

## Model Fertility Schedules Revisited: The Log-Multiplicative Model Approach

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This paper reconsiders Coale and Trussell's (1974) specification of model fertility schedules by age. It formally presents model fertility schedules within the framework of categorical data analysis. Specifically, births are assumed to follow an independent Poisson distribution for each age interval of each population. Identification and estimation problems are discussed. It shows that the Coale-Trussell specification corresponds to Goodman's (1979) log-multiplicative model. Following Goodman's algorithm, the paper simultaneously estimates Coale and Trussell's  $v$  (age),  $m$ , and  $M$  through an iterative maximum likelihood procedure. This is demonstrated with the same data that were used in Coale and Trussell's article. The new estimates are superior to those of Coale and Trussell according to an array of conventional goodness-of-fit criteria. © 1991 Academic Press, Inc.

The two-parameter specification of model schedules of marital fertility (Coale and Trussell, 1974) provides a convenient tool for comparing fertility across different populations. The framework of model fertility schedules has been an important advance over such traditional methods as crude or standardized fertility rates in several respects. First, model fertility schedules are based on the well-understood demographic theory (Henry, 1961; Coale, 1971) that marital fertility is the combined result of natural fertility and voluntary control of fertility. Second, the two parameters in model fertility schedules have clear interpretations: one as a scale factor or underlying level of marital fertility, and the other as an index of the degree of voluntary control as an increasing function of age (for a review, see Wilson, Oeppen, and Pardoe, 1988). Finally, model

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fertility schedules facilitate extrapolation of age-specific fertility rates to age intervals in which data are not available.

Coale and Trussell's method places a strong emphasis on treating marital fertility as a function of age. Other important factors, such as marriage duration (Page, 1977), parity specific fertility limitation (David, Mroz, Sanderson, Wachter, and Weir, 1988; Pullum 1990), and proportions of childless and one-child families (Ewbank, 1989) are omitted. This omission makes the Coale-Trussell method attractive in being simple and even applicable when only age-specific grouped data are available (Lively and Freedman, 1990). Another reason for a continuing interest in the Coale-Trussell method is that age can be used as a proxy to capture the effect of marriage duration (Trussell, Menken, and Coale, 1979).

The important idea of characterizing marital fertility by one parameter for fertility level and another parameter for fertility control was initially presented by Coale (1971) and later implemented by Coale and Trussell (1974). Improvements have been made since then. In particular, estimation methods based on the least squares (Coale and Trussell, 1978) and the maximum likelihood (Broström, 1985; Trussell, 1985) principles have been developed. These innovations have made the use of the Coale-Trussell method a common standard among demographers interested in fertility control (e.g., Lively, 1986; Lively and Freedman, 1990). The aforementioned innovations are concerned with estimation of the level parameter ( $M$ ) and the control parameter ( $m$ ) with raw data on fertility and a set of known values decreasing with age ( $v$ ). The  $v$  values were obtained by Coale and Trussell (1974) as simple averages from 43 fertility schedules reported in the United Nations' (1966) *Demographic Yearbook 1965*. The initial estimation of the  $v$  values and their applicability to other populations have not been seriously questioned.

This paper reconsiders Coale and Trussell's (1974) specification of model fertility schedules by age. It formally presents model fertility schedules within the framework of categorical data analysis. As in Broström (1985) and Trussell (1985), births are assumed to follow an independent Poisson distribution for each age interval of each population. Identification and estimation problems are discussed. It shows that the Coale-Trussell specification corresponds to Goodman's (1979) log-multiplicative model. Following Goodman's algorithm, the paper simultaneously estimates Coale and Trussell's  $v$ ,  $m$ , and  $M$  through an iterative maximum likelihood procedure. This is demonstrated with the same data that were used in Coale and Trussell's (1974) article. The new estimates are superior to those of Coale and Trussell according to an array of conventional goodness-of-fit criteria.

## THE COALE-TRUSSELL METHOD

The Coale-Trussell method assumes that the observed marital fertility rate for the  $a$ th age of the  $i$ th population,  $r_{ia}$ , can be modelled by the expected fertility rate,  $R_{ia}$ , in the following way:

$$R_{ia} = n_a \cdot M_i \cdot \exp(m_i \cdot v_a), \quad (1)$$

where  $n_a$  is the standard age pattern of natural fertility,  $M_i$  measures the fertility level of the  $i$ th population,  $m_i$  measures the age-specific fertility control of the  $i$ th population through a typical age pattern  $v_a$ , and  $v_a$  is a set of common values describing the deviation of realized fertility from natural fertility.

The  $n_a \cdot M_i$  product represents the underlying natural fertility level of the  $i$ th population. What Henry (1961) observes in all populations under natural fertility is a common *age pattern*. Let us call  $n_{ia}$  the natural fertility at the  $a$ th age of the  $i$ th population. Henry's observation can be formalized into proportional constraints:  $n_{ia} = n_a \cdot M_i$ . This specification has been supported recently by formal statistical testing in a loglinear analysis (Xie, 1990).

It is worth noting that the Coale-Trussell method builds on natural fertility as the theoretical baseline from which controlled fertility deviates in a log-multiplicative fashion. As shown in Eq. (1), the ratio of controlled fertility to natural fertility for the  $a$ th age is  $\exp(m_i \cdot v_a)$ . Coale and Trussell (1974) did not provide an explicit reason for their specification of  $\exp(m_i \cdot v_a)$  to measure the degree of fertility control. It may well have been the result of a trial-and-error process. However, this specification is a powerful one. It varies in the desirable range between 0 and 1 if  $m_i \cdot v_a$  is kept negative. Taking the natural logarithm on both sides of Eq. (1) changes it into a loglinear expression. Under the assumption that births follow an independent Poisson distribution, the  $M_i$  and  $m_i$  parameters can be estimated in the conventional framework of loglinear analysis (Broström, 1985; Wilson, Oeppen, and Pardoe, 1988). Moreover, it is enormously parsimonious because it uses one population-specific parameter to measure the degree of fertility control.

Coale and Trussell (1974, 1978) utilized the simple method of averaging in their sequential estimation of  $n_a$  and  $v_a$  and then  $M_i$ , and  $m_i$  of Eq. (1). Fertility rates rather than numbers of births were used. This method thus implicitly presumes that data on marital fertility are complete enumerations of populations and consequently that the fertility rates are exact. As such, the Coale-Trussell method is not a statistical model, but a summary description of population data.

Recent developments by Broström (1985), Trussell (1985), and Wilson, Oeppen, and Pardoe (1988) have advanced beyond this limitation of Coale and Trussell's (1974, 1978) original method. The common approach is to

treat fertility data as samples but not as populations. Births are viewed as realizations of random variables distributed as Poisson. Sampling variability, which is an inverse function of sample size, is taken into serious consideration. Observed births, rather than rates, are used in estimation. As a result, the Coale-Trussell formulation of Eq. (1) has become a statistical model; to be more precise, a Poisson regression model. Also common to the new developments is the application of the conventional estimation method—maximum likelihood. Standard errors of the estimates are now routinely reported in applied work (e.g., Lavelly and Freedman, 1990).<sup>1</sup> This is in sharp contrast to earlier applications (e.g., Lavelly, 1986). In the following, we briefly summarize the work of Broström (1985), Trussell (1985), and Wilson, Oeppen, and Pardoe (1988).

Let us call  $b_{ia}$  the number of observed births,  $B_{ia}$  the number of expected births under some model,  $T_{ia}$  the total women-years at risk of giving birth (exposure), at the  $a$ th age in the  $i$ th sample, which represents the  $i$ th population. The number of births thus can be seen as the product of exposure and fertility rate:

$$b_{ia} = T_{ia} \cdot r_{ia}; B_{ia} = T_{ia} \cdot R_{ia}. \quad (2)$$

Substituting Eq. (1) into Eq. (2), we obtain

$$B_{ia} = T_{ia} \cdot n_a \cdot M_i \cdot \exp(m_i \cdot v_a). \quad (3)$$

Taking the natural logarithm on both sides changes Eq. (3) into

$$\log(B_{ia}) = \log(T_{ia} \cdot n_a) + \log(M_i) + m_i \cdot v_a, \quad (4)$$

where  $T_{ia} \cdot n_a$  can be interpreted as the number of births expected under the natural fertility standard. In Eq. (4),  $\log(T_{ia} \cdot n_a)$  is taken as known and is included as a control. Assuming that observed  $b_{ia}$  follows an independent Poisson distribution, Broström (1985), Trussell (1985), and Wilson, Oeppen, and Pardoe (1988) propose to estimate Eq. (4) through maximum likelihood for any given sample. This is conditional on prior knowledge of  $n_a$  and  $v_a$ . As a convention,  $n_a$  and  $v_a$  are normally taken from Coale and Trussell (1974, 1975). The  $n_a$  values are used to calculate expected births under the natural fertility standard; the  $v_a$  values are used as a predictor in a regression. Thus  $\log(M_i)$  and  $m_i$  are estimated respectively as the constant and the slope parameters in the loglinear regression model specified by Eq. (4).

The new developments, however, are restricted to better estimation of the  $M_i$  and  $m_i$  parameters. The  $n_a$  and  $v_a$  values of Coale and Trussell (1974, 1975) are accepted as exactly known. Coale and Trussell's  $n_a$  values are simple age-specific averages of the ten natural fertility schedules re-

<sup>1</sup> There are two ways to compute standard errors. One is based on the asymptotic properties of maximum likelihood estimates. The other is through the jackknife technique.

ported by Henry (1961) that are believed to be reliable.<sup>2</sup> Recently, based on the same classical data on natural fertility, Xie (1990) reestimates the standard age pattern of natural fertility, the  $n_a$  values, through an explicit loglinear model that simultaneously describes births by population and age. Xie's (1990) results confirm Henry's (1961) hypothesis that populations under natural fertility have the same age pattern with different levels of fertility. Furthermore, Xie (1990) demonstrates the superiority of the new  $n_a$  values and recommends their replacement of the traditional  $n_a$  values estimated by Coale and Trussell (1974).

Coale and Trussell (1974, 1975) obtained their  $v_a$  values also by the averaging method. Specifically, let  $m_i$  in Eq. (1) be 1,  $r_{ia}$  be  $R_{ia}$ , and  $v_a$  vary with  $i$ . Solving Eq. (1) for  $v_{ia}$ , we have

$$v_{ia} = \log(r_{ia}/(M_i \cdot n_a)). \quad (5)$$

Using the 43 fertility schedules reported in the United Nations' (1966) *Demographic Yearbook 1965*, Coale and Trussell (1974, 1975) calculated the average for each 5-year age interval across the 43 schedules. The reason for setting  $m_i$  to 1 in Eq. (5) is for normalization, since it is not possible to identify scales of both  $m_i$  and  $v_a$ . To see this, define a new  $v_a$ , called  $v_a^*$ , by multiplying a constant, say  $c$ , to  $v_a$ :

$$v_a^* = c \cdot v_a. \quad (6)$$

This can leave Eq. (1) intact if we redefine the  $m_i$  parameter so that

$$m_i^* = m_i/c. \quad (7)$$

Thus the scales of  $v_a$  and  $m_i$  cannot be jointly determined. Also for normalization purposes, Coale and Trussell set  $v_a$  for the first age interval (20–24) to be zero. This is necessary because the scale of  $M_i$  and the location of  $v_a$  cannot be jointly identified:

$$\begin{aligned} R_{ia} &= n_a \cdot M_i \cdot \exp(m_i \cdot (v_a + c)) \\ &= n_a \cdot M_i \cdot \exp(m_i \cdot c) \cdot \exp(m_i \cdot v_a) \\ &= n_a \cdot M_i^* \cdot \exp(m_i \cdot v_a), \end{aligned} \quad (8)$$

where  $M_i^*$  is redefined as  $M_i \cdot \exp(m_i \cdot c)$ .

The  $M_i$  parameter is often interpreted as representing the level of natural fertility or the extent of birth spacing (e.g., Lavelly, 1986; Wilson, Oeppen, and Pardoe, 1988). This interpretation should be qualified.<sup>3</sup> Under the normalization of Coale and Trussell (1974),  $M_i$  measures the level of natural fertility or the extent of birth spacing *only if* married women

<sup>2</sup> Henry (1961) adjusted fertility rates for two populations (see Wilson, Oeppen and Pardoe, 1988, p. 10). However, the effect of the adjustment is minor (Xie, 1990).

<sup>3</sup> A similar critique is made by Ewbank (1989).

TABLE 1  
Comparison of Estimated Parameters

Parameter	Age					
	20-24	25-29	30-34	35-39	40-44	45-49
Panel A: old estimates (Coale and Trussell, 1974, 1974)						
$n_a$	0.460	0.431	0.395	0.322	0.167	0.024
$v_a$	0.000	-0.279	-0.677	-1.042	-1.414	-1.671
Panel B: new estimates						
$n_a$ (Xie, 1990)	0.460	0.436	0.392	0.333	0.199	0.043
$v_a$ (from Model A3, Table 2)	0.000	-0.320	-0.787	-1.216	-1.657	-1.671
$v_a$ (from Model B3, Table 2)	0.000	-0.228	-0.533	-0.856	-1.279	-1.671

Note. New estimates of  $v_a$  are estimated from models reported in Table 2. Also, see text for an explanation.

in the first age category (20-24) do not practice fertility control to stop having children. Coale and Trussell's normalization solution is such that the  $m_i \cdot v_a$  product is always zero for the first age group (20-24), which is the age group least likely to be affected by the fertility control behavior. That  $m_i \cdot v_a$  is equal to zero for ages 20-24 is a result of normalization, whether or not fertility control is present for this age group. If stopping is present for the first age group (20-24), as is likely to be true in many modern populations, the  $M_i$  parameter cannot be interpreted as measuring the level of natural fertility.

The  $v_a$  and  $n_a$  values estimated by different methods are reported in Table 1. Panel A presents the original estimates of Coale and Trussell (1974, 1975); Panel B presents the new estimates. The new  $n_a$  estimates are taken from Xie (1990). This paper is focused on the new estimation of  $v_a$  values.

### THE LOG-MULTIPLICATIVE APPROACH

A more fruitful way of considering the Coale-Trussell method is to view  $v_a$  not as a set of known values, but as a set of parameters to be estimated. Until now we have purposely used the term "values" and avoided the term "parameters" when referring to  $v_a$ . Within the traditional approach of the Coale-Trussell method in the Poisson regression form,  $v_a$ 's are always assumed to be exactly known and free from estimation. In contrast, this paper proposes to treat  $v_a$ 's as unknown parameters, along with  $M_i$  and  $m_i$ , to be estimated from controlled fertility data.

If we treat  $v_a$ 's as unknown parameters, Eq. (4) becomes the "log-multiplicative" model (Clogg, 1982). It is "log-multiplicative" because the age effect ( $v_a$ ) and the population effect ( $m_i$ ) on the logged births

are multiplicative. This model has been extensively discussed by Goodman (1979), referred to as "Goodman's Association Model II," or simply RC (row and column) Model. We demonstrate the model with the empirical data used by Coale and Trussell (1974).<sup>4</sup> Let us form two sets of frequencies,  $b$  and  $T$ , in two  $43 \times 6$  two-way tables indexed by  $i$  ( $i = 1, \dots, 43$ ) for  $P$  (population) and  $a$  ( $a = 1, \dots, 6$ ) for  $A$  (age). As in Coale and Trussell (1974), we use six 5-year age intervals (20–24, 25–29, 30–34, 35–39, 40–44, 45–49).

Unfortunately, the data used by Coale and Trussell (1974) were reported by the United Nations (1966) in the form of rates. We are unable to reconstruct the data into births and women at risk of giving birth. Lacking necessary information about sample sizes, we choose to give an equal base of 10,000 to all rates. That is, we assume  $T_{ia} = 10,000$ . From this, we derive  $b_{ia}$  as

$$b_{ia} = T_{ia} \cdot r_{ia} = 10,000 \cdot r_{ia}. \quad (9)$$

This assumption is obviously unrealistic. The sample sizes for different age–population combinations were surely different from each other and from 10,000, which is an arbitrary number. Changing 10,000 to another arbitrary number would affect statistical inferences, but not point estimates. The significance of the assumption in Eq. (9) is to assign an equal weight to all age intervals of all populations. This is reasonable in the absence of additional information.<sup>5</sup> Concerning statistical inference, we need to be cautious because an arbitrary number of 10,000 is used as the common base of rates.

The log–multiplicative model is a general method for analyzing a cross-classified table of two ordinal variables. Several pioneering papers on this method by Leo Goodman and Clifford Clogg are collected in Goodman's (1984) book. Readers interested in technical details should consult the book and textbooks by Agresti (1984, 1990). One important property of Goodman's Association Model II is that the correct order either of the row categories or of the column categories is not assumed *a priori* (Goodman, 1979; Clogg, 1982). The model implicitly assumes the existence of an order both of the row categories and of the column categories. But the categories can be arbitrarily ordered before Goodman's Association Model II is applied. The model itself reveals the orders of the categories through estimating association parameters. This property makes Good-

<sup>4</sup> We are grateful to James Trussell at Princeton University for providing the data used by Coale and Trussell (1974).

<sup>5</sup> We contemplated but finally abandoned the assumption that  $T_{ia}$  is proportional to the number of women in the age–population classification because  $T_{ia}$  refers to the *sample* size, not the population size.

man's Association Model II analogous to canonical correlation analysis (Goodman, 1981).

It can be proven that the Coale-Trussell method is essentially the same as Goodman's Association Model II. From Eq. (3), the odds-ratio of a  $2 \times 2$  subtable of adjacent  $P$  (population) categories and adjacent  $A$  (age) categories can be written as

$$\begin{aligned}\Theta_{ia} &= (B_{ia} \cdot B_{i+1,a+1}) / (B_{i,a+1} \cdot B_{i+1,a}) \\ &= \frac{T_{ia} \cdot n_a \cdot M_i \cdot \exp(m_i \cdot v_a) \cdot T_{i+1,a+1} \cdot n_{a+1} \cdot M_{i+1} \cdot \exp(m_{i+1} \cdot v_{a+1})}{T_{i,a+1} \cdot n_{a+1} \cdot M_i \cdot \exp(m_i \cdot v_{a+1}) \cdot T_{i+1,a} \cdot n_a \cdot M_{i+1} \cdot \exp(m_{i+1} \cdot v_a)} \\ &= S_{ia} \cdot \exp(m_i \cdot v_a + m_{i+1} \cdot v_{a+1} - m_i \cdot v_{a+1} - m_{i+1} \cdot v_a),\end{aligned}\quad (10)$$

where  $S_{ia}$  is a constant factor for exposure:

$$S_{ia} = (T_{ia} \cdot T_{i+1,a+1}) / (T_{i,a+1} \cdot T_{i+1,a}). \quad (11)$$

Taking the logarithm on both sides of Eq. (10) gives

$$\log(\Theta_{ia}) = \log(S_{ia}) + (m_{i+1} - m_i)(v_{a+1} - v_a). \quad (12)$$

This is of the same form as Goodman's Association Model II (see Goodman, 1979, 1981; Clogg, 1982).<sup>6</sup>

## RESULTS

With the modification of the data by Eq. (9), we reanalyze the same data used by Coale and Trussell (1974). We focus on three models. The models are (1) Natural Fertility Model, (2) Traditional Coale-Trussell Model, and (3) Log-Multiplicative Model. We will first describe the three models before we interpret the empirical results.

### *Natural Fertility Model*

The Natural Fertility Model assumes no fertility control in the 43 populations. From Coale and Trussell's (1974) work, and in fact from common sense, we know that this model is an unrealistic one. But this model is still useful because we use it as our baseline model. Furthermore, it will be reassuring to reconfirm earlier observations.

Should the Natural Fertility Model be true, all  $m_i$ 's in Eq. (4) would be zero. That is, the expected births for the  $a$ th age in the  $i$ th population,  $B_{ia}$ , are determined solely by the exposure ( $T_{ia}$ ), a common natural fertility age pattern ( $n_a$ ), and the level of fertility for the  $i$ th population ( $M_i$ ).

<sup>6</sup>  $\log(S_{ia})$  here is inconsequential because it is a controlled factor in estimation.



### *Traditional Coale-Trussell Model*

The Traditional Coale-Trussell Model utilizes the Coale-Trussell formulation to account for fertility control as expressed by the  $m_i \cdot v_a$  product in Eq. (4). Thus the Traditional Coale-Trussell Model is nested with the Natural Fertility Model. The model is characterized by its reliance on the  $v_a$ 's estimated by Coale and Trussell (1974) and its treatment of the 43 populations separately, as if they were unrelated. For each population,  $M_i$  and  $m_i$  are estimated in a single Poisson regression model by using  $v_a$  as the independent variable (Broström, 1985; Trussell, 1985). Test statistics for the goodness of fit are summed over the 43 separately estimated models.

Note that the assumption that  $v_a$  can be used as the independent variable is difficult to justify, particularly with the data in question. The 43 populations are related through  $v_a$ 's, which should be parameters to be estimated, but not predetermined variables. Because there are six age categories and two normalization constraints, we know that four degrees of freedom are used in estimating  $v_a$ 's. In reporting the degrees of freedom for the Traditional Coale-Trussell Model, we deduct four degrees of freedom.<sup>7</sup>

### *Log-Multiplicative Model*

The Log-Multiplicative Model is similar to the Traditional Coale-Trussell Model. The difference is that the former treats  $v_a$ 's in Eq. (4) as parameters to be estimated simultaneously with  $M_i$  and  $m_i$  parameters, whereas the latter treats  $v_a$ 's as known variables. This difference is significant. The estimated  $v_a$ 's from the Log-Multiplicative Model should be superior to the old estimates that have been in use. In addition, the Log-Multiplicative Model, as a general approach, allows more flexible applications of the Coale-Trussell method. For example, we may pool fertility data from homogeneous populations to test whether we should use different sets of  $v_a$ 's for different groups of populations.

The Log-Multiplicative Model implementation of the Coale-Trussell formulation as shown in Eq. (4) is slightly different from the normal case discussed by Goodman (1979) and Clogg (1982) in two respects. First, there is a known factor of  $\log(T_{ia} \cdot n_a)$ , which can be easily controlled in estimation.<sup>8</sup> Second, there are no marginal effects of age. If we arrange the table so that column represents age, there are no marginal column

<sup>7</sup> Normally, four degrees of freedom (six minus two, six for the number of age categories and two for the  $M_i$  and  $m_i$  parameters) are reported for each population. The total degrees of freedom for the 43 populations would be 172 if we did not deduct the four degrees of freedom for estimating  $v_a$ 's.

<sup>8</sup> Following Broström (1985), we use the OFFSET command in GLIM to control for this factor.

effects. This is so because the Coale–Trussell specification of Eq. (4) aims at explaining the age pattern of fertility by the standard natural fertility age pattern ( $n_a$ ) and the fertility control age pattern ( $v_a$ ). One consequence of this is an increase in the degrees of freedom. A normal Goodman's Association Model II would leave 164 degrees of freedom.<sup>9</sup> In our case, of 43  $m_i$ 's and 6  $v_a$ 's, 47 are free to be estimated. To be consistent with the traditional practice of giving a unique  $m_i$  parameter to each population, we add two normalization constraints on  $v_a$ . After Coale and Trussell (1974), we constrain the location of  $v_a$  by fixing  $v_1$  at 0. Furthermore, we constrain the scale of  $v_a$  by fixing  $v_6$  at  $-1.671$ . The value of  $-1.671$  was estimated by Coale and Trussell (1974, 1975) for the last age category. We use it to normalize the scale of  $v_a$  so that the resulting  $m_i$ 's in our model will be comparable to  $m_i$ 's estimated by other implementations of the Coale–Trussell method. Since four degrees of freedom are used to estimate the  $v_a$  parameters, our Log–Multiplicative Model has 168 degrees of freedom.<sup>10</sup>

### *Empirical Results*

The three models can be estimated by assuming that observed births for each age interval in each population follow a Poisson distribution. The Poisson assumption is reasonable in the analysis of fertility. First, births are discrete counts. Second, the Poisson distribution presumes larger variance at higher fertility. Similar justifications are provided by Broström (1985), Rodriguez and Cleland (1988), and Wilson, Oeppen, and Pardoe (1988).

With the Poisson assumption, the three models are estimated according to the maximum likelihood principle.<sup>11</sup> The goodness of fit of the models can be measured by the log-likelihood ratio test statistic,  $L^2$ . Asymptotically,  $L^2$  approaches the chi-square distribution with the degrees of freedom equal to the difference between the number of cells and the number of parameters fitted (Bishop, Fienberg, and Holland, 1975; pp. 125–130). Furthermore, the difference in  $L^2$  between two nested models asymptotically follows a chi-square distribution with the degrees of freedom equal to the difference in the degrees of freedom. However, it is well known that with large samples the log-likelihood ratio test is likely to reject a good model. With our data, this problem is particularly acute because

<sup>9</sup>  $(I - 2)(J - 2) = (6 - 2)(43 - 2) = 164$ .

<sup>10</sup> There are altogether 258 cells in the  $P \times A$  table. We use 86 degrees of freedom to estimate the  $M_i$  and  $m_i$  parameters and 4 degrees of freedom to estimate the  $v_a$  parameters. Therefore,  $258 - 86 - 4 = 168$ .

<sup>11</sup> The implementation of the maximum likelihood estimation in this paper is through a GLIM macro kindly provided by Mark Becker of the Department of Biostatistics, University of Michigan. All models in this paper were estimated with the general computer program GLIM Release 3.77 (Baker et al., 1987).

TABLE 2  
Models for Comparative Fertility

Model	Description	$L^2$	$X^2$	$DF$	$D$	$BIC$
Panel A: old natural fertility standard (Coale and Trussell 1974) is used						
A1:	Natural Fertility	66,948	63,597	215	16.29%	64,170
A2:	Traditional Coale-Trussell	2,121	2,439	168	2.04%	-49
A3:	Log-Multiplicative	2,039	2,254	168	2.00%	-131
Panel B: new natural fertility standard (Xie, 1990) is used						
B1:	Natural Fertility	80,548	74,722	215	17.56%	77,770
B2:	Traditional Coale-Trussell	1,780	1,692	168	2.04%	-390
B3:	Log-Multiplicative	1,395	1,421	168	1.87%	-775

Note.  $L^2$  is the log-likelihood ratio chi-square statistic, and  $X^2$  is the Pearson chi-square statistic, both with the degrees of freedom reported in column  $DF$ .  $D$  is the index of dissimilarity.  $BIC = L^2 - (DF) \log(N)$ , where  $N$  is the total number of births (408,032). See text for explanation of the three models.

we arbitrarily assign 10,000 to each cell of the age-population classifications due to the lack of information about actual sample sizes. Raftery (1986) proposes a Bayesian statistic,  $BIC$ , for large samples:  $BIC = L^2 - (DF) \log N$ , where  $L^2$  is the log-likelihood ratio statistic,  $DF$  is the associated degrees of freedom, and  $N$  is the sample size. If  $BIC$  is negative, we should accept the null hypothesis. When comparing several models, we should select the model with the lowest  $BIC$  value. Another test statistic for the goodness of fit is the Pearson chi-square statistic ( $X^2$ ), which is the sum of all squared relative differences between observed and predicted births:  $X^2 = \sum (b_{ia} - B_{ia})^2 / B_{ia}$ . As a purely descriptive measurement of the goodness of fit, we also use the Index of Dissimilarity, denoted as  $D$ , which can be interpreted as the proportion of misclassified births.

In Table 2, we compare the goodness-of-fit statistics for three models. There are two sets of parallel models in Table 2, as shown in two panels. Models in Panel A are based on Coale and Trussell's (1974) natural fertility standard, while models in Panel B take advantage of the newly estimated natural fertility standard (Xie, 1990).

Test statistics for the goodness of fit in Table 2 give strong support to rejection of the Natural Fertility Model. In both Panels A and B, the log-likelihood ratio chi-square statistic ( $L^2$ ) of the model is enormous (66,948 and 80,548, respectively) compared to its degrees of freedom (215). So is the Pearson chi-square statistic  $X^2$  (63,597 and 74,722). Large percentages (16.29% and 17.56%) of births are misclassified. Furthermore, the model fits the data poorly even by the  $BIC$  criterion, which

adjusts for the large sample size. This result reconfirms the well-known fact that these 43 populations were not governed solely by natural fertility. Fertility control was exercised.

The Traditional Coale–Trussell Model remarkably improves a model's fit to the data. The log-likelihood ratio chi-square statistic ( $L^2$ ) is reduced by 96.8% (from 66,948 to 2,121) for Panel A and by 97.8% (from 80,548 to 1,780) for Panel B. This tremendous improvement is achieved at the expense of only 47 degrees of freedom. A similar story holds true with the Pearson chi-square statistic. The percentage of misclassifications drops to 2.04% for both panels. According to the *BIC* criterion, the Traditional Coale–Trussell Model fits the data well. The *BIC* statistic is  $-49$  for Panel A and  $-390$  for Panel B. These results demonstrate Coale and Trussell's (1974) wisdom in providing a powerful and parsimonious model.

The Log–Multiplicative Model fits the data better than the Traditional Coale–Trussell Model. In Panel A, the  $L^2$ ,  $X^2$  statistics of Model A3 are smaller (2,039 and 2,254) than those of Model A2 (2,121 and 2,439) for the same degrees of freedom. The reduction in misclassified births is 0.04% (from 2.04% to 2.00%). The *BIC* statistic changes from  $-49$  to  $-131$ , indicating that Model A3 should be preferred to Model A2. The improvement in the goodness of fit of the Log–Multiplicative Model is even more salient in Panel B, where the new natural fertility standard of Xie (1990) is used: the  $L^2$  and  $X^2$  statistics change from 1,780 and 1,692 in Model B2 to 1,395 and 1,421 in Model B3. Misclassifications ( $D$ ) fall from 2.04% to 1.87%. Furthermore, the *BIC* statistic (from  $-390$ ) reaches a far larger negative value of  $-775$ .

The results also show the advantage of using Xie's (1990) newly estimated  $n_a$  standard. For the same degrees of freedom (168), Model B2 fits the data better than Model A2. And Model B3 fits better than Model A3. This is true for all the statistics of  $L^2$ ,  $X^2$ ,  $D$ , and *BIC*.<sup>12</sup> Here we would like to reiterate Xie's (1990) recommendation that the new natural fertility standard for the age pattern should replace the old standard of Coale and Trussell (1974). They are both reproduced in Table 1.

It should be noted that Models A2, A3, B2, and B3 are essentially identical in their mathematical expression of the Coale–Trussell formulation. What distinguishes them is their statistical estimation methods. The Log–Multiplicative Model is superior to the Traditional Coale–Trussell Model in that the former estimates the  $v_a$  parameters simultaneously, in an iterative fashion, along with  $M_i$  and  $m_i$  parameters, whereas the latter takes simple averages as estimates. Model B3 fits the data better than Model A3 because Xie's (1990) new estimates of the standard age pattern of natural fertility are more accurate.

<sup>12</sup> Models B2 and A2 appear to have the same value of  $D$  due to rounding errors. Actually, Model B2 has a slightly smaller  $D$  than Model A2.

## CONCLUSION

The preceding findings have two implications for future researchers using the Coale-Trussell method. First of all, the researchers may wish to reestimate the  $v_a$  parameters when comparing fertility of multiple populations, especially of homogeneous populations. A minor disadvantage of doing this is that the resulting  $M_i$  and  $m_i$  are no longer truly comparable to a common standard, for the scales of  $M_i$  and  $m_i$  depend on values of  $v_a$ . The second implication is that the researchers may still apply the Coale-Trussell method to single populations with the new  $v_a$  estimates reported in this paper and the newly estimated natural fertility standard (Xie, 1990).

This paper arbitrarily assigns the number of 10,000 to each age-population classification as the total exposure. The effect of this assumption is to give each an equal weight, and we are aware that this assumption cannot be correct. Observed rates for some of the 43 populations are more reliable than others. Our residual analysis, unreported here, confirms this. However, giving each classification an equal weight is the best that we could do without knowing more about the actual sample sizes. Future research can apply the same framework to more recent and more detailed data, which will almost surely yield better estimates of the  $v_a$  parameters. We shall welcome such endeavors.

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