|w) = \sum_{i \in R(\bar{F}, \pi_0)} \pi_0(\sigma_i|w)$ for all $w \in [l, h]$.

Proof of claim 7.4.

By claim 7.2, $\pi$ can be chosen such that $\pi(\sigma|w) = \sum_{i \in R(\pi_0)} \pi_0(\sigma_i|w)$ for all $w \in [l, h]$ for all $w \in [l, h]$. By claim 7.3, $R(\bar{F}, \pi_0) = R(\pi_0)$ for all $\bar{F} \in \mathcal{C}_{a,g}$. Then $\pi$ can be chosen such that $\pi(\sigma|w) = \sum_{i \in R(\bar{F}, \pi_0)} \pi_0(\sigma_i|w)$ for all $w \in [l, h]$, as required. ■

If $\pi_0$ is sender-optimal, claim 7.4 implies that $U(\pi) = \int_{[l, h]} \pi(\sigma|w)f_s(w)dw = \int_{[l, h]} \left( \sum_{i \in R(\bar{F}, \pi_0)} \pi_0(\sigma_i|w) \right) f_s(w)dw$, as required. ■

Lemma 8. Suppose that $\pi$ is a signal structure that is a sender-optimal in the class of all signal structures with two realizations and that there exists a partition $(A, B)$ of $\Omega_1^\pi$ such that $\pi$ satisfies

$$\pi(\sigma|w') = \min \left\{ \frac{\int_{[w^*, l]} (w - w^*)g(w)dw - \int_{\Omega_1 \setminus \Omega_2} g(w)\pi(\sigma|w)(w^* - w)dw - \int_{\Omega_2} g(w)(w^* - w)dw}{(w^* - w')\frac{a}{1-a}}, 1 \right\}$$

for all $w' \in A$,

$$\pi(\sigma|w') < \min \left\{ \frac{\int_{[w^*, l]} (w - w^*)g(w)dw - \int_{\Omega_1 \setminus \Omega_2} g(w)\pi(\sigma|w)(w^* - w)dw - \int_{\Omega_2} g(w)(w^* - w)dw}{(w^* - w')\frac{a}{1-a}}, 1 \right\}$$

for all $w' \in B$, and $\pi(\sigma|w') = 1$ for all $w' \in \Omega_2^\pi$.

Then there exists a constant $c_1 > 0$ such that

1. $\pi(\sigma|w) = \frac{c_1}{w^* - w}$ for all $w \in [l, w^* - c_1] \cap A$
2. $\pi(\sigma|w) = 1$ for all $w \in \Omega_2^\pi = [w^* - c_1, w^*) \cap A$

where $c_1$ satisfies

$$c_1 = \frac{\int_{[w^*, l]} (w - w^*)g(w)dw - \int_B g(w)\pi(\sigma|w)(w^* - w)dw - \int_{\Omega_2} g(w)(w^* - w)dw}{\frac{a}{1-a} + \int_{A \setminus \Omega_2} g(w)dw}$$

Proof of lemma 8.

Observe that if $w \in A$ and $\pi(\sigma|w) < 1$, then $w \in A \setminus \Omega_2^\pi$. Observe also that for
all \( w \in A \setminus \Omega_2^c \), \( \pi(\sigma|w) \) has the form \( \pi(\sigma|w) = \frac{c_1}{w^* - w} \) for some constant \( c_1 > 0 \). Then \((w^* - w)\pi(\sigma|w) = c_1 \) for all \( w \in A \setminus \Omega_2^c \).

Thus, substituting in \( \pi(\sigma|w) = \frac{c_1}{w^* - w} \) for all \( w \in A \setminus \Omega_2^c \), we have that for all \( w' \in A \setminus \Omega_2^c \),

\[
\int_{[w^* - w]}(w - w^*)g(w)dw - \int_{\Omega_1 \setminus \Omega_2^c} g(w)\pi(\sigma|w)(w^* - w)dw - \int_{\Omega_2^c} g(w)(w^* - w)dw = \frac{c_1}{w^* - w'}(w^* - w') - \int_{\Omega_2^c} g(w)(w^* - w)dw.
\]

Solving for \( c_1 \), we obtain

\[
c_1 = \frac{\int_{[w^* - w]}(w - w^*)g(w)dw - \int_B g(w)\pi(\sigma|w)(w^* - w)dw - \int_{\Omega_2^c} g(w)(w^* - w)dw}{\frac{a}{1-a} + \int_{A \setminus \Omega_2^c} g(w)dw}.
\]

Moreover, since \( \pi(\sigma|w) \leq 1 \) for \( w \in A \setminus \Omega_2^c \) is equivalent to \( \frac{c_1}{w^* - w} \leq 1 \), which simplifies to \( w \leq w^* - c_1 \), \( \Omega_2^c \) must have the form \( \Omega_2^c = [w, w^*) \cap A \) for \( w = w^* - c_1 \).

Then, substituting in the formula for \( \Omega_2^c \), we find that \( c_1 \) satisfies

\[
c_1 = \frac{\int_{[w^* - w]}(w - w^*)g(w)dw - \int_B g(w)\pi(\sigma|w)(w^* - w)dw - \int_{A \setminus \Omega_2^c} g(w)(w^* - w)dw}{\frac{a}{1-a} + \int_{A \setminus [w, w^* - c_1]} g(w)dw}.
\]

(28)

We next show that there exists a constant \( c_1 \) satisfying the above equation.

**Claim 8.1.** There exists \( c_1 > 0 \) satisfying equation (28).

**Proof of claim 8.1**

Consider a mapping \( c \mapsto L(c) \) given by

\[
L(c) = \frac{\int_{[w^* - w]}(w - w^*)g(w)dw - \int_B g(w)\pi(\sigma|w)(w^* - w)dw - \int_{A \setminus [w^* - c_1, w^*)} g(w)(w^* - w)dw}{\frac{a}{1-a} + \int_{A \setminus [w^* - c_1, w^*)} g(w)dw} - c.
\]

We have \( L(0) \leq 0 \) is equivalent to

\[
\frac{\int_B g(w)\pi(\sigma|w)wdw + \int_{[w^* - w]} g(w)wdw}{\int_B g(w)\pi(\sigma|w)wdw + \int_{[w^* - w]} g(w)wdw} \leq w^*.
\]

This, in turn, implies that \( E_{F,\pi}[w|\sigma < w^* \) for all \( F \in F_{\alpha, g} \), which contradicts the hypothesis that \( \pi \) is sender-optimal. Therefore, we must have \( L(0) > 0 \). Moreover, \( L(\tilde{c}) =
\[
\int_{[w^*, h]} (w - w^*) g(w) dw - \int_B g(w) \pi|w| (w^* - w) dw - \int_A g(w) (w^* - w) dw - c < 0 \quad \text{for } c \text{ sufficiently large.}
\]

Then, because \( L(0) \geq 0 \), \( L(c) < 0 \) and \( c \mapsto L(c) \) is continuous, there exists \( c \in [0, c] \) such that \( L(c) = 0 \). Therefore, there exists \( c_1 > 0 \) satisfying equation (28). ■ ■ ■

**Lemma 9.** If Assumption 7 holds and \( \pi \) is a signal structure that is sender-optimal in the class of all signal structures with two realizations, then it satisfies

\[
\pi|w'| = \min \left\{ \frac{\int_{[w^*, h]} (w - w^*) g(w) dw - \int_{\Omega^*_1} g(w) \pi|w| (w - w^*) dw}{(w^* - w') \frac{a}{1-a}}, 1 \right\}
\]

\( \mu_{F_s} - \text{a.e.} \ w' \in \Omega^*_1 \) and \( \pi|w| = 1 \ \mu_{F_s} - \text{a.e.} \ w \in [w^*, h] \).

Suppose that (for all parameter values) there exists a signal structure satisfying \( \pi|w'| = Z_\pi(w') \) \( \mu_{F_s} - \text{a.e.} \ w' \in \Omega^*_1 \) and \( \pi|w| = 1 \ \mu_{F_s} - \text{a.e.} \ w \in [w^*, h] \). If Assumption 7 fails, then there exists a signal structure \( \pi_0 \) that is sender-optimal in the class of all signal structures with two realizations and satisfies the above condition.

**Proof of lemma 9.**

Suppose that Assumption 7 holds and \( \pi \) is a signal structure that is sender-optimal in the class of all signal structures with two realizations. The fact that \( \pi|w| = 1 \ \mu_{F_s} - \text{a.e.} \ w \in [w^*, h] \) follows from lemma 5.

Since \( \pi \) is a signal structure that is sender-optimal in the class of all signal structures with two realizations, by lemma 5 for all \( w' \in \Omega^*_1 \)

\[
\pi|w'| \leq \min \left\{ \frac{\int_{[w^*, h]} (w - w^*) g(w) dw - \int_{\Omega^*_1} g(w) \pi|w| (w^* - w) dw}{(w^* - w') \frac{a}{1-a}}, 1 \right\} \quad (29)
\]

Observe that we can write the inequality (29) as \( \pi|w| \leq Z_\pi(w) \).

Suppose for the sake of contradiction that there exists a signal structure \( \pi \) that is sender-optimal in the class of all signal structures with two realizations such that \( \pi|w'| < Z_\pi(w') \) for some subset \( B \subseteq \Omega^*_1 \) of a strictly positive measure under \( \mu_{F_s} \).

Define \( A = \Omega^*_1 \setminus B \). Note that, because \( \Omega^*_2 = \{ w \in \Omega^*_1 : \pi|w| = 1 \} \), we have \( \Omega^*_2 = \{ w \in A : \pi|w| = 1 \} \). Note also that we then have \( A = \{ w \in \Omega^*_1 : \pi|w| = Z_\pi(w) \} \) and \( B = \{ w \in \Omega^*_1 : \pi|w| < Z_\pi(w) \} \).

Fix \( x \in B \) such that for all intervals \( I \) satisfying \( x \in \text{int}(I) \) (where \( \text{int} \) denotes the
interior), we have \(\mu(I \cap B) > 0\). Because, by the hypothesis, \(\mu_F(B) > 0\), such \(x\) exists. Given \(\epsilon > 0\) sufficiently small, fix an interval \(I_{x,\epsilon} = [x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}]\). Note that then \(\mu(I_{x,\epsilon} \cap B) > 0\).

We define a new signal structure \(\pi_1\) with two realizations as follows. For all \(w \in I_{x,\epsilon} \cap B\) we let \(\pi_1(\sigma|w) = \pi(\sigma|w) + \eta\) for some \(\eta > 0\). We let \(\pi_1(\sigma|w) = \pi(\sigma|w)\) for all \(w \in B \setminus I_{x,\epsilon}\).

We choose \(\eta\) small enough such that \(\pi_1(\sigma|w) < 1\) and \(E_{F_{\pi,\pi_1}}[\omega|\sigma] > w^*\) for all \(w \in B\). Note that this is feasible because \(\pi(\sigma|w) < 1\) for all \(w \in B\) and because, by lemma 5 if \(\pi(\sigma|w) < Z_\pi(w)\), then \(E_{F_{\pi,\pi_1}}[\omega|\sigma] > w^*\), which implies that \(E_{F_{\pi,\pi_1}}[\omega|\sigma] > w^*\) for all \(w \in B\).

To complete the construction of the signal structure \(\pi_1\), we require that \(\pi_1(\sigma|w) = Z_{\pi_1}(\sigma|w)\) for all \(w \in A\).

Observe that, because \(\pi(\sigma|w) = Z_\pi(\sigma|w)\) for all \(w \in A\) and \(\pi(\sigma|w) < Z_\pi(\sigma|w)\) for all \(w \in B\), by lemma 8 there exists a constant \(c\) such that \(\Omega_2^\pi = [w^* - c, w^*] \cap A\), \(\pi(\sigma|w) = \frac{c}{w^* - w}\) for all \(w \in A \setminus \Omega_2^\pi\) and \(\pi(\sigma|w) = 1\) for all \(w \in \Omega_2^\pi\). Similarly, because \(\pi_1(\sigma|w) = Z_{\pi_1}(\sigma|w)\) for all \(w \in A\) and \(\pi_1(\sigma|w) < Z_{\pi_1}(\sigma|w)\) for all \(w \in B\), by lemma 8 there exists a constant \(c_1\) such that \(\Omega_2^{\pi_1} = [w^* - c_1, w^*] \cap A\), \(\pi_1(\sigma|w) = \frac{c_1}{w^* - w}\) for all \(w \in A \setminus \Omega_2^{\pi_1}\) and \(\pi_1(\sigma|w) = 1\) for all \(w \in \Omega_2^{\pi_1}\).

Thus to ensure that \(\pi_1(\sigma|w) = Z_{\pi_1}(\sigma|w)\) for all \(w \in A\), we choose the constant \(c_1\) such that

\[
\frac{c_1}{w^* - w} = \frac{\int_{[w^*,w^*]} g(u)du - \int_B g(w)\pi_1(\sigma|w)(w^* - w)dw - c_1 \int_{A \setminus \Omega_2^{\pi_1}} g(w)dw - \int_{\Omega_2^{\pi_1}} g(w)(w^* - w)dw}{\int_{(w^* - w)^2} f_s(u)du}
\]

for all \(w' \in A \setminus \Omega_2^{\pi_1}\) and \(\pi(\sigma|w') = 1\) for all \(w' \in A \cap \Omega_2^{\pi_1}\) where \(\Omega_2^{\pi_1} = [w^* - c_1, w^*] \cap \Omega_2^{\pi_1}\).

**Claim 9.1.** \(U(\pi_1) - U(\pi) = \eta \int_{I_{x,\epsilon} \cap B} f_s(w)dw - \int_{A \setminus \Omega_2^{\pi_1}} \frac{c}{w^* - w} f_s(w)dw - \int_{[w^*, w^* - c) \cap A} (\frac{c_1}{w^* - w} - 1) f_s(w)dw\).

**Proof of claim 9.1**

\[
U(\pi_1) - U(\pi) = \int_{A \setminus \Omega_2^{\pi_1}} \frac{c_1}{w^* - w} f_s(w)dw + \int_{\Omega_2^{\pi_1}} f_s(w)dw + \int_{I_{x,\epsilon} \cap B} (\pi(\sigma|w) + \eta) f_s(w)dw
\]

\[
- \int_{A \setminus \Omega_2^{\pi}} \frac{c_1}{w^* - w} f_s(w)dw - \int_{\Omega_2^{\pi}} f_s(w)dw + \int_{I_{x,\epsilon} \cap B} \pi(\sigma|w) f_s(w)dw
\]

(30)

Using an argument similar to the one in lemma 11 it can be shown that \(c > c_1\). This implies that \([w^* - c_1, w^*] \subseteq [w^* - c, w^*]\). Because, by lemma 8 \(\Omega_2^{\pi_1} = [w^* - c, w^*] \cap \Omega_2^{\pi_1}\) and \(\Omega_2^{\pi_1} = [w^* - c_1, w^*] \cap \Omega_2^{\pi_1}\), it follows that \(\Omega_2^{\pi_1} \subseteq \Omega_2^{\pi_1}\) and \(\Omega_2^{\pi} \setminus \Omega_2^{\pi_1} = [w^* - c, w^* - c_1) \cap A\).

Then (30) is equivalent to

\[
U(\pi_1) - U(\pi) = \eta \int_{I_{x,\epsilon} \cap B} f_s(w)dw - \int_{A \setminus \Omega_2^{\pi_1}} \frac{c_1}{w^* - w} f_s(w)dw - \int_{[w^*, w^* - c) \cap A} (\frac{c_1}{w^* - w} - 1) f_s(w)dw
\]

\[
\square
\]
Define $D_0 = \frac{a}{1-a} + \int_{A \setminus \Omega_2^{\pi_1}} g(w)dw$, $D_1 = \frac{a}{1-a} + \int_{A \setminus \Omega_2^{\pi_1}} g(w)dw$, $A_1 = \int_{[w^*, h]} (w - w^*) g(w) dw - \int_B g(w) \pi(\sigma|w)(w^* - w) dw - \int_{\Omega_2^0} g(w) (w^* - w) dw$.

Claim 9.2. $A_1 \frac{\int_{[w^*, h]} g(w) dw}{D_1 D_0} - \frac{\int_{A \setminus \Omega_2^{\pi_1}} g(w) (w^* - w) dw}{D_1} = O((c - c_1)^2)$.

Proof of claim 9.2.

Multiplying both sides of the expression in the claim by $D_1 D_0$, we obtain

$$\int_{\Omega_2^0 \setminus \Omega_2^{\pi_1}} g(w) (A_1 - D_0 (w^* - w)) dw = O((c - c_1)^2)$$

Observe that we have $w^* - w \leq c = \frac{A_1}{D_0}$ for all $w \in [w^* - c, w^* - c_1]$. This implies that $A_1 - D_0 (w^* - w) \geq 0$ for all $w \in [w^* - c, w^* - c_1]$. Therefore, because $\Omega_2^0 \setminus \Omega_2^{\pi_1} = [w^* - c, w^* - c_1] \cap A \subseteq [w^* - c, w^* - c_1]$, we have

$$\int_{\Omega_2^0 \setminus \Omega_2^{\pi_1}} g(w) (A_1 - D_0 (w^* - w)) dw \leq \int_{[w^* - c, w^* - c_1]} g(w) (A_1 - D_0 (w^* - w)) dw$$

$$= (c - c_1) g(w^* - c) (A_1 - D_0 c) + O((c - c_1)^2) = O((c - c_1)^2)$$

where the first equality follows from the fact that $g$ is $C^1$ and the second equality follows from the fact that $c = \frac{A_1}{D_0}$.

Claim 9.3. $c - c_1 = \frac{\eta \int_{Z_\pi \cap B} g(w) (w^* - w) dw}{D_1} + O((c - c_1)^2)$.

Proof of claim 9.3.

We first show that

$$c - c_1 = \frac{\eta \int_{Z_\pi \cap B} g(w) (w^* - w) dw}{D_1} - A_1 \frac{\int_{A \setminus \Omega_2^{\pi_1}} g(w) dw}{D_1 D_0} + \frac{\int_{A \setminus \Omega_2^{\pi_1}} g(w) (w^* - w) dw}{D_1}$$

Recall that, by lemma 8, we have $\pi(\sigma|w) = \frac{e}{w^* - w}$ for all $w \in A \setminus \Omega_2^0$ and $\pi_1(\sigma|w) = \frac{c_1}{w^* - w}$ for all $w \in A \setminus \Omega_2^{\pi_1}$. This implies that

$$\int_{A \setminus \Omega_2^0} g(w) \pi(\sigma|w) (w^* - w) dw = \int_{A \setminus \Omega_2^0} \frac{c}{w^* - w} g(w) (w^* - w) dw = c \int_{A \setminus \Omega_2^0} g(w) dw$$

(31)

Recall also that $\pi(\sigma|w) = Z_\pi (w)$ for all $w \in A$. This implies that $\pi(\sigma|w') = \frac{\int_{[w^*, h]} (w - w^*) g(w) dw - \int_{\Omega_2^0} g(w) \pi(\sigma|w) (w^* - w) dw}{(w^* - w')^{\frac{1}{1-a}}} = \frac{\int_{[w^*, h]} (w - w^*) g(w) dw - \int_{\Omega_2^0} g(w) \pi(\sigma|w) (w^* - w) dw}{(w^* - w')^{\frac{1}{1-a}}}$ for all $w' \in A \setminus \Omega_2^0$. Substituting in $\pi(\sigma|w') = \frac{e}{w^* - w'}$ on the left hand side of the expression, we obtain

$$\frac{e}{w^* - w'} = \frac{\int_{[w^*, h]} (w - w^*) g(w) dw - \int_{\Omega_2^0} g(w) \pi(\sigma|w) (w^* - w) dw}{(w^* - w')^{\frac{1}{1-a}}}$$

(32)
for all $w' \in A \setminus \Omega^*_2$. Multiplying both sides by $(w^* - w')\frac{a}{1-a}$, we get

\[
c\left(\frac{a}{1-a}\right) = \int_{[w^*, h]} (w - w')g(w)dw - \int_{A \setminus \Omega^*_2} g(w)\pi(\sigma | w)(w^* - w)dw - \int_{B \setminus I_{x, c}} g(w)\pi(\sigma | w)(w^* - w)dw
\]

\[
- \int_{I_{x, c} \cap B} \pi(\sigma | w)(g(w)w^* - w)dw - \int_{\Omega^*_2} g(w)(w^* - w)dw
\]

\[
= \int_{[w^*, h]} (w - w')g(w)dw - c\int_{A \setminus \Omega^*_2} g(w)dw - \int_{B \setminus I_{x, c}} g(w)\pi(\sigma | w)(w^* - w)dw
\]

\[
- \int_{I_{x, c} \cap B} \pi(\sigma | w)(g(w)w^* - w)dw - \int_{\Omega^*_2} g(w)(w^* - w)dw
\]

where the second equality follows from (31).

Solving the above for $c$, we obtain

\[
c = \left(\int_{[w^*, h]} (w - w')g(w)dw - \int_{B \setminus I_{x, c}} g(w)\pi(\sigma | w)(w^* - w)dw
\]

\[
- \int_{I_{x, c} \cap B} g(w)\pi(\sigma | w)(w^* - w)dw - \int_{\Omega^*_2} g(w)(w^* - w)dw\right)\frac{a}{1-a} + \frac{1}{\int_{A \setminus \Omega^*_2} g(w)dw}
\]

Similarly, $c_1 = \frac{\int_{[w^*, h]} (w - w')g(w)dw - \int_{B \setminus I_{x, c}} g(w)\pi(\sigma | w)(w^* - w)dw - \int_{I_{x, c} \cap B} g(w)\pi(\sigma | w)(w^* - w)dw - \int_{\Omega^*_2} g(w)(w^* - w)dw}{\int_{A \setminus \Omega^*_2} g(w)dw}$. \[\square\]

Observe that $c_1 = \frac{A_1}{D_1} - \frac{\eta \int_{I_{x, c} \cap B} g(w)(w^* - w)dw}{D_1} + \frac{\int_{\Omega^*_2 \setminus \Omega^*_{21}} g(w)(w^* - w)dw}{D_1}$ and \(\frac{A_1}{D_1} = c + A_1 \frac{\int_{\Omega^*_2 \setminus \Omega^*_{21}} g(w)dw}{D_1 D_0}\). This implies that

\[
c_1 - c = A_1 \frac{\int_{\Omega^*_2 \setminus \Omega^*_{21}} g(w)dw}{D_1 D_0} - \frac{\eta \int_{I_{x, c} \cap B} g(w)(w^* - w)dw}{D_1} + \frac{\int_{\Omega^*_2 \setminus \Omega^*_{21}} g(w)(w^* - w)dw}{D_1}
\]

By claim 9.2, we have $A_1 \frac{\int_{\Omega^*_2 \setminus \Omega^*_{21}} g(w)dw}{D_1 D_0} - \frac{\eta \int_{I_{x, c} \cap B} g(w)(w^* - w)dw}{D_1} = O((c - c_1)^2)$. This implies that we can write (32) as $c - c_1 = \frac{\eta \int_{I_{x, c} \cap B} g(w)(w^* - w)dw}{D_1} + O((c - c_1)^2)$, as required. \[\square\]

Define the derivative of the sender’s payoff with respect to the transformation of the signal structure from $\pi$ to $\pi_1$ given $x \in B$ as $E_x^{\pi} = \lim_{\epsilon \to 0} \frac{U(\pi_1) - U(\pi)}{\epsilon}$.

Claim 9.4. $E_x^{\pi} = \frac{f_s(x)}{g(x)(w^* - x)} - \frac{1}{1-a^\pi} + \int_{A \setminus \Omega^*_2} f_s(w)\frac{g(w)(w^* - w)dw}{w^* - w} D_1$.

Proof of claim 9.4.
Approximating the integrals, we have
\[ \int_{I_{x,\epsilon} \cap B} f_s(w) \, dw \leq \int_{I_{x,\epsilon}} f_s(w) \, dw = \epsilon f_s(x) + O(\epsilon^2) \quad (33) \]
and
\[ -\int_{[w^*-c, w^*-c_1] \cap A} \left( \frac{c_1}{w^*-w} - 1 \right) f_s(w) \, dw \leq -\int_{[w^*-c, w^*-c_1]} \left( \frac{c_1}{w^*-w} - 1 \right) f_s(w) \, dw \]
\[ = (c - c_1) \left( \frac{1}{c_1 - 1} - 1 \right) f_s(w^* - c_1) + O((c - c_1)^2) \]
\[ = (c - c_1) \left( \frac{1}{c_1 - 1} - 1 \right) f_s(w^* - c_1) + O((c - c_1)^2) = O((c - c_1)^2) \quad (34) \]
where the first inequality in (34) follows because \(\frac{c_1}{w^*-w} - 1 \leq 0\) for all \(w \in [w^*-c, w^*-c_1]\) and the first equality follows from the fact that \(f_s\) is \(C^1\).

By claim 9.1
\[ U(\pi_1) - U(\pi) = \eta \int_{I_{x,\epsilon} \cap B} f_s(w) \, dw - \int_{A \setminus \Omega^2} \frac{c - c_1}{w^* - w} f_s(w) \, dw - \int_{[w^*-c, w^*-c_1] \cap A} \left( \frac{c_1}{w^*-w} - 1 \right) f_s(w) \, dw \quad (35) \]

(33) and (34) imply that we can write (35) as
\[ U(\pi_1) - U(\pi) = \eta \left( \epsilon f_s(x) + O(\epsilon^2) \right) - (c - c_1) \int_{A \setminus \Omega^2} \frac{f_s(w)}{w^* - w} \, dw + O((c - c_1)^2) \quad (36) \]

Observe that \(\int_{I_{x,\epsilon} \cap B} g(w)(w^* - w) \, dw \leq \int_{I_{x,\epsilon}} g(w)(w^* - w) \, dw = \epsilon g(x)(w^* - x) + O(\epsilon^2)\). Then claim 9.3 implies that
\[ c - c_1 = \frac{n(\epsilon g(x)(w^* - x) + O(\epsilon^2))}{D_1} + O((c - c_1)^2) \]
\[ = \frac{n(\epsilon g(x)(w^* - x) + O(\epsilon^2))}{\frac{\alpha}{1-\alpha} + \int_{A \setminus \Omega^2} g(w) \, dw} + O((c - c_1)^2) \quad (37) \]

Note that (37) implies that \(c - c_1 = O(\eta \epsilon), (c - c_1)^2 = O((\eta \epsilon)^2)\) and
\[ \frac{(c - c_1)^2}{\eta \epsilon} = O(\eta) \quad (38) \]
Substituting (37) into (36) and dividing both sides by $\eta\epsilon$, we obtain

$$\frac{U(\pi_1) - U(\pi)}{\eta\epsilon} = f_s(x) + O(\epsilon) - \left(\frac{g(x)(w^* - x) + O(\epsilon)}{1 - a + \int_{A \setminus \Omega^\pi_2} g(w) dw} + \frac{O((c - c_1)^2)}{\eta\epsilon}\right) \int_{A \setminus \Omega^\pi_2} \frac{f_s(w)}{w^* - w} dw + \frac{O((c - c_1)^2)}{\eta\epsilon}$$

Because, by (38), we have $\frac{(c - c_1)^2}{\eta\epsilon} = O(\eta\epsilon)$, we can write the above as

$$\frac{U(\pi_1) - U(\pi)}{\eta\epsilon} = f_s(x) + O(\epsilon) - \left(\frac{g(x)(w^* - x) + O(\epsilon)}{1 - a + \int_{A \setminus \Omega^\pi_2} g(w) dw} + O(\eta\epsilon)\right) \int_{A \setminus \Omega^\pi_2} \frac{f_s(w)}{w^* - w} dw + O(\eta\epsilon)$$

Note that $\lim_{\epsilon \to 0} \Omega^\pi_{2, \epsilon} = \Omega^\pi_2$. Then, taking the limit of both sides as $\epsilon \to 0$, we obtain

$$\lim_{\epsilon \to 0} \frac{U(\pi_1) - U(\pi)}{\eta\epsilon} = \frac{f_s(x)}{g(x)(w^* - x)} - \frac{1}{1 - a + \int_{A \setminus \Omega^\pi_2} g(w) dw} \int_{A \setminus \Omega^\pi_2} \frac{f_s(w)}{w^* - w} dw$$

Observe that, in order for the sender to not have a strictly improving deviation, we need that for all $x \in B$ such that for all intervals $I$ satisfying $x \in \text{int}(I)$ we have $\mu(I \cap B) > 0$, either $E^\pi_x = 0$ or $E^\pi_x > 0$ and $\pi(\sigma|x) = 1$ (note that if $\pi(\sigma|x) = 0$, then $x \notin B$). Thus, because $E^\pi_x = \frac{f_s(x)}{g(x)(w^* - x)} - \frac{1}{1 - a + \int_{A \setminus \Omega^\pi_2} g(w) dw} \int_{A \setminus \Omega^\pi_2} \frac{f_s(w)}{w^* - w} dw$ by claim 9.4 we need that

$$\frac{f_s(x)}{g(x)(w^* - x)} - \frac{1}{1 - a + \int_{A \setminus \Omega^\pi_2} g(w) dw} \int_{A \setminus \Omega^\pi_2} \frac{f_s(w)}{w^* - w} dw = 0$$

(39)

for almost all $x \in B$ satisfying $\pi(\sigma|x) < 1$.

However, by Assumption 1, $\mu\left(\left\{x \in [l, w^*): \frac{f_s(x)}{g(x)(w^* - x)} = c_0\right\}\right) = 0$ for all $c_0 > 0$. This implies that almost everywhere on $\Omega^\pi_1$, (39) fails. Thus if $\pi(\sigma|w) \leq Z_\pi(w)$ holds strictly on a set of a strictly positive measure, then the sender has a strictly improving deviation, which contradicts the optimality of $\pi$. Therefore, the inequality $\pi(\sigma|w) \leq Z_\pi(w)$ must be satisfied with equality almost everywhere on $\Omega^\pi_1$.

We now prove the second part of the lemma. Fix the sender’s prior $f_s$ (and $g$) such that Assumption 1 fails. Observe that there exists a sequence of functions $\{f^n_s\}$ converging to $f_s$ such that for each $f^n_s$ Assumption 1 is satisfied. The first part of the lemma and the assumption that for all parameter values there exists a signal structure satisfying $\pi(\sigma|w') = Z_\pi(w') \mu_{F_s}$-a.e. $w' \in \Omega^\pi_1$ and $\pi(\sigma|w) = 1 \mu_{F_s}$-a.e. $w \in [w', h]$ imply that for each $f^n_s$ there
exists a sender-optimal signal structure \( \pi_n \) with two realizations satisfying \( \pi_n(\sigma|w) = Z_{\pi_n}(w) \) for almost all \( w \in \Omega_1^n \). Because \( \{\pi_n\} \) is an infinite sequence in a compact set, it has a convergent subsequence. Without loss of generality, assume that this is the sequence itself, and let \( \pi^* \) denote the signal structure that it converges to. Observe that, because \( \pi_n(\sigma|w) = Z_{\pi_n}(w) \) for almost all \( w \in \Omega_1^n \) for all \( n \), we have \( \pi^*(\sigma|w) = Z_{\pi^*}(w) \) for almost all \( w \in \Omega_1^{\pi^*} \).

Let \( U(\pi, f_s) = \int_{[t,h]} \pi(\sigma|w)f_s(w)dw \) denote the sender’s payoff from the signal structure \( \pi \) given that the sender’s prior is \( f_s \). Suppose for the sake of contradiction that there exists a signal structure \( \tilde{\pi} \) with two realizations that is sender-optimal given \( f_s \) such that \( \tilde{\pi}(\sigma|w) < Z_{\tilde{\pi}}(w) \) for a subset \( B \subseteq \Omega_1^\pi \) of a strictly positive measure and \( U(\tilde{\pi}, f_s) > U(\pi^*, f_s) \).

Observe that the fact that \( f_s \mapsto U(\pi, f_s) \) is continuous implies that for all \( \delta > 0 \) there exists \( N \) such that for all \( n > N \), \( |U(\tilde{\pi}, f_s) - U(\tilde{\pi}, f_{s_0}^n)| < \delta \) and \( |U(\pi^*, f_s) - U(\pi^*, f_{s_0}^n)| < \delta \). Moreover, because \( \{\pi_n\} \) converges to \( \pi^* \) and \( \pi \mapsto U(\pi, f_s) \) is continuous, for all \( \delta > 0 \) there exists \( N \) such that for all \( n > N \), \( |U(\pi^*, f_{s_0}^n) - U(\pi_n, f_{s_0}^n)| < \delta \).

Note that, because \( U(\tilde{\pi}, f_s) > U(\pi^*, f_s) \) and for all \( \delta > 0 \) there exists \( N \) such that for all \( n > N \), \( |U(\pi^*, f_s) - U(\pi^*, f_{s_0}^n)| < \delta \), there exists \( N_1 \) and \( \delta_1 > 0 \) such that for all \( n > N_1 \), we have \( U(\tilde{\pi}, f_s) - U(\pi^*, f_{s_0}^n) > \delta_1 \).

Fix \( \delta > 0 \). Then there exists \( N_2 \) such that for all \( n > N_2 \), we have \( U(\tilde{\pi}, f_s) - U(\tilde{\pi}, f_{s_0}^n) + U(\pi_n, f_{s_0}^n) - U(\pi^*, f_{s_0}^n) = |U(\tilde{\pi}, f_s) - U(\tilde{\pi}, f_{s_0}^n)| + |U(\pi_n, f_{s_0}^n) - U(\pi^*, f_{s_0}^n)| < 2\delta \). This implies that \( U(\tilde{\pi}, f_{s_0}^n) - U(\pi_n, f_{s_0}^n) > U(\tilde{\pi}, f_s) - U(\pi^*, f_{s_0}^n) - 2\delta \). Therefore, because \( U(\tilde{\pi}, f_s) - U(\pi^*, f_{s_0}^n) > \delta_1 \) for all \( n > N_1 \), we have \( U(\tilde{\pi}, f_{s_0}^n) - U(\pi_n, f_{s_0}^n) > \delta_1 - 2\delta \) for all \( n > \max\{N_2, N_1\} \).

This implies that there exists \( \delta_0 > 0 \) and \( N \) such that for all \( n > N \), \( U(\tilde{\pi}, f_{s_0}^n) > U(\pi_n, f_{s_0}^n) + \delta_0 \). However, this contradicts the hypothesis that the signal structure \( \pi_n \) is sender-optimal given that the sender’s prior is \( f_{s_0}^n \).

\[ \begin{proof} \]

**Lemma 10.** If \( \pi \) is a signal structure that is sender-optimal in the class of all signal structures with two realizations and \( \pi \) satisfies the condition in lemma 9, then there exists a threshold \( t \) such that \( \Omega_1^\pi = \Omega_1 \mu_{\pi^*}-a.e. \) for some \( \Omega_1 \in \Omega_2(t) \).

Either the threshold \( t \geq t \) satisfies \( t = \frac{\int_{\Omega_1^\pi \cap [t, u^* - c]} f_s(w)dw}{\int_{\Omega_1^\pi \cap [t, u^* - c]} g(w)dw} \), where \( c \) is a constant such that \( \pi(\sigma|w) = \min\{\frac{c}{u^* - w}, 1\} \) \( \mu_{\pi^*}-a.e. \) \( w \in \Omega_2^\pi \), or \( t > \frac{\int_{\Omega_1^\pi \cap [t, u^* - c]} f_s(w)dw}{\int_{\Omega_1^\pi \cap [t, u^* - c]} g(w)dw} \) and \( t = t \).

**Proof of lemma 10.**
Let $\Omega_1 = \Omega_1^\pi$. Given $x \in [l, w^*)$ and $\epsilon > 0$ satisfying $x - \frac{\epsilon}{2} \geq l$, $x + \frac{\epsilon}{2} < w^*$, define $I_{x,\epsilon} = [x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}]$. Consider a signal structure $\pi_1$ obtained by adding an interval $I_{x,\epsilon}$ to the support of the signal structure $\pi$ below the threshold $w^*$. That is, $\pi_1$ is a signal structure with two signal realizations such that $\Omega_1^{\pi_1} = \Omega_1^\pi \cup I_{x,\epsilon}$ and $\mu(I_{x,\epsilon} \cap \Omega_1^{\pi_1}) < \epsilon$.

Because $\pi$ satisfies the condition in lemma 9 by lemmas 8 and 9, there exist constants $c$ and $c_1$ such that $\pi(\sigma|w) = \min \left\{ \frac{c}{w^* - w}, 1 \right\}$ $\mu_{w^*}$-a.e. $w \in \Omega_1^\pi = \Omega_1$ and $\pi_1(\sigma|w) = \min \left\{ \frac{c_1}{w^* - w}, 1 \right\} \mu_{w^*}$-a.e. $w \in \Omega_1^{\pi_1} = \Omega_1 \cup I_{x,\epsilon}$.

Define $D_0$, $D_1$, $D_2$, $q$ and $p$ as follows.

\[
D_0 = \frac{a}{1-a} + \int_{\Omega_1 \cap [l, w^* - c]} g(w)dw \\
D_1 = \frac{a}{1-a} + \int_{(\Omega_1 \cup I_{x,\epsilon}) \cap [l, w^* - c_1]} g(w)dw \\
D_2 = \frac{a}{1-a} + \int_{\Omega_1 \cap [l, w^* - c_1]} g(w)dw
\]

\[
q = \int_{[w^*, h]} (w - w^*)g(w)dw - \int_{\Omega_1 \cap [w^* - c, w^*)} g(w)(w^* - w)dw \\
p = \int_{\Omega_1 \cap [w^* - c, w^*))} g(w)(w^* - w)dw
\]

Observe that, because $g$ is $C^1$, we have $p \leq \int_{[w^* - c, w^*)} g(w)(w^* - w)dw = (c_1 - c)g(w^* - c)c + O((c_1 - c)^2)$. Observe also that $\int_{I_{x,\epsilon} \setminus \Omega_1} g(w)dw \leq \int_{I_{x,\epsilon}} g(w)dw = \epsilon g(x) + O(\epsilon^2)$.

**Claim 10.1.** Suppose that $I_{x,\epsilon} \subseteq [l, w^* - c_1]$. Then $c - c_1 = \frac{\epsilon}{D_1} + O((c - c_1)^2) + O(\epsilon^2)$ and $c_1 = \frac{q + (c_1 - c)g(w^* - c)}{D_1} + O((c - c_1)^2)$.

**Proof of claim 10.1.**

Recall that, by lemma 8, \[
c = \frac{\int_{[w^*, h]} (w - w^*)g(w)dw - \int_{\Omega_1 \cap [w^* - c_1, w^*)} g(w)(w^* - w)dw}{\frac{a}{1-a} + \int_{(\Omega_1 \cup I_{x,\epsilon}) \cap [l, w^* - c_1]} g(w)dw} = \frac{q}{D_1} \quad \text{and} \quad c_1 = \frac{\int_{[w^*, h]} (w - w^*)g(w)dw - \int_{\Omega_1 \cap [w^* - c_1, w^*)} g(w)(w^* - w)dw}{\frac{a}{1-a} + \int_{(\Omega_1 \cup I_{x,\epsilon}) \cap [l, w^* - c_1]} g(w)dw} = \frac{q}{D_1} - \frac{p}{D_1}, \quad \text{so that}
\]

\[
c_1 = c - c_1 = \frac{D_0 - D_1}{D_1 D_0} - \frac{p}{D_1}, \quad \text{(40)}
\]

We have

\[
D_0 - D_1 = \int_{\Omega_1 \cap [l, w^* - c]} g(w)dw - \int_{(\Omega_1 \cup I_{x,\epsilon}) \cap [l, w^* - c_1]} g(w)dw
\]

\[
= \int_{\Omega_1 \cap [w^* - c, w^*))} g(w)dw - \int_{I_{x,\epsilon} \setminus \Omega_1} g(w)dw
\]

\[
= (c_1 - c)g(w^* - c) - \epsilon g(x) + O((c - c_1)^2) + O(\epsilon^2)
\]
Substituting the expressions for $D_0 - D_1$ in (41) and $p = (c_1 - c)g(w^* - c) + O((c - c_1)^2)$ into the formula for $c_1 - c$ in (40) and multiplying both sides by $D_1D_0$, we obtain
\begin{equation}
D_1D_0(c_1 - c) = q(c_1 - c)g(w^* - c) - \epsilon g(x)q + O((c_1 - c)^2) + O(\epsilon^2)
\end{equation}

Observe that the fact that $c = \frac{q}{D_0}$ implies that
\begin{equation}
q(c_1 - c)g(w^* - c) - D_0(c_1 - c)g(w^* - c)c = 0
\end{equation}

Then (43) implies that (42) is equivalent to $c - c_1 = \frac{qg(x)q}{D_1D_0} + O((c_1 - c)^2) + O(\epsilon^2)$.

**Claim 10.2.** Suppose that $\Omega_{\pi,\epsilon} \subseteq [l, w^* - c)$. Then $U(\pi_{\epsilon}) > U(\pi)$ if and only if $c_1 \epsilon_{w^*-x} \geq (c - c_1)\int_{\Omega_{\pi,\epsilon} \cap [l, w^* - c]} \frac{f_s(w)}{w^* - w}dw - (c_1 - c)\epsilon_{w^* - c} + O((c_1 - c)^2) + O(\epsilon^2)$.

**Proof of claim 10.2.**

Observe that $U(\pi_{\epsilon}) > U(\pi)$ if and only if $\int_{\Omega_1} \pi_1(\sigma|w)f_s(w)dw + \int_{\Omega_{\pi,\epsilon} \cap \Omega_1} \pi_1(\sigma)w|f_s(w)dw > \int_{\Omega_1} \pi(\sigma|w)f_s(w)dw$.

Observe also that, by lemma 8, $\pi(\sigma|w) = \frac{\epsilon}{w^* - w} \mu_{Fs}$-a.e. $w \in \Omega_1 \cap [l, w^* - c)$, $\pi_1(\sigma|w) = \frac{c_1}{w^* - w} \mu_{Fs}$-a.e. $w \in (\Omega_1 \cup \Omega_{\pi,\epsilon}) \cap [l, w^* - c)$, $\pi_1(\sigma|w) = 1$ $\mu_{Fs}$-a.e. $w \in \Omega_1 \cap [w^* - c, w^*)$ and $\pi_1(\sigma|w) = 1$ for $\mu_{Fs}$-a.e. $w \in (\Omega_1 \cup \Omega_{\pi,\epsilon}) \cap [w^* - c, w^*)$.

If $\Omega_{\pi,\epsilon} \subseteq [l, w^* - c)$, then $U(\pi_{\epsilon}) > U(\pi)$ is equivalent to
\begin{align*}
c_1 \int_{\Omega_{\pi,\epsilon} \cap \Omega_1} \frac{f_s(w)}{w^* - w}dw > & \int_{\Omega_{\pi,\epsilon} \cap [l, w^* - c]} \epsilon_{w^* - c} \frac{f_s(w)}{w^* - w}dw \nonumber \\
- & c_1 \int_{\Omega_{\pi,\epsilon} \cap [w^* - c, w^* - c_1]} \frac{f_s(w)}{w^* - w}dw + \int_{\Omega_{\pi,\epsilon} \cap [w^* - c, w^* - c_1]} f_s(w)dw \nonumber
\end{align*}

Approximating the integrals, because $\int_{\Omega_{\pi,\epsilon} \cap \Omega_1} \frac{f_s(w)}{w^* - w}dw \leq \int_{\Omega_{\pi,\epsilon} \cap [l, w^* - c]} \frac{f_s(w)}{w^* - w}dw = (c_1 - c)\epsilon_{w^* - c} + O((c_1 - c)^2)$ and $\int_{\Omega_{\pi,\epsilon} \cap [w^* - c, w^* - c_1]} f_s(w)dw \lesssim \int_{[w^* - c, w^* - c_1]} f_s(w)dw = (c_1 - c)\epsilon_{w^* - c} + O((c_1 - c)^2)$, we find that the above is equivalent to $c_1 \epsilon_{f_s(w)} \geq (c_1 - c)\int_{\Omega_{\pi,\epsilon} \cap [l, w^* - c)} \frac{f_s(w)}{w^* - w}dw - (c_1 - c)\epsilon_{w^* - c} + O((c_1 - c)^2) + O(\epsilon^2)$.

**Claim 10.3.** If $\frac{f_s(x)}{w^* - x} g(x) > \frac{\int_{\Omega_{\pi,\epsilon} \cap [l, w^* - c]} \frac{f_s(w)}{w^* - w}dw}{\int_{\Omega_{\pi,\epsilon} \cap [l, w^* - c]} g(w)dw}$, then $U(\pi_{\epsilon}) > U(\pi)$. If $U(\pi_{\epsilon}) \geq U(\pi)$, then
\[ f_s(x) \frac{1}{w^*-x} g(x) \geq \frac{f_{\Omega_1}([l,w^*-c])}{f_{\Omega_1}([l,w^*-c]) g(x) dw} f_s(w) \frac{f_s(w)}{w^*-w} dw. \]

**Proof of claim 10.4.**

We will prove the second statement of the claim. The proof of the first statement of the claim is analogous. Moreover, we will prove the statement of the claim for the case in which \( I_{x,\epsilon} \subseteq [l, w^* - c_1] \). The proof of the statement in the other cases is analogous.

Suppose that \( U(\pi_1) \geq U(\pi) \). By claim 10.2 this implies that

\[ c_1 \frac{f_s(w_1)}{w^*-w_1} \geq (c - c_1) \int_{\Omega_1 \cap [l,w^*-c]} \frac{f_s(w)}{w^*-w} - (c - c_1)^2 \frac{f_s(w^*-c)}{c_1} + O((c - c_1)^2) + O(c^2) \quad (44) \]

By claim 10.1 \( c - c_1 = \frac{g(x)q}{D_1 D_2} + O((c - c_1)^2) + O(c^2) \) and \( c_1 = \frac{g(c_1)q}{D_1 D_2} + O((c_1 - c_1)^2) \).

Substituting the formulas for \( c_1 \) and \( c - c_1 \) from claim 10.1 into (44) and dividing both sides by \( \epsilon \), we obtain

\[ \frac{g(x)q}{D_1 D_2} \int_{\Omega_1 \cap [l,w^*-c]} f_s(w) \frac{f_s(w)}{w^*-w} dw + O((c - c_1)^2) + O(c^2). \]

Taking the limit as \( \epsilon \to 0 \), because \( \lim_{\epsilon \to 0} \frac{O((c - c_1)^2) + O(c^2)}{\epsilon} = 0 \), we get

\[ \frac{g(x)q}{D_1 D_2} \int_{\Omega_1 \cap [l,w^*-c]} f_s(w) \frac{f_s(w)}{w^*-w} dw. \]

Equivalently, \( f_s(w_1) \frac{1}{w^*-w_1} g(w_1) \geq \frac{f_{\Omega_1}([l,w^*-c])}{f_{\Omega_1}([l,w^*-c]) g(w) dw} f_s(w) \frac{f_s(w)}{w^*-w} dw \), as required.

**Claim 10.4.** If \( w_1 \notin \Omega_1^\pi \), then \( w_2 \notin \Omega_1^\pi \ \mu_{F_{\pi}} - a.e. \ w_2 \in [l, w^*] \) such that \( \frac{f_s(w_1)}{w^*-w_1} g(w_1) > \frac{f_s(w_2)}{w^*-w_2} g(w_2) \).

**Proof of claim 10.4.**

If \( w_1 \in [l, w^*] \) is such that \( w_1 \notin \Omega_1^\pi \) and \( \pi \) is a sender-optimal signal structure, then adding an interval \( I_{w_1,\epsilon} \) to the support does not strictly benefit the sender. That is, then \( U(\pi_1) \leq U(\pi) \), where \( \pi_1 \) is a signal structure obtained by adding an interval \( I_{w_1,\epsilon} \) to \( \Omega_1^\pi \). By claim 10.3 this implies that

\[ \frac{f_s(w_1)}{w^*-w_1} g(w_1) \leq \frac{f_{\Omega_1 \cap [l,w^*-c]} f_s(w)}{f_{\Omega_1 \cap [l,w^*-c]} g(w) dw} \]

\[ (45) \]

implies that for any \( w_2 \in [l, w^*] \) such that \( \frac{f_s(w_1)}{w^*-w_1} g(w_1) > \frac{f_s(w_2)}{w^*-w_2} g(w_2) \) we have

\[ \frac{f_{\Omega_1 \cap [l,w^*-c]} f_s(w_1)}{f_{\Omega_1 \cap [l,w^*-c]} g(w) dw} > \frac{f_{\Omega_1 \cap [l,w^*-c]} f_s(w_2)}{f_{\Omega_1 \cap [l,w^*-c]} g(w) dw}. \]

By claim 10.3 this implies that \( U(\pi_2) < U(\pi) \), where \( \pi_2 \) is a signal structure obtained by adding an interval \( I_{w_2,\epsilon} \) to \( \Omega_1^\pi \). Therefore, \( w_2 \notin \Omega_1^\pi \ \mu_{F_{\pi}} - a.e. \ w_2 \in [l, w^*] \) such that \( \frac{f_s(w_1)}{w^*-w_1} g(w_1) > \frac{f_s(w_2)}{w^*-w_2} g(w_2) \), as required.
Thus claim 10.4 implies that for \( \mu_F \)-a.e. \( w_1, w_2 \in [l, w^*], \) if \( w_2 \in \Omega_1^\pi \) and
\[
\frac{f_s(w_1)}{w^*-w_1} \frac{1}{g(w_1)} > \frac{f_s(w_2)}{w^*-w_2} \frac{1}{g(w_2)},
\]
then \( w_1 \in \Omega_1^\pi \). Therefore, \( \Omega_1^\pi = \Omega_1 \) \( \mu_F \)-a.e. for some \( \Omega_1 \in \Omega_1(t) \), as required. Observing that the threshold \( t \geq t \) satisfies either
\[
t > \frac{f_s(w_1)}{w^*-w_1} \frac{1}{g(w_1)} d\omega w
\]
and \( t = t \) concludes the proof.

Lemma 11. \( \Omega_1 \mapsto \hat{c}(\Omega_1) \) is a well-defined function. If Assumption \( \text{I} \) is satisfied, then \( t \mapsto c(t) \) is a continuous and strictly increasing function for \( t \geq t \).

Proof of lemma 11

Let \( \hat{x}(c, \Omega_1) = \int_{[w^*, l]} (w - w^*) g(w) dw - \int_{\Omega_1 \cap [w^*-c, w^*]} g(w)(w^* - w) dw - c \left( \frac{a}{1-a} + \int_{\Omega_1 \cap [l, w^*-c]} g(w) dw \right) \), so that \( \Omega_1 \mapsto \hat{c}(\Omega_1) \) is the mapping satisfying \( \hat{x}(c, \Omega_1) = 0 \).

We have
\[
\hat{x}(c + \eta, \Omega_1) - \hat{x}(c, \Omega_1) = -\eta \left( \frac{a}{1-a} + \int_{\Omega_1 \cap [l, w^*-c-\eta]} g(w) dw \right) + O(\eta^2)
\]
because \( g \) is \( C^1 \). Then \( \lim_{\eta \to 0} \frac{\hat{x}(c + \eta, \Omega_1) - \hat{x}(c, \Omega_1)}{\eta} = -\lim_{\eta \to 0} \left( \frac{a}{1-a} + \int_{\Omega_1 \cap [l, w^*-c-\eta]} g(w) dw \right) = -\left( \frac{a}{1-a} + \int_{\Omega_1 \cap [l, w^*-c]} g(w) dw \right) < 0. \)

We have \( \hat{x}(0, \Omega_1) = \int_{[w^*, l]} (w - w^*) g(w) dw > 0 \) and \( \hat{x}(\infty, \Omega_1) = -\infty. \) Because \( c \mapsto \hat{x}(c, \Omega_1) \) is continuous and strictly decreasing, there exists a unique \( c \in (0, \infty) \) such that \( \hat{x}(c, \Omega_1) = 0 \), which implies that \( \Omega_1 \mapsto \hat{c}(\Omega_1) \) is a well-defined function, as required.

Recall that
\[
x(c, t) = \int_{[w^*, l]} (w - w^*) g(w) dw - \int_{\Omega_1(t) \cap [w^*-c, w^*]} g(w)(w^* - w) dw - c \left( \frac{a}{1-a} + \int_{\Omega_1(t) \cap [l, w^*-c]} g(w) dw \right)
\]
and, because \( c(t) = \frac{f_s(w(t)) g(w(t)-w^*) - \int_{\Omega_1(t) \cap [w^*-c(t), w^*]} g(w(t))(w^* - w) dw}{\frac{a}{1-a} + \int_{\Omega_1(t) \cap [l, w^*-c(t)]} g(w) dw} \), \( t \mapsto c(t) \) is the mapping satisfying \( x(c, t) = 0 \).
We have

\[ x(c + \eta, t) - x(c, t) = -\int_{\Omega_1(t) \cap [w^* - c, w^*]} g(w)(w^* - w) dw + \int_{\Omega_1(t) \cap [w^* - c, w^*]} g(w)(w^* - w) dw \]

\[ -(c + \eta) \left( \frac{a}{1 - a} + \int_{\Omega_1(t) \cap [l, w^* - c]} g(w) dw \right) + c \left( \frac{a}{1 - a} + \int_{\Omega_1(t) \cap [l, w^* - c]} g(w) dw \right) \]

\[ = -\int_{\Omega_1(t) \cap [w^* - c - \eta, w^*]} g(w)(w^* - w) dw + \int_{\Omega_1(t) \cap [w^* - c - \eta, w^*]} g(w)(w^* - w) dw \]

\[ -\eta \left( \frac{a}{1 - a} + \int_{\Omega_2(t) \cap [l, w^* - c - \eta]} g(w) dw \right) = -g(w^* - c) c + g(w^* - c) \]

\[ = -\eta \left( \frac{a}{1 - a} + \int_{\Omega_2(t) \cap [l, w^* - c - \eta]} g(w) dw \right) + O(\eta^2) \]

where the penultimate equality follows from the fact that \( g \) is \( C^1 \). Then \( \lim_{\eta \to 0} \frac{x(c + \eta, t) - x(c, t)}{\eta} = -\lim_{\eta \to 0} \left( \frac{a}{1 - a} + \int_{\Omega_1(t) \cap [l, w^* - c - \eta]} g(w) dw \right) = -\left( \frac{a}{1 - a} + \int_{\Omega_1(t) \cap [l, w^* - c]} g(w) dw \right) < 0. \]

Observe that, because \( \mu \left( \left\{ w \in [l, w^*] : \frac{f_s(w)}{g(w)(w^* - w)} = c_0 \right\} \right) = 0 \) for all \( c_0 > 0 \) by Assumption 1 and \( f_s \) and \( g \) are continuous, we have \( \Omega_1(t + \eta) \cap [l, w^* - c] \subset \Omega_1(t) \cap [l, w^* - c] \) for all \( t \geq t \) and for all \( \eta > 0 \) sufficiently small. Then

\[ x(c, t + \eta) - x(c, t) = -\int_{\Omega_1(t) \cap [l, w^*]} g(w)(w^* - w) dw - c \left( \frac{a}{1 - a} + \int_{\Omega_1(t) \cap [l, w^* - c]} g(w) dw \right) \]

\[ = \int_{\Omega_1(t) \cap [l, w^*]} g(w)(w^* - w) dw + c \left( \frac{a}{1 - a} + \int_{\Omega_1(t) \cap [l, w^* - c]} g(w) dw \right) \]

Because \( t \mapsto c(t) \) is the mapping satisfying \( x(c, t) = 0 \), the fact that \( \lim_{\eta \to 0} \frac{x(c + \eta, t) - x(c, t)}{\eta} < 0 \) and \( x(c, t + \eta) - x(c, t) > 0 \) for all \( \eta > 0 \) by the arguments above implies that \( t \mapsto c(t) \) is a strictly increasing function, as required.

Observe that \((c, t) \mapsto x(c, t)\) is continuous because

\[ \mu \left( \left\{ w \in [l, w^*] : \frac{f_s(w)}{g(w)(w^* - w)} = c_0 \right\} \right) = 0 \] for all \( c_0 > 0 \) by Assumption 1 and \( f_s \) and \( g \) are continuous. The continuity of \( t \mapsto c(t) \) then follows from the fact that \((c, t) \mapsto x(c, t)\) is continuous.

**Lemma 12.** Let \( T = \{ t : |y(t)| > 1 \} \) and let \( y_0(t, \Omega_1) = t \left( \frac{a}{1 - a} + \int_{\Omega_1 \cap [l, w^* - c(\Omega_1)]} g(w) dw \right) - \)
\[ \int_{\Omega \cap \{w = c(\Omega)\}} \frac{f_k(w)}{w^r - w} \, dw \]. Then

1. \( T \) is at most countable

2. for any selection \( t \mapsto \hat{y}(t) \) from the correspondence \( t \mapsto \bar{y}(t) \), if \( t \leq t_1 < t_2 \), then \( \hat{y}(t_1) < \hat{y}(t_2) \)

3. \( \bar{y}(t) \) is an interval

4. if \( |\bar{y}(t)| = 1 \) for all \( t \in (t_1, t_2) \), then \( t \mapsto \bar{y}(t) \) is continuous on \((t_1, t_2)\)

5. \( \inf \bar{y}(t_0) = \lim_{t \uparrow t_0} \bar{y}(t), \sup \bar{y}(t_0) = \lim_{t \downarrow t_0} \bar{y}(t) \)

6. \( \bar{y}(t) = y_0(t, \Omega_1), \Omega_1 \in \{ \Omega(t, z) \}_{z \in \{ w \in \{l, w^r\} : \frac{f_k(w)}{g(w)(w^r - w)} = \tau \}} \)

**Proof of lemma 12.**

Note that \( |T| \leq \left| \left\{ c_0 : \mu \left( \left\{ w : \frac{f_k(w)}{g(w)(w^r - w)} = c_0 \right\} \right) > 0 \right\} \right| \), which implies that \( T \) is at most countable.

Fix \( \Omega_1, \Omega_2 \) such that \( \Omega_1 \subset \Omega_2, \Omega_1 \in \overline{\Omega_1}(t + \eta) \) and \( \Omega_2 \in \overline{\Omega_1}(t) \) for some \( \eta \geq 0 \) and \( t \geq t \). We will show that \( y_0(t + \eta, \Omega_1) - y_0(t, \Omega_2) > 0 \) for all \( \eta \geq 0 \).

A proof similar to the proof that \( t \mapsto c(t) \) is strictly increasing for \( t \geq t \) in lemma 11 can be used to show that \( \tilde{c}(\Omega_2) > \tilde{c}(\Omega_1) \). Then a proof similar to the proof that \( y(t + \eta) - y(t) > 0 \) for all \( t \geq t \) and for all \( \eta > 0 \) in the proof of the second part of lemma 13 can be used to establish that \( y_0(t + \eta, \Omega_1) - y_0(t, \Omega_2) > 0 \).

Let \( \Omega_0(t) = \left\{ w : \frac{f_k(w)}{g(w)(w^r - w)} > t \right\} \). Observe that the argument above implies that \( \inf \bar{y}(t) = y_0(t, \Omega_1(t)) \) and \( \sup \bar{y}(t) = y_0(t, \Omega_0(t)) \). It can be shown that for any \( y' \in (y_0(t, \Omega_1(t)), y_0(t, \Omega_0(t))) \) there exists \( \Omega_1 \) such that \( \Omega_0(t) \subseteq \Omega_1 \subseteq \Omega_1(t) \) and \( y_0(t, \Omega_1) = y' \). Then \( \bar{y}(t) \) is an interval, as required.

Suppose that \( |\bar{y}(t)| = 1 \) for all \( t \in (t_1, t_2) \). Observe that then for all \( t \in (t_1, t_2) \) we have \( \Omega_1 = \Omega_2 \) a.e. for all \( \Omega_1, \Omega_2 \in \overline{\Omega_1}(t) \). This implies that \( \Omega_1 = \Omega_1(t) \) a.e. for all \( \Omega_1 \in \overline{\Omega_1}(t) \) and \( \bar{y}(t) = y(t) \) for all \( t \in (t_1, t_2) \). Moreover, \( t \mapsto \tilde{c}(\Omega_1(t)) \) and \( t \mapsto \mu(\Omega_1(t)) \) are continuous on \((t_1, t_2)\). Therefore, \( t \mapsto \bar{y}(t) \) is continuous on \((t_1, t_2)\), as required.

Observe that \( \lim_{t \uparrow t_0} \Omega_1(t) = \Omega_1(t_0) \) and \( \lim_{t \downarrow t_0} \Omega_1(t) = \Omega_0(t_0) \) a.e. Because \( \Omega_1 \mapsto \tilde{c}(\Omega_1) \) is continuous, this implies that \( \inf \bar{y}(t_0) = y_0(t_0, \Omega_1(t_0)) = \lim_{t \uparrow t_0} y(t) = \lim_{t \downarrow t_0} \bar{y}(t), \sup \bar{y}(t_0) = y_0(t_0, \Omega_0(t)) = \lim_{t \downarrow t_0} y(t) = \lim_{t \uparrow t_0} \bar{y}(t), \) as required.

To prove the last claim, define \( y_1(t) = y_0(t, \Omega_1(t)), y_2(t) = y_0(t, \Omega_0(t)) \). Fix \( t \geq 0 \) and \( \Omega_1 \in \overline{\Omega_1}(t) \). Note that \( \bar{y}(t) = \bar{y}(t) \) for all \( t < t \). Observe that this and the second claim
in the lemma imply that \( y_0(t, \Omega_1) \in [y_1(t), y_2(t)] \). Let \( Z(t) = \left\{ w \in [l, w^*] : \frac{f_s(w)}{g(w)(w-w')} = t \right\} \) and \( z(t) = \inf Z(t), \quad \Omega(t) = \sup Z(t) \). Observe that \( y_1(t) = y_0(t, \Omega(t, z(t))) \) and \( y_2(t) = y_0(t, \Omega(t, \pi(t))) \). Because \( z \mapsto y_0(t, \Omega(t, z)) \) is continuous, the Intermediate Value Theorem implies that there exists \( z \in Z(t) \) such that \( y_0(t, \Omega(t, z)) = y_0(t, \Omega_1) \).

**Lemma 13.** If \( \pi \) is sender-optimal in the class of all signal structures with two realizations and \( \pi \) satisfies the condition in lemma 9, then there exists a threshold \( t \geq t \) such that \( \Omega_1^\pi = \Omega_1 \mu_{F_s}\text{-a.e. for some } \Omega_1 \in \Omega_1(t) \) and either \( t = j \) for some \( j \in T(t) \) or \( t = \underline{t} \) and \( \underline{t} > j \) for all \( j \in T(t) \) for all \( t' \). Moreover, \( \Omega_1 \) can be chosen such that \( \Omega_1 = \Omega(t, z) \) for some \( z \in [l, w^*] \). If Assumption 1 is satisfied, then there exists a unique threshold \( t \geq t \) satisfying either \( t = J(t) \) or \( t = \underline{t} \) and \( \underline{t} > J(t') \) for all \( t' \). Finally, \( y(t + \eta) - y(t) > 0 \) for all \( t \geq t \) and for all \( \eta > 0 \).

**Proof of Lemma 13.**

Because \( \pi \) satisfies the condition in lemma 9, lemma 10 applies. Then lemma 10 implies that \( \int_{\Omega_1^\pi \cap [l, w^* - c]} \frac{f_s(w)}{w-w'} dw \) and either \( t \) satisfies \( t = \frac{\int_{\Omega_1^\pi \cap [l, w^* - c]} g(w)dw}{\int_{\Omega_1^\pi \cap [l, w^* - c]} g(w)dw} \) where \( c \) is a constant such that \( \pi(\sigma|w) = \frac{c}{w-w'} \mu_{F_s} \text{-a.e. } w \in [l, w^* - c] \cap \Omega_1^\pi \) and \( \pi(\sigma|w) = 1 \mu_{F_s} \text{-a.e. } w \in [w^* - c, w^*) \cap \Omega_1^\pi \) or \( t > \frac{\int_{\Omega_1^\pi \cap [l, w^* - c]} f_s(w)}{\int_{\Omega_1^\pi \cap [l, w^* - c]} g(w)dw} \). Note, because \( \pi \) satisfies the condition in lemma 9, lemma 11 implies that \( c = \hat{c}(\Omega_1^\pi) \). Then, because \( \Omega_1^\pi = \Omega_1 \) for some \( \Omega_1 \in \Omega_1(t) \) by lemma 10, we have \( c = \hat{c}(\Omega_1) \) for some \( \Omega_1 \in \Omega_1(t) \). Because \( J(t) = \frac{\int_{\Omega_1^\pi \cap [l, w^* - c]} g(w)dw}{\int_{\Omega_1^\pi \cap [l, w^* - c]} g(w)dw} \), \( \Omega_1 \in \Omega_1(t) \), either \( t \) satisfies \( t = j \) for some \( j \in J(t) \) or \( t > j \) for all \( j \in J(t') \) for all \( t' \). Thus a necessary condition for \( \pi \) to be sender-optimal is that there is a threshold \( t \geq t \) such that \( \Omega_1^\pi = \Omega_1 \mu_{F_s} \text{-a.e. for some } \Omega_1 \in \Omega_1(t) \) and either \( t = j \) for some \( j \in J(t) \) or \( t > j \) for all \( j \in J(t') \) for all \( t' \) and \( t = \underline{t} \).

I next show that a threshold \( t \geq \underline{t} \) satisfying the above properties exists. Note that, because \( y(t) = t \left( \frac{a}{T-a} + \int_{\Omega_1 \cap [l, w^* - \hat{c}(\Omega_1)]} g(w)dw \right) - \int_{\Omega_1 \cap [l, w^* - \hat{c}(\Omega_1)]} f_s(w) dw, \Omega_1 \in \Omega_1(t) \), we have that \( t = j \) for some \( j \in J(t) \) is equivalent to \( 0 \in \Omega(t) \). Observe that it follows from lemma 12 that \( t \mapsto \bar{y}(t) \) is a convex-valued and upper hemi-continuous correspondence, which implies that the Intermediate Value Theorem for correspondences (see Theorem 9.39 on page 145 in Moore, 1999) applies to it. Next, observe that, because \( y' \leq 0 \) for all \( y' \in \bar{y}(0) \) and \( \lim_{t \to \infty} \bar{y}(t) = \{ \infty \} \), the Intermediate Value Theorem for correspondences implies that there exists \( t^* \in [0, \infty) \) such that \( 0 \in \bar{y}(t^*) \). This implies that there exists \( t^* \geq 0 \) such that \( \Omega_1^{\pi_3} = \Omega_1 \mu_F \text{-a.e. for some } \Omega_1 \in \Omega_1(t^*) \) and \( t^* = j \) for some \( j \in J(t^*) \). Moreover, because the last claim of lemma 12 implies that \( J(t) = \frac{\int_{\Omega_1^\pi \cap [l, w^* - c]} f_s(w)}{\int_{\Omega_1^\pi \cap [l, w^* - c]} g(w)dw} \),
\(\Omega_1 \in \{\Omega(t, z)\}_{z \in \{w^* : g(w) = z\}}\). \(\Omega_1\) can be chosen such that \(\Omega_1 = \Omega(t, z)\) for some \(z \in [l, w^*]\), as required. Note that if \(t^* < \bar{t}\) for all fixed points \(t^*\), then \(j < \bar{t}\) for all \(j \in \mathcal{J}(t')\) for all \(t'\). To conclude the proof, if \(t^* \geq \bar{t}\) for some fixed point \(t^*\), set \(t = t^*\), otherwise set \(t = \bar{t}\).

In the rest of the proof I suppose that Assumption 1 is satisfied.

Observe that, because \(y(t) = t \left(\frac{a}{1-a} + \int_{\Omega_1(t) \cap [l, w^* - c(t)]} g(w)dw\right) - \int_{\Omega_1(t) \cap [l, w^* - c(t)]} \frac{f_s(w)}{w^*-w} dw\), \(t = J(t)\) is equivalent to \(y(t) = 0\). Because, provided that Assumption 1 is satisfied, \(t \mapsto c(t)\) is a strictly increasing function for \(t \geq \bar{t}\) by lemma 1, we have \(c(t) < c(t + \eta)\) for all \(\eta > 0\) sufficiently small.

Define

\[S^1_\eta = \left(\Omega_1(t) \cap [w^* - c(t + \eta), w^* - c(t)]\right) \cup \left((\Omega_1(t) \setminus \Omega_1(t + \eta)) \cap [l, w^* - c(t)]\right)\]

\[\cup \left((\Omega_1(t) \setminus \Omega_1(t + \eta)) \cap [w^* - c(t + \eta), w^* - c(t)]\right)\]

Observe that \(\Omega_1(t + \eta) \cap [l, w^* - c(t + \eta)] = \Omega_1(t) \cap [l, w^* - c(t)] \setminus S^1_\eta\).

Then we have

\[y(t + \eta) - y(t) = (t + \eta) \left(\frac{a}{1-a} + \int_{\Omega_1(t+\eta) \cap [l, w^* - c(t+\eta)]} g(w)dw\right)\]

\[- \int_{\Omega_1(t+\eta) \cap [l, w^* - c(t+\eta)]} \frac{f_s(w)}{w^*-w} dw\]

\[-t \left(\frac{a}{1-a} + \int_{\Omega_1(t) \cap [l, w^* - c(t)]} g(w)dw\right) + \int_{\Omega_1(t) \cap [l, w^* - c(t)]} \frac{f_s(w)}{w^*-w} dw\]

\[= \eta \left(\frac{a}{1-a} + \int_{\Omega_1(t+\eta) \cap [l, w^* - c(t+\eta)]} g(w)dw\right)\]

\[+ t \int_{\Omega_1(t+\eta) \cap [l, w^* - c(t+\eta)]} g(w)dw - \int_{\Omega_1(t+\eta) \cap [l, w^* - c(t+\eta)]} \frac{f_s(w)}{w^*-w} dw\]

\[-t \int_{\Omega_1(t) \cap [l, w^* - c(t)]} g(w)dw + \int_{\Omega_1(t) \cap [l, w^* - c(t)]} \frac{f_s(w)}{w^*-w} dw\]

\[= \eta \left(\frac{a}{1-a} + \int_{\Omega_1(t+\eta) \cap [l, w^* - c(t+\eta)]} g(w)dw\right) - t \int_{S^1_\eta} g(w)dw + \int_{S^1_\eta} \frac{f_s(w)}{w^*-w} dw\]

We claim that \(\int_{S^1_\eta} f_s(w)dw - t \int_{S^1_\eta} g(w)dw \geq 0\) for all \(\eta > 0\). This is equivalent to \(t \leq \frac{f_s}{g} \int_{S^1_\eta} \frac{f_s}{g} dw\). Observe that the definition of \(\Omega_1(t)\) implies that for all \(w \in \Omega_1(t)\) we have

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\[ \frac{f_s(w)}{g(w)(w^* - w)} \geq t. \]  
Because \( S^t_\eta \subseteq \Omega_1(t) \), this implies that for all \( w \in S^t_\eta \) we have \( \frac{f_s(w)}{g(w)(w^* - w)} \geq t. \)

Because \( \int_{S^t_\eta} \frac{f_s(w)}{g(w)dw} \geq \inf_{w \in S^t_\eta} \frac{f_s(w)}{g(w)(w^* - w)} \geq t \), the inequality \( t \leq \frac{\int_{S^t_\eta} f_s(w)}{\int_{S^t_\eta} g(w)dw} \) is satisfied.

Because \( \frac{\int \omega}{1-\alpha} + \int_{\Omega(t+\eta)} g(w)dw > 0 \), this implies that \( y(t+\eta) - y(t) > 0 \) for all \( \eta > 0 \).

Consider a point \( t^* \) such that \( t^* = J(t^*) \). By the argument above, \( t^* = J(t^*) \) is equivalent to \( y(t^*) = 0 \). The fact that \( y(t+\eta) - y(t) > 0 \) for all \( t \geq \hat{t} \) and for all \( \eta > 0 \) implies that the function \( t \mapsto y(t) \) can intersect the zero function at \( t \geq \hat{t} \) at most one point.

The argument in the first part of the proof implies that there exists \( t^* \) such that \( y(t^*) = 0 \). If there exists a fixed point \( t^* \) such that \( t^* \geq \hat{t} \), set \( t = t^* \), and if \( t^* < \hat{t} \) for all fixed points \( t^* \), set \( t = \hat{t} \). Note that if \( t^* < \hat{t} \) for all fixed points \( t^* \), then \( \hat{t} > J(t') \) for all \( t' \).

**Proof of Theorems 1 and 2**

We will prove the following statement:

There exist \( t \geq \hat{t} \), \( c \), \( z \geq 0 \) and a sender-optimal signal structure that induces a distribution \( s \) over the receiver’s actions satisfying \( s = S^{t,c,z} \). A sender-optimal signal structure inducing the distribution \( s \) is given by \( \pi(\sigma|\omega) = s(\omega) \), \( \pi(\sigma_0|\omega) = 1 - s(\omega) \) for all \( \omega \in [l, h] \). We have \( \Omega^t_\eta = \Omega_1 \mu_{F_\eta} \)-a.e. for some \( \Omega_1 \subseteq \Omega_1(t) \), \( c = c(\Omega_1) \) for some \( \Omega_1 \subseteq \Omega_1(t) \) and either \( t = j \) for some \( j \in J(t) \) or \( t = \hat{t} \) and \( \hat{t} > j \) for all \( j \in J(t') \) for all \( t' \) Moreover, \( \Omega_1 \) can be chosen such that \( \Omega_1 = \Omega(t, z) \) for some \( z \in [l, h^*] \).

If Assumption 1 is satisfied, then there exist unique \( t \geq \hat{t} \), \( c \geq 0 \) such that any sender-optimal signal structure induces a distribution \( s \) satisfying \( s = S^{t,c} \mu_{F_\eta} \)-a.e. Moreover, \( \Omega^t_\eta = \Omega_1(t) \mu_{F_\eta} \)-a.e., \( c = c(t) \) and either \( t = J(t') \) or \( t = \hat{t} \) and \( \hat{t} > J(t') \) for all \( t' \).

Lemma 7 shows that there exists a sender-optimal signal with two realizations.

Lemma 8 shows that if a signal structure \( \pi \) with two realizations satisfies \( \pi(w) \leq Z^t_\eta \) for all \( w \in [l, h^*] \) and satisfies \( \pi(\sigma|\omega) = Z^t_\eta \) for all \( w \) in some \( A \subseteq [l, w^*] \), then \( \pi(\sigma|\omega) \) is given by \( \frac{c}{w^* - w} \). For \( \mu_{F_\eta} \)-a.e. \( w \in \Omega_1^t \cap A \) and \( \pi(\sigma|\omega) = 1 \) for \( \mu_{F_\eta} \)-a.e. \( w \in [w^*, h] \), where \( c \) is given by \( c = \frac{\int_{[w^*, h]} (w^* - w) g(w)dw - \int_{\Omega_1^t \cup A \mu_{F_\eta} g(w)\pi(\sigma|\omega)(w^* - w)dw - \int_{[w^*, h]} \cap \mu_{F_\eta} g(w)(w^* - w)dw}{\frac{e}{w^* - w} \mu_{F_\eta} g(w)dw} \).

Lemma 9 shows that if Assumption 1 is satisfied and \( \pi \) is a sender-optimal signal structure with two realizations, then \( A = \Omega_1^t \mu_{F_\eta} \)-a.e., while if Assumption 1 fails and (for all parameter values) there exists a signal structure satisfying \( \pi(\sigma|\omega) = Z^t_\eta \mu_{F_\eta} \)-a.e.
$w' \in \Omega_1^\pi$ and $\pi(\sigma|w) = 1$ $\mu_{F_1}$-a.e. $w \in [w^*, h]$, then there exists a sender-optimal signal structure with two realizations satisfying $A = \Omega_1^\pi$ $\mu_{F_1}$-a.e.

The proof of lemma 13 shows that if $\pi$ satisfies the condition in lemma 9 then $\pi$ must be such that there is a threshold $t \geq t$ satisfying $\Omega_1^\pi = \Omega_1$ $\mu_{F_1}$-a.e., $c = \hat{c}(\Omega_1)$ for some $\Omega_1 \in \Omega_1(t)$ and either $t = j$ for some $j \in J(t)$ or $t = t$ and $t > j$ for all $j \in J(t')$ for all $t'$, and that $\Omega_1$ can be chosen such that $\Omega_1 = \Omega(t, z)$ for some $z \in [l, w^*)$. The proof of lemma 13 also shows that $\pi$ satisfying these requirements exists. Therefore, there exist $t \geq t$, $c$, $z \geq 0$ and a sender-optimal signal structure that induces a distribution $s$ satisfying $s = S^{t, c, z}$ and the properties in the first part of the statement to be proven.

Now suppose that Assumption 1 is satisfied. Observe that then any selection $t : \Omega(t)$ from $t \mapsto \Omega(t)$ satisfies $\Omega(t) = \Omega_1(t)$ a.e. This is because $\Omega(t) = \left\{ w : \frac{f_s(w)}{g(w)(w^*-w)} > t \right\} \cup \Omega_1$, $\Omega_1(t) = \left\{ w : \frac{f_s(w)}{g(w)(w^*-w)} > t \right\} \cup \Omega_2'$ for some $\Omega_1$, $\Omega_2' \subseteq \left\{ w : \frac{f_s(w)}{g(w)(w^*-w)} = t \right\}$, and Assumption 1 implies that $\mu(\Omega_1') = \mu(\Omega_2') = 0$. Then the correspondence $t : \Omega(t)$ is a function equal to $t \mapsto J(t)$. Note that $c(t) = \hat{c}(\Omega_1(t))$. Then the result in the paragraph above implies that if $\pi$ satisfies the condition in lemma 9 then there exists a sender-optimal signal structure $\pi$ with two realizations such that there is a threshold $t \geq t$ satisfying $\Omega_1^\pi = \Omega_1(t)$ $\mu_{F_1}$-a.e., $c = c(t)$ and either $t = J(t)$ or $t = t$ and $t > J(t')$ for all $t'$. By lemma 13 if Assumption 1 is satisfied, then the threshold $t \geq t$ is unique. Observe that, because Assumption 1 is satisfied, by lemma 9 any sender-optimal signal structure $\pi$ with two realizations satisfies the condition in lemma 9. Then there exist unique $t \geq t$, $c \geq 0$ such that any optimal sender-optimal signal structure induces a distribution $s$ satisfying $s = S^{t, c, z}$ $\mu_{F_1}$-a.e., $\Omega_1^\pi = \Omega_1(t)$ $\mu_{F_1}$-a.e., $c = c(t)$ and either $t = J(t)$ or $t = t$ and $t > J(t')$ for all $t'$, as required.

We next show that, provided that Assumption 1 is satisfied, if a sender-optimal signal structure $\pi_0$ induces a distribution $s_0$ over the receiver’s actions, where the probability of action 1 in state $w$ is given by $s_0(w)$, then $s_0(w) = \pi(\sigma|w)$ $\mu_{F_1}$-a.e. $w \in [l, h]$.

Lemma 7 shows that for any sender-optimal signal structure $\pi_0$ the sender’s payoff from $\pi_0$ is $\int_{[l, h]} \left( \sum_{i \in R(\hat{F}, \pi_0)} \pi_0(\sigma_i|w) \right) f_s(w)dw$ and that there exists a sender-optimal signal structure $\pi_1$ with two realizations satisfying $\pi_1(\sigma|w) = \sum_{i \in R(\hat{F}, \pi_0)} \pi_0(\sigma_i|w)$ for all $w \in [l, h]$.

Note that, because the sender’s payoff from $\pi_0$ is $\int_{[l, h]} \left( \sum_{i \in R(\hat{F}, \pi_0)} \pi_0(\sigma_i|w) \right) f_s(w)dw$, we have $s_0(w) = \sum_{i \in R(\hat{F}, \pi_0)} \pi_0(\sigma_i|w)$. Then the fact that $\pi_1(\sigma|w) = \sum_{i \in R(\hat{F}, \pi_0)} \pi_0(\sigma_i|w)$ for all $w \in [l, h]$ implies that $s_0(w) = \pi_1(\sigma|w)$ for all
\[ w \in [l, h]. \]

The fact that, provided that Assumption 1 is satisfied, the optimal signal structure with two realizations in unique \( \mu_{F_s} \)-a.e. by the arguments above implies that \( \pi_1(\sigma|w) = \pi(\sigma|w) \) \( \mu_{F_s} \)-a.e. \( w \in [l, h] \). Therefore, \( s_0(w) = \pi(\sigma|w) \mu_{F_s} \)-a.e. \( w \in [l, h] \), as required. \[ \blacksquare \]

**Proof of Proposition 1**

Suppose that \( \mu_G(\{l, w^*\}) = 0 \). Then \( g(w) = 0 \) a.e. \( w \in [l, w^*] \), so that \( \frac{f_s(w)}{g(w)(w^* - w)} = \infty \) a.e. \( w \in [l, w^*] \). This implies that \( \left\{ w \in [l, w^*) : \frac{f_s(w)}{g(w)(w^* - w)} > t \right\} = [l, w^*) \) for all \( t < \infty \). Thus, because \( f_s \) has full support, the proof of Theorems 1 and 2 implies that an optimal signal structure induces the probability of action 1 in state \( w \) given by \( s(w) = 1 \) a.e. \( w \in [w^*, h] \) and \( s(w) = \min \left\{ \frac{c}{w^* - w}, 1 \right\} \) a.e. \( w \in [l, w^*] \), where \( c \) satisfies \( c = \int_{[w^*, h]} g(w)(w - w^*)dw - \int_{[w^*, c, w^*]} g(w)(w^* - w)dw \). Observe that \( s \) is independent of \( F_s \), as required.

Next suppose that a sender-optimal signal structure with two realizations induces a distribution over the receiver’s actions that is independent of \( F_s \). Suppose for the sake of contradiction that \( \mu_G(\{l, w^*\}) > 0 \). Let \( A \subseteq [l, w^*] \) denote the set satisfying \( \mu_G(A) = \mu_G(\{l, w^*\}) \) (note that \( A \) is unique up to sets of measure zero).

For simplicity, I will provide a proof allowing for discontinuous density functions \( f_s \). Fix \( \overline{\eta} > 1 \), \( A_1 \subset A \) such that \( \mu(A_1) = \frac{1}{\overline{\eta}} \) and the sender’s prior \( F^\eta_s \) for \( \eta \in (0, \overline{\eta}] \) such that \( f^\eta_s(w) = \eta \) for \( w \in A_1 \) and \( f^\eta_s(w) = \epsilon_\eta \) for \( w \in [l, h] \setminus A_1 \) where \( \epsilon_\eta \) is given by \( \epsilon_\eta = \frac{\epsilon - \eta}{\overline{\eta} - \eta} \) (which ensures that \( f^\eta_s \) is a probability density function). Let \( A_2 = A \setminus A_1 \).

Let \( \pi^\eta \) denote a signal structure that is sender-optimal when the sender’s prior is \( F^\eta_s \) and satisfies the condition in lemma 1 and let \( \Omega^\eta = \Omega^\eta_{\pi^\eta} = \{ w \in [l, w^*) : \pi^\eta(\sigma|w) > 0 \} \). Let \( A_\varnothing = \{ w \in [l, w^*) : g(w) = 0 \} \). Observe that, because for all \( w \in A_\varnothing \) we have \( g(w) = 0 \) and \( f^\eta_s(w) = 0 \), adding a subset of \( A_\varnothing \) to the support of the signal realization recommending action 1 under \( \pi^\eta \) does not change the sender’s payoff. Then, without loss of generality, we may assume that \( A_\varnothing \subseteq \Omega^\eta \).

Note that, because \( f^\eta_s(w) = 0 \) for \( w \in [l, h] \setminus A_1 \) and \( \mu_G(A') > 0 \) for all \( A' \subseteq A \) such that \( \mu(A') > 0 \), we must have \( \pi^\eta(\sigma|w) = 0 \) a.e. \( w \in A \setminus A_1 \). Let \( A_0 = \Omega^\eta \cap A_1 \). It can be shown that \( \mu(A_0) > 0 \). Then, because \( A_\varnothing \subseteq \Omega^\eta \) by the argument above, we have \( \Omega^\eta = A_0 \cup A_\varnothing \).

We claim that there exists \( \tilde{\epsilon} > 0 \) such that for all \( \epsilon_\eta \in (0, \tilde{\epsilon}) \), there is a set \( A^\eta_3 \subseteq A \setminus A_1 = A_2 \) satisfying \( \mu(A^\eta_3) > 0 \) and \( \mu(A^\eta_3 \cap \Omega^\eta) = 0 \). Suppose for the sake of contradiction that for all \( \tilde{\epsilon} > 0 \) there exists \( \epsilon \in (0, \tilde{\epsilon}) \) such that if \( \epsilon_\eta = \epsilon \), then \( \mu(A_2 \setminus \Omega^\eta) = 0 \). Fix \( \tilde{\epsilon} > 0 \)
and $\epsilon \in (0, \bar{\epsilon})$ such that if $\epsilon_\eta = \epsilon_1$, then $\mu(A_2 \setminus \Omega^\eta) = 0$. Fix $\eta \in (0, \bar{\eta})$ such that $\epsilon_\eta = \epsilon$. We will show that the sender’s payoff given $F_s^\eta$ is strictly higher under the signal structure $\pi^\eta$ than under $\pi^\eta$, which would lead to a contradiction.

Because $\pi$ satisfies the condition in lemma 2, the proof of Theorems 1 and 2 implies that $\pi^\eta(\sigma|w) = \frac{\hat{c}(\Omega^\eta)}{w^s - w^s - \hat{c}(\Omega^\eta)}$ for $w \in \Omega^\eta \cap [l, w^* - \hat{c}(\Omega^\eta))$. Note that, because $\mu(A_2 \setminus \Omega^\eta) = 0$, we have $A_0 \cup A_0 \cup A_2 \subseteq \Omega^\eta$ and $\hat{c}(\Omega^\eta) \leq \hat{c}(A_0 \cup A_0 \cup A_2) = c$. Observe that $c$ is the constant of proportionality for the signal structure resulting from adding $A_2$ to the support of the signal realization recommending action 1 under $\pi^\eta$. $\hat{c}(\Omega^\eta) \leq c$ then follows from the fact that $\Omega^\eta = A_0 \cup A_0$ and $A_0 \cup A_0 \cup A_2 \subseteq \Omega^\eta$.

Thus, because $\Omega^\eta = A_0 \cup A_0$, we have $\Omega^\eta \cup A_2 = A_0 \cup A_0 \cup A_2 \subseteq \Omega^\eta$, so that $\Omega^\eta$ is obtained by adding $A_2$ and possibly other sets to the support of the signal realization recommending action 1 under $\pi^\eta$. The gain in the sender’s payoff from adding $A_2$ and possibly other sets to the support is at most $\int_{[l, w^*]} f_s^\eta(w) dw = \epsilon \mu([l, w^*]) \setminus A_0)$. Observe that the loss in the sender’s payoff from adding $A_2$ and possibly other sets to the support is greater than the loss in the sender’s payoff from adding only $A_2$ to the support. The loss in the sender’s payoff from adding $A_2$ to the support is at least $\Delta U = \int_{[l, w^*]} f_s^\eta(w) dw = \epsilon \mu([l, w^*]) \setminus A_0) = \Delta U_0$.

We claim that for $\epsilon_\eta \in (0, \bar{\epsilon})$ where $\bar{\epsilon}_1 = \frac{\frac{\epsilon_\eta - \hat{c}(\Omega^\eta)}{w^s - l} \eta \mu([l, w^* - \hat{c}(\Omega^\eta)) \setminus A_0)}{\frac{\epsilon_\eta - \hat{c}(\Omega^\eta)}{w^s - l} \eta \mu([l, w^* - \hat{c}(\Omega^\eta)) \setminus A_0)} > 0$. Note that the definition of $\eta \mapsto \hat{c}(\Omega^\eta)$ and the fact that $\mu([l, w^* - \hat{c}(\Omega^\eta)) \setminus A_0) > 0$. Observe also that $\mu([l, w^* - \hat{c}(\Omega^\eta)) \setminus A_0) > 0$. Then the claim follows from the observation that $\eta \mapsto \frac{\epsilon_\eta - \hat{c}(\Omega^\eta)}{w^s - l} \eta \mu([l, w^* - \hat{c}(\Omega^\eta)) \setminus A_0)$ is strictly increasing, while $\eta \mapsto \epsilon_\eta$ is strictly decreasing, that $\epsilon_\eta < \frac{\Delta U_0}{\mu([l, w^s]) \setminus A_0)$ holds at $\eta = \bar{\eta}$ because $\epsilon_\eta = 0$ and $\bar{\eta} > 0$.

Then the gain from adding $A_2$ and possibly other sets to the support is strictly less than the loss. Therefore, the sender’s payoff given $F_s^\eta$ is strictly higher under $\pi^\eta$ than under $\pi^\eta$, which is a contradiction. Thus there exists $\bar{\epsilon} > 0$ such that for all $\epsilon_\eta \in (0, \bar{\epsilon})$, there is a set $A_3^\eta \subseteq A \setminus A_1 = A_2$ satisfying $\mu(A_3^\eta) > 0$ and $\mu(A_3^\eta \cap \Omega^\eta) = 0$.

Fix $A_3^\eta \subseteq A_3^\eta$ such that $\mu(A_3^\eta) > 0$ and consider the sender’s prior $F_s'^\eta$ with full support constructed as above with $A_1'$ instead of $A_1$. Then the above argument implies that there exists $A_0' \subseteq A_1$ such that $\mu(A_0') > 0$ and under the sender’s prior $F_s'^\eta$ we have $\mu(A_0' \cap \Omega^\eta') = 0$ for some $\eta' \in (0, \bar{\eta})$. Because $A_0' \subseteq A_3^\eta$ and $\mu(A_3^\eta \cap \Omega^\eta) = 0$, this contradicts the hypothesis that the distribution over the receiver’s actions induced by a sender-optimal signal structure
is independent of the sender’s prior. □

Proof of Proposition 2.

Let us write the fixed point equation pinning down the values of the threshold \( t > \underline{t} \) as \( t^* = J(t^*) \). Observe that \( \lim_{a \to 1} J(t) = 0 \). Then \( \lim_{a \to 1} t^* = 0 \).

The fact that \( g \) is \( C^1 \) and \( [l, h] \) is a compact set implies that \( g \) attains a maximum \( M \) on \( [l, h] \), which implies that \( g(w) \leq M \) for all \( w \in [l, w^*] \). It follows that if there exists \( m > 0 \) such that \( f_s(w) \geq m \) for all \( w \in [l, w^*] \), then \( \underline{t} = \min_{w \in [l, w^*]} \left\{ \frac{f_s(w)}{g(w)(w^*-w)} \right\} > 0 \).

Thus, because \( \lim_{a \to 1} t^* = 0 \), there exists \( \alpha \in (0, 1) \) such that for all \( a \in (\alpha, 1) \) we have \( t^* < \underline{t} \). Therefore, for all \( a \in (\alpha, 1) \) the threshold is \( t = \underline{t} \), so that \( \Omega_1(t) = \{l, w^*\} \), as required. □

Proof of Proposition 3.

Because \( f_s \) and \( g \) are uniform probability density functions on \([l, h] \), we have \( f_s(w) = g(w) = \frac{1}{h-l} \) for all \( w \in [l, h] \). Then \( \Omega_1^* = \Omega_1(t) = \{w \in [l, w^*] : \frac{1}{w^*-w} \geq t\} \), so that \( \Omega_1^* = [w^*, w^*] \) for some state \( w^* \) satisfying \( \frac{1}{w^*-w^*} = t \). This implies that the constant \( c \) satisfies \( c = \frac{\int_{w^*}^{h} \frac{(w^*-w)dw}{c(w^*-w)_{w^*}}(w^*-w)dw}{\int_{w^*}^{h} (h-l)c(w^*-w)_{w^*} + f_s(w)_{w^*} dw} \).

We first conjecture that \( w^* \geq w^* - c \). Observe that then \( \Omega_1(t) \cap [l, w^* - c) = \emptyset \), which implies that \( J(t) = 0 \). Because the threshold satisfies either \( t = J(t) \) or \( t = \underline{t} \) by lemma 13, this implies that \( t = \underline{t} \), so that \( \Omega_1^* = \{l, w^*\} \). Because \( \Omega_1^* \cap [l, w^* - c) = \emptyset \), this implies that \( \Omega_1^* \cap [w^* - c, w^*] = \{l, w^*\} \). Recall that we must have \( \pi(\sigma|w) = 1 \) \( \mu_{F_s} \)-a.e. \( w \in \Omega_1^* \cap [w^* - c, w^*] \). Then \( \pi(\sigma|w) = 1 \) \( \mu_{F_s} \)-a.e. \( w \in [l, h] \) and \( E_{\pi, F}[w|\sigma] \geq w^* \) for all \( F \in C_{a,g} \), which is a contradiction to our assumption on the parameters. Therefore, we must have \( w^* < w^* - c \).

We can then write the formula for \( c \) as \( c = \frac{\int_{w^*}^{h} \frac{(w^*-w)dw}{c(w^*-w)_{w^*}}(w^*-w)dw}{\int_{w^*}^{h} (h-l)c(w^*-w)_{w^*} + f_s(w)_{w^*} dw} \). Observe that \( \int_{w^*}^{h} (w - w^*)dw = \frac{h^2}{2} - w^*h + \frac{(w^*)^2}{2} \) and \( \int_{w^*}^{h} (w^* - w)dw = \frac{c^2}{2} \), which implies that \( c = \frac{\frac{h^2}{2} - w^*h + \frac{(w^*)^2}{2} - \frac{c^2}{2}}{\int_{w^*}^{h} (h-l)c(w^*-w)_{w^*} + f_s(w)_{w^*} dw} \). Equivalently, \( c \) satisfies \( -c^2 + 2c \left( \frac{a}{1-a} (h-l) + (w^* - w') \right) - (h - w^*)^2 = 0 \).

Observe that we have \( J(t) = \frac{\int_{w^*}^{h} \frac{1}{w^*-w} \frac{dw}{c(w^*-w)_{w^*} + f_s(w)_{w^*}}}{\int_{w^*}^{h} (h-l)c(w^*-w)_{w^*} + f_s(w)_{w^*} dw} \). Equivalently, \( J(t) = \frac{\ln(w^*-w') - \ln(c)}{\int_{w^*}^{h} (h-l)c(w^*-w)_{w^*} + f_s(w)_{w^*} dw} \). This implies that either \( t = \underline{t} \) and \( \frac{1}{w^*-w} > \frac{\ln(w^*-w) - \ln(c)}{\int_{w^*}^{h} (h-l)c(w^*-w)_{w^*} + f_s(w)_{w^*} dw} \) for all \( w \in [l, w^*] \) or the equation \( \frac{1}{w^*-w} = \frac{\ln(w^*-w) - \ln(c)}{\int_{w^*}^{h} (h-l)c(w^*-w)_{w^*} + f_s(w)_{w^*} dw} \) determines the threshold. Equivalently, either
Recall also that, because $t$ with unknown beliefs, and let $\alpha$ if persuasion model, let a sender-optimal signal structure with two realizations, then there exists a unique threshold follows from the formula for the signal $s = \frac{a}{w^*-w'}$ for all $w \in [l, w^*)$ or $\ln(w^*-w') - \ln(c) = 1 + \frac{a}{w^*-w'}$.

Lemma 14. Suppose that $F_s \in C(\alpha, \beta)$ admits a density and that $E_{F_s}[w] < w^*$. Consider the standard persuasion model such that $F_s$ is the common prior of the sender and the receiver. Given $b \in \Omega$, let $p_b$ denote the signal with two realizations, $\sigma$ and $\sigma_0$, satisfying $p_b(\sigma|w) = 1$ if $w \geq b$ and $p(\sigma|w) = 0$ if $w < b$. Let $w'$ denote the threshold that is optimal in the standard persuasion model, let $\pi$ a signal with two realizations, $\sigma$ and $\sigma_0$, that is optimal in the model with unknown beliefs, and let $\omega = w^* - \frac{\beta}{1-\beta}(\alpha - w^*)$. Then $\omega > w'$.

Proof of lemma 14.
Suppose for the sake of contradiction that $\omega \leq w'$. Note that, because $F_s \in C(\alpha, \beta)$, we have $\mu_{F_s}([w^*, h]) > 0$. Then the signal $p_{w'}$ optimal in the standard persuasion model must be such that $E_{F_s,p_{w'}}[\omega|\sigma] = w^*$.

Next, note that the fact that $E_{F_s}[w] < w^*$ and $E_{F_s,p_{w'}}[\omega|\sigma] = w^*$ implies that $\mu_{F_s}([l, w^*)) > 0$.

We have

$$w^* = E_{F_s,p_{w'}}[\omega|\sigma] \geq E_{F_s,p_{\omega}}[\omega|\sigma] > E_{F_s,\pi}[\omega|\sigma]$$

where the equality follows from the form of the optimal solution in the standard persuasion model, the first inequality follows from the assumption that $\omega \leq w'$, and the second inequality follows from the formula for the signal $\pi$ and from the fact that $\mu_{F_s}([l, w^*)) > 0$.

Therefore, we have $E_{F_s,\pi}[\omega|\sigma] < w^*$, which is a contradiction because the optimal signal $\pi$ must satisfy $E_{F_s,\pi}[\omega|\sigma] \geq w^*$ for every feasible prior $F \in C(\alpha, \beta)$.


Proof of Proposition 5.
Recall that, provided that Assumption 1 is satisfied, lemmas 10 and 13 imply that if $\pi$ is a sender-optimal signal structure with two realizations, then there exists a unique threshold $t \geq l$ satisfying either $t = l$ or $t = J(t)$ where $J(t) = \frac{\int_{\Omega_1 \cap [l,w^*-c(t)]} f_s(w) \frac{dw}{w^*-w}}{\frac{\mu_{\pi}}{\mu_{\pi_0}} + \int_{\Omega_1 \cap [l,w^*-c(t)]} g(w) dw}$. Recall also that, because $y(t) = t \left( \frac{a}{1-a} + \int_{\Omega_1 \cap [l,w^*-c(t)]} g(w) dw \right) - \int_{\Omega_1 \cap [l,w^*-c(t)]} \frac{f_s(w)}{w^*-w} dw$, $t = J(t)$ is equivalent to $y(t) = 0$.

Let us write $y_a(t) = t \left( \frac{a}{1-a} + \int_{\Omega_1 \cap [l,w^*-c(t)]} g(w) dw \right) - \int_{\Omega_1 \cap [l,w^*-c(t)]} \frac{f_s(w)}{w^*-w} dw$, making
the dependence of \( y(t) \) on \( a \) explicit. Then
\[
y_a'(t) - y_a(t) = t \left( \frac{a'}{1-a'} + \int_{\Omega(t) \cap [l,w^*-c(t)]} g(w) dw \right) - t \left( \frac{a}{1-a} + \int_{\Omega(t) \cap [l,w^*-c(t)]} g(w) dw \right)
\]

**Claim 4.1.** \( y_a'(t) - y_a(t) < 0 \) for all \( t \geq t \).

**Proof of claim 4.1.**
\[
y_a'(t) - y_a(t) = t \left( \frac{a'}{1-a'} - \frac{a}{1-a} \right) < 0
\]
because \( t > 0 \) and \( a' < a \).

**Claim 4.2.** \( t_{a'} - t_a > 0 \).

**Proof of claim 4.2.**
Because \( \mu(\Omega^{a'}_1) < \mu([l, w^*]) \), we have \( t_a > t \). Then, because \( t = J(t) \) is equivalent to \( y(t) = 0 \) and, by lemma 13, we have \( y(t + \eta) > y(t) \) for all \( t \geq t \) and for all \( \eta \), to show that \( t_{a'} - t_a > 0 \) it is enough to show that \( y_a'(t) - y_a(t) < 0 \) for all \( t \geq t \). Claim 4.1 implies that this is satisfied.

The fact that \( t_{a'} - t_a > 0 \) by claim 4.2 implies that \( \Omega^{a'}_1 \subseteq \Omega^a_1 \) \( \mu_{F_1} \)-a.e. Finally, the fact that \( c(t_{a'}) > c(t_a) \) follows from the fact that, provided that Assumption 1 is satisfied, \( t \mapsto c(t) \) is strictly increasing by lemma 11.

**Proof of Proposition 6.**

Let \( d \in \{0, 1\} \) denote the actions available to the receiver. Let us write the receiver’s utility function as \( u(d, w) = 1_{d=1}(w - w^*) \). Let \( \pi_a \) denote the sender-optimal signal with two realizations when the receiver’s set of priors is \( C_{a,g} \). Then the expected payoff of a receiver with belief \( F \) is \( V(a) = \int_{[l,h]} \pi_a(\sigma|w)(w - w^*)dF(w) \).

Observe that \( V(a) = \int_{[l,h]} \pi_a(\sigma|w)dF(w) - w^*\int_{[l,h]} \pi_a(\sigma|w)dF(w) = \int_{[l,h]} \pi_a(\sigma|w)dF(w) \left( \int_{[l,h]} \pi_a(\sigma|w)dF(w) - w^* \right) = \int_{[l,h]} \pi_a(\sigma|w)dF(w)(E_{F,\pi_a}[w|\sigma] - w^*) \), so that \( V(a) = \int_{[l,h]} \pi_a(\sigma|w)dF(w)(E_{F,\pi_a}[w|\sigma] - w^*) \). This implies that \( V(a') = 0 \) if and only if \( E_{F,\pi_a}[w|\sigma] - w^* = 0 \). Then, because \( \int_{[l,h]} \pi_a(\sigma|w)dF(w) > 0 \), to prove the first part of the Proposition it is enough to show that if \( E_{F,\pi_a}[w|\sigma] = w^* \), then \( E_{F,\pi_a}[w|\sigma] > w^* \).

Observe that \( \mu_F = \int_{[l,h]} \tau(\cdot) d\omega = \int_{[l,h]} \tau_\omega d\omega \) for some \( \{\tau_\omega\}_{\omega \in [l,h]} \) such that \( \int_{[l,h]} \tau_\omega d\omega = 81 \).
1. Then

\[
E_{F_{l,w}}[w]\sigma = \frac{\int_{[l,h]} \tau_\omega \left( a'\omega \pi_a(\sigma|\omega) + (1 - a') \int_{[l,h]} w\pi_a(\sigma|w)gw\right) dw}{\int_{[l,h]} \tau_\omega \left( a'\omega \pi_a(\sigma|\omega) + (1 - a') \int_{[l,h]} \pi_a(\sigma|w)gw\right) dw}
\] (46)

Let \( \mu_{F_{l,w}} = a'\delta_w + (1 - a')\mu_G \) and note that

\[
E_{F_{l,w},a}[w]\sigma = \frac{a'\omega \pi_a(\sigma|\omega) + (1 - a') \int_{[l,h]} w\pi_a(\sigma|w)gw\left( a'\omega \pi_a(\sigma|\omega) + (1 - a') \int_{[l,h]} \pi_a(\sigma|w)gw\right) dw}{a'\pi_a(\sigma|\omega) + (1 - a') \int_{[l,h]} \pi_a(\sigma|w)gw\right) dw}
\] (47)

Observe that, because \( E_{F_{l,w},a}^*[w]\sigma = w^* \) for some \( \omega \in [l, h] \) and \( a > a' \), we have \( E_{F_{l,w},a}[w]\sigma = w^* \) for all \( \omega \in [l, h] \). Then (46) and (47) imply that \( E_{F_{l,w}}[w]\sigma > w^* \), as required.

Observe that

\[
V(a) = \int_{[w^*,h]} (w - w^*)dF(w) - \int_{[l,w^*)} (w^* - w)\pi_a(\sigma|w)dF(w)
\] (48)

Then we have

\[
V(a) - V(a') = \int_{[l,w^*)} (w^* - w)(\pi_a'(\sigma|w) - \pi_a(\sigma|w))dF(w)
\] (49)

We next consider the case in which \( \Omega_1^a = \Omega_1^{a'} = [l, w^*) \). Observe that \( \pi_a(\sigma|w) = \text{min}\{\frac{e_0}{w^* - w}, 1\} \) for some constant \( c_a \) if \( w \in [l, w^*) \) and \( \pi_a(\sigma|w) = 1 \) if \( w \in [w^*, h] \). Moreover, \( c_a < c_{a'} \). Then we have \( \pi_a(\sigma|w) < \pi_{a'}(\sigma|w) \) for \( w \in [l, w^* - c_a] \) and \( \pi_a(\sigma|w) \leq \pi_{a'}(\sigma|w) \) for \( w \in [w^* - c_a, w^*) \). This and (49) imply that \( V(a) - V(a') \geq 0 \). Moreover, \( V(a) - V(a') > 0 \) if \( \mu_F([l, w^* - c_u]) > 0 \) and \( V(a) - V(a') = 0 \) if \( \mu_F([l, w^* - c_a]) = 0 \). Therefore, if \( \mu_F([l, w^*]) = 0 \), then \( V(a) = V(a') \), while if \( g \) has full support, then \( V(a) > V(a') \), as required.

We now provide an example of the parameters under which \( V(a) < V(a') \). For simplicity, suppose that \( \mu_G = (1 - \kappa)\mu_{G_0} + \kappa\delta_h \) for some \( \kappa \in (0, 1) \) and \( \mu_{G_0} \) that is uniform on \([w_0, w_1]\) for some \( l < w_b < w_1 < w^* \).

**Claim 7.1.** There exist parameters such that \( \Omega_1^a = [l, w^*) \) and \([w_0, w_1] \cap \Omega_1^{a'} = \emptyset \).

**Proof of claim 7.1.**
First observe that, because \( \mu_G([l, w_0]) = 0 \), we have \([l, w_0) \subseteq \Omega_1^{a}, \Omega_1^{a'}\).

Fix \(a' \in (0, 1)\). Consider \(F_s\) such that \(f_s(w) = \eta_1\) for \(w \in [w_0, w_1]\) and \(f_s(w) = \eta_0\) for \(w \in [l, h] \setminus [w_0, w_1]\). Then for \(\eta_1\) sufficiently small we have \([w_0, w_1] \cap \Omega_1^{a'} = \emptyset\). This is because \(\lim_{\eta_1 \to 0} \frac{f_s(w)}{\eta_1} = 0\) for \(w \in [w_0, w_1]\) but \(\lim_{\eta_1 \to 0} J(t) = \lim_{\eta_1 \to 0} \frac{\int_{\Omega_1(t) \cap [l, w^* - c(t)]} f_s(w) dw - \int_{\Omega_1(t) \cap [l, w^* - c(t)]} g(w) dw}{\eta_1} > 0\) because \(\mu(\Omega_1(t) \cap [l, w_0]) > 0\) and \(f_s(w) \geq \eta_0\) for all \(w \in [l, w_0]\).

Observe next that \(l < w^* - c(t)\) for \(t\) such that \(\Omega_1(t) = \Omega_1^{a}\) for \(a > a'\) sufficiently large. This is because \(\lim_{a \to 1} c(t) = \lim_{a \to 1} \frac{\int_{[w^*, h]} g(w)(w-w^*)dw - \int_{[w^*, h]} g(w)(w^*-w)dw}{1/a + \int_{[l, w^* - c(t)]} g(w) dw} = 0\) and \(l < w^*\).

The fact that \(l < w^* - c(t)\) for \(t\) such that \(\Omega_1(t) = \Omega_1^{a}\) for \(a > a'\) sufficiently large and that \([l, w_0) \subseteq \Omega_1(t)\) implies that \(\mu(\Omega_1(t) \cap [l, w^* - c(t)]) > 0\) for \(a > a'\) sufficiently large. Because \(\lim_{a \to 1} \frac{a}{1-a} = \infty\), the fact that \(\mu(\Omega_1(t) \cap [l, w^* - c(t)]) > 0\) for \(a > a'\) sufficiently large implies that \(\lim_{a \to 1} J(t) = \lim_{a \to 1} \frac{\int_{\Omega_1(t) \cap [l, w^* - c(t)]} f_s(w) dw - \int_{\Omega_1(t) \cap [l, w^* - c(t)]} g(w) dw}{\eta_1} = 0\). Because the threshold \(t\) satisfies \(t = J(t)\) and \(\frac{f_s(w)}{g(w)(w^*-w)} = \frac{\eta_1}{g(w)(w^*-w)} > 0\) for \(w \in [w_0, w_1]\), this implies that as \(a \to 1\), \(\Omega_1^{a} \to [l, w^*]\).

Consider \(\mu_F = (1 - a)(1 - \kappa)\mu_{G_0} + (1 - a)\kappa\delta_h + ah\). Because \(V(a) = \int_{[w^*, h]} (w - w^*)dF(w) - \int_{[w^*, h]} (w - w^*)\pi_a(\sigma|w)dF(w)\) by (48), we have \(V(a') = \int_{[w^*, h]} (w - w^*)dF(w)\) because \([w_0, w_1] \cap \Omega_1^{a'} = \emptyset\) and \(V(a) = \int_{[w^*, h]} (w - w^*)dF(w) - \int_{[w^*, h]} (w - w^*)\pi_a(\sigma|w)dF(w)\). Note that \(\int_{[l, w^*]} (w - w^*)\pi_a(\sigma|w)dF(w) > 0\) because \(\Omega_1^{a} = [l, w^*]\). This implies that \(V(a') > V(a)\), as required.

**Proof of Proposition 7.**

Because \(a = 0\), the revelation principle applies. By the revelation principle, because the receiver has two actions, there exists a sender-optimal signal structure with two realizations, \(\sigma\) and \(\sigma_0\). Then the sender’s problem is \(\max_{\{(\pi(\sigma|w)\in [0, 1]\}_{w \in [l, h]} f_s(w)\pi(\sigma|w)dw\) subject to the constraint that \(E_{G,\pi}[w|\sigma] \geq w^*\).

The constraint that \(E_{G,\pi}[w|\sigma] \geq w^*\) is equivalent to \(\frac{\int_{[l, h]} w g(w)\pi(\sigma|w)dw}{\int_{[l, h]} g(w)\pi(\sigma|w)dw} \geq w^*\). Equivalently, \(\int_{[l, h]} (w^* - w)g(w)\pi(\sigma|w)dw \leq 0\). Observe that the constraint must bind.

We can write the Lagrangian for the constrained optimization problem as \(\int_{[l, h]} f_s(w)\pi(\sigma|w)dw - \lambda \int_{[l, h]} (w^* - w)g(w)\pi(\sigma|w)dw\). Equivalently, \(\int_{[l, h]} (f_s(w) - \lambda(w^* - w))g(w)\pi(\sigma|w)dw\).

Then the solution is \(\pi(\sigma|w) = 1\) if \(f_s(w) \geq \lambda(w^* - w)\) and \(\pi(\sigma|w) = 0\) if \(f_s(w) < 83\).
\[ \lambda(w^* - w) \text{ where } \lambda > 0 \text{ is such that } E_{G,\pi} [\omega | \sigma] = w^* . \] This implies that \( \pi(\sigma | w) = 1 \) for all \( w \in [w^*, h] \). Moreover, for \( w \in [l, w^*), \) we have \( \pi(\sigma | w) = 1 \) if \( \frac{f_s(w)}{\bar{g}(w)(w^*-w)} \geq \lambda \) and \( \pi(\sigma | w) = 0 \) if \( \frac{f_s(w)}{\bar{g}(w)(w^*-w)} \leq \lambda \) where \( \lambda \) is such that \( E_{G,\pi} [\omega | \sigma] = w^* . \)

Next, we prove the second part of the Proposition. Let \( \pi_a \) denote the sender-optimal signal with two realizations when the set of priors is \( C_{a,g} \). Without loss of generality, suppose that the limits \( \lim_{a \to 0} \int_I \pi_a(\sigma | w) - \pi(\sigma | w) dF_s \) and \( \bar{\pi} = \lim_{a \to 0} \pi_a(\sigma | w) \) exist. Suppose for the sake of contradiction that \( \lim_{a \to 0} \int_I \pi_a(\sigma | w) - \pi(\sigma | w) dF_s \neq 0 . \)

Note that the sender’s payoff from \( \pi_a \) is \( U_s(\pi_a) = \int_I \pi_a(\sigma | w) dF_s(w) \). Suppose first that \( \lim_{a \to 0} U_s(\pi_a) = U_s(\pi) \). Note that we must have \( E_{G,\pi_a} [w | \sigma] \geq w^* \) for all \( a > 0 \). This implies that \( \lim_{a \to 0} E_{G,\pi_a} [w | \sigma] = E_{G,\bar{\pi}} [w | \sigma] \geq w^* \), so under the limit signal \( \bar{\pi} \) the receiver with prior \( G \) takes action 1 after the signal realization \( \sigma \). Then \( \bar{\pi} \) is a signal with two realizations such that \( \int_I |\pi_a(\sigma | w) - \bar{\pi}(\sigma | w)| dF_s \neq 0 \) and \( U_s(\bar{\pi}) = U_s(\pi) \) given that the receiver’s prior is \( G \). This contradicts the fact that, because of Assumption 1, \( \bar{\pi} \) is unique up to sets of measure zero under \( F_s \).

Next, suppose that \( \lim_{a \to 0} U_s(\pi_a) \neq U_s(\pi) \). Because \( U_s(\pi_a) < U_s(\pi) \) for all \( a > 0 \), this implies that \( \lim_{a \to 0} U_s(\pi_a) < U_s(\pi) \). We will show that there exists a sequence of signals \( \{\pi_a^0\} \) with two realizations such that \( E_{F,\pi_a^0} [w | \sigma] \geq w^* \) for all \( F \in C_{a,g} \) and \( \lim_{a \to 0} U_s(\pi_a^0) = U_s(\pi) \). Because \( \lim_{a \to 0} U_s(\pi_a) < U_s(\pi) \), this would establish that \( \{\pi_a\}_{a \in (0,1)} \) was not a collection of sender-optimal signals, which is a contradiction.

Let \( \pi_a^0(\sigma | w) = \pi(\sigma | w) - \epsilon_a \) if \( \pi(\sigma | w) = 1 \) and \( w < w^* \), \( \pi_a^0(\sigma | w) = \pi(\sigma | w) = 1 \) if \( w \geq w^* \) and \( \pi_a^0(\sigma | w) = 0 \) if \( \pi(\sigma | w) = 0 \). Choose the minimal \( \epsilon_a > 0 \) such that \( E_{F,\pi_a^0} [w | \sigma] \geq w^* \) for all \( F \in C_{a,g} \). Note that this is feasible because if \( \epsilon_a = 1 \), then \( E_{F,\pi_a^0} [w | \sigma] > w^* \) for all \( F \in C_{a,g} \). Observe that \( \lim_{a \to 0} \epsilon_a = 0 \). This implies that \( \lim_{a \to 0} U_s(\pi_a^0) = U_s(\pi) \), as required. \[ \blacksquare \]

**Lemma 15.** Suppose that \( \Omega = \{0,1,2,3,4\} \) and \( w^* = 2.5 \). Two priors of the receiver are possible: \( f_1 = \left( \frac{1}{4}, \frac{1}{24}, \frac{1}{2}, \frac{5}{24}, 0 \right) \) and \( f_2 = \left( \frac{1}{4}, \frac{1}{2}, \frac{3}{16}, 0, \frac{1}{16} \right) \). The sender’s prior \( f_s \) satisfies \( f_s = f_2 \). Then there does not exist a sender-optimal signal structure with two realizations.

**Proof of lemma 15.**

We first show that the signal structure \( \pi_1 \) with two realizations satisfying \( \pi_1(\sigma | 2) = 1, \pi_1(\sigma | 4) = 1 \) maximizes the sender’s payoff in the setting where the sender and the receiver have a common prior \( f_2 \). Observe that in a setting with a common prior the optimal signal structure is a monotone partitional signal structure with two realizations satisfying \( E_{F_2,\pi_1} [\sigma | w] = w^* \). It can be verified that \( \pi_1 \) is monotone partitional given the prior \( f_2 \) and
that $E_{F_2,\pi_1}[\sigma|w] = \frac{2f_2(2)+4f_2(4)}{f_2(2)+f_2(4)} = \frac{\frac{2}{16}+\frac{1}{16}}{\frac{1}{16}+\frac{1}{16}} = \frac{6+4}{4} = 2.5 = w^*$. Moreover, $\pi_1$ yields a payoff of $f_2(2) + f_2(4) = \frac{3}{16} + \frac{1}{16} = \frac{1}{4}$ to the sender.

Because $\pi_1$ is the optimal signal structure with two realizations conditional on Nature choosing the prior $f_2$, any other signal structure with two realizations (that differs from $\pi_1$ on states $w$ satisfying $f_s(w) > 0$) yields a payoff to the sender conditional on Nature choosing the prior $f_2$ that is strictly lower than $\frac{1}{4}$.

Consider a sender-optimal signal structure $\pi_0$ with two realizations, $\sigma$ and $\sigma_0$. Observe that this signal structure must satisfy $E_{F_1,\pi_0}[\omega|\sigma] \geq w^*$ and $E_{F_2,\pi_0}[\omega|\sigma] \geq w^*$.

Observe next that $\pi_1$ fails the requirement that $E_{F_1,\pi_1}[\omega|\sigma] \geq w^*$ because $E_{F_1,\pi_1}[\omega|\sigma] = \frac{2f_1(2)+4f_1(4)}{f_1(2)+f_1(4)} = \frac{\frac{2}{2}+\frac{4}{2}}{\frac{2}{2}+\frac{2}{2}} = \frac{2+4}{2+2} = \frac{3}{2} < 2.5 = w^*$. Moreover, a signal structure $\pi_2$ with two realizations satisfying $\pi_2(\sigma|w) = 1$ for $w = 2, 3, 4$ also fails the requirement that $E_{F_1,\pi_2}[\omega|\sigma] \geq w^*$ because $E_{F_1,\pi_2}[\omega|\sigma] = \frac{2f_1(2)+3f_1(3)+4f_1(2)}{f_1(2)+3f_1(3)+f_1(4)} = \frac{\frac{2}{2}+\frac{3}{2}+\frac{4}{2}}{\frac{2}{2}+\frac{3}{2}+\frac{1}{2}} = \frac{3+1}{2} < 2.5 = w^*$.

Then we must have $\pi_0(\sigma|w) \neq \pi_1(\sigma|w)$ on some $w$ such that $f_s(w) > 0$. By the argument above, this implies that $\pi_0$ yields a payoff to the sender conditional on Nature choosing the prior $f_2$ that is strictly lower than $\frac{1}{4}$. Therefore, $\pi_0$ yields a payoff to the sender that is strictly lower than $\frac{1}{4}$.

However, the signal structure $\pi$ with three realizations $\sigma_1$, $\sigma_2$ and $\sigma_0$ such that $\pi(\sigma_2|1) = 1$, $\pi(\sigma_2|3) = 1$ and $\pi(\sigma_1|2) = 1$, $\pi(\sigma_1|4) = 1$ guarantees a payoff of $\frac{1}{4}$ to the sender. Therefore, there does not exist a sender-optimal signal structure with two realizations, as required. ■
References


